Chapter Ten

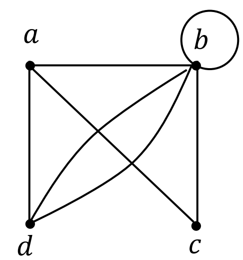
Introduction to Graph Theory

Reference: Discrete Mathematics and its Applications, by, Kenneth H. Rosen, **6th** Edition, McGrow-Hill Education, 2007.

10.1 Introduction to Graphs

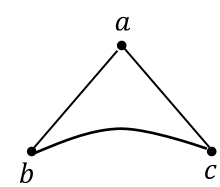
Basic Graph Terminology

Definition: A **graph** $G = (V, E)$ consists of V, nonempty set of **vertices** (or nodes) and E, a set of **edges**. Each edge has either one or two vertices associated with it, called its **endpoints**. An edge is said to **connect** its endpoints.



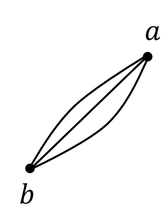
We write the edge e as $\{u, v\}$, which is the edge between the two vertices u and v. Note that, the notations $\{u, v\}$ and $\{v, u\}$ are the same.

Definition: A **simple graph** is a graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices.



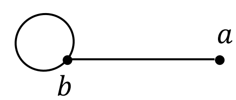
In another word, If there is an edge $\{u, v\}$ between the vertices u and v in a simple graph G, then there is no other edge is associated with these two vertices. Which means, each pair of vertices has exactly one edge is associated with.

Definition: A **multiple edge** is more than one edge connecting the same pair of vertices in a graph. If there are m different multiple edges associated with same unordered pair of vertices $\{u, v\}$, then we say that the edge $\{u, v\}$ has multiplicity m.

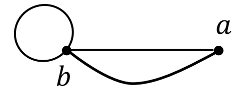


Definition: A **multigraph** is a graph that has multiple edges.

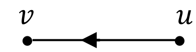
Definition: A **loop** is an edge that connects a vertex to itself.



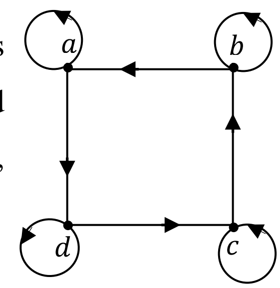
Definition: A **pseudograph** is a graph contains loops and possibly multiple edges



Definition: When the edge **starts** at the vertex u and **ends** at vertex v is called **directed edge**, and denoted by (u, v).



Definition: **Directed graph** (or a digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. Directed graphs can be simple graphs, multigraphs or pseudographs.

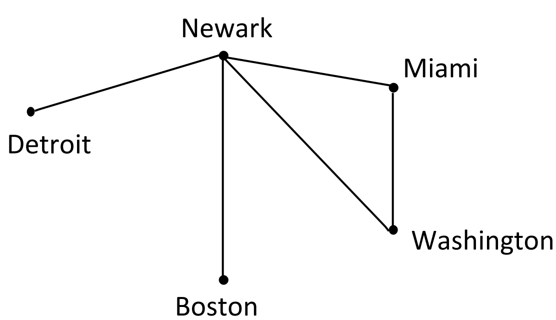


Example 1: Draw graph models, stating the type of graph used, to represent airline routes where every day there are four flights from Boston to Newark, two flights from Newark to Boston, three flights from Newark to Miami, two flights from Miami to Newark, one flight from Newark to Detroit, two flights from Detroit to Newark, three flights from Newark to Washington, two flights from Washington to Newark, and one flight from Washington to Miami, with

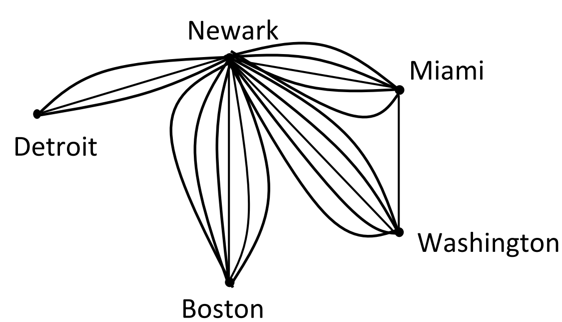
a. an edge between vertices representing cities that have a flight between them (in either direction).
b. an edge between vertices representing cities for each flight that operates between them (in either direction).
c. an edge between vertices representing cities for each flight that operates between them (in either direction). Plus a loop for a special sightseeing trip takes off and lands in Miami.
d. an edge from a vertex representing a city where a flight starts to the vertex representing the city where it ends.
e. an edge for each flight from a vertex representing a city where the flight begins to the vertex representing the city where the flight ends.

Solution:

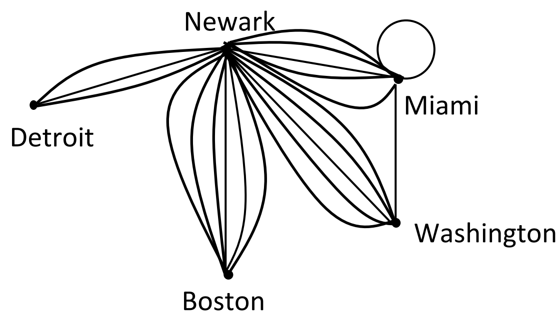
a. The type of the graph is undirected simple graph



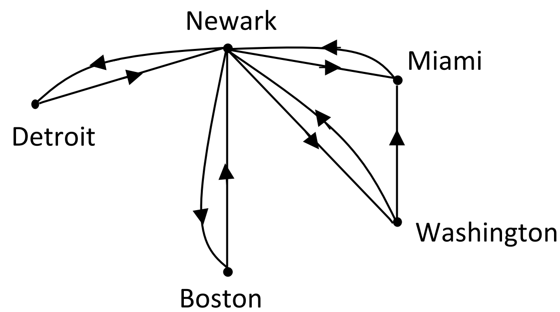
b. The type of the graph is undirected multigraph.



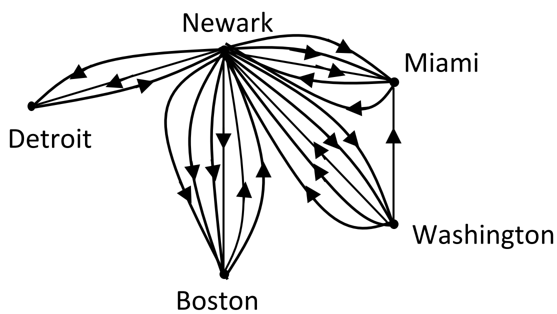
c. The type of the graph is undirected pseudograph.



d. The type of the graph is directed multigraph.



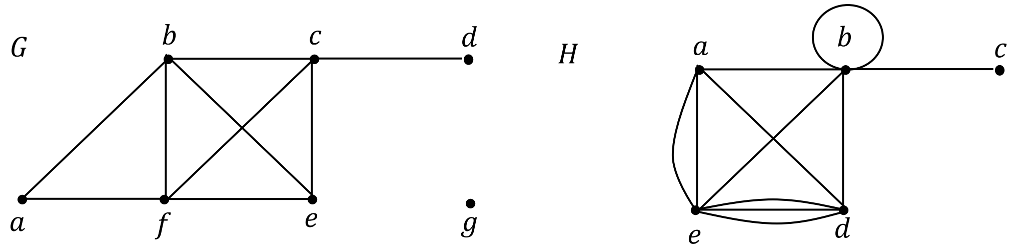
e. The type of the graph is directed multigraph



Definition: Two vertices u and v in an undirected graph G are called **adjacent** (or **neighbors**) in G if u and v are endpoints of an edge of G. If e is associated with $\{u, v\}$, the edge e is called **incident** with the vertices u and v. The edge e is also said to **connect** u and v. The vertices u and v are called **endpoints** of an edge associated with $\{u, v\}$.

Definition: The **degree of a vertex in undirected graph** is the number of edges incident with it, except that a loop at a vertex contributions twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

Example 2: What are the degrees of the vertices in the graphs G and H displayed in Figure below?



Solution: In G, $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$, $\deg(e) = 3$, and $\deg(g) = 0$. In H, $\deg(a) = 4$, $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, and $\deg(d) = 5$.

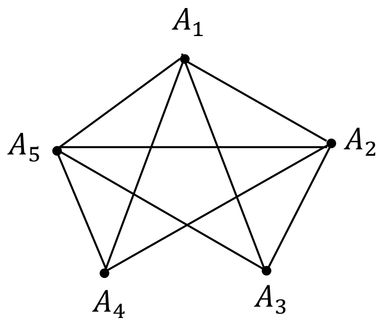
Definition: A vertex of degree zero is called **isolated**. A vertex is **pendent** if and only if it has degree one.

Definition: The **intersection graph** of a collection of sets $A_1, A_2, \dots, A_n$ is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these two sets have a nonempty intersection.

Example 3: Construct the intersection graph of these collections of sets.

$A_1 = \{\dots, -4, -3, -2, -1, 0\}$, $A_4 = \{\dots, -5, -3, -1, 1, 3, 5\}$,
$A_2 = \{\dots, -2, -1, 0, 1, 2, \dots\}$, $A_5 = \{\dots, -6, -3, 0, 3, 6, \dots\}$
$A_3 = \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$,

Solution:



Theorem 1: **THE HANDSHAKING THEOREM** Let $G = (V, E)$ be an undirected graph with e edges. Then

$$2e = \sum_{v \in V} \deg(v)$$

Example 4: How many edges are there in a graph with 10 vertices each of degree six?

Solution: We have 10 vertices each of degree 6, therefore $\sum_{v \in V} \deg(v) = 6 \cdot 10 = 60$. By applying the theorem above we find that the total number of edges is $e = 30$.

Theorem 2: An undirected graph has an even number of vertices of odd degree.

