

Creating a Grid and Graphical Representation

Our grid relies on several key assumptions. Our grid always assumes that a grid is $n \times n$, and that the start is always at the top left corner and the goal at the bottom right corner. Each tile of the grid has a number on it that represents the number of tiles you can move in any direction from that grid. Every tile on a grid can go to at least one other tile on the grid.



2	4	2	2	1
3	3	2	2	2
2	2	1	2	1
2	3	1	1	1
2	1	4	1	0

Figure 1: A 5x5 grid with a solution.

Puzzle Evaluations

In order to evaluate our generated puzzles, we implemented a Breadth First Search (BFS) algorithm equivalent to the one described in the task. Evaluated grids are filled with either numbers or X's that represent the number of moves to reach that cell or that a cell is unreachable, respectively. The bottom most and right hand most cell contains either the value of the puzzle or an X which indicates that the puzzle is not solvable. Below are examples of evaluated $n \times n$ grids for varying values of n ($n = 5, 7, 9, 11$) with their corresponding values and if they are solvable or not.

2	4	3	1	3
1	2	3	1	4
4	1	2	1	2
4	3	3	2	3
1	1	1	2	0

(a) Puzzle

0	4	1	9	3
10	11	9	8	9
1	X	3	6	2
8	6	2	7	4
7	6	4	5	3

(b) Evaluation

Figure 2: Solvable 5x5 puzzle. Value: 3.

4	1	4	3	2
4	2	1	1	1
1	1	2	3	1
3	1	3	3	3
2	1	3	2	0

(a) Puzzle

0	X	2	6	1
3	5	4	5	3
2	3	5	3	2
3	4	5	4	3
1	5	3	X	X

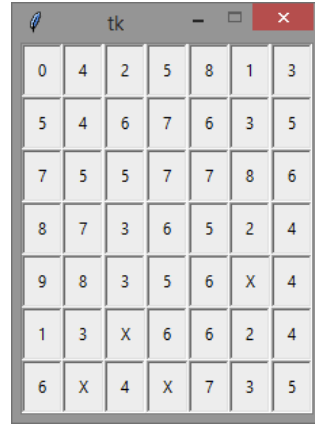
(b) Evaluation

Figure 3: Unsolvable 5x5 puzzle. Value: -3.



5	2	4	5	1	3	5
5	1	1	2	1	4	3
3	5	1	2	3	4	6
6	1	4	2	2	3	2
1	1	4	1	2	4	3
5	5	3	3	5	4	5
5	3	4	1	6	3	0

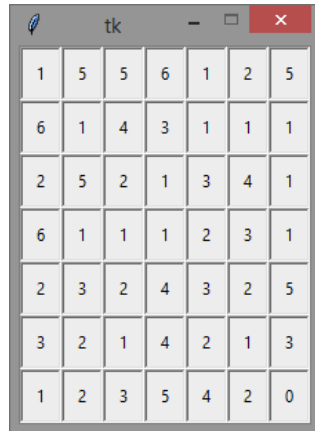
(a) Puzzle



0	4	2	5	8	1	3
5	4	6	7	6	3	5
7	5	5	7	7	8	6
8	7	3	6	5	2	4
9	8	3	5	6	X	4
1	3	X	6	6	2	4
6	X	4	X	7	3	5

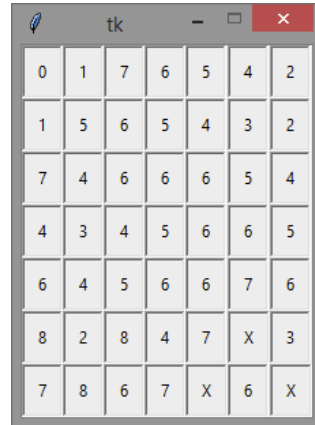
(b) Evaluation

Figure 4: Solvable 7x7 puzzle. Value: 5.



1	5	5	6	1	2	5
6	1	4	3	1	1	1
2	5	2	1	3	4	1
6	1	1	1	2	3	1
2	3	2	4	3	2	5
3	2	1	4	2	1	3
1	2	3	5	4	2	0

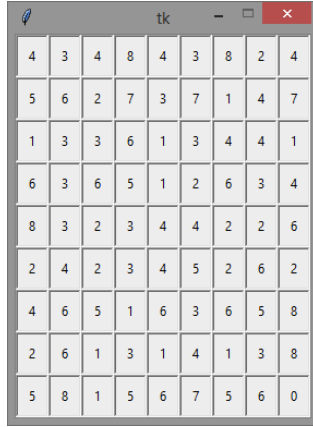
(a) Puzzle



0	1	7	6	5	4	2
1	5	6	5	4	3	2
7	4	6	6	6	5	4
4	3	4	5	6	6	5
6	4	5	6	6	7	6
8	2	8	4	7	X	3
7	8	6	7	X	6	X

(b) Evaluation

Figure 5: Unsolvable 7x7 puzzle. Value: -3.



4	3	4	8	4	3	8	2	4
5	6	2	7	3	7	1	4	7
1	3	3	6	1	3	4	4	1
6	3	6	5	1	2	6	3	4
8	3	2	3	4	4	2	2	6
2	4	2	3	4	5	2	6	2
4	6	5	1	6	3	6	5	8
2	6	1	3	1	4	1	3	8
5	8	1	5	6	7	5	6	0

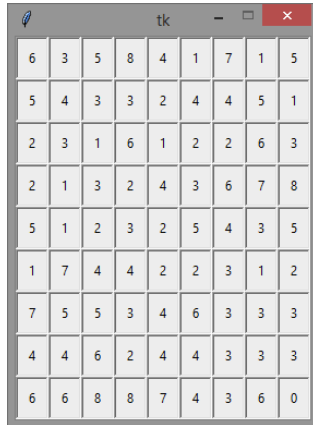
(a) Puzzle



0	10	9	X	1	8	10	8	2
7	6	6	8	5	8	X	7	8
9	10	5	5	4	6	11	8	7
8	11	7	7	5	6	9	7	8
1	12	4	9	2	11	10	10	3
7	9	6	X	7	7	9	8	8
8	12	5	X	9	X	10	6	X
8	8	7	8	10	10	9	9	9
X	9	8	6	3	9	10	X	8

(b) Evaluation

Figure 6: Solvable 9x9 puzzle. Value: 8.



6	3	5	8	4	1	7	1	5
5	4	3	3	2	4	4	5	1
2	3	1	6	1	2	2	6	3
2	1	3	2	4	3	6	7	8
5	1	2	3	2	5	4	3	5
1	7	4	4	2	2	3	1	2
7	5	5	3	4	6	3	3	3
4	4	6	2	4	4	3	3	3
6	6	8	8	7	4	3	6	0

(a) Puzzle



0	X	7	6	8	7	1	8	9
6	8	7	5	5	7	6	7	X
X	7	5	6	4	5	7	6	8
4	6	6	X	5	5	X	3	6
5	7	4	6	5	6	3	6	8
5	7	8	4	X	8	6	5	6
1	X	5	X	3	6	X	2	4
7	5	X	3	6	4	2	7	7
X	8	X	5	X	X	4	7	X

(b) Evaluation

Figure 7: Unsolvable 9x9 puzzle. Value: -15.

0	2	X	3	8	5	4	3	4	1	6
5	5	6	X	X	6	5	4	6	7	6
7	6	8	4	7	8	9	6	7	5	5
8	9	5	8	10	7	8	8	6	7	9
6	5	7	6	6	6	7	8	5	4	4
5	4	5	X	5	7	6	6	6	8	7
2	3	4	5	9	5	8	7	3	6	5
5	4	6	X	X	X	4	5	6	3	7
8	6	7	7	4	7	4	5	3	2	3
1	6	5	2	7	6	X	X	6	3	X
7	5	6	9	7	X	7	6	5	4	10

(a) Puzzle

9	6	7	2	6	4	1	1	10	8	5
5	7	8	9	1	4	4	7	3	1	4
8	3	4	4	3	7	3	4	3	5	2
2	2	6	7	1	2	5	5	1	6	7
2	2	5	4	5	1	1	3	3	5	2
1	1	6	2	3	4	5	6	3	1	8
8	1	3	6	3	3	2	1	6	9	2
7	6	8	1	4	6	6	5	6	3	9
8	4	6	4	3	5	8	8	4	1	4
3	8	3	9	5	6	5	6	8	9	1
7	1	2	4	10	3	6	8	1	8	0

(b) Evaluation

Figure 8: Solvable 11x11. Value: 10.

