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11 April 2015

MATH 400

Assignment 2

Problem 1: We have the Lagrange interpolating polynomial:

$$p(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$$

with nodes:

$$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n$$

and:

$$L_i(x) = \frac{(x - x_0)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}$$

We wish to prove:

$$L_0(x) = 1 + \frac{(x - x_0)}{(x_0 - x_1)} + \frac{(x - x_0)(x - x_1)}{(x_0 - x_1)(x_0 - x_2)} + \cdots + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

and thus, a general result for $L_i(x)$.

We begin with a base case, n=1, $x_0 < x$. Note that $L_{n\,i}$ indicates n is the index of the last node. We want to show:

$$L_{1\,0}(0) = L_0(x) = \frac{x - x_1}{x_0 - x_1} = 1 + \frac{x - x_0}{x_0 - x_1}$$

We begin with the right hand side:

$$1 + \frac{x - x_0}{x_0 - x_1} = \frac{x_0 - x_1}{x_0 - x_1} + \frac{x - x_0}{x_0 - x_1}$$

$$= \frac{x - x_1}{x_0 - x_1}$$

We assume the following:

$$L_{n\,0}(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

We wish to show that

$$L_{n+1\,0}(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)(x-x_{n+1})}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)(x_0-x_{n+1})}$$

$$= 1 + \frac{(x-x_0)}{(x_0-x_1)} + \frac{(x-x_0)(x-x_1)}{(x_0-x_1)(x_0-x_2)} + \cdots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)(x_0-x_{n+1})}$$

We can rewrite the right hand side using our induction hypothesis:

$$L_{n+1\,0}(x) = L_{n\,0}(x) + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)(x_0-x_{n+1})}$$

$$= \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)(x_0-x_{n+1})} + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)(x_0-x_{n+1})}$$

We use a common factor to make the denominators the same and simplify:

$$= \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})(x-x_n) + (x-x_0)(x-x_1)\dots(x-x_{n-1})(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)(x_0-x_{n+1})}$$

We simply further:

$$= \frac{(x-x_1)(x-x_2)\dots(x-x_n)(x-x_{n+1})}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)(x_0-x_{n+1})}$$

Which is what we intended to prove. So we can make a general statement:

$$L_{n\,i}(x) = \frac{(x-x_0)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_{n-1})(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}$$

$$= 1 + \frac{(x-x_0)}{(x_i-x_1)} + \frac{(x-x_0)(x-x_1)}{(x_i-x_1)(x_i-x_2)} + \cdots + \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})}{(x_0-x_1)(x_0-x_2)\dots(x_i-x_{i-1})(x_i-x_{i+1})} + \cdots$$

$$+ \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})(x-x_{n-1})}{(x_0-x_1)(x_0-x_2)\dots(x_i-x_{i-1})(x_i-x_{i+1})(x-x_n)}$$

Problem 2:

Proof:

First we define A_i, B_i, C_i $\quad such\ that\ A_i(x) = \delta_{ij}\ , A_i^{'}(x) = o\ ,\ A_i^{''}(x) = o$

$$B_i\,(x) = 0\ , B_i^{'(x)} = \delta_{ij}\ ,\ B_i^{''(x)} = 0$$

$$C_i(x) = o, C_i^{'(x)} = o\ , C_i^{''(x)} = \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1\ if\ i = j \\ 0\ if\ i \neq j \end{cases}$$

Therefore, $A_i \ = r_k(x_k)[L_k(x_k)\,]^2, where\ L_k(x_k) is\ the\ kth\ lagrange\ polynomial\ and$

$$r(x) is\ the\ cubic\ equation\ s.t\ \ r(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2$$

$$L_k(x) = 1\ when\ \ x = x_k\ \ and\ r(x) = a_k\ \ , therefore\ A_i = a_k\ \ (1)$$

$$Since\ \ we\ want\ \ A_i = \delta_{ij} = 1\ then, a_k = 1$$

If we take $A_k^{'} we\ get\ = r^{'}(x)[L_k(x)]^2 + r(x)L_k^{'}(x)L(x)$

$$where\ r^{'}(x) = z_k + 2(x - x_k)$$

Solving for $x_k = x\ \ we\ arrive\ at\ \ A_k^{'} = b_k[1]^2 + 2L_k^{'}(x)1(1) = b_k + 2L_k^{'}(x_k) = 0$

