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MATH 400
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Assignment 4

Problem 1:
We will approximate solutions of the following differential equation:

$$\frac{dx}{dt} = t - 2x$$

on the interval [0.2] with the initial condition $t_0 = 0, x_0 = 1$. We will use Euler's method and a 4th order Runga Kutta method, each with step sizes h=0.2, h=0.1, and h=0.05. We will also prove that $\frac{dx}{dt} = t - 2x$ is the exact solution to the equation.

Euler's Method, Step size (h = 0.2):

t	Exact Solution	*x*	Relative Error
0	1.0	1.0	0.0
0.2	0.687900057545	0.6	0.127780273574
0.4	0.511661205147	0.4	0.218232697776
0.6	0.42649276489	0.32	0.24969418864
0.8	0.402370647493	0.312	0.224595526677
1.0	0.419169104046	0.3472	0.171694677282
1.2	0.463397441612	0.40832	0.118855730882
1.4	0.526012578282	0.484992	0.0779840254306
1.6	0.600952754973	0.5709952	0.0498501000704
1.8	0.684154653059	0.66259712	0.0315097368157
2.0	0.772894548611	0.757558272	0.019842650771

Euler's Method, Step size (h = 0.1):

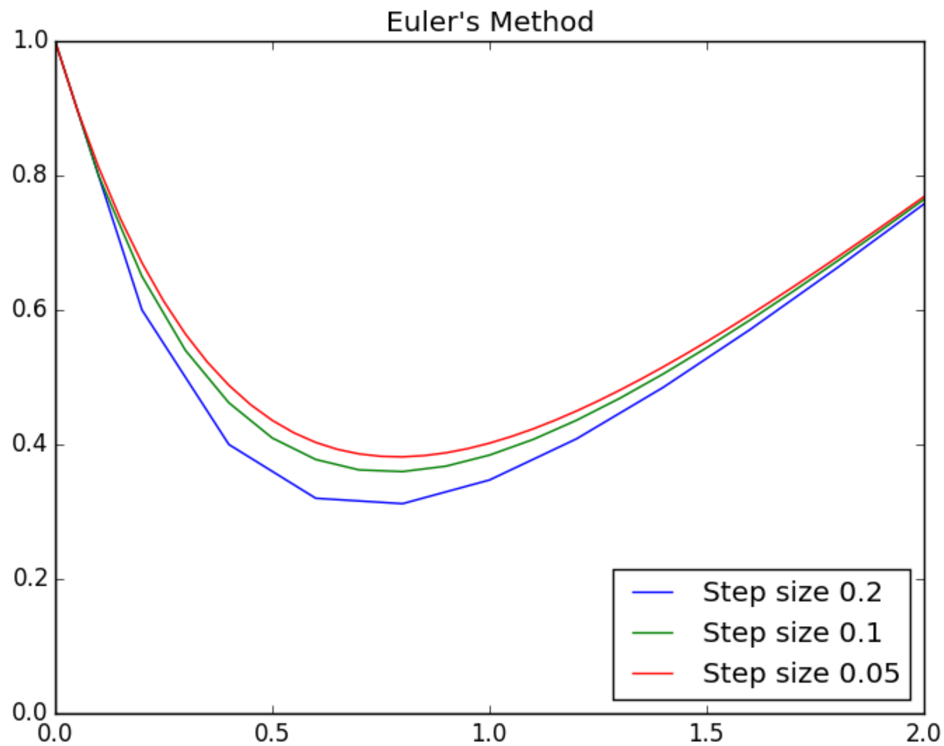
t	Exact Solution	*x*	Relative Error
0	1.0	1.0	0.0
0.1	0.823413441347	0.8	0.0284346115472
0.2	0.687900057545	0.65	0.0550952963717
0.3	0.586014545118	0.54	0.0785211655596
0.4	0.511661205147	0.462	0.0970587659315
0.5	0.459849301464	0.4096	0.109273410451
0.6	0.42649276489	0.37768	0.114451566143
0.7	0.408246204927	0.362144	0.112927454978
0.8	0.402370647493	0.3597152	0.106010335891
0.9	0.406623610277	0.36777216	0.0955464692533
1.0	0.419169104046	0.384217728	0.0833825196285