3	6	2	3	4	5	4	5	3	5	6
9	8	8	5	2	3	9	8	4	3	7
6	7	1	6	5	7	3	6	8	1	10
3	4	2	3	7	3	6	5	8	5	1
3	9	7	6	3	6	2	6	3	3	7
6	3	6	4	4	1	6	5	7	3	4
5	2	7	5	5	4	6	7	2	9	3
1	8	7	6	2	1	4	1	8	4	6
10	7	2	7	5	7	6	5	3	7	4
2	5	9	8	1	2	4	7	6	3	7
4	8	2	5	8	7	3	7	7	2	0

(a) Puzzle

0	X	3	1	4	6	2	11	5	X	3
3	6	6	4	6	7	6	X	5	4	7
7	5	4	5	6	4	4	7	7	6	6
1	X	5	2	7	5	3	8	6	6	5
X	6	X	8	5	5	3	6	4	5	7
7	8	6	9	7	7	5	10	7	X	X
2	4	9	3	X	3	4	5	4	4	4
6	7	7	X	6	6	7	X	5	7	8
8	5	8	6	8	6	8	9	5	7	7
7	6	6	6	7	5	4	6	X	X	5
4	9	9	8	5	4	7	7	8	8	X

(b) Evaluation

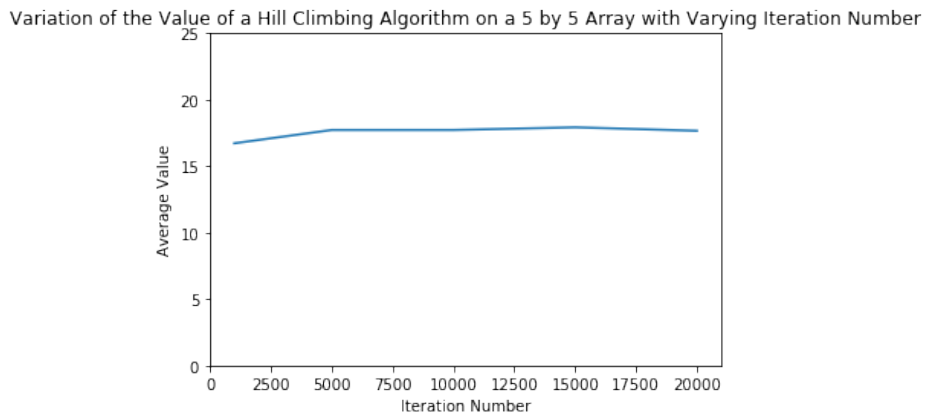
Figure 9: Unsolvable 11x11. Value: -14.

Hill Climbing Approach

The most basic algorithm we tested was a hill-climbing algorithm. This algorithm randomly changes a tile in the grid. If the new value is greater than or equal to the best value of the grid so far, then the change is kept and the hill-climbing algorithm is repeated. Otherwise, the change is discarded.

Varying Iteration Number

To measure the effect of iteration number on the output of the hill climbing algorithm, we compared the average final value of grids of differing sizes with different iteration numbers to see what the optimal number of iterations would be for the different sized grids. We tested average final value averaged over 50 runs for iteration numbers of 1,000; 5,000; 10,000; 15,000; and 20,000 for grids of size 5 by 5, 7 by 7, 9 by 9, and 11 by 11.

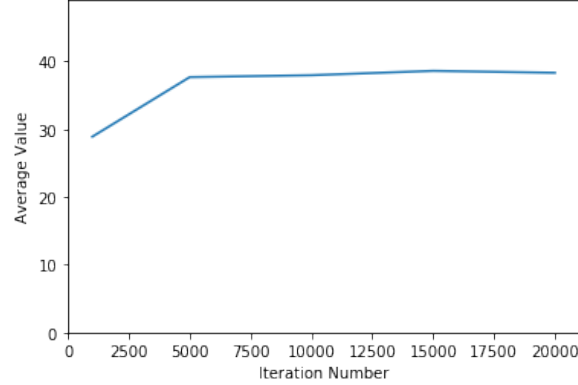


Grid Side Length	Iteration Number	Average Value
5	1,000	16.72
5	5,000	17.72
5	10,000	17.72
5	15,000	17.92
5	20,000	17.66

Figure 10: A graph showing the effects of iteration size on final value for a puzzle of size 5 by 5.

For a 5 by 5 grid, the number of iterations does not seem to make much of a difference, with the largest change being a nominal increase in average value when increasing the iteration number from 1,000 to 5,000. This suggests that the optimal number of iterations that maximizes average value while minimizing computation time is 1,000 iterations for a 5 by 5 grid.

Variation of the Value of a Hill Climbing Algorithm on a 7 by 7 Array with Varying Iteration Number

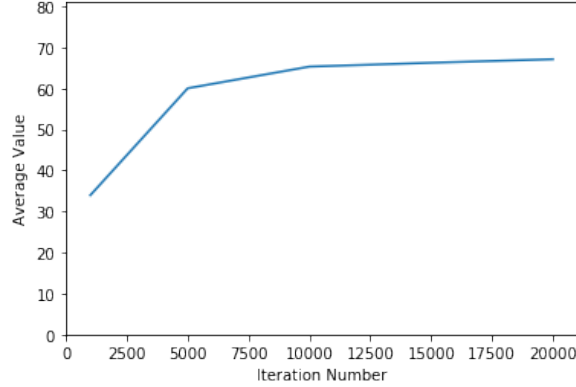


Grid Side Length	Iteration Number	Average Value
7	1,000	28.88
7	5,000	37.66
7	10,000	37.94
7	15,000	38.58
7	20,000	38.30

Figure 11: A graph showing the effects of iteration size on final value for a puzzle of size 7 by 7.

For a 7 by 7 grid, there is a much more marked increase in average value when iteration number increases from 1,000 to 5,000, but once again the average value plateaus as iteration number increases beyond that. This suggests that the optimal number of iterations that maximizes average value while minimizing computation time is 5,000 iterations for a 7 by 7 grid.

Variation of the Value of a Hill Climbing Algorithm on a 9 by 9 Array with Varying Iteration Number

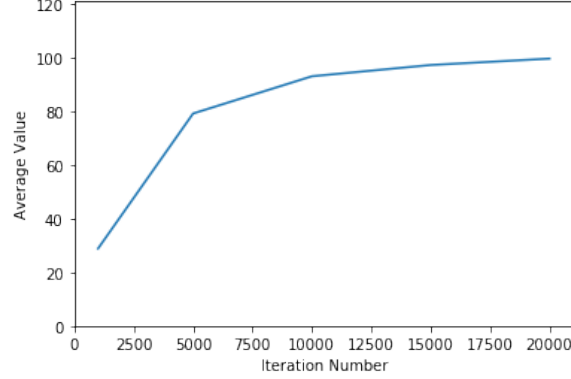


Grid Side Length	Iteration Number	Average Value
9	1,000	33.96
9	5,000	60.02
9	10,000	65.30
9	15,000	66.24
9	20,000	67.10

Figure 12: A graph showing the effects of iteration size on final value for a puzzle of size 9 by 9.

For a 9 by 9 grid, there is a large increase in average value when iteration number increases from 1,000 to 5,000, and another smaller increase when iteration number increases from 5,000 to 10,000. Average value beyond that point increases very slightly. This suggests that the optimal number of iterations that maximizes average value while minimizing computation time is 10,000 iterations for a 9 by 9 grid.