$b_k = -2\,L_k{'}(x_k)$

$$A_k^{''} we\ get\ = r^{''}(x)[L_k(x)]^2 + 2r^{'(x)L_k^{'}(x)}L(x) + 2r^{'(x)L_k^{'}(x)}L(x) +$$
$$2r(x)L_k^{''}(x)L(x) + 2r(x)[L_k^{'}(x)]^2$$
$$where\ r_k^{''} = 2\,c_k$$

Substituting $r^{''}(x) = 2c_k\ \ and\ b_k = -2\,L_k^{'}(x_k),\ A''_i = 2c_k - 6[L^{'}{}_k(x_k)]^2 + 2L''_k$

Solving for $C_k = 3[L^{'}{}_k(x_k)]^2 - L''_k$

Therefore, the final equation for $1 - 2L_k^{'}(x - x_k)L(x) + \big(3r(x)[L_k^{'}(x)]^2 - 2L^{''}{}_k(x_k)\big)(x - x_{k)}$

Finally, $A_i = [\,1 - 2L_k^{'}(x - x_k)L(x) + \big(3r(x)[L_k^{'}(x)]^2 - 2L^{''}{}_k(x_k)\big)\big(x - x_{k)}\big][L_k(x)\,]^2$

Following a similar process

$B_k = [(x - x_k) - 2L^{'}{}_k(x)(x - x_k)^2]\ [L_k(x)\,]^2$
$C_k = [(x - x_k) - 2L^{'}{}_k(x)(x - x_k)^2]\ [L_k(x)\,]^2$

Therefore

$$\sum_{k=o}^{n} A_k f(x_k) + \sum_{k=0}^{n} B^{'}{}_k f(x_k) + \sum_{k=0}^{n} C^{''}{}_k f(x_k) \quad \blacksquare$$

Problem 3: For this problem we took the set of nodes

$$x_0 = 0 < {}^{1}/_{6} < {}^{1}/_{3} < {}^{1}/_{2} < {}^{7}/_{12} < {}^{2}/_{3} < {}^{3}/_{4} < {}^{5}/_{6} < {}^{11}/_{12} < 1 = x_9$$

and the function $f(x) = 1.6e^{-2x}\sin(3pix)$ to compute Lagrange and Hermite interpolating polynomials and linear and cubic interpolating functions. Plots were produced using these results so that a comparison could be made. In the plots below the purple or blue graphs represent the interpolant.

A quick review of the four graphs shows the Hermite interpolating polynomials to be the one with the best fit. Lagrange also seems to offer a reasonable fit for these data. For Lagrange, the error in the model seems to be larger between points x_0 and x_1.