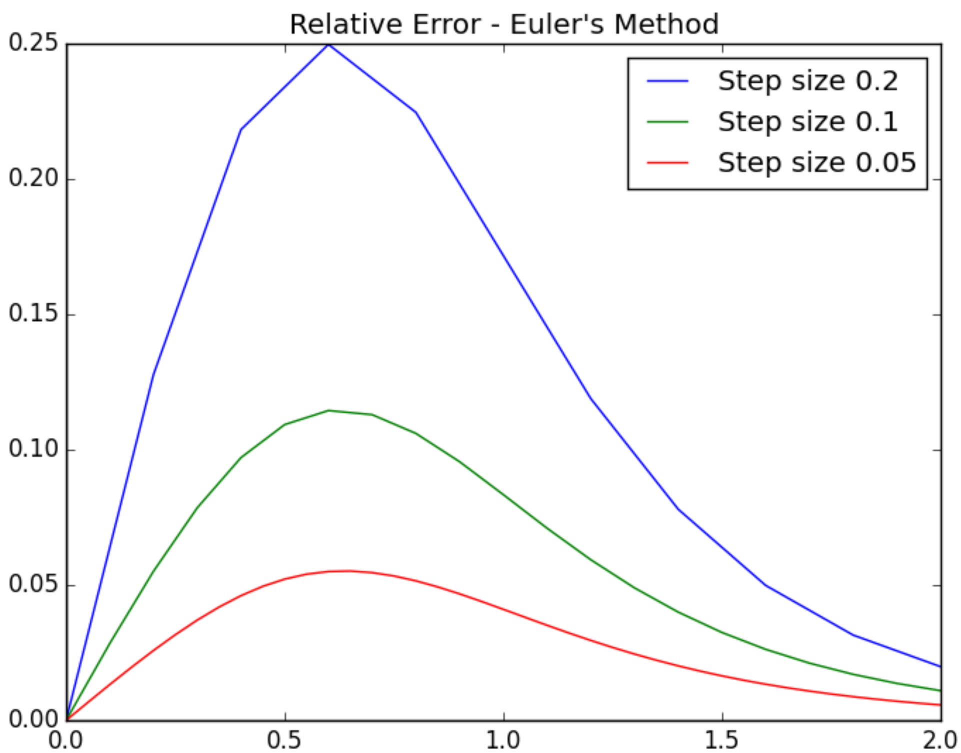
1.1	0.438503947953	0.4073741824	0.0709908444342
1.2	0.463397441612	0.43589934592	0.0593401974688
1.3	0.492841972768	0.468719476736	0.0489457013907
1.4	0.526012578282	0.504975581389	0.0399933343067
1.5	0.56223383546	0.543980465111	0.0324657983877
1.6	0.600952754973	0.585184372089	0.0262389726208
1.7	0.64171658745	0.628147497671	0.0211449883714
1.8	0.684154653059	0.672517998137	0.0170088076873
1.9	0.72796346482	0.718014398509	0.0136669857644
2.0	0.772894548611	0.764411518808	0.010975662616

<u>Euler's Method, Step size (h = 0.05):</u>

t	**Exact Solution**	***x***	**Relative Error**
0	1.0	1.0	0.0
0.05	0.906046772545	0.9	0.00667379734488
0.10	0.823413441347	0.8125	0.0132539023526
0.15	0.751022775852	0.73625	0.0196702101816
0.20	0.687900057545	0.670125	0.0258395930479
0.25	0.633163324641	0.6131125	0.0316676975126
0.30	0.586014545118	0.56430125	0.037052484957
0.35	0.545731629739	0.522871125	0.0418896459239
0.40	0.511661205147	0.4880840125	0.0460796957232
0.45	0.483212074676	0.45927561125	0.0495361450597
0.50	0.459849301464	0.435848050125	0.0521937323007
0.55	0.441088854623	0.417263245113	0.0540154421505
0.60	0.42649276489	0.403036920601	0.0549970508762
0.65	0.415664741293	0.392733228541	0.055168289425
0.70	0.408246204927	0.385959905687	0.0545903402678
0.75	0.403912700186	0.382363915118	0.0533501052513
0.80	0.402370647493	0.381627523606	0.0515522790145
0.85	0.403354405066	0.383464771246	0.0493105655232
0.90	0.406623610277	0.387618294121	0.0467393325803
0.95	0.411960774028	0.393856464709	0.0439466824527
1.0	0.419169104046	0.401970818238	0.041029469113
1.05	0.428070535316	0.411773736414	0.0380703588716
1.10	0.438503947953	0.423096362773	0.0351367080089
1.15	0.450323554654	0.435786726496	0.0322808523064
1.20	0.463397441612	0.449708053846	0.0295413537849
1.25	0.47760624828	0.464737248461	0.0269447894887
1.30	0.492841972768	0.480763523615	0.0245077526266
1.35	0.509006890925	0.497687171254	0.0222388338404
1.40	0.526012578282	0.515418454128	0.0201404388232
1.45	0.543779025071	0.533876608716	0.0182103683636
1.50	0.56223383546	0.552988947844	0.0164431363478
1.55	0.581311502992	0.57269005306	0.0148310327388
1.60	0.600952754973	0.592921047754	0.0133649561514