Definition: When (u, v) is an edge of the grapg G with directed edges, u is said to be **adjacent to** v and v is said to be **adjacent from** u. The vertex u is called the **initial vertex** of (u, v), and v is called the **terminal** or the **end vertex** of (u, v). The initial vertex and the terminal vertex of a loop are the same.

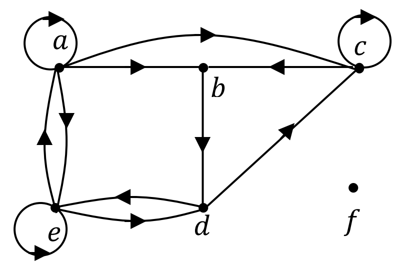
Definition: In a graph with directed edges, the **in-degree of a vertex** v, denoted by $\deg^{-}(v)$, is the number of the edges with v as their terminal vertex. The **out-degree of** v, denoted by $\deg^{+}(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

Theorem2: Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|.$$

Example 5: Find the in-degree and out-degree of each vertex in the directed graph G in the figure here.

Solution: The in-degrees in G $\deg^{-}(a) = \deg^{-}(b) = 2$, $\deg^{-}(c) = 3$, $\deg^{-}(d) = 2$, $\deg^{-}(e) = 3$, $\deg^{-}(f) = 0$. The out-degrees in G $\deg^{+}(a) = 4, \deg^{+}(b) = 1$, $\deg^{+}(c) = 2$, $\deg^{+}(d) = 2$, $\deg^{+}(e) = 3$, $\deg^{+}(f) = 0$.



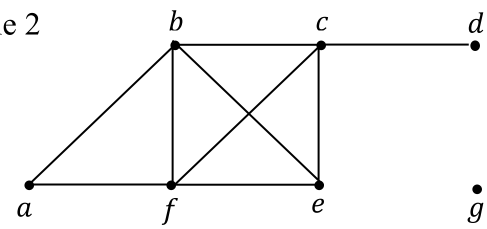
G

Definition: The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in a nonincreasing order.

Note: We use "non-increasing order" not "decreasing order" because the non-increasing means that the elements could be less than or equal the element before it. While decreasing means the element is less than the element before it.

Example 6: The degree sequence of the graph G in example 2 is 4,4,4,3,2,1,0

G



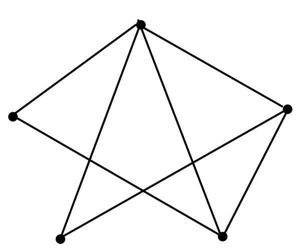
Definition: A sequence $d_1, d_2, \dots, d_n$ is called **graphic** if it is the degree sequence of a simple graph.

Example 7: Draw the graphs of the following degree sequences and determine whether or not these sequences are graphics.

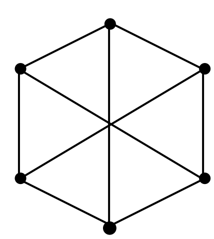
(a) 4,3,3,2,2 (b) 3,3,3,3,3,3

Solution:

(a) Is a graphic



(b) Is a graphic



Exercises 10.1

1. Determine the type of the graph for each of the following graphs.

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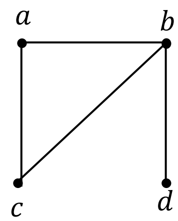
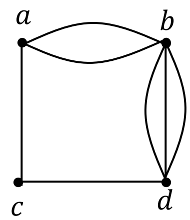
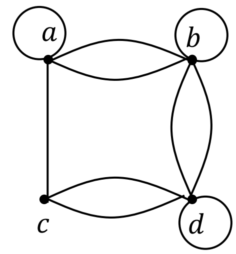
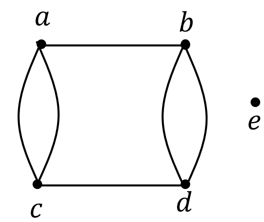
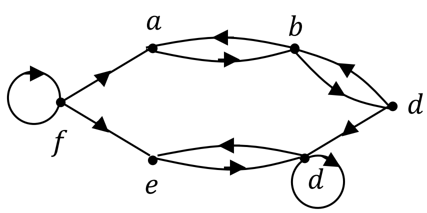
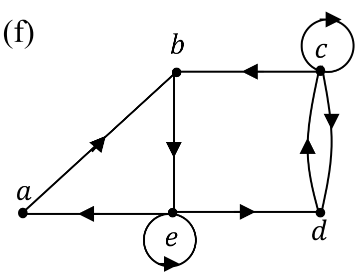


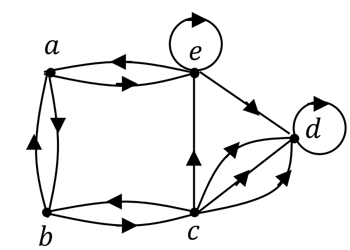
(b)


(c)


(d)


(e)


(f)


(g)


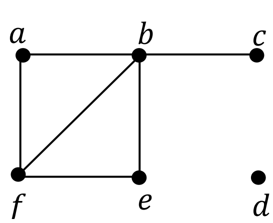
2. For each undirected graph in exercise 1 that is not simple, find a set of edges to remove to make it simple.

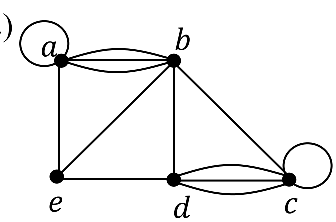
3. Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

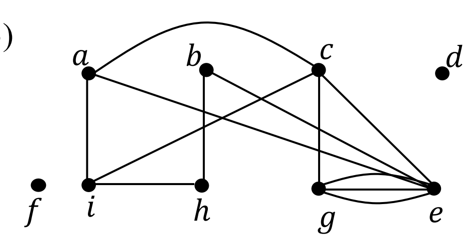
4. Let G be an undirected graph with loop at every vertex.. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, reflexive relation on G.

5. Draw the call graph for the telephone numbers (1) $555 - 0011$, (2) $555 - 1221$, (3) $555 - 1333$, (4) $555 - 8888$, (5) $555 - 2222$, (6) $555 - 0091$, and (7) $555 - 1200$. If there were three calls from the first number to the fourth number and two calls from fourth number to the first number, two calls from the fifth number to the sixth number, two calls from the second number to each of the other numbers, and one call from third number to the first, second, and seventh numbers.

6. In each of the following graphs, find the number of the vertices and the edges and the degree of each vertex. Identify the isolated and the pendant vertices.

(1)


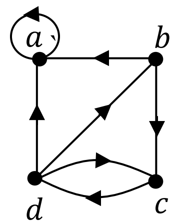
(2)


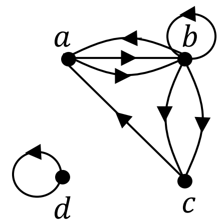
(3)


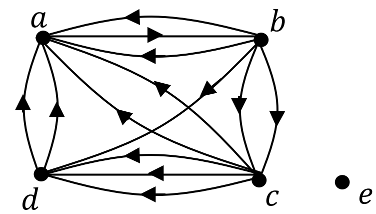
7. Find the sum of the degrees of the vertices of each graph in exercise (6) and verify that it equals twice the number of edges in the graph.

8. Can a simple graph exist with 15 vertices each of degree five?

9. In each of the following graphs, determine the number of vertices and edges and fine the in-degree and out-degree of each vertex.

(1) 

(2) 

(3) 

10. For each graph in exercise (9), determine the sum of the in-degrees of the vertices and the sum of the out-degrees of the vertices directly. Show that they are both equal to the number of the edges in the graph.

11. Construct the underlying undirected graph for the directed graph in example 4.

12. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree. (Mathematical induction proof can be used here.)

13. How many edges does a graph have if its degree sequence is 5,2,2,2,2,1? Draw such a graph.

14. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

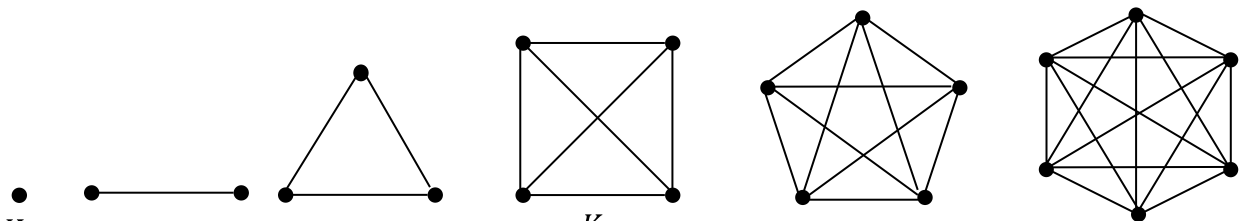
a. 3,3,3,3,2	c. 4,4,3,2,1	e. 3,2,2,1,0
b. 5,4,3,2,1	d. 4,4,3,3,3	f. 1,1,1,1,1

10.2Some Special Types of Graphs

1. Complete Graphs

Definition: The **complete graph on $\boldsymbol{n}$ vertices**, denoted by $\boldsymbol{K_n}$, is the simple graph that contains exactly one edge between each pair of distinct vertices.

Example1: The graphs K_n, for $n = 1, 2, 3, 4, 5, 6$ are displayed in the following figures.

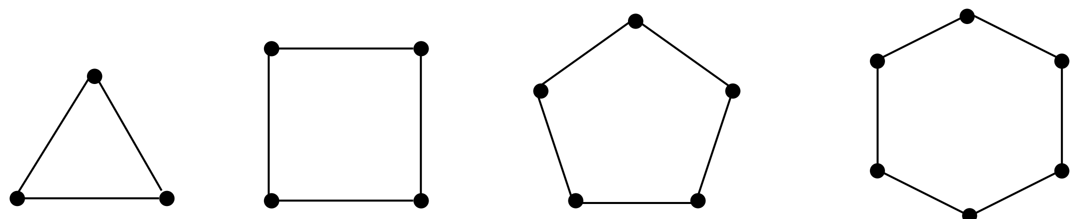


K_1 K_2 K_3 K_4 K_5 K_6

2. Cycles Graphs

Definition: The **cycles graph**, denoted by $\boldsymbol{C_n}$, $n \geq 3$, is the simple graph consists of n vertices $v_1, v_2, \dots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.