Variation of the Value of a Hill Climbing Algorithm on a 11 by 11 Array with Varying Iteration Number



Grid Side Length	Iteration Number	Average Value
11	1,000	28.86
11	5,000	79.20
11	10,000	93.08
11	15,000	97.28
11	20,000	99.66

Figure 13: A graph showing the effects of iteration size on final value for a puzzle of size 11 by 11.

For an 11 by 11 grid, increasing iteration number from 1,000 to 5,000 once again resulted in a steep increase in average value, while subsequent increases in iteration number led to diminishing returns. After 10,000 iterations, subsequent increases in iteration number lead to very small gains. This suggests that the optimal number of iterations that maximizes average value while minimizing computation time is 10,000 iterations for a 11 by 11 grid.

Hill Climbing with Random Restarts

The next algorithm we tested was hill-climbing with random restarts. The idea was that a basic hill-climbing algorithm might get stuck at local maxima, where every change would result in a worse grid. In order to circumvent this problem, random restarts would cause the algorithm to start again with a random grid of the same size and run hill-climbing again, repeating the process and keeping track of the best grid.

Varying Restart Number

All random restart tests were performed with a total of 10,000 iterations. This means that for 2 restarts, there were 5,000 iterations per restart. We tested the effects of having 2, 5, 10, 50, and 100 restarts on grids of size 5 by 5, 7 by 7, 9 by 9, and 11 by 11.

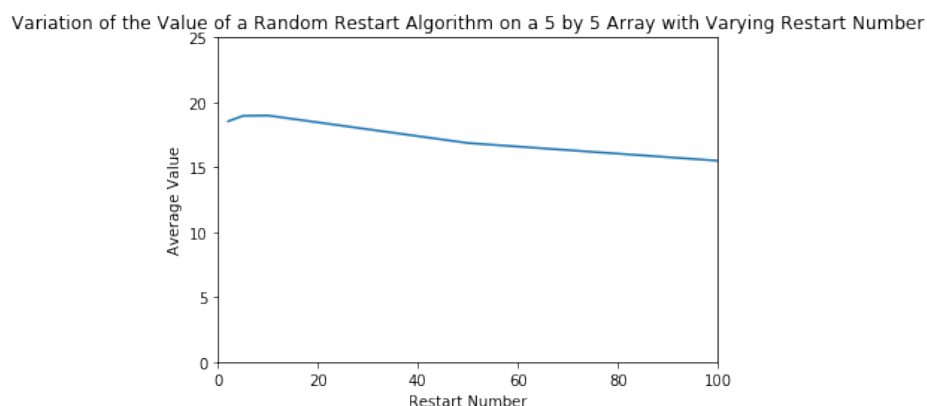


Figure 14: A graph showing the effects of restart number on final value for a puzzle of size 5 by 5.

For a 5 by 5 grid, there is a small increase in average value when restart number increases from 2 to 5, but then average value stays the same until 10 restarts, then plummets. This suggests that the optimal number of restarts for a 5 by 5 grid would be around 2 or 5.

Variation of the Value of a Random Restart Algorithm on a 7 by 7 Array with Varying Restart Number

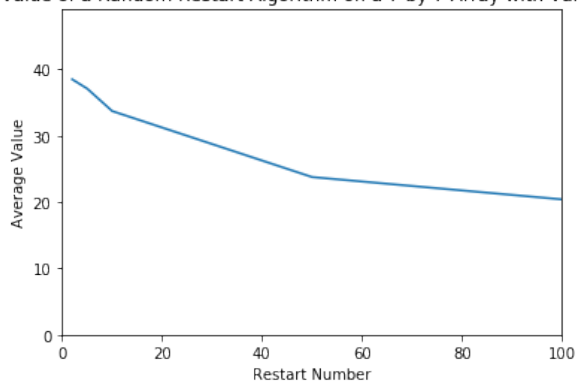


Figure 15: A graph showing the effects of restart number on final value for a puzzle of size 7 by 7.

Variation of the Value of a Random Restart Algorithm on a 9 by 9 Array with Varying Restart Number

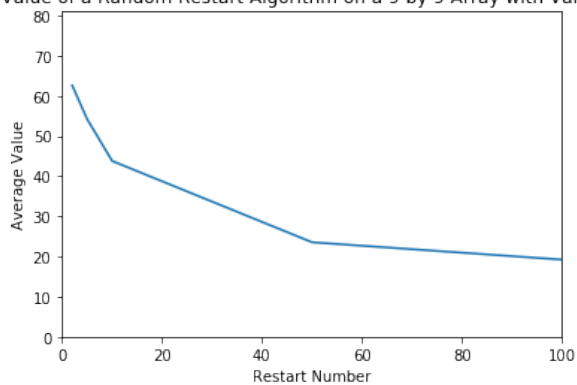


Figure 16: A graph showing the effects of restart number on final value for a puzzle of size 9 by 9.

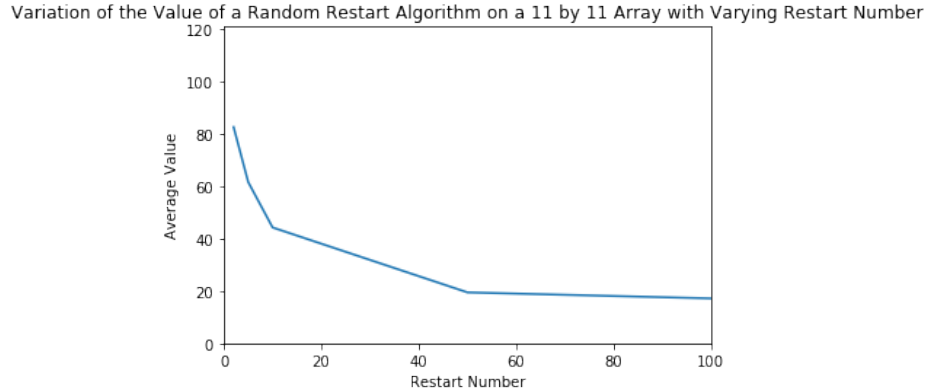


Figure 17: A graph showing the effects of restart number on final value for a puzzle of size 11 by 11.

For grids of larger size, an increase in restart number leads to a decrease in average value, with larger grids leading to a steeper decline. For the hill climbing algorithm, the larger the grid was, the higher the optimal number of iterations was before the average value plateaued. Because an increase from 2 restarts to 5 decreases the number of iterations per restart from 5,000 to just 2,000, this suggests that perhaps there are not enough total iterations for an increase in the number of restarts to outweigh the cost of decreasing the number of iterations per restart. For a 5 by 5 grid, which had an optimal iteration number of 1,000, having 5 restarts improved the average value, because having 2,000 iterations per restart was enough to reach the optimal value achievable by a hill climbing algorithm. For an 11 by 11 grid on the other hand, the optimal number of

Hill Climbing with Random Walk

Another variation of the Hill Climbing algorithm we tested is the Hill Climbing with Random Walk. In order to escape potential local maxima in the optimization of the puzzles, we adjusted the hill climb to, based on random chance, accept a randomization regardless of the modified graph's evaluation function value. We believed that allowing for reverse walks, in terms of efficiency, would allow more beneficial changes to occur and generate a favorable, final puzzle solution.

Varying Walk Threshold

As a variation on the Hill Climbing algorithm, Random Walk's main difference is the presence of a condition in which a random probability threshold is used to determine if the algorithm will accept a change regardless if it makes the value better or worse. We examine the effect this addition has on the effectiveness of the algorithm by testing a number of different probabilities over grids of increasing size.

As a note, walk threshold means the likelihood of accepting a change (walking backwards). Zero is equivalent to Hill Climbing. One is purely random.