1.65	0.621103959252	0.613628942978	0.0120350485002
1.70	0.64171658745	0.63476604868	0.0108311658228
1.75	0.662746729278	0.656289443812	0.00974321740149
1.80	0.684154653059	0.678160499431	0.00876140153566
1.85	0.705904408088	0.700344449488	0.0078763619212
1.90	0.72796346482	0.722810004539	0.0070792842361
1.95	0.750302389307	0.745529004085	0.00636194858227
2.0	0.772894548611	0.768476103677	0.00571675003021





Modified Euler's Method, Step size (h = 0.2):

t	Exact Solution	*x*	Relative Error
0	1.0	1.0	0.0
0.2	0.687900057545	0.7	0.0175896808304
0.4	0.511661205147	0.528	0.0319328389355
0.6	0.42649276489	0.44304	0.0387983958274
0.8	0.402370647493	0.4172672	0.0370219661883
1.0	0.419169104046	0.431741696	0.029994080749
1.2	0.463397441612	0.47358435328	0.0219830986395
1.4	0.526012578282	0.53403736023	0.0152558746315
1.6	0.600952754973	0.607145404957	0.0103047201839
1.8	0.684154653059	0.688858875371	0.00687596333722
2.0	0.772894548611	0.776424035252	0.00456658239781

Modified Euler's Method, Step size (h = 0.1):

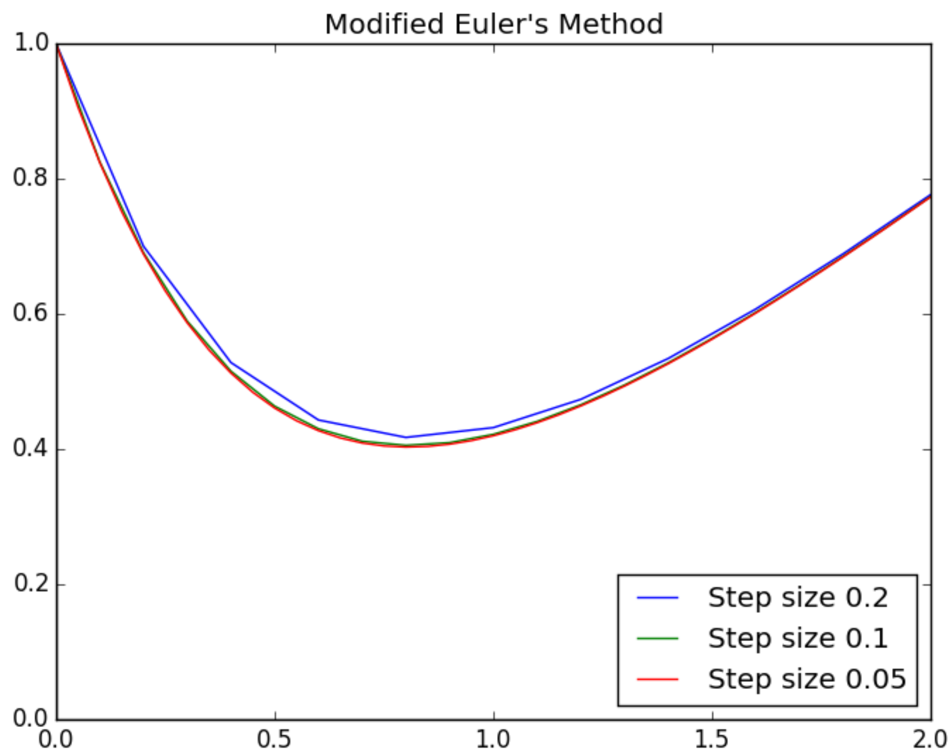
t	Exact Solution	*x*	Relative Error
0	1.0	1.0	0.0
0.1	0.823413441347	0.825	0.00192680684193