Example2: The graphs C_n, for $n = 3, 4, 5, 6$ are displayed in the following figures.

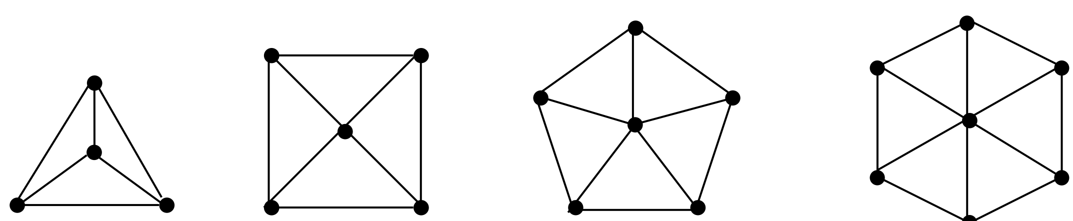


C_3 C_4 C_5 C_6

3. Wheel Graphs

Definition: The **Wheel graph**, denoted by $\boldsymbol{W_n}$, $n \geq 3$, is the simple graph has a total of $n + 1$ vertices, that are, the n vertices of $\boldsymbol{C_n}$ with added one additional vertex, and connects this new vertex to each of the n vertices in $\boldsymbol{C_n}$ by an edge for each.

Example3: The graphs W_n, for $n = 3, 4, 5, 6$ are displayed in the following figure.

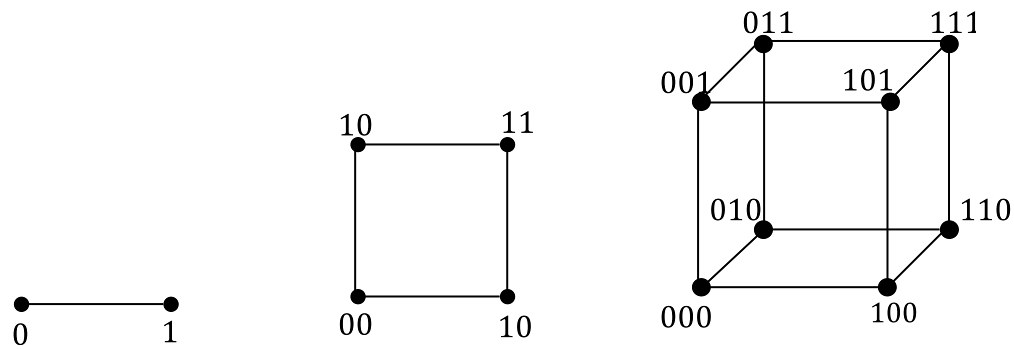


W_3 W_4 W_5 W_6

4. n −Cube Graphs

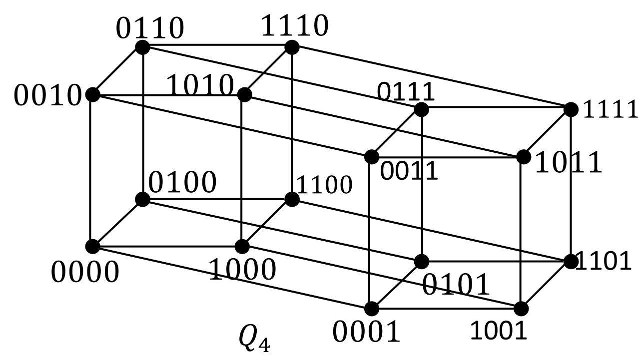
Definition: The $\boldsymbol{n}$ **−dimensional hypercube**, **or** $\boldsymbol{n}$ **−cube**, denoted by $\boldsymbol{Q_n}$, is the simple graph that has vertices representing the 2^n bit string of length n. Two vertices are adjacent if and only if the bit strings that they represented differ in exactly one bit position.

Example4: The graphs $Q_{\boldsymbol{n}}$, for $n = 1, 2, 3$ are displayed in the following figures.



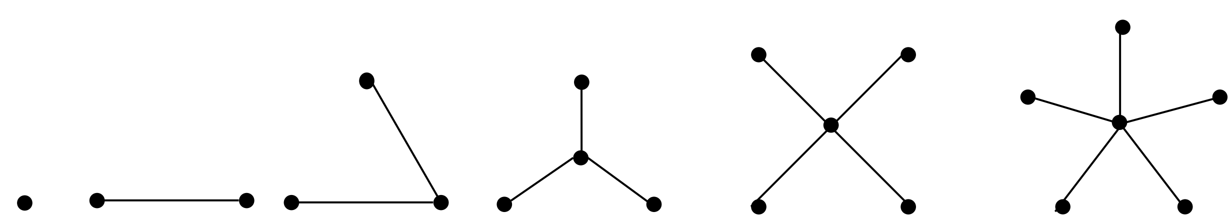
Q_1 Q_2 Q_3

Easy way to draw or construct $n + 1 -$ cube Q_{n+1} from the $n -$ cube $Q_{\boldsymbol{n}}$ by making two copies of $Q_{\boldsymbol{n}}$, and the labels of the vertices will be the same as the labels on $Q_{\boldsymbol{n}}$ but with adding 0 to each lable in one copy and 1 to each label in the other copy, as it is shown in the figure here.



5. Star graphs

Definition: The **Star graph,** denoted by S_n, $n \geq 1$ is a simple graph of n vertices that has one vertex of degree $n - 1$ and every other vertex is a pendent.

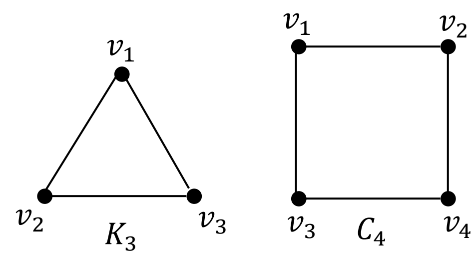


S_1 S_2 S_3 S_4 S_5 S_6

6. Bipartite graphs

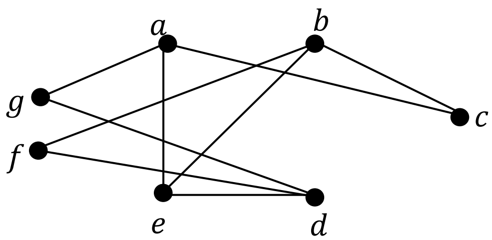
Definition: A simple graph G is called **bipartite** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a **bipartition** of the vertex set V of G.

Example 5: From the figure here, we can see that C_4 is bipartite because its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_4\}, V_2 = \{v_2, v_3\}$ and every edge connects a vertex in $V_1$and a vertex in V_2. While K_3 is not bipartite because there are no disjoint subsets of vertices that edges do not connect two vertices in the same set.

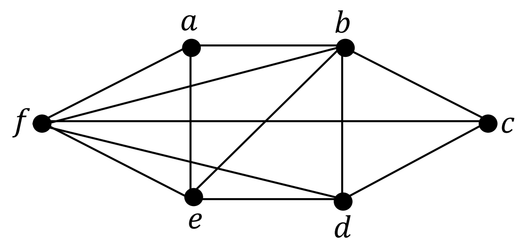


Theorem: If a part- which will be called later subgraph- of a graph forms C_n where n is odd (which will be called later odd cycles) then the graph itself is not bipartite.

Example6: Are the graphs G and H displayed here bipartite?



G

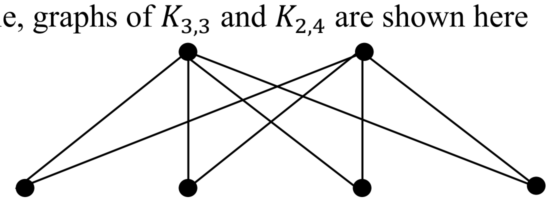


H

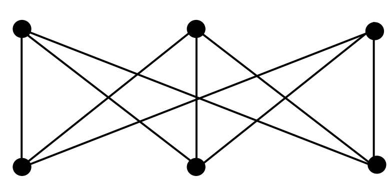
Solution: graphs G is bipartite because its vertex set is the union of two disjoint sets, $\{a, b, d\}$ and $\{c, e, f, g\}$ and each edge connects a vertex in one of these subsets to a vertex in the other subset. For graph H is not bipartite because we cannot partition its vertex set into two disjoint subsets of vertices that edges do not connect two vertices in the same set.

Definition: The **complete bipartite graph $K_{m,n}$** is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively. There is an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

For example, graphs of $K_{3,3}$ and $K_{2,4}$ are shown here



$K_{2,4}$

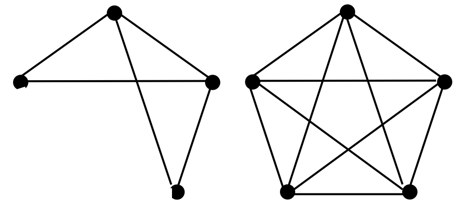


$K_{3,3}$

7. New Graphs from Old

Definition: A **subgraph of a graph** $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a proper subgraph of G if $H \neq G$.

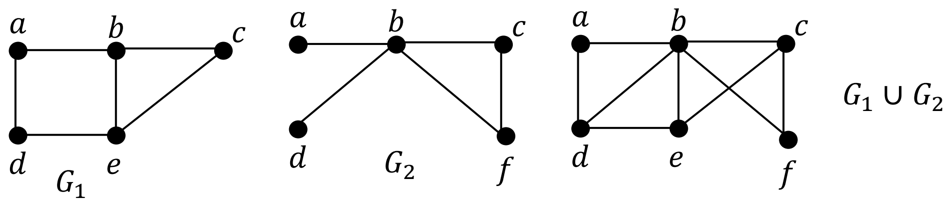
For example, the graph H here is a proper subgraph of the graph G



H G

Definition: Two or more graphs can be combined in various ways. The new graph that contains all the vertices and the edges of these graphs is called the union of the graphs. Which means, the **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. The union of G_1 and $G_2$is denoted by $G_1 \cup G_2$.

