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MATH 400
25 April 2015

Assignment 3

Problem 1:

We intend to find the exact values of the Gauss points $0 < x_1 < x_2 < x_3 < 1$ and weights for $n = 3$ over the interval [0,1]. We will first use the Gram-Schmidt process on the standard power function basis to find orthogonal basis polynomials. Note the standard power function basis for polynomials is as follows:

$$\vec{v} = \{1, x, x^2\}$$

For reference, we will define the following inner product space:

$$< f(x), g(x) > = \int_a^b f(x)\, g(x) dx$$

Using the Gram-Schmidt process, we will find the orthogonal basis defined as $P = p_0, p_1, p_2$:

$$p_0 = v_0$$

$$p_1 = v_1 - proj_{p_0}(v_1) = v_1 - \frac{< p_0, v_1 >}{< p_0, p_0 >}(p_0)$$

$$p_2 = v_2 - proj_{p_0}(v_2) - proj_{p_1}(v_2) = v_2 - \frac{< p_0, v_2 >}{< p_0, p_0 >}(p_0) - \frac{< p_1, v_2 >}{< p_1, p_1 >}(p_1)$$

We already have $p_0 = v_0 = 1$.

Next we solve for p_1. First we find the inner products $< p_0, v_1 >$ and $< p_0, p_0 >$:

$$< p_0, v_1 > = \int_0^1 (1)\,(x)\; dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$< p_0, p_0 > = \int_0^1 (1)(1)\; dx = [x]_0^1 = 1$$

Returning to the Gram-Schmidt:

$$p_1 = v_1 - \frac{< p_0, v_1 >}{< p_0, p_0 >}(p_0) = x - \frac{1}{2}$$

So $p_1 = x - \frac{1}{2}$.

Lastly, we solve for p_2:

$$p_2 = v_2 - \frac{< p_0, v_2 >}{< p_0, p_0 >}(p_0) - \frac{< p_1, v_2 >}{< p_1, p_1 >}(p_1)$$

$$= x^2 - \frac{\frac{1}{3}}{1}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}\left(x - \frac{1}{2}\right)$$

$$= x^2 - x + \frac{1}{6}$$

Now we have the orthogonal basis $P = \{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\}$.

We need to find the zeros, as these will be our Gauss points.

$p_0 = 1$ has no real roots.

$p_1 = x - \frac{1}{2}$ has a root at $\frac{1}{2}$

Using the quadratic formula, we can find the roots of p_2:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(\frac{1}{6})}}{2(1)}$$

$$= \frac{1 \pm \sqrt{\frac{1}{3}}}{2}$$

Finally, we have the Gauss points:

$$\boldsymbol{x_1} = \frac{1 - \sqrt{\frac{1}{3}}}{2} \sim 0.2113248$$

$$\boldsymbol{x_2} = \frac{1}{2} = .5$$

$$\boldsymbol{x_3} = \frac{1 + \sqrt{\frac{1}{3}}}{2} \sim .7886751$$

We can find the weights over the interval $[0,1]$. We already have the weights calculated over the interval $[-1,1]$, represented by the formula:

$$W_i = \frac{b-a}{2} w_i$$

We rewrite to solve for w_i and arrive at:

$$w_i = \frac{2}{b-a} W_i$$

But $W_i = \frac{b-a}{2} w_i$, so we have:

$$w_i = 2W_i$$

We have $W_1 = \frac{5}{9}, W_2 = \frac{8}{9}, W_3 = \frac{5}{9}$. Using the equation we have found, we have the following weights:

$$\left\{\frac{10}{9}, \frac{16}{9}, \frac{10}{9}\right\}$$

Problem 2: We will now estimate three integrals within seven decimal place accuracy using Gaussian quadrature, the trapezoidal method, and Romberg integration.
Here is our output:
a.

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(a)

Gauss Quadrature:
Result: 0.31782370393
Failed at n = 2
E = 0.000486182253483
Result: 0.318312471355
Failed at n = 3
E = 2.58517141793e-06
Result: 0.318309878925
Success at n = 4
Error = 7.25837207005e-09

Trapezoid Rule
Failed at n = 800
E = 1.02521524437e-07
Failed at n = 801
E = 1.02265380886e-07
Failed at n = 802
E = 1.0201019579e-07
Failed at n = 803
E = 1.01755964932e-07
Failed at n = 804
E = 1.0150268337e-07
Failed at n = 805
E = 1.01250345885e-07
Failed at n = 806
E = 1.00998948704e-07
Failed at n = 807
E = 1.00748485998e-07
Failed at n = 808
E = 1.00498954436e-07
Failed at n = 809
E = 1.00250348412e-07
Failed at n = 810
E = 1.00002664094e-07
Success at n = 811
Error = 9.97558963767e-08

Romberg:
Result: 0.319035593729
Failed at n = 1
E = 0.00072570754507
Result: 0.31830720139
Failed at n = 2
E = 2.68479332011e-06
Result: 0.318309888776
Success at n = 9
Error = 2.59233395861e-09
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b.