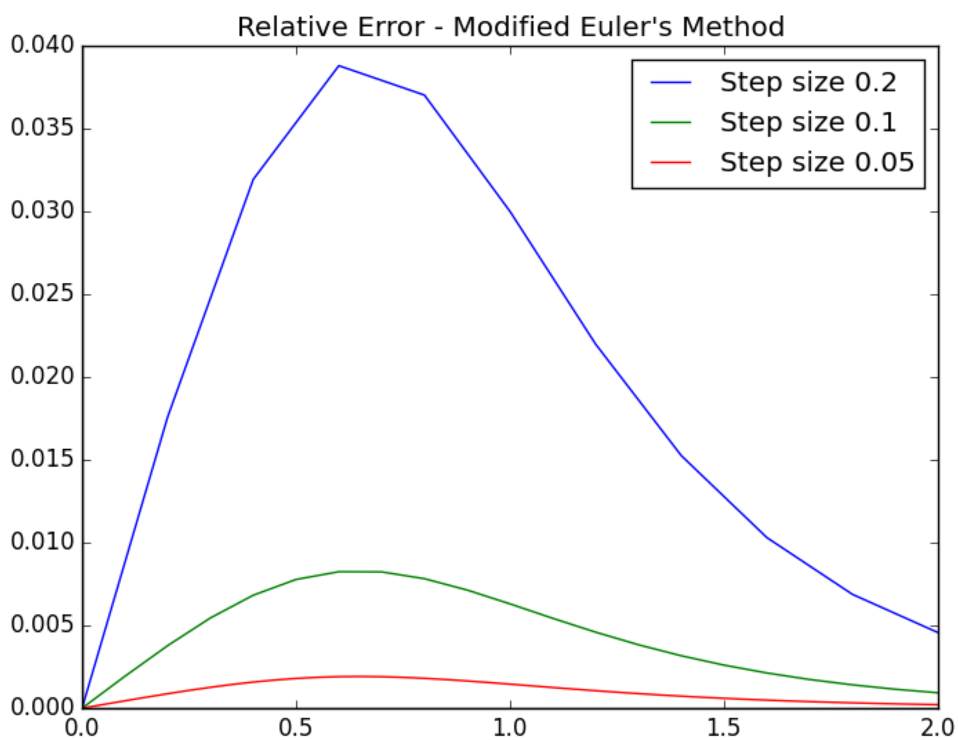
0.2	0.687900057545	0.6905	0.00377953516203
0.3	0.586014545118	0.58921	0.00545285933445
0.4	0.511661205147	0.5151522	0.00682286407169
0.5	0.459849301464	0.463424804	0.00777537885632
0.6	0.42649276489	0.43000833928	0.00824298717155
0.7	0.408246204927	0.41160683821	0.00823187880753
0.8	0.402370647493	0.405517607332	0.00782104723135
0.9	0.406623610277	0.409524438012	0.00713393827077
1.0	0.419169104046	0.42181003917	0.00630040501243
1.1	0.438503947953	0.440884232119	0.00542819324103
1.2	0.463397441612	0.465525070338	0.00459136916835
1.3	0.492841972768	0.494730557677	0.0038320293593
1.4	0.526012578282	0.527679057295	0.00316813529271
1.5	0.56223383546	0.563696826982	0.00260210508503
1.6	0.600952754973	0.602231398125	0.00212769330325
1.7	0.64171658745	0.642829746463	0.00173465831193
1.8	0.684154653059	0.685120392099	0.00141158002216
1.9	0.72796346482	0.728798721522	0.00114738821617
2.0	0.772894548611	0.773614951648	0.000932084510159

Modified Euler's Method, Step size (h = 0.05):