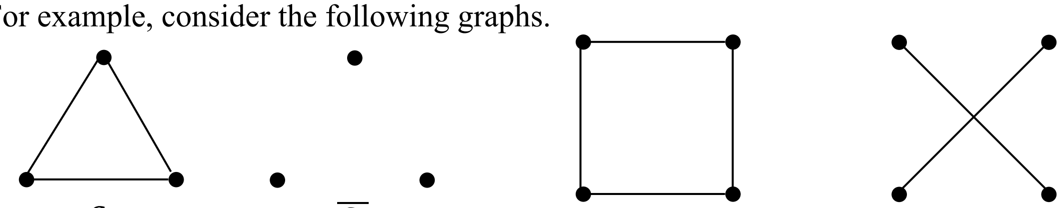
For example



Definition: A simple graph is called **regular** if every vertex has the same degree. A regular graph is called $\boldsymbol{n}$ **−regular** if every vertex in this graph has degree n.

For example C_3 is regular because each vertex has the same degree and it is equal to 3.

Definition: The **complementary graph** $\overline{G}$ of a simple graph G has the same set of vertices as G. But two vertices are adjacent in $\overline{G}$ if they are not adjacent in G.
For example, consider the following graphs.



C_3 $\overline{C_3}$ C_4 $\overline{C_4}$

Theorem: If G is a simple graph with n vertices then $G \cup \bar{G}$ forms the complete graph K_n.

Results:

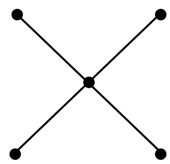
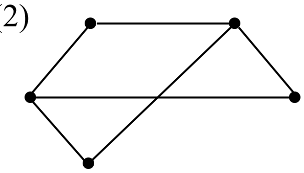
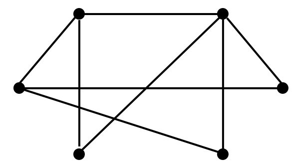
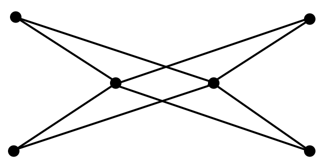
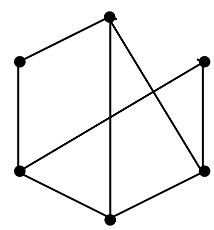
1. $|V_{K_n}| = |V_G| = |V_{\bar{G}}|$
2. $|V_{K_n}| = |E_G| + |E_{\bar{G}}|$
3. If the degree sequence of G is $d_n, d_{n-1}, \dots, d_1$ then the degree sequence of $\bar{G}$ is
$$n - 1 - d_1, n - 1 - d_2, \dots, n - 1 - d_n$$

Exercises 10.2

1. Draw these graphs.

    a. K_7 b. $K_{1,8}$ c. $K_{4,4}$ d. C_7 e. W_7

2. For each of the following graphs, determine whether the graph is bipartite.

(1)

(2)

(3)

(4)

(5)


3. For which values of n are these graphs bipartite?

    a. K_n b. C_n c. W_n d. Q_n

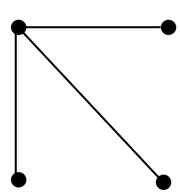
4. Suppose that a new company has five employees: Zamora, Agraharam, Smith, Chou, and Macintyre. Each employee will assume one of six responsibilities: planning, publicity, sales, marketing, development, and industry relation. Each employee is capable of doing one or more of these jobs: Zamora could do planning, sales, marketing, or industry relations; Agraharam could do planning or development; Smith could do Planning, sales, or industry relations; Chou could do planning, sales, or industry relations; and Macintyre could do planning, publicity, sales, or industry relations.

    a. Model the capabilities of these employees using a bipartite graph.
    b. Find the assignment of responsibilities such that each employee is assigned a responsibility.

5. How many vertices and how many edges do these graphs have?

    a. K_n b. C_n c. W_n d. $K_{m,n}$ e. Q_n

6. Draw all the subgraphs of this graph



7. Let G be a graph with $|V|$ vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Show that $m \leq \frac{2e}{|V|} \leq M$.

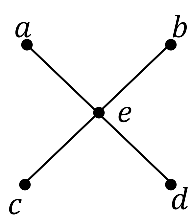
8. For which values of n are these graphs regular?

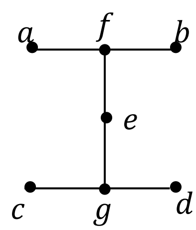
   a. K_n b. C_n c. W_n d. Q_n

9. For which values of m and n is $K_{m,n}$ regular?

10. How many vertices does a regular graph of degree four with 10 edges have?

11. Find the union of these simple graphs (assume the edges with the same endpoints are the same.)





12. Describe each of these graphs.

   a. $\overline{K_n}$ b. $\overline{C_n}$ c. $\overline{Q_n}$ d. $\overline{K_{m,n}}$

13. If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, how many vertices does G have?

14. If a simple graph G with v vertices and e edges, how many edges does $\overline{G}$ have?

15. If the degree sequence of a simple graph G is 4,3,3,2,2, what is the degree sequence of $\overline{G}$?

16. If the degree sequence of a simple graph G is $d_1, d_2, \ldots, d_n$, what is the degree sequence of $\overline{G}$?

17. Show that if G is a simple graph with n vertices, then the union of G and $\overline{G}$ is K_n.

10.3 Representing Graphs and Graphs Isomorphism

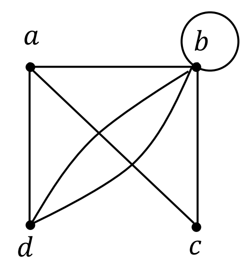
Representing Graphs

1. Adjacency Matrix

Definition: Suppose that $G = (V, E)$ is an undirected graph where $|V| = n$. Suppose that the vertices of G are listed arbitrary as $v_1, v_2, \dots, v_n$. Then the **adjacency matrix** $\boldsymbol{A}$ or $\boldsymbol{A_G}$ of the undirected graph G, with respect to this listing of vertices, is the $n \times n$ matrix where $A = [a_{ij}]$ such that each entry a_{ij} represent the number of edges between v_i and v_j.

Note: The loop in an undirected graph is counted as one edge.

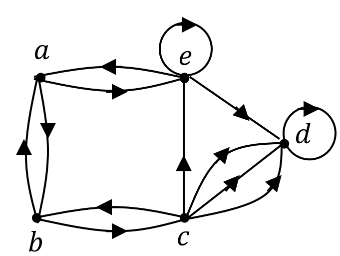
Example: Find the adjacency matrix of the given graph



Solution: $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$

Definition: Suppose that $G = (V, E)$ is a directed graph where $|V| = n$. Suppose that the vertices of G are listed arbitrary as $v_1, v_2, \dots, v_n$. Then the **adjacency matrix** $\boldsymbol{A}$ or $\boldsymbol{A_G}$ of the directed graph G, with respect to this listing of vertices, is the $n \times n$ matrix where $A = [a_{ij}]$ such that each entry a_{ij} represent the number of edges started with vertex v_i and ended with the vertex v_j.

Example: Find the adjacency matrix of the given graph



Solution: $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$

Note: The adjacency matrix of an undirected graph is a symmetric matrix while it is not necessary to be symmetric if the graph is directed.

2. Incidence Matrix

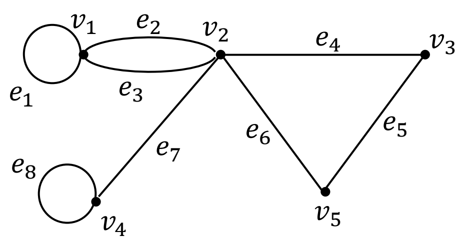
Definition: Suppose that $G = (V, E)$ is an undirected graph where $|V| = n$. Suppose that the vertices and the edges of G are listed arbitrary as $v_1, v_2, \dots, v_n$ and $e_1, e_2, \dots, e_m$ respectively. Then the **incidence matrix** $\boldsymbol{M}$ or $\boldsymbol{M_G}$ of the undirected graph G, with respect to this ordering of vertices and edges, is the $n \times m$ matrix where $M = [m_{ij}]$ where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incedent with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Example: Find the incidence matrix of the given pseudograph

Solution:

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

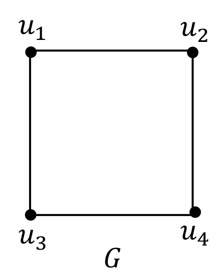


Isomorphism of Graphs

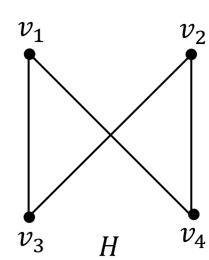
Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a 0ne-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such function f is called an **isomorphism**.

Example: Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in the following figure, are isomorphic.

Solution: Define a function f with, $f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3, f(u_4) = v_2$ is a one-to one corresponding between V and W. To see that this corresponding preserves adjacency, follow the table below



Pairs of vertices adjacent in G	Pairs of vertices adjacent in H
u_1, u_2	$f(u_1), f(u_2)$ which they are v_1, v_4
u_1, u_3	$f(u_1), f(u_3)$ which they are v_1, v_3
u_2, u_4	$f(u_2), f(u_4)$ which they are v_4, v_2
u_3, u_4	$f(u_3), f(u_4)$ which they are v_3, v_2



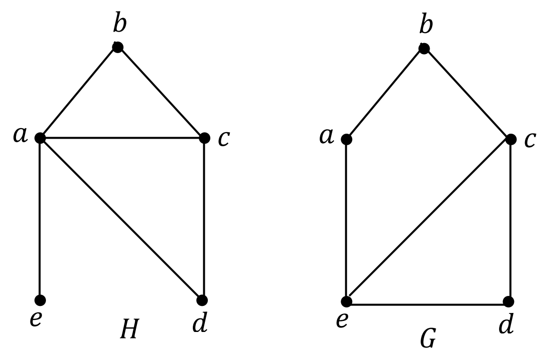
Note: Isomorphism of graphs preserves properties of graphs. Therefore it is easier to identify that two graphs are not isomorphic if one of them has a property that the other one does not have.