Piecewise linear and cubic spline interpolating functions for these data do not seem to generate good results. The interpolants seem to deviate more dramatically between (x_0, x_1), (x_1, x_2) and (x_2, x_3).

```
3-Lagrange_Interpolation-Group_7.py for group 7:
Albert Que,  Yefa Qi,  Sylvia Isela Ramirez,

The value of f(0.0) is: 0.0
The value of f(0.166666666667) is: 1.14645009692
The value of f(0.333333333333) is: 1.00600741032e-16
The value of f(0.5) is: -0.588607105874
The value of f(0.583333333333) is: -0.352312530101
The value of f(0.666666666667) is: -1.03300285267e-16
The value of f(0.75) is: 0.252442958925
The value of f(0.833333333333) is: 0.30220096454
The value of f(0.916666666667) is: 0.180883284204
The value of f(1.0) is: 7.95542022852e-17

The value of f'(0.0) is: 15.0796447372
The value of f'(0.166666666667) is: -2.29290019384
The value of f'(0.333333333333) is: -7.74214775702
The value of f'(0.5) is: 1.17721421175
The value of f'(0.583333333333) is: 4.02509242921
The value of f'(0.666666666667) is: 3.97495119654
The value of f'(0.75) is: 1.87433291778
The value of f'(0.833333333333) is: -0.60440192908
The value of f'(0.916666666667) is: -2.06655135884
The value of f'(1.0) is: -2.04080799162

In [36]:
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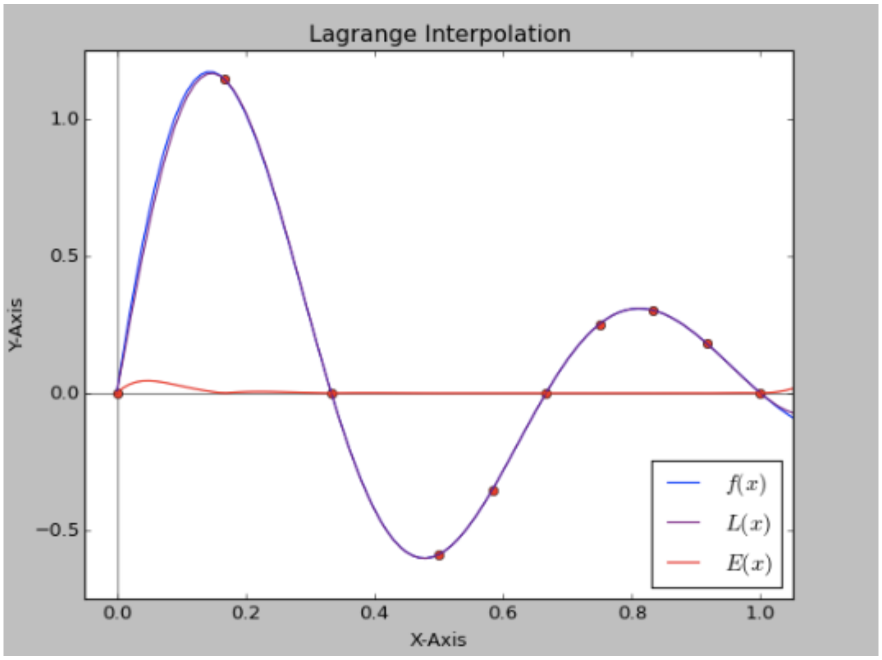

Lagrange Interpolation
1.0
0.5
0.0
−0.5
0.0
0.2
0.4
0.6
0.8
1.0
Y-Axis
X-Axis
$f(x)$