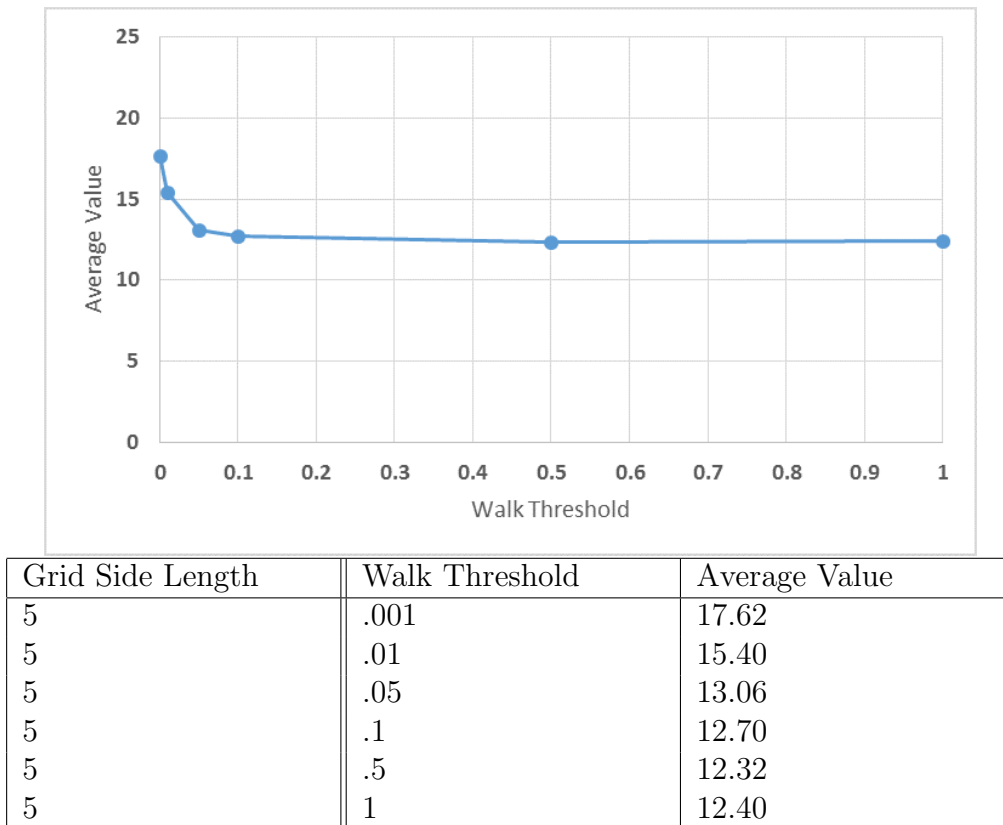


Figure 18: A graph showing the effects of varying Walk probability on a 5x5 grid. 10,000 iterations for each condition

For the smallest possible grid, there is an exponential decrease in the average value produced by the Random Walk algorithm. The best value is produced at with a probability of .001 which is approximately equivalent to hill climbing performed for an equivalent number of iterations. The average value quickly degenerates to an average value of approximately 12 after two increases to the threshold. The average values appear to follow an exponential decrease over the course of consecutive increases to the walk threshold, and the algorithm performs better at lower thresholds approaching Hill Climbing.

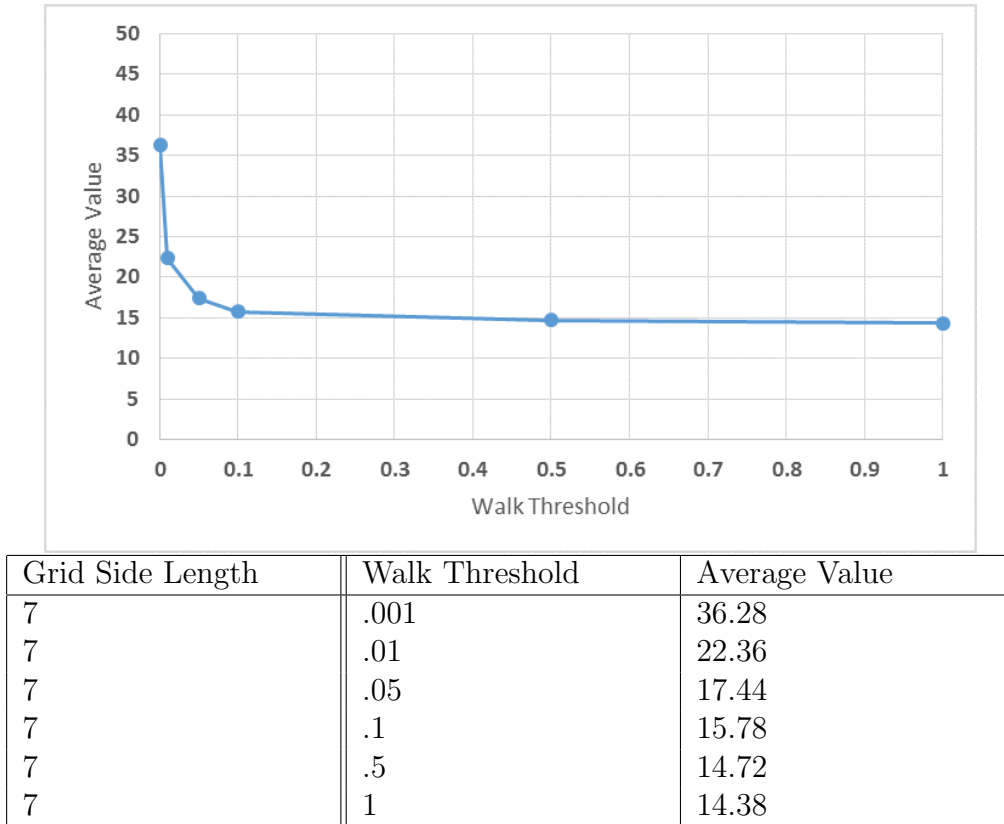


Figure 19: A graph showing the effects of varying probability on a 7x7 grid. 10,000 iterations for each condition

The graphs of 5x5 and 7x7 experience similar trends. Given that the average values are growing with the graphs, the values and curves for these graphs are experience steeper and steeper exponential decreases. By a thresh-

old probability of .1, the 7x7 grid has already effectively approached its lower limit at a value approximately between 14 and 15. Barring insignificant variations, there is no significant variation in the trend observed in the 5x5 grid.

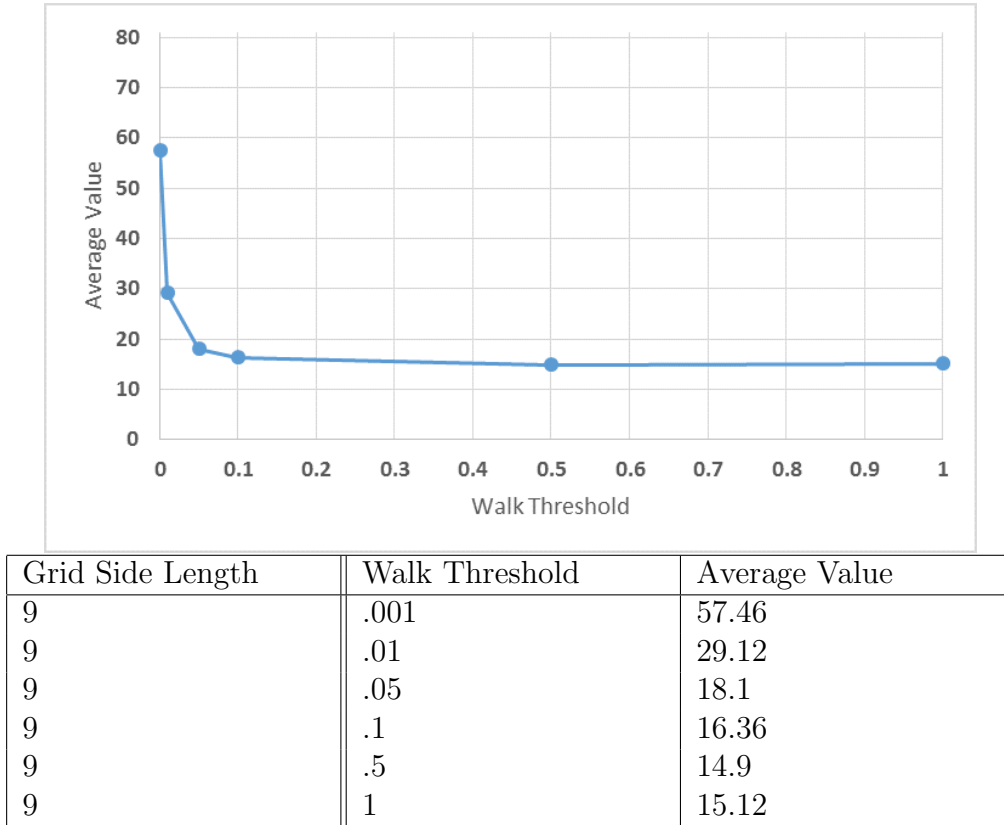


Figure 20: A graph showing the effects of varying probability on a 9x9 grid. 10,000 iterations for each condition

As before, the graph of 9x9 has experienced an even more drastic decrease from an average roughly equivalent to Hill Climbing to a lower limit that is apparently the average value of a grid resulting from random changes. However the transition from 7x7 to 9x9 has widened a gap between Random Walk and Hill Climbing at our lowest Walk Threshold. For 10,000 iterations the difference between the average values of a 7x7 grid for Hill Climbing and Random Walk is approximately 1 whereas for a 9x9 grid it is 8. This may be a result of the increase in grid size, but it may also be a deficiency with the Random Walk algorithm.