Definition: A property preserved by isomorphism is called **graph invariant.** (We will re-visit the topic after completing section10.6)

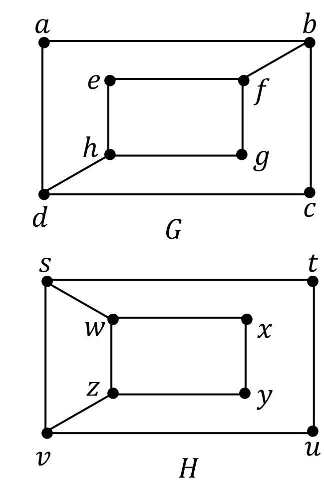
Note: There is still a possibility (but not definitely) that two graphs are isomorphic if they have the same number of vertices, the same number of edges, because, one-to-one correspondence between vertices established a one-to-one correspondence between edges. Moreover, the corresponding vertices must have the same degrees. In other words, a vertex v of degree d in graph G must corresponds to a vertex $f(v)$ of degree d in graph H.

Example: Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in the following figure, are not isomorphic.

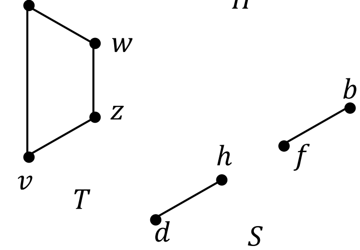
Solution: Both of the graphs G and H have five vertices and six edges. Nevertheless, H has a vertex of degree one, namely, e, while G has no vertices of degree one. It follows that, G and H are not isomorphic.



Example: Determine whether or not the graphs $G = (V, E)$ and $H = (W, F)$, displayed in the following figure, are isomorphic.

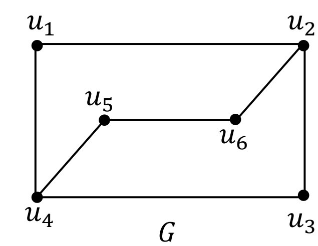


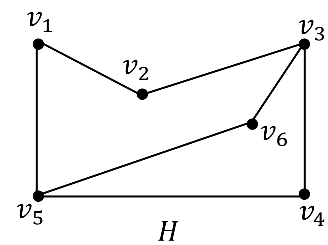
Solution: Both of the graphs G and H have eight vertices and 10 edges. They also both have four vertices of degree two and four vertices of degree three. Now, because these invariants all agree, it is still conceivable that these graphs are isomorphic. Nevertheless, G and H are not isomorphic. To see this, note that because $\deg(a) = 2$ in G, a must correspond to either t, u, x or y in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G. Another way to look into the matter, G and H are not isomorphic because, the subgraphs S and T of G and H respectively, do not have the same properties and therefore there is no one-to one corresponding between them. (see the graphs below). Because the number of the degrees do not match.



Example: Determine whether or not the graphs $G = (V, E)$ and $H = (W, F)$, displayed in the following figure, are isomorphic.

Solution: Both of the graphs G and H have six vertices and seven edges. They also both have four vertices of degree two and two vertices of degree three. Now, because these invariants all agree, it is reasonable to try to find an isomorphism between them.



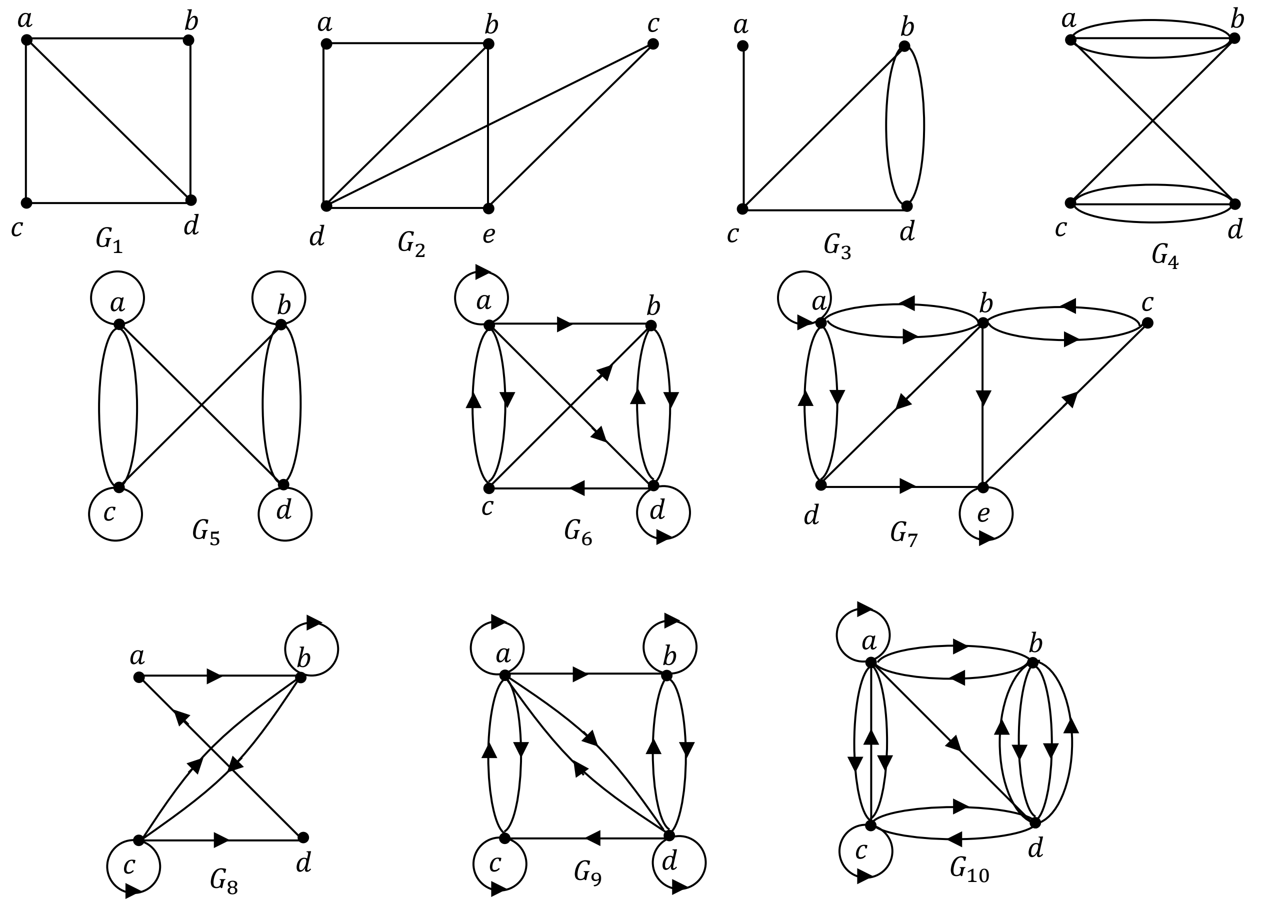


We will define the following function f from G to H.
The degree of u_1 is two and it is adjacent to vertices of degree three, therefore it must be corresponds to vertex in H of the same properties which is v_6. Therefore, $f(u_1) = v_6$. The degree of u_2 is three and it is adjacent to vertices of degree two, v_3 has the same properties, therefore $f(u_2) = v_3$. The degree of u_3 is two and it is connected to vertices of degree three, v_4 has the same properties, therefore, $f(u_3) = v_4$. The degree of u_4 is three and it is connected to vertices of degree two, v_5 has the same properties, therefore, $f(u_4) = v_5$. The degree of u_5 is two and it is connected to one vertex of degree two and other vertex of degree three, v_1 has the same properties, therefore, $f(u_5) = v_1$. u_6 in G and v_2 in H have the same situation of u_5 and v_1. Therefore $f(u_6) = v_2$. To see that this isomorphism preserves adjacency, follow the table below.

Pairs of vertices adjacent in G	Pairs of vertices adjacent in H
u_1, u_2	$f(u_1), f(u_2)$ which they are v_6, v_3
u_1, u_4	$f(u_1), f(u_4)$ which they are v_6, v_5
u_2, u_3	$f(u_2), f(u_3)$ which they are v_3, v_4
u_2, u_6	$f(u_2), f(u_6)$ which they are v_3, v_2
u_3, u_4	$f(u_3), f(u_4)$ which they are v_4, v_5
u_4, u_5	$f(u_4), f(u_5)$ which they are v_5, v_1
u_5, u_6	$f(u_5), f(u_6)$ which they are v_1, v_2

Exercises 10.3

1. For each of the following graphs, find the adjacency matrix.



2. Represent each of the following graphs with an adjacency matrix.

a. K_4	c. $K_{2,3}$	e. W_4
b. $K_{1,4}$	d. C_4	f. C_3

3. Draw the graph of each of the following adjacency matrices.

a. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

4. Draw the undirected graph of each of the following adjacency matrices.

a. $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

5. Draw the graph of each of the following adjacency matrices.

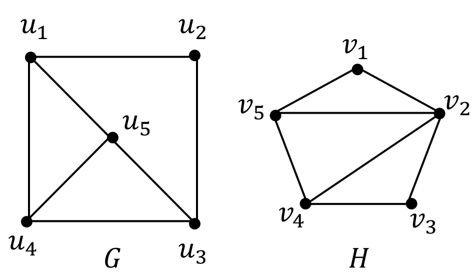
a. $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

6. Use the incidence matrix to represent the graphs G_1, G_2, G_3, G_4 and G_5 in exercise 1.

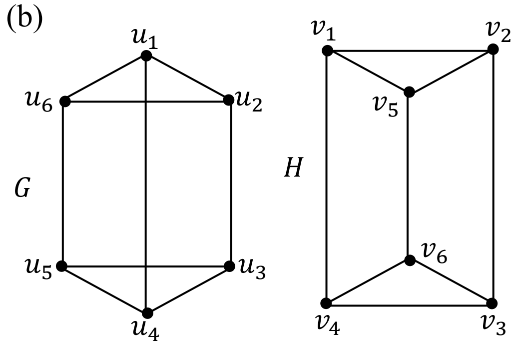
7. What is the sum of the entries in a raw of the incidence matrix for an undirected graph?