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Gaussian Quaddrature:
The correct answer: 0.34066362143045939975
Estimate Result: 0.34105629377
Failed at n = 2
E = 0.00039267233969
Estimate Result: 0.340660213412
Failed at n = 3
E = 3.40801850796e-06
Estimate Result: 0.340663634418
Success at n = 4
Error = 1.29876729926e-08

Trapezoid Rule:
The correct answer: 0.34066362143045939975
result: 0.34066372533692235
Failed at n = 520
E = 1.03906462934e-07
result: 0.3406637249376661
Failed at n = 521
E = 1.03507206695e-07
result: 0.3406637245407076
Failed at n = 522
E = 1.03110248173e-07
result: 0.34066372414602697
Failed at n = 523
E = 1.02715567551e-07
result: 0.34066372375360865
Failed at n = 524
E = 1.02323149231e-07
result: 0.34066372336343437
Failed at n = 525
E = 1.0193297495e-07
result: 0.3406637229754879
Failed at n = 526
E = 1.01545028497e-07
result: 0.3406637225897526
Failed at n = 527
E = 1.01159293164e-07
result: 0.3406637222062105
Failed at n = 528
E = 1.00775751077e-07
result: 0.3406637218248453
Failed at n = 529
E = 1.0039438586e-07
result: 0.3406637214456408
Failed at n = 530
E = 1.00015181359e-07
result: 0.3406637210685815
Success at n = 531
Error = 9.96381220864e-08

Romberg:
The correct answer: 0.34066362143045939975
Estimate Result: 0.340079302064
Failed at n = 3
E = 0.000584319366685
Estimate Result: 0.340667156935
Failed at n = 5
E = 3.53550501803e-06
Estimate Result: 0.340663616753
Success at n = 9
Error = 4.67776306579e-09
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C.

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Gaussian Quaddrature:
The correct answer:
0.4514116667901403133978501889534646984347655748004997
Estimate Result: -0.257524719551
Failed at n = 2
E = 0.708936386341
Estimate Result: 0.663311891667
Failed at n = 3
E = 0.211900224876
Estimate Result: 0.417079737028
Failed at n = 4
E = 0.0343319297624
Estimate Result: 0.45489707584
Failed at n = 5
E = 0.00348540904957
Estimate Result: 0.451169493952
Failed at n = 6
E = 0.000242172838171
Estimate Result: 0.451423912526
Failed at n = 7
E = 1.22457361666e-05
Estimate Result: 0.451411195633
Failed at n = 8
E = 4.71157328708e-07
Estimate Result: 0.451411681055
Success at n = 9
Error = 1.42645579793e-08

Trapezoid Rule:
The correct answer:
0.4514116667901403133978501889534646984347655748004997
Estimate Result: 0.45141176702655095
Failed at n = 2574
E = 1.00236410616e-07
Estimate Result: 0.4514117669488585
Failed at n = 2575
E = 1.00158718153e-07
Estimate Result: 0.451411766871259
Failed at n = 2576
E = 1.0008111867e-07
Estimate Result: 0.4514117667937467
Failed at n = 2577
E = 1.00003606396e-07
Estimate Result: 0.45141176671632655
Success at n = 2578
Error = 9.99261862145e-08

Romberg:
The correct answer:
0.4514116667901403133978501889534646984347655748004997
Estimate Result: 0.266666666667
Failed at n = 5
E = 0.184745000124
Estimate Result: 0.460387370678
Failed at n = 9
E = 0.00897570388798
Estimate Result: 0.451300348606
Failed at n = 17
E = 0.000111318184369
Estimate Result: 0.451412023508
Failed at n = 33
E = 3.56718070749e-07
Estimate Result: 0.451411666496
Success at n = 65
Error = 2.94153812419e-10
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Note the success points in bold for each integral. These indicate how many points before the integral is approximated to seven decimal points. For the first integral, it takes 4 points for Gaussian quadrature, 811 for trapezoidal, and 9 for Romberg. For the second integral, it takes 4 points for Gaussian quadrature, 531 for trapezoidal, and 9 for Romberg. For the last integral, it takes 9 for Gaussian quadrature, 2578 for trapezoidal, and 65 for Romberg.

Problem 3: We will approximate the function using the centered difference formula and values of h in the table below:

h	**Centered Difference Estimate**	**Relative Error**
10^{-1}	-0.5245244542619476	-0.0009110841300118411
10^{-2}	-0.5254264308094225	-9.107582536915793e-06
10^{-3}	-0.5254354473163969	-9.10755625360693e-08
10^{-4}	-0.5254355374811892	-9.107702370059201e-10
10^{-5}	-0.5254355383843556	-7.603806473355235e-12
10^{-6}	-0.5254355383566001	-3.535938208898415e-11
10^{-7}	-0.5254355384121112	2.015176914227368e-11
10^{-8}	-0.5254355339712191	-4.4207403293583525e-09
10^{-9}	-0.5254355839312552	4.553929577877369e-08
10^{-10}	**-0.525435528420104**	**-9.971855452484135e-09**
10^{-11}	-0.5254380264219094	2.488029949954118e-06
10^{-12}	-0.5254685575550866	3.301916312714592e-05
10^{-13}	-0.5248579348915428	-0.0005776035004166902
10^{-14}	-0.5245803791353865	-0.0008551592565729793
10^{-15}	-0.5828670879282072	0.05743154953624774
10^{-16}	0.0	-0.5254355383919594
10^{-17}	0.0	-0.5254355383919594
10^{-18}	0.0	-0.5254355383919594
10^{-19}	0.0	-0.5254355383919594
10^{-20}	0.0	-0.5254355383919594

Above are the estimates for *g'(1.4)* when using the centered difference formula with $h = 10^{-1}, 10^{-2}, 10^{-3}, \ldots$ The relative errors decrease to a minimum at 10^{-10}.