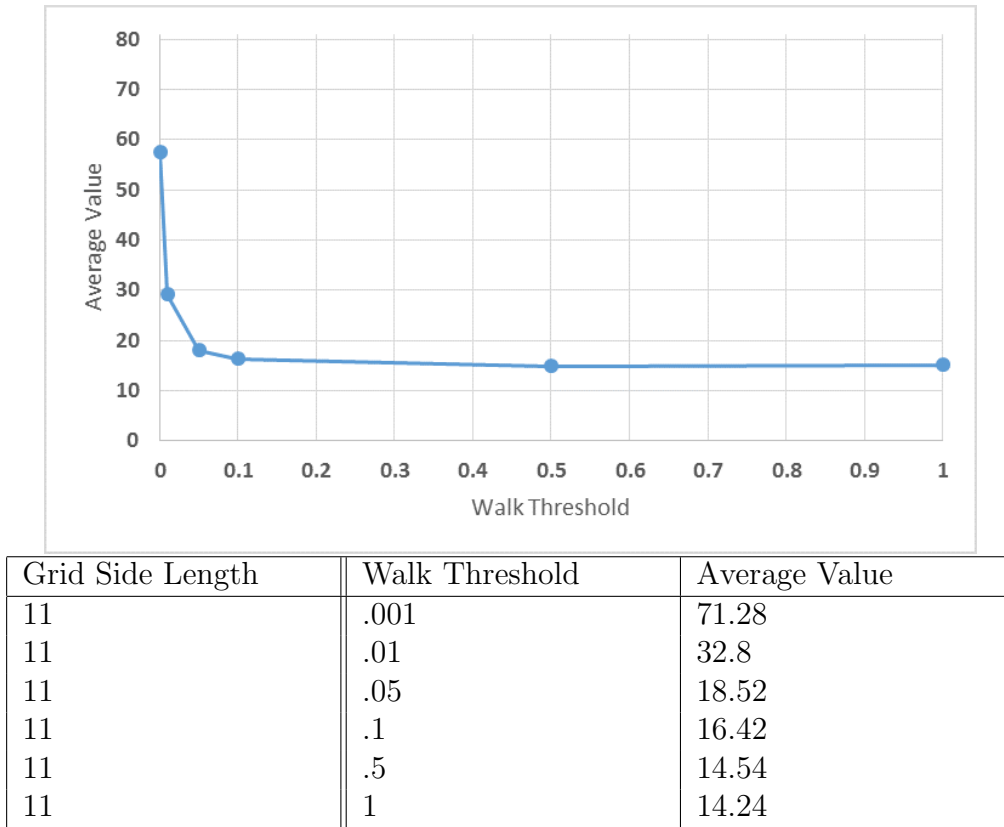


Figure 21: A graph showing the effects of varying probability on a 9x9 grid. 10,000 iterations for each condition

At the largest grid size, the trend remains consistent. As before, the exponential decrease in average value as a result of the initial two threshold probability increases has increased. The lower limit of the average value is again fairly consistent as well. This is notable if only because it represents highlights the effects of the algorithm and illustrates what the average of purely randomized changes can produce as a best puzzle's value across all 4 grid sizes.

Overall, Random Walk seems to perform the best with threshold values approaching zero. It is possible that the average value could be improved over pure Hill Climbing at some probability incredibly close to zero, but this is likely only to be relevant at iteration numbers greatly surpassing 10000. A re-evaluation of threshold probabilities may give a slightly less homogenous

curve, though the inverse correlation of average value and walk threshold as the probability approaches 1 is likely to remain consistent. For example, a range of probabilities much closer to zero: .00001, .0001, .001, .01. Due to resource constraints we were unable to perform this analysis.

Genetic Algorithm based upon a Population

The genetic algorithm we have implemented takes inspiration from the biological principles of natural selection and meiotic cross-over events. Inside a population of graphs, their values are evaluated and set as their fitness values. The probability of each member of the population surviving is assigned by evaluation using their fitness values with a selection algorithm designed to select for maximum fitness inside a population. Using a random number generator, members of the population are selected based upon their assigned probability to progress to the next iteration. Surviving members are then crossed at either a random or pre-determined point in their solution (in this case, their grids) and subsequently randomly mutated to further introduce variation. The cycle is then repeated.

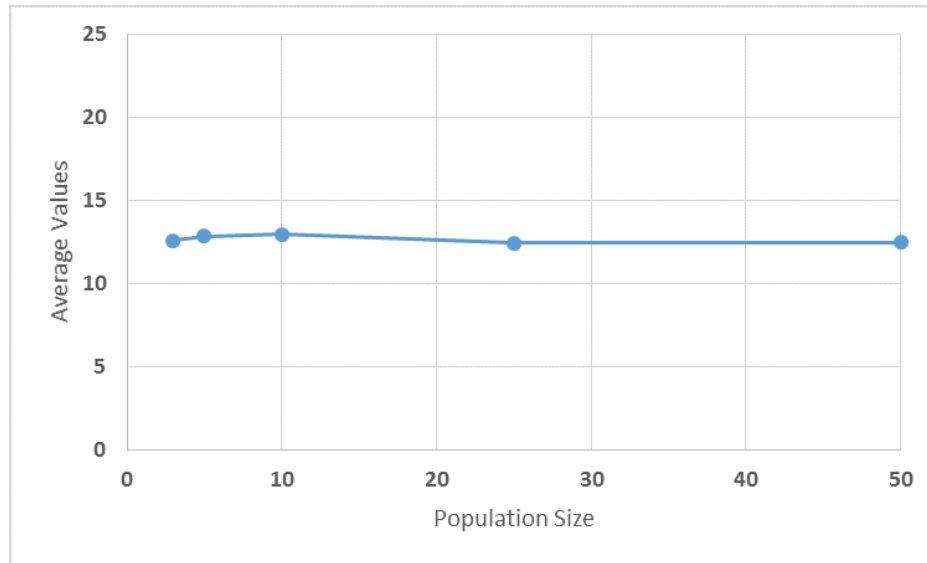
Genetic algorithm is surprisingly, far less effective for an experiment of this nature compared to Hill Climbing and its variants. We find that the reason for this is simply due to the limited size of both population and iterations that can be tested with our hardware.

Selection Probability Equation

The Selection Probability Equation assigns probabilities based upon the distance a particular member is from the ideal. In this instance, the ideal is the maximum possible moves a puzzle may have or n -squared, if n is the length of one side of a puzzle. The reciprocal of the difference is divided by the total of the population's reciprocals to produce the normalized selection probability. The base probability of each member of a population is represented in the equation (1).