8. Determine whether the given pair of graphs is isomorphic. Give reasons for those that are not.

(a)



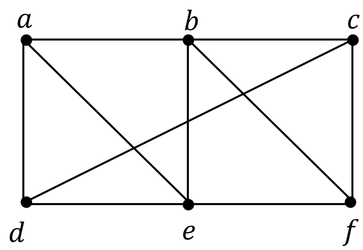
(b)



10.4 Connectivity
10.6 The Shortest Path Problem

Definition: let n be nonnegative integer and G an undirected graph. A **path** of length n from u to v in G is a sequence of n edges $e_1, e_2, \dots, e_n$ of G such that e_1 is associated with $\{x_0, x_1\}$, e_2 is associated with $\{x_1, x_2\}$, and so on, with e_n associated with $\{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \dots, x_n$ (because listing these vertices uniquely determines the path). The path is a **circuit** if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero. The path or circuit is said to **pass through** the vertices $x_0, x_1, \dots, x_{n-1}$or **traverse** the edges $e_1, e_2, \dots, e_n$. A path or a circuit is **simple** if it does not contain the same edge more than once.

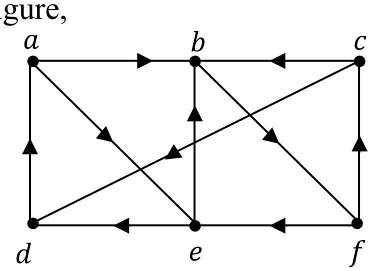
Example: Discuss the path lengths in the simple graph shown in this figure,



a, d, c, f, e is a simple path of length 4, because $\{a, d\}$, $\{d, c\}$, $\{c, f\}$, and $\{f, e\}$ are all edges. However, d, e, c, a is not a path, because $\{e, c\}$ is not an edge. Note that b, c, f, e, b is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at b. The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twise.

Definition: let n be nonnegative integer and G a directed graph. A **path** of length n from u to v in G is a sequence of n edges $e_1, e_2, \dots, e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2), and so on, with e_n associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, \dots, x_n$. A path of length greater than zero that begins and ends at the same vertex is called a circuit or a cycle. A path or circuit is simple if it does not contain the same edge more than once.

Example: In the simple graph shown in this figure,



d, a, e, b, fc, d is a circuit of length 6, because (d, a), (a, e), (e, b), (b, f), (f, c), and (c, d) are all edges and begins and ends at the same vertex. while a, b, e is not a path, because (b, e)

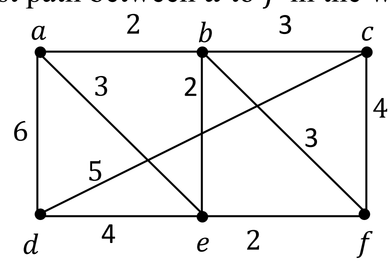
is not an edge. Note that f, c, b is a simple path of length 2. The path f, e, d, a, e, d which is of length 5, is not simple because it contains the edge (e, d) twise.

Definition Graphs that have a number assigned to each edge are called **weighted graphs**.

Definition: The **length of weighted path** is the sum of the weights of the edges that construct the path.

Definition: In a weighted graph, the **shortest path** is a path between two vertices in a graph such that the total sum of the edges weights is minimum. And, in an unweighted graph, the shortest path is the minimum number edges between two vertices.

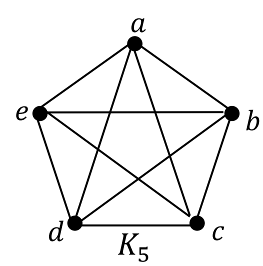
Example: Find the length of the shortest path between a to f in the weighted graph shown below.

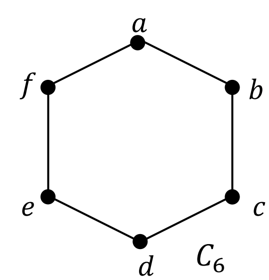


Solution: With respect the edges weights, the shortest path from a to f is a, b, f.

Definition: The **diameter** of a graph is the maximum distance- number of edges- between two distinct vertices. The diameter is the longest path of the shortest paths between any two vertices.

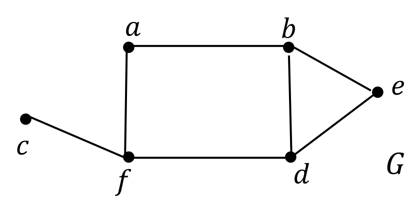
Example: What is the diameter of k_5 and C_6





Solution: The lengths of the shortest paths between any two different vertices in k_5 is 1. Therefore, the diameter is 1. And in C_6 is 3.

Example: Find the diameter of the following graph.

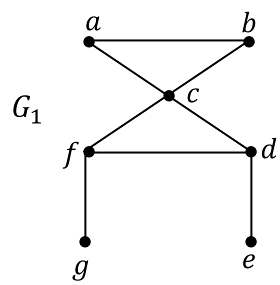


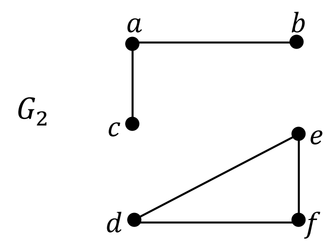
Solution:
The shortest path between c to e is 3. Every other shortest path is less than 3. Therefore, the diameter is 3

Connectedness in Undirected Graphs

Definition: An undirected graph is called **connected** if there is a path between every pair of distinct vertices of the graph.

Example: Given the following graphs. Which one is connected and why?

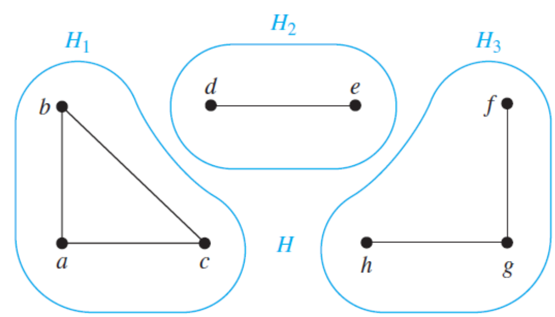




Graph G_1 is connected because there is a path for every pair of vertices, (the proof is left for the student), while graph G_2 is not connected. For instance, there is no path between a and f.

Connected Components A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

Example: What are the connected components of the graph H?
Solution: The graph H is the union of three disjoint connected subgraphs H_1, H_2, and H_3. These three subgraphs are the connected components of H.

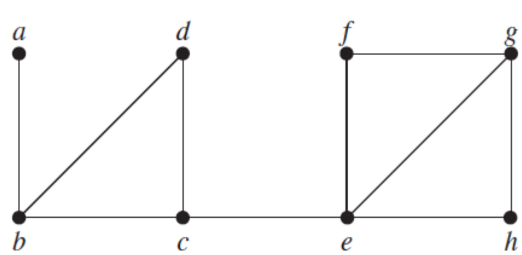


How Connected is a Graph?

Suppose that a graph represents a computer network. Knowing that this graph is connected tells us that any two computers on the network can communicate. However, we would also like to understand how reliable this network is. For instance, will it still be possible for all computers to communicate after a router or a communications link fails? To answer this and similar questions, we need the following definitions.

Definition: A **Cut vertex or (articulation point)** is a removal a vertex and all incident edges produces a subgraph with more connected components. And an edge whose removal produces a graph with more connected components than in the original graph is called a **cut edge or bridge**.

Example: Find the cut vertices and cut edges in the graph G_1.



G_1

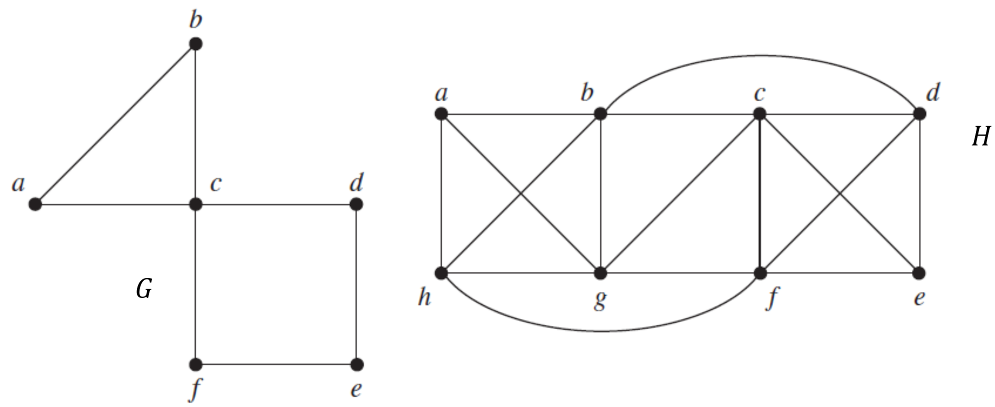
Solution: The cut vertices of G_1 are b, c, and e. The removal of one of these vertices (and its adjacent edges) disconnects the graph. The cut edges are $\{a, b\}$ and $\{c, e\}$. Removing either one of these edges disconnects G_1.

<u>Vertex Connectivity</u>

Not all graphs have cut vertices. For example, the complete graph K_n, where $n \geq 3$, has no cut vertices. When you remove a vertex from K_n and all edges incident to it, the resulting subgraph is the complete graph K_{n-1}, a connected graph. Connected graphs without cut vertices are called **nonseparable graphs**, and can be thought of as more connected than those with a cut vertex.