$L(x)$
$E(x)$

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3-Hermite_Interpolation-Group_7.py for group 7:
Albert Que,  Yefa Qi,  Sylvia Isela Ramirez,

The value of f(0) is: 0.0
The value of f(0.166666666667) is: 1.14645009692
The value of f(0.333333333333) is: 1.00600741032e-16
The value of f(0.5) is: -0.588607105874
The value of f(0.583333333333) is: -0.352312530101
The value of f(0.666666666667) is: -1.03300285267e-16
The value of f(0.75) is: 0.252442958925
The value of f(0.833333333333) is: 0.30220096454
The value of f(0.916666666667) is: 0.180883284204
The value of f(1) is: 7.95542022852e-17

The value of f'(0) is: 15.0796447372
The value of f'(0.166666666667) is: -2.29290019384
The value of f'(0.333333333333) is: -7.74214775702
The value of f'(0.5) is: 1.17721421175
The value of f'(0.583333333333) is: 4.02509242921
The value of f'(0.666666666667) is: 3.97495119654
The value of f'(0.75) is: 1.87433291778
The value of f'(0.833333333333) is: -0.60440192908
The value of f'(0.916666666667) is: -2.06655135884
The value of f'(1) is: -2.04080799162

In [33]:
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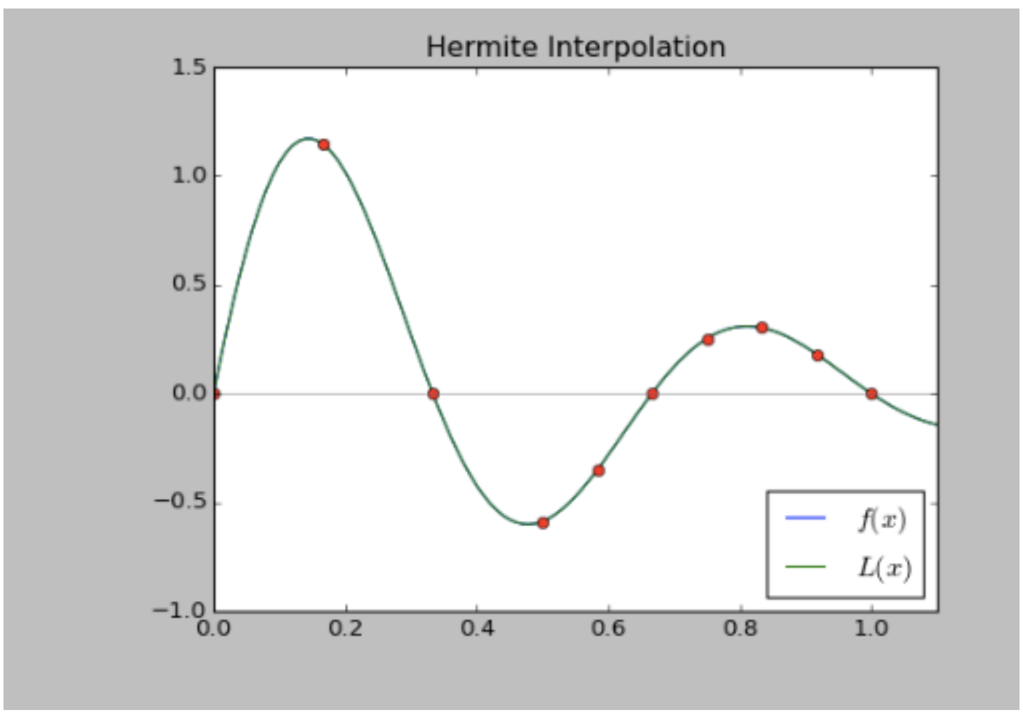

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3-Linear_Interpolation-Group_7.py for group 7:
Albert Que,  Yefa Qi,  Sylvia Isela Ramirez,

The value of f(0) is: 0.0
The value of f(0.166666666667) is: 1.14645009692