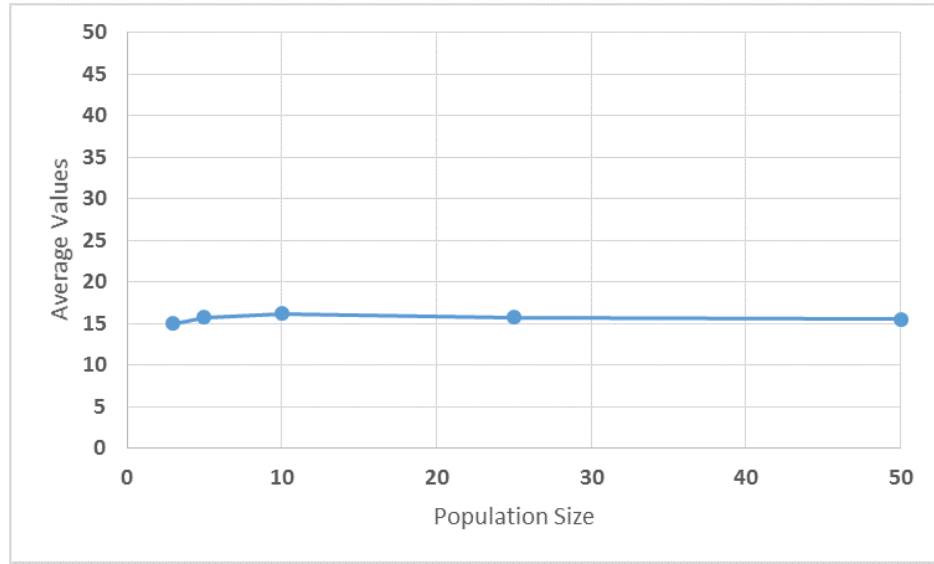
$$\frac{1}{Ideal - Member} \tag{1}$$

Varying Population Size



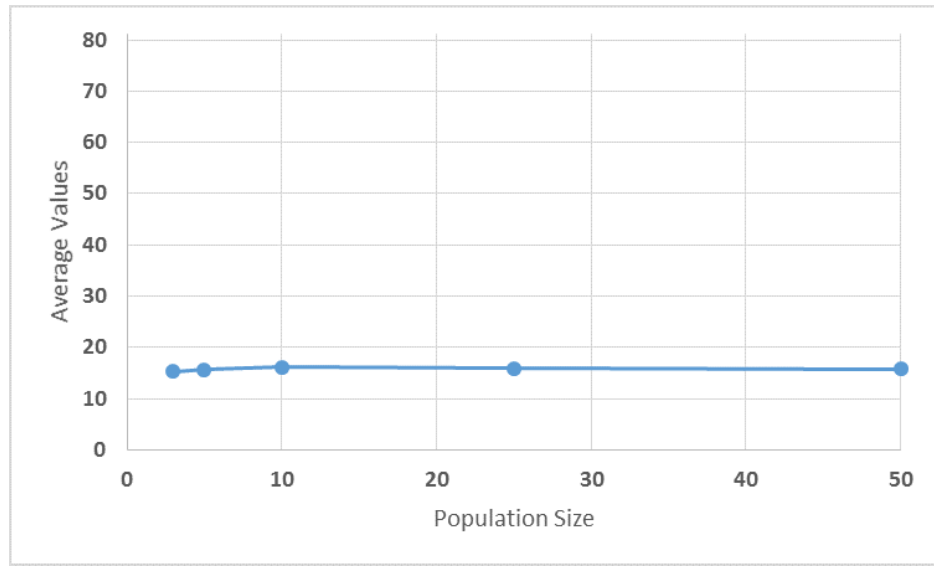
Grid Side Length	Population Size	Average Value
5	3	12.60
5	5	12.86
5	10	12.98
5	25	12.44
5	50	12.50

Figure 22: A graph showing the effects of varying population size on a 5x5 grid. 10,000 iterations for each condition. Random Crossover position.



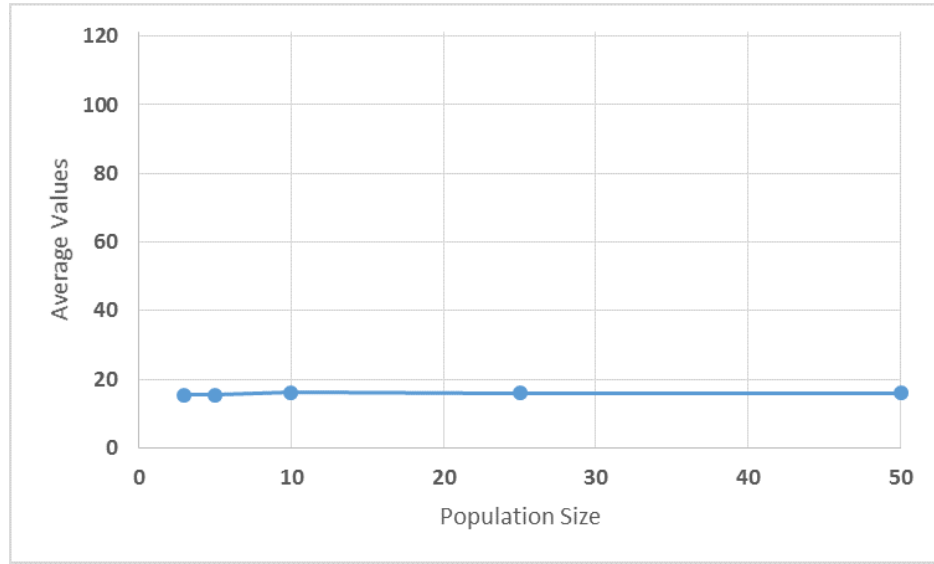
Grid Side Length	Population Size	Average Value
7	3	14.94
7	5	15.7
7	10	16.18
7	25	15.72
7	50	15.52

Figure 23: A graph showing the effects of varying population size on a 7x7 grid. 10,000 iterations for each condition. Random Crossover position.



Grid Side Length	Population Size	Average Value
9	3	15.28
9	5	15.62
9	10	16.1
9	25	15.95
9	50	15.75

Figure 24: A graph showing the effects of varying population size on a 7x7 grid. 10,000 iterations for each condition. Random Crossover position.

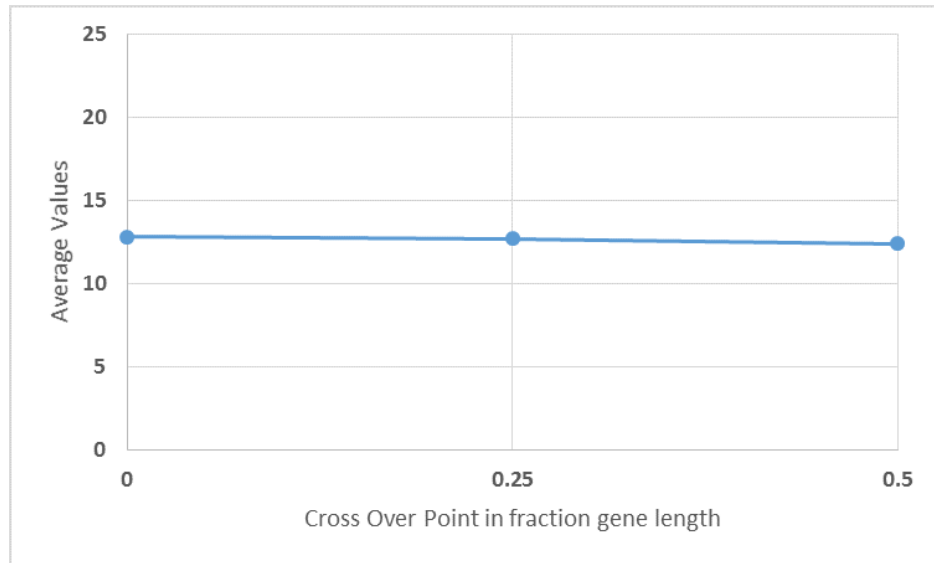


Grid Side Length	Population Size	Average Value
11	3	15.42
11	5	15.52
11	10	16.14
11	25	15.90
11	50	15.96

Figure 25: A graph showing the effects of varying population size on a 11x11 grid. 10,000 iterations for each condition. Random Crossover position.