Definition: We define the **vertex connectivity** of a noncomplete graph G, denoted by $\kappa(G)$, as the minimum number of vertices in a vertex cut.

Example: Find the vertex connectivity for each of the following graphs

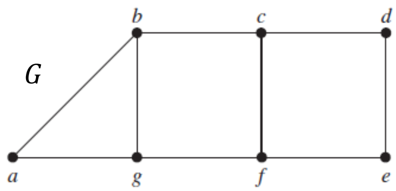


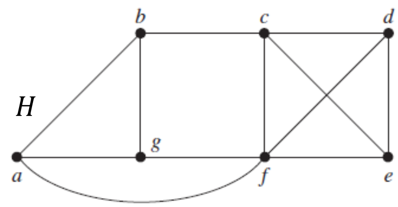
Solution: For graph G, the vertex connectivity is $k(G) = 1$ which is the cut vertex set $\{c\}$. The graph H does not have a cut vertex. However, the vertex connectivity is $k(H) = 3$ which is the minimum number of vertices that we must cut to get more connected components. For example, the vertices $\{b, c, f\}$.

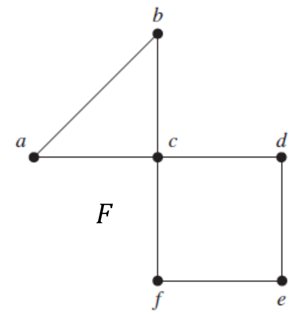
<u>Edge Connectivity</u>

We can also measure the connectivity of a connected graph $G = (V, E)$ in terms of the minimum number of edges that we can remove to disconnect it. If a graph has a cut edge, then we need only to remove it to disconnect G. If G does not have a cut edge, we look for the smallest set of edges that can be removed to disconnect it. A set of edges E is called an edge cut of G if the subgraph $G - E$ is disconnected. The edge connectivity of a graph G, denoted by $\lambda(G)$, is the minimum number of edges in an edge cut of G. This defines $\lambda(G)$ for all connected graphs with more than one vertex because it is always possible to disconnect such a graph by removing all edges incident to one of its vertices. Note that $\lambda(G) = 0$ if G is not connected. We also specify that $\lambda(G) = 0$ if G is a graph consisting of a single vertex. It follows that if G is a graph with n vertices, then $0 \leq \lambda(G) \leq n - 1$.

Example: Find the cut edge and edge connectivity of each of the following graphs







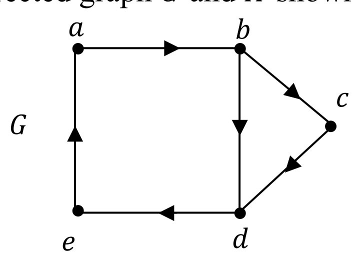
Solution: For graph G, it does not have cut edge. However, the removal of the edges $\{b, c\}$ and $\{g, f\}$ or $\{c, d\}$ and $\{f, e\}$ or $\{a, b\}$ and $\{a, g\}$ will disconnect the graph. Therefore $\lambda(G) = 2$. For the graph H, also there is not cut edge. But the graph will be disconnect when we remove the edges set $\{b, c\}$, $\{a, f\}$ and $\{g, f\}$ or the edges set $\{a, f\}$, $\{a, g\}$ and $\{b, g\}$ or the edges set $\{a, b\}$, $\{b, g\}$ and $\{b, c\}$ or the edges set $\{c, d\}$, $\{d, f\}$ and $\{d, e\}$ or the edges set $\{d, e\}$, $\{e, f\}$ and $\{c, e\}$ either way $\lambda(G) = 3$. For the graph F, it does not have cut edge, but it can be disconnected by removing either the pair of $\{a, c\}$ and $\{b, c\}$ or the pair $\{c, f\}$ and $\{c, d\}$ or the pair $\{f, c\}$ and $\{d, e\}$, either way, $\lambda(G) = 2$.

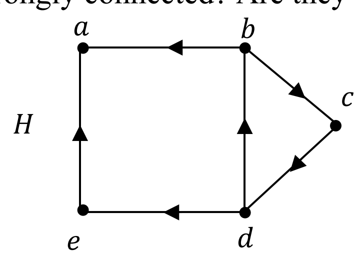
<u>**Connectedness in Directed Graphs**</u>

Definition: A directed graph is **strongly connected** if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

Definition: A directed graph is **weakly connected** if there is a path between every two vertices in the underlying undirected graph.

Example: Are the directed graph G and H shown below strongly connected? Are they weakly connected?

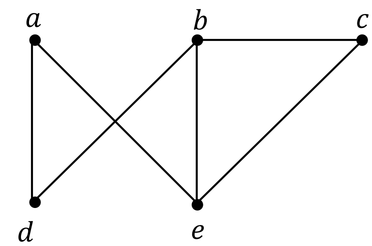




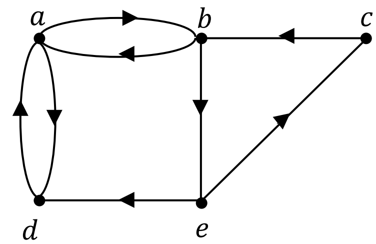
Solution: The graph G is strongly connected because there is a path between any two vertices in this directed graph. While, the graph H is weakly connected because there is no directed path from a to b, but there is a path between any two vertices.

Exercises 10.4, 10.6

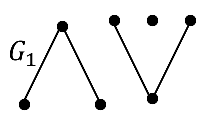
9. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?
   a. a, e, b, c, b
   b. a, e, a, d, b, c, a
   c. e, b, a, d, b, e
   d. c, b, d, a, e, c

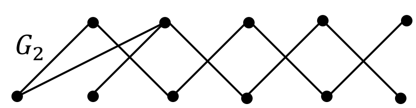


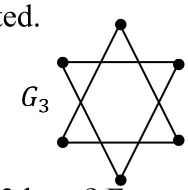
10. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?
    a. a, e, b, c, b
    b. a, d, a, d, a
    c. a, d, b, e, a
    d. $a, b. e. c. b. d. a$



11. In each of the following, determine whether the given graph is connected.



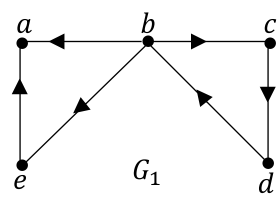


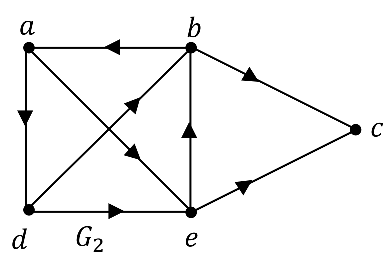


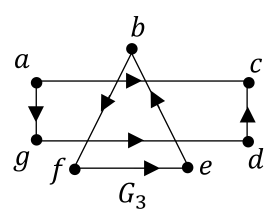
12. How many connected components does each of the graphs in exercise 3 have? For each graph find each of its connected components.

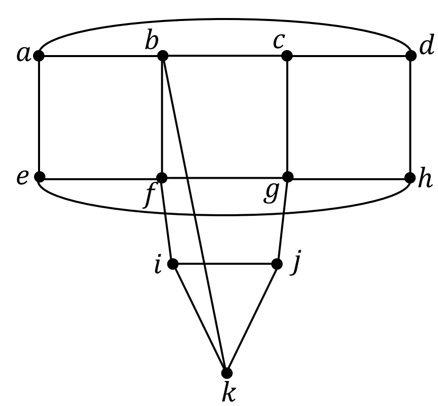
13. Find the cut vertex, cut edge, vertex connectivity and cut connectivity of the graphs in exercise 3.

14. In each of the following, determine whether the given graph is strongly connected and if not, whether it is weakly connected.





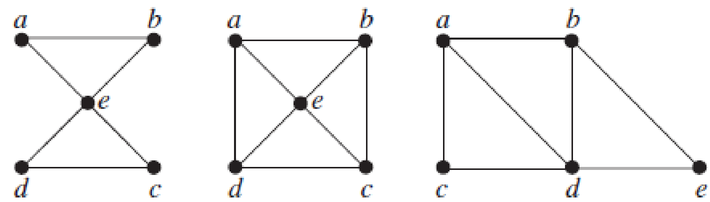




10.5 Euler and Hamilton Paths

Definition: In a graph G, an **Euler circuit** is a simple circuit containing every edge of G. An **Euler path** is a simple path containing every edge of G.

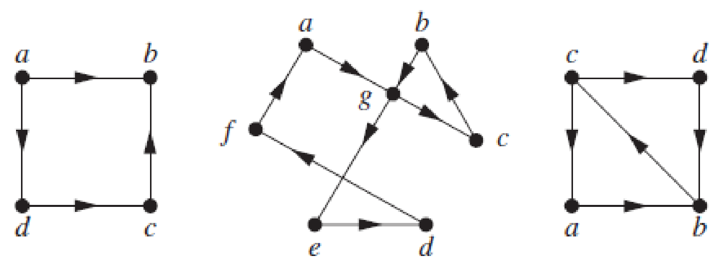
Example: Which of the following undirected graphs have an Euler circuit? Of those that do not, which have an Euler path?



G_1 G_2 G_3

Solution: The graph G_1 has an Euler circuit, for example, a, e, c, d, e, b, a. Neither of the graphs G_2 or G_3 has an Euler circuit. However, G_3 has an Euler path, namely, a, c, d, e, b, d, a, b. G_2 does not have an Euler path.

Example: Which of the directed graphs in Figure 4 have an Euler circuit? Of those that do not, which have an Euler path?



H_1 H_2 H_3

Solution: The graph H_2 has an Euler circuit, for example, a, g, c, b, g, e, d, f, a. Neither H_1 nor H_3 has an Euler circuit. H_3 has an Euler path, namely, c, a, b, c, d, b, but H_1 does not.