The value of f(0.333333333333) is: 1.00600741032e-16
The value of f(0.5) is: -0.588607105874
The value of f(0.583333333333) is: -0.352312530101
The value of f(0.666666666667) is: -1.03300285267e-16
The value of f(0.75) is: 0.252442958925
The value of f(0.833333333333) is: 0.30220096454
The value of f(0.916666666667) is: 0.180883284204
The value of f(1) is: 7.95542022852e-17

The value of f'(0) is: 15.0796447372
The value of f'(0.166666666667) is: -2.29290019384
The value of f'(0.333333333333) is: -7.74214775702
The value of f'(0.5) is: 1.17721421175
The value of f'(0.583333333333) is: 4.02509242921
The value of f'(0.666666666667) is: 3.97495119654
The value of f'(0.75) is: 1.87433291778
The value of f'(0.833333333333) is: -0.60440192908
The value of f'(0.916666666667) is: -2.06655135884
The value of f'(1) is: -2.04080799162

In [38]:
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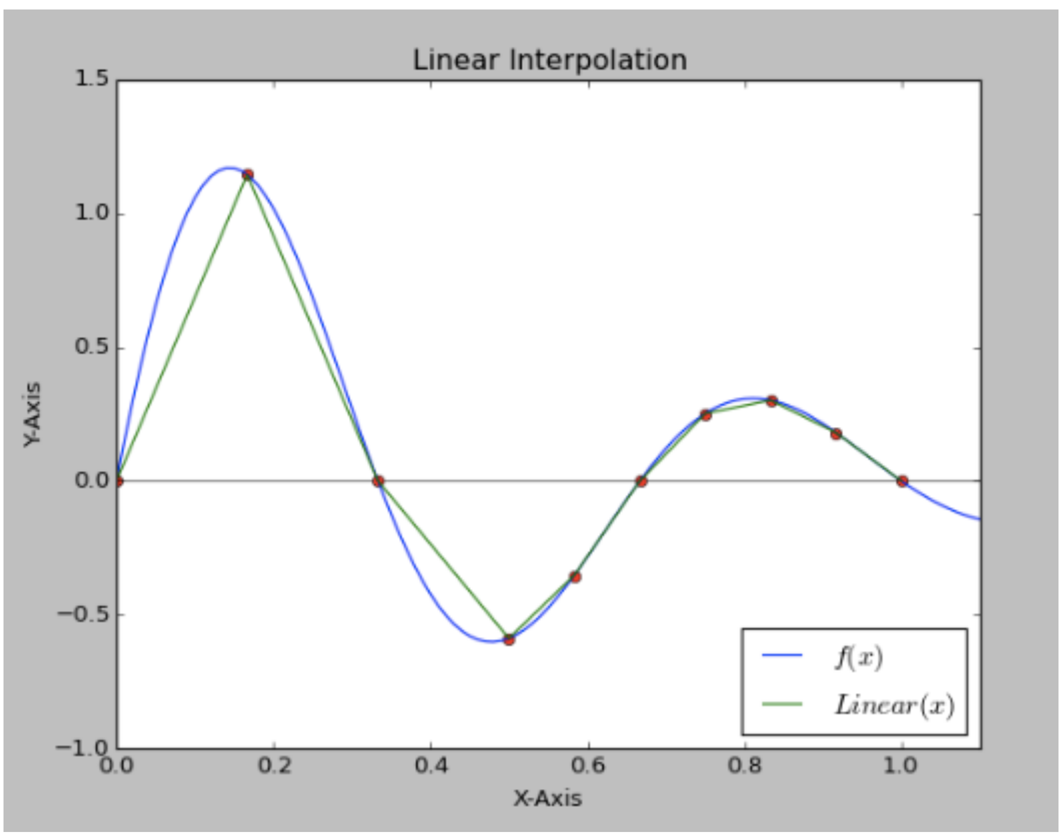

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3-Splines_Interpolation-Group_7.py for group 7:
Albert Que,  Yefa Qi,  Sylvia Isela Ramirez,

The value of f(0.0) is: 0.0
The value of f(0.166666666667) is: 1.14645009692
The value of f(0.333333333333) is: 1.00600741032e-16
The value of f(0.5) is: -0.588607105874
The value of f(0.583333333333) is: -0.352312530101
The value of f(0.666666666667) is: -1.03300285267e-16
The value of f(0.75) is: 0.252442958925
The value of f(0.833333333333) is: 0.30220096454