t	**Exact Solution**	***x***	**Relative Error**
0	1.0	1.0	0.0
0.05	0.906046772545	0.90625	0.000224301284667
0.10	0.823413441347	0.82378125	0.000446687695456
0.15	0.751022775852	0.75152203125	0.000664767319854
0.20	0.687900057545	0.688502438281	0.00087568060228
0.25	0.633163324641	0.633844706645	0.00107615519918
0.30	0.586014545118	0.586754459513	0.00126262121296
0.35	0.545731629739	0.54651278586	0.00143139242387
0.40	0.511661205147	0.512469071203	0.00157890816859
0.45	0.483212074676	0.484034509439	0.00170201616632
0.50	0.459849301464	0.460676231042	0.00179826211544
0.55	0.441088854623	0.441911989093	0.00186614207484
0.60	0.42649276489	0.427305350129	0.00190527320922
0.65	0.415664741293	0.416461341867	0.00191644971345
0.70	0.408246204927	0.40902251439	0.0019015717798
0.75	0.403912700186	0.404665375523	0.00186346043742
0.80	0.402370647493	0.403097164848	0.00180559232904
0.85	0.403354405066	0.404052934187	0.00173179990754
0.90	0.406623610277	0.40729290544	0.00164598204734
0.95	0.411960774028	0.412600079423	0.00155185987302
1.0	0.419169104046	0.419778071878	0.00145279751289

1.05	0.428070535316	0.428649155049	0.00135169250221
1.10	0.438503947953	0.43905248532	0.00125092914032
1.15	0.450323554654	0.450842499214	0.00115238156061
1.20	0.463397441612	0.463887461789	0.00105745119218
1.25	0.47760624828	0.478068152919	0.000967124363843
1.30	0.492841972768	0.493276678392	0.000882038559446
1.35	0.509006890925	0.509415393944	0.000802549095121
1.40	0.526012578282	0.52639593152	0.000728791009941
1.45	0.543779025071	0.544138318025	0.000660733382947
1.50	0.56223383546	0.562570177813	0.000598225029999
1.55	0.581311502992	0.581626010921	0.000541031662235
1.60	0.600952754973	0.601246539883	0.0004888652358
1.65	0.621103959252	0.621378118594	0.000441406528943
1.70	0.64171658745	0.641972197328	0.000398322066855
1.75	0.662746729278	0.662984838582	0.000359276467605
1.80	0.684154653059	0.684376278916	0.000323941167893
1.85	0.705904408088	0.706110532419	0.000292000346079
1.90	0.72796346482	0.72815503184	0.000263154716677
1.95	0.750302389307	0.750480303815	0.000237123738464
2.0	0.772894548611	0.773059674952	0.000213646663391


Modified Euler's Method
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
Step size 0.2
Step size 0.1
Step size 0.05


Relative Error - Modified Euler's Method
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
0.0
0.5
1.0
1.5
2.0
Step size 0.2
Step size 0.1
Step size 0.05

4th order Runge-Kutta Method, Step size (h=0.2):

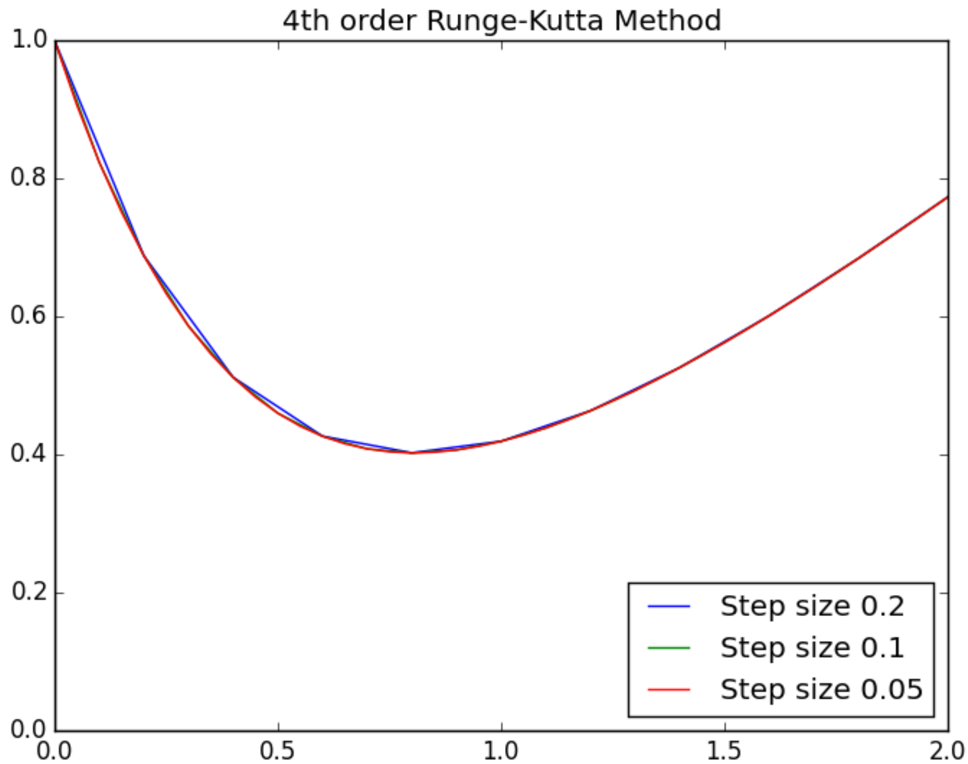
t	**Exact Solution**	***x***	**Relative Error**
0	1.0	1.0	0.0
0.2	0.687900057545	0.688	0.000145286301919
0.4	0.511661205147	0.5117952	0.000261881987778
0.6	0.42649276489	0.42662750208	0.00031591905148
0.8	0.402370647493	0.402491077394	0.000299300910399
1.0	0.419169104046	0.419270018285	0.000240748276739
1.2	0.463397441612	0.463478620258	0.000175181473528
1.4	0.526012578282	0.526076067021	0.000120698139819
1.6	0.600952754973	0.601001395331	8.09387388285e-05
1.8	0.684154653059	0.68419133543	5.36170742245e-05
2.0	0.772894548611	0.772921871272	3.53510855476e-05