Python estimates the numerator of the centered difference formula to be 0 at 10^{-16} and beyond. This is most likely machine error, as 10^{-16} is a very high order of accuracy already. Note that at 10^{-10} our estimate begins to drift away from maximum accuracy. So our best estimate is -0.525435528420104.

Problem 4:

We wish to estimate $h'(1.4)$ and $\int_1^{1.8} h(x)\,dx$ using the centered difference formula with Richardson extrapolation and Romberg integration. We have the following data:

x	h(x)
1.0	0.24197072
1.1	0.21785218
1.2	0.19418605
1.3	0.17136859
1.4	0.14972747
1.5	0.12951760
1.6	0.11092083
1.7	0.09404908
1.8	0.07895016

To estimate the derivative, we will use the central difference formula with Richardson extrapolation. Recall that the formula is as follows:

$$f'(x) \approx D(h) = \frac{[f(x+h) - f(x-h)]}{2h}$$

With Richardson extrapolation:

$$D_j(h) = D_{j-1}\left(\frac{h}{2}\right) + \frac{\left[D_{j-1}\left(\frac{h}{2}\right) - D_{j-1}(h)\right]}{(2^{2(j-1)} - 1)}$$

We choose $h = 0.4$. So we have:

$$D_1(h) = \frac{[h(1.8) - h(1.0)]}{(2)(0.4)} \approx -0.20377570$$

$$D_1\left(\frac{h}{2}\right) = \frac{[h(1.6) - h(1.2)]}{(2)(0.2)} \approx -0.20816305$$

$$D_1\left(\frac{h}{4}\right) = \frac{[h(1.5) - h(1.3)]}{(2)(0.1)} \approx -0.20925495$$

$$D_2(h) = D_1\left(\frac{h}{2}\right) + \frac{\left[D_1\left(\frac{h}{2}\right) - D_1(h)\right]}{3} \approx -0.20962550$$

$$D_2\left(\frac{h}{2}\right) = D_1\left(\frac{h}{4}\right) + \frac{\left[D_1\left(\frac{h}{4}\right) - D_1(\frac{h}{2})\right]}{3} \approx -0.20961892$$

$$D_3(h) = D_2\left(\frac{h}{2}\right) + \frac{\left[D_2\frac{h}{2} - D_2(h)\right]}{15} \approx -0.20961848$$

Using Richardson extrapolation, we can estimate the derivative $h'(1.4)$ within about six decimal places to be -0.20961848.

Next we wish to approximate the integral $\int_1^{1.8} h(x)\, dx$ using Romberg integration. Recall that Romberg integration is from the trapezoidal method:

$$I = \int f(x)\ dx \approx T(h) = \frac{h}{2}[f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)]$$

Using this we then can use Romberg integration:

$$T_j(h) = T_{j-1}\left(\frac{h}{2}\right) + \frac{\left[T_{j-1}\left(\frac{h}{2}\right) - T_{j-1}(h)\right]}{(2^{2(j-1)} - 1)}$$

We choose h=0.4. So we have:

$$T(h) = \left(\frac{0.4}{2}\right)[h(1.0) + 2h(1.1) + 2h(1.2) + \cdots + 2h(1.7) + h(1.8)] \approx 0.491232896$$

$$T\left(\frac{h}{2}\right) = \left(\frac{0.2}{2}\right)[h(1.0) + 2h(1.1) + 2h(1.2) + \cdots + 2h(1.7) + h(1.8)] \approx 0.245616448$$

$$T\left(\frac{h}{4}\right) = \left(\frac{0.1}{2}\right)[h(1.0) + 2h(1.1) + 2h(1.2) + \cdots + 2h(1.7) + h(1.8)] \approx 0.122808224$$

$$T_2(h) = T\left(\frac{h}{2}\right) + \frac{\left[T\left(\frac{h}{2}\right) - T(h)\right]}{3} \approx 0.163744299$$

$$T_2\left(\frac{h}{2}\right) = T\left(\frac{h}{4}\right) + \frac{\left[T\left(\frac{h}{4}\right) - T\left(\frac{h}{2}\right)\right]}{3} \approx 0.081872149$$

$$T_3(h) = T_2\left(\frac{h}{2}\right) + \frac{\left[T_2\left(\frac{h}{2}\right) - T_2(h)\right]}{15} \approx 0.076414006$$

Using Romberg integration we have approximated the integral to be 0.076414006.