The value of f(0.916666666667) is: 0.180883284204
The value of f(1.0) is: 7.95542022852e-17

The value of f'(0.0) is: 15.0796447372
The value of f'(0.166666666667) is: -2.29290019384
The value of f'(0.333333333333) is: -7.74214775702
The value of f'(0.5) is: 1.17721421175
The value of f'(0.583333333333) is: 4.02509242921
The value of f'(0.666666666667) is: 3.97495119654
The value of f'(0.75) is: 1.87433291778
The value of f'(0.833333333333) is: -0.60440192908
The value of f'(0.916666666667) is: -2.06655135884
The value of f'(1.0) is: -2.04080799162

In [41]: |
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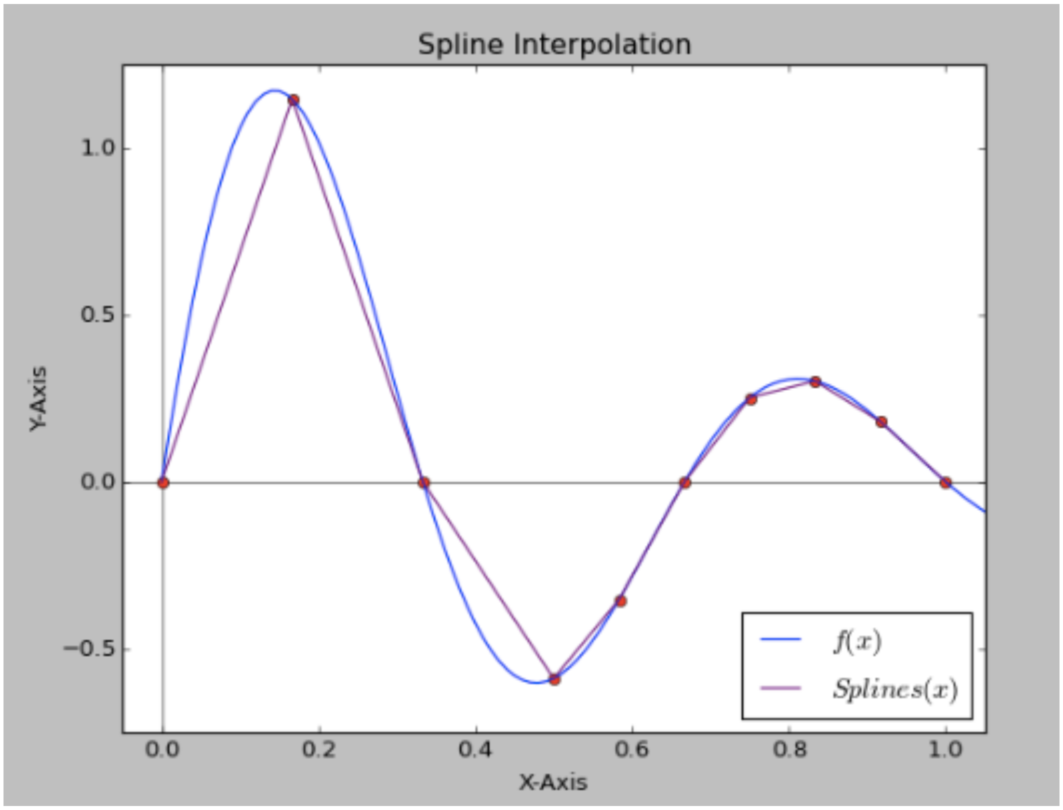


Spline Interpolation
1.0
0.5
0.0
−0.5
Y-Axis
0.0
0.2
0.4
0.6
0.8
1.0
X-Axis
$f(x)$
$Splines(x)$

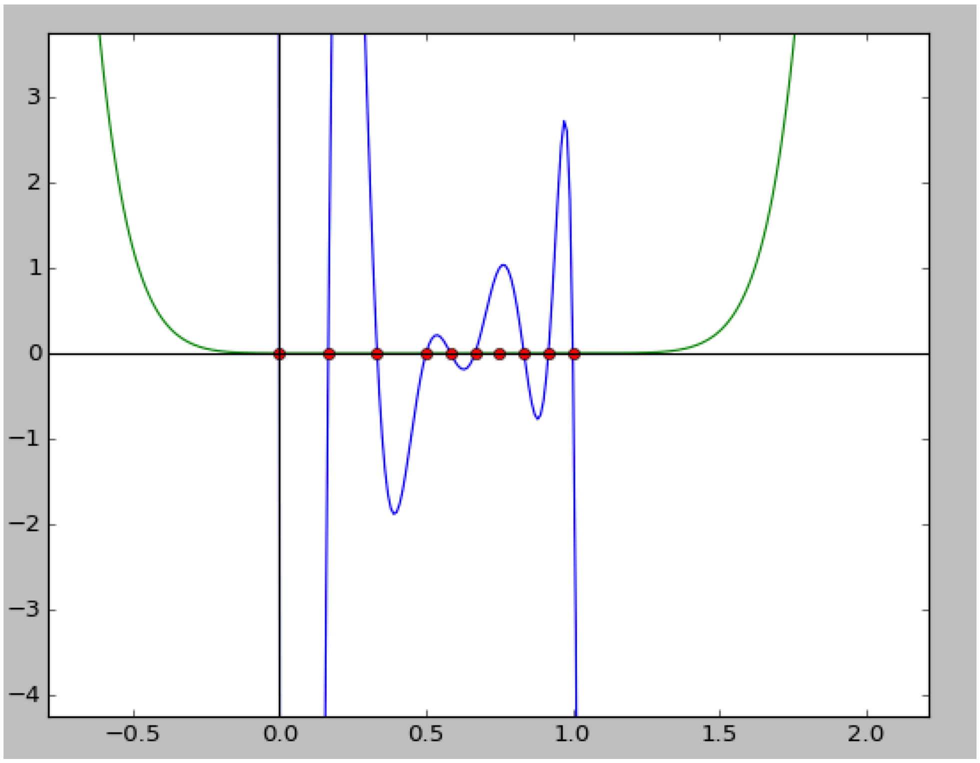
4. Using the nodes $x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n$ such that:

$$x_0 = 0 < \frac{1}{6} < \frac{1}{3} < \frac{1}{2} < \frac{7}{12} < \frac{2}{3} < \frac{3}{4} < \frac{5}{6} < \frac{11}{12} < 1 = x_9$$

We intend to create a graph of $L_{9\,6}(x)$ and the polynomial:

$$\pi(x) = (x - x_0)(x - x_1)(x - x_8)(x - x_9)$$

For our plot, $\pi(x)$ is graphed in green, $L_{9\,6}(x)$ is in blue, and the data points are red.



If we zoom in on the data points we find that the plot of $L_{9\,6}(x)$ moves toward a high critical point in the first interval and before the last data point.