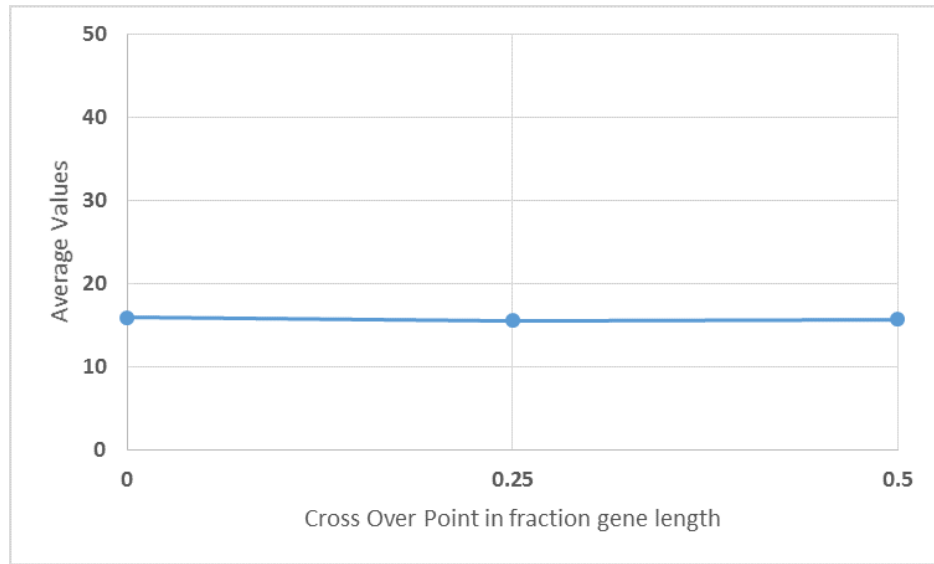
One variable in the genetic algorithm is the population size, or the number of members of a population. This variable has a bearing on the number of available cross overs as well as affecting the relative fitness levels of the members of a population as a result of our selection algorithm. Despite this, the average values of all grid sizes remains relatively consistent over all population sizes, value roughly between 12 and 17. What this seems to suggest is that despite its theorized effect on the aforementioned properties of the genetic algorithm, population size is negligible in the overall efficiency.

Varying Crossover Location



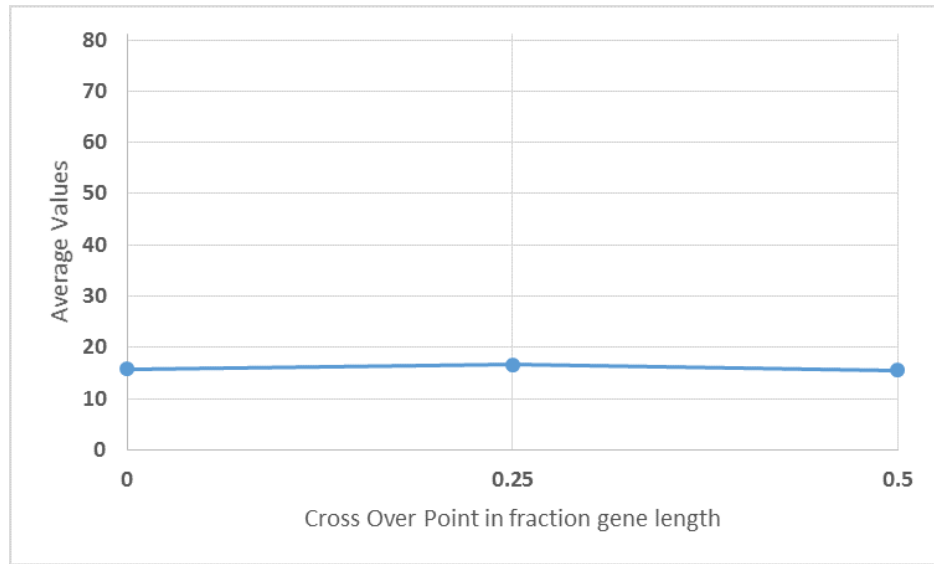
Grid Side Length	Crossover Point	Average Value
5	0	12.84
5	.25	12.7
5	.50	12.42

Figure 26: A graph showing the effects of varying cross over point on a 5x5 grid. 10,000 iterations for each condition. Population size of 5.



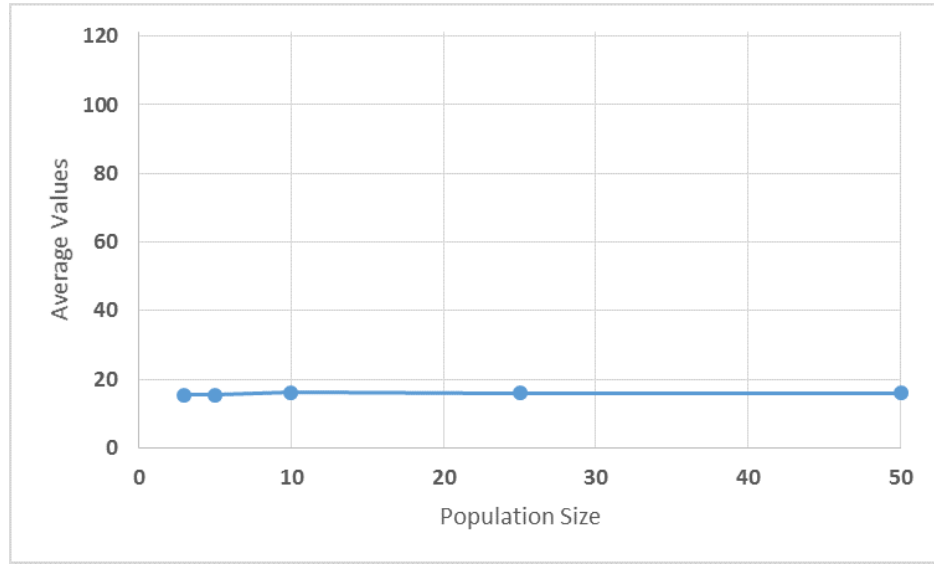
Grid Side Length	Crossover Point	Average Value
7	0	15.92
7	.25	15.56
7	.50	15.74

Figure 27: A graph showing the effects of varying cross over point on a 7x7 grid. 10,000 iterations for each condition. Population size of 5.



Grid Side Length	Crossover Point	Average Value
9	0	15.74
9	.25	16.60
9	.50	15.52

Figure 28: A graph showing the effects of varying cross over point on a 9x9 grid. 10,000 iterations for each condition. Population size of 5.



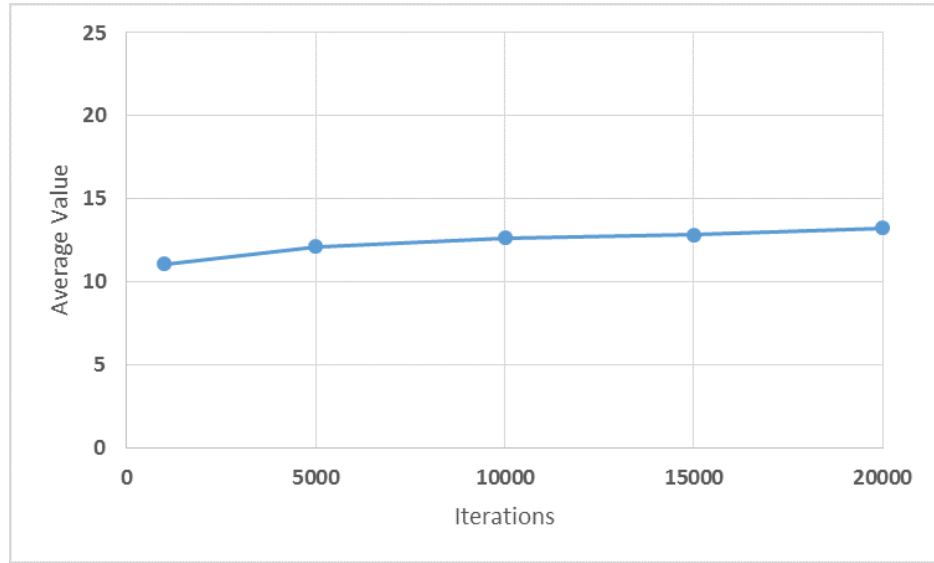
Grid Side Length	Crossover Point	Average Value
11	0	15.62
11	.25	15.88
11	.50	15.86

Figure 29: A graph showing the effects of varying cross over point on a 11x11 grid. 10,000 iterations for each condition. Population size of 5.