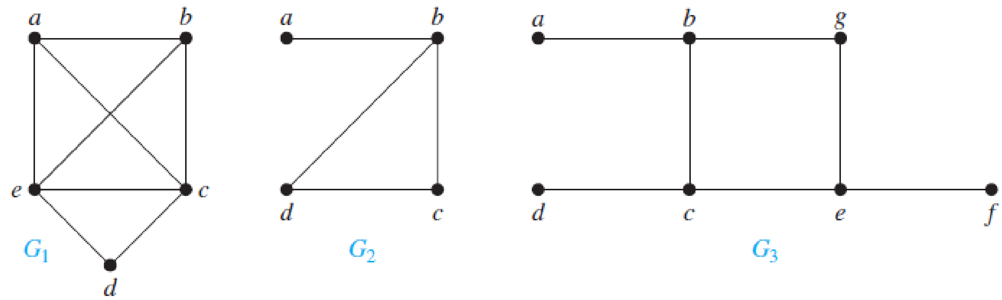
Theorem: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

<u>Hamilton Paths and Circuits</u>

Definition: A simple path in a graph G that passes through every vertex exactly once is called a **Hamilton path**, and a simple circuit in a graph G that passes through every vertex exactly once is called a **Hamilton circuit**. That is, the simple path $x_0, x_1, \dots, x_{n-1}, x_n$, in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ and $x_i = x_j$ for $0 \le i < j \le n$, and the simple circuit $x_0, x_1, \dots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \dots, x_{n-1}, x_n$ is a Hamilton path.

Example: Which of the following simple graphs have a Hamilton circuit or, if not, a Hamilton path?



Solution: G_1 has a Hamilton circuit: a, b, c, d, e, a. There is no Hamilton circuit in G_2 but G_2 does have a Hamilton path, namely, a, b, c, d. G_3 has neither a Hamilton circuit nor a Hamilton path.

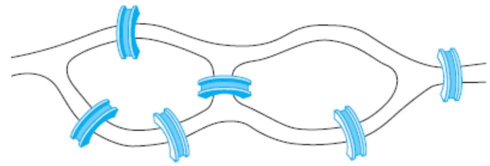
Note: a graph with a vertex of degree one cannot have a Hamilton circuit

Note: To complete the topic of isomorphic graphs and isomorphism. Here is a helpful- incomplete- list of graph invariants:

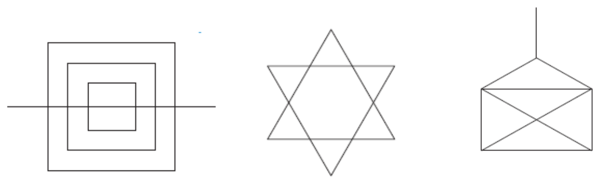
1. Total number of vertices.
2. Total number of edges.
3. Degree sequences.
4. The sum of the degrees.
5. Euler circuit and Euler path.
6. Hamilton circuit and Hamilton path.
7. Diameter.
8. Bipartite.
9. Number of circuits of the same length.
10. Edge connectivity and vertex connectivity

Exercises 10.5

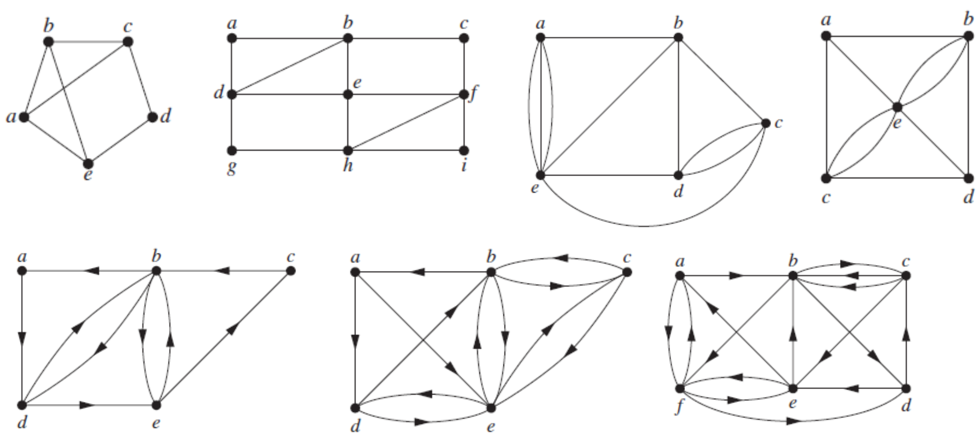
1. Can someone cross all the bridges shown in this map exactly once and return to the starting point?



2. Determine whether the pictures shown here can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture.



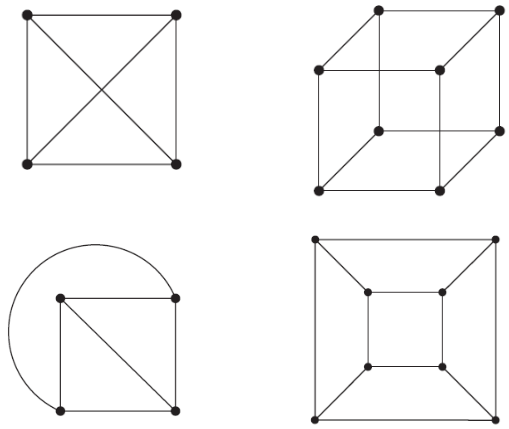
3. Determine which of the following graphs has an Euler circuit or Hamilton circuit, and construct such circuit if one exists. If no Euler or Hamilton circuit exists, determine whether the graph has an Euler or Hamilton path and construct such a path if one exists.



4. For which values of n do these graphs have an Euler or Hamilton circuit?
a) K_n b) C_n c) W_n d) Q_n

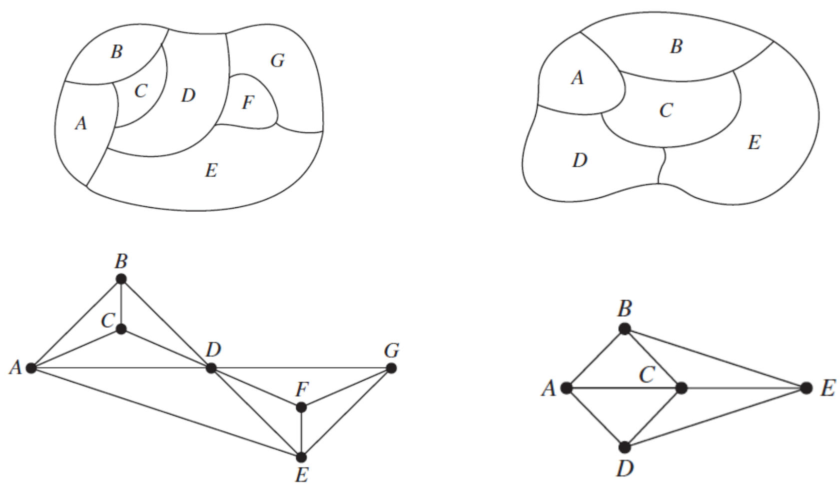
10.8 Graph Coloring

Definition: A graph is called **planar** if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a **planar representation of the graph**.



Dual graph of the map

Definition: **Duel graph of a map** is the graph representation of a map. Each region of the map is represented by a vertex. Edge is connecting two vertices if the regions represented by these vertices have a common border. Two regions that touch at only one point are not considered adjacent.

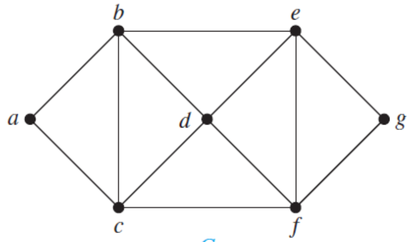


Definition: A **coloring** of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

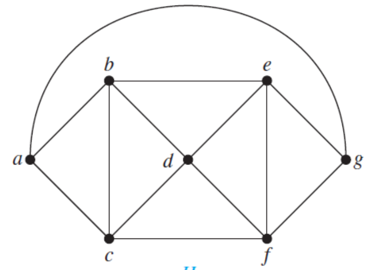
Definition: The **chromatic number** of a graph is the least number of colors needed for coloring a graph. The chromatic number of a graph G is denoted by $\chi(G)$.

Theorem: The **Four-Color theorem**- The chromatic number of a planar graph is no greater than four.

Example: What are the chromatic numbers of the graphs G and H



G



H

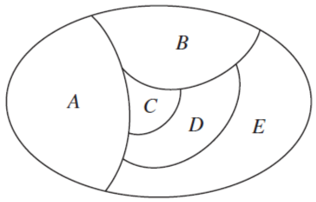
Solution: For the graph G, $\chi(G) = 3$, In H, $\chi(H) = 4$

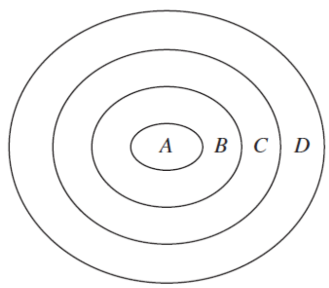
Example: What is the chromatic number in C_n , $K_{n,m}$, K_n, W_n graphs?

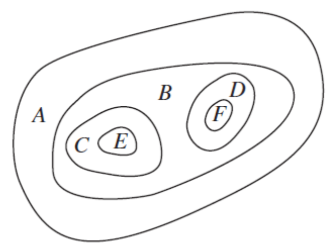
Solution: If n is even, then the needed colors are two, if n is odd, then it needs three colors. $K_{n,m}$ graphs need only two colors. K_n graphs need n colors, one color for each vertex. W_n needs three colors if n is even and four colors if n is odd. Q_n graphs need two colors only.

Exercises 10.8

5. Construct the dual graph for each of the following maps. Then find the number of colors needed to color the map so that no two adjacent regions have the same color. Then find their chromatic numbers.







6. Construct the map of each of the dual graphs. Then find the number of colors needed to color the map so that no two adjacent regions have the same color. Then find their chromatic numbers.