Crossing over in our algorithm is the swapping of portions of grids based on the location provided on a range of 0 to n-squared, 0 being a simple swap of the members to points representing .25, or one quarter, and .50, or one half, of each member with another member in the population. These notations appear in the tabular data. Members were swapped in a round robin style where, if a population were imagined to be a queue, the swaps would occur with an immediate partner with the ends of the queue wrapping around to complete the cross.

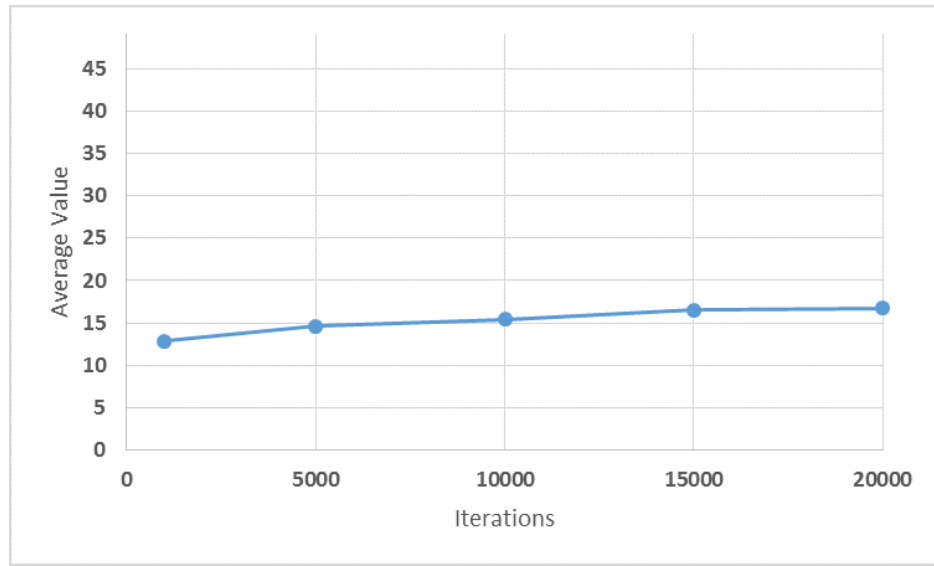
As before, the crossover point seems to be a negligible point of the genetic algorithm with all values approaching a consistent value despite variations in the position. This value is roughly between 12 and 17. One of our hypothesized explanations is that the crossing over is indiscriminate and will cross fitter members with less fit members despite the contradiction in doing so.

0.1 Varying Iterations



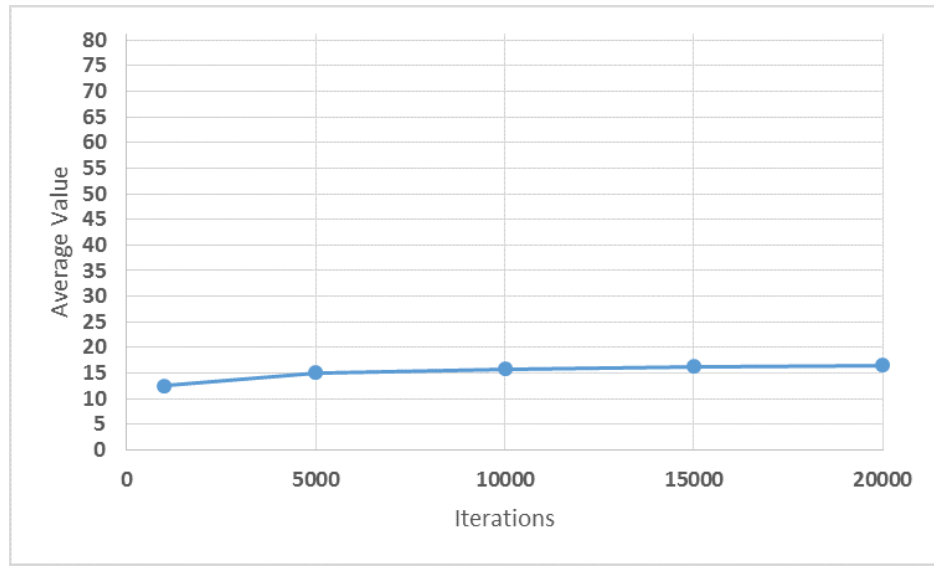
Grid Side Length	Iterations	Average Value
5	1000	11.08
5	5000	12.14
5	10000	12.66
5	15000	12.82
5	20000	13.24

Figure 30: A graph showing the effects of varying iterations on a 5x5 grid. Population of 5. Random Crossover position.



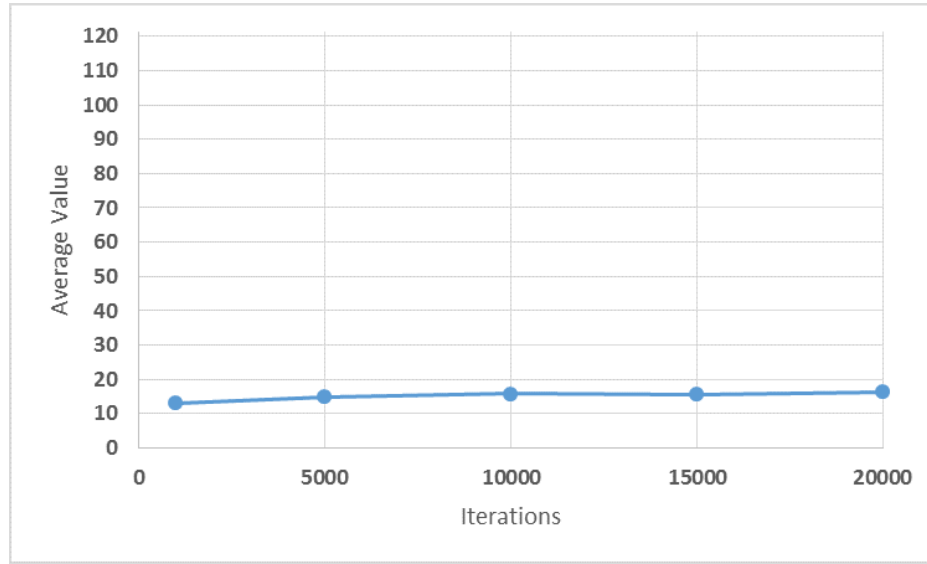
Grid Side Length	Iterations	Average Value
7	1000	12.84
7	5000	14.64
7	10000	15.40
7	15000	16.52
7	20000	16.76

Figure 31: A graph showing the effects of varying iterations on a 7x7 grid. Population of 5. Random Crossover position.



Grid Side Length	Iterations	Average Value
9	1000	12.56
9	5000	15.10
9	10000	15.82
9	15000	16.34
9	20000	16.54

Figure 32: A graph showing the effects of varying iterations on a 9x9 grid. Population of 5. Random Crossover position.



Grid Side Length	Iterations	Average Value
11	1000	13.1
11	5000	14.86
11	10000	15.76
11	15000	15.70
11	20000	16.30

Figure 33: A graph showing the effects of varying iterations on a 11x11 grid. Population of 5. Random Crossover position.

Given the biological inspiration of the genetic algorithm, another variable worth investigating was iteration number. In an algorithm utilizing some principles of natural selection, undergoing an increasing number of iterations would logically produce a more and more fit population. However, as with the previous two variations, a consistent average value was observed instead. This value was found to be between 12 and 17 as before. We hypothesize this is because of there simply being far too few iterations for the effects to be appreciable, but we are unable to test this hypothesis because of the prohibitively high costs of running more than the provided number of iterations.