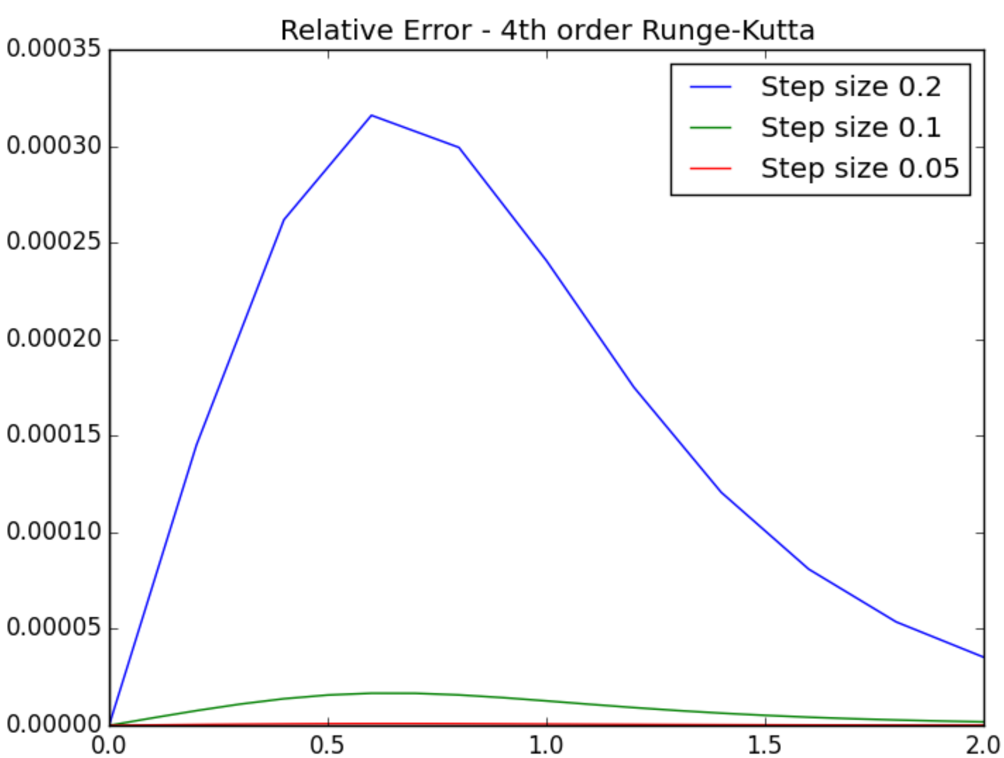
4th order Runge-Kutta Method, Step size (h=0.1):

t	**Exact Solution**	***x***	**Relative Error**
0	1.0	1.0	0.0
0.1	0.823413441347	0.823416666667	3.91701061377e-06
0.2	0.687900057545	0.687905338889	7.67748785863e-06
0.3	0.586014545118	0.586021031126	1.10679996211e-05
0.4	0.511661205147	0.511668285551	1.38380713737e-05
0.5	0.459849301464	0.459856547657	1.5757754337e-05
0.6	0.42649276489	0.426499884118	1.66924937604e-05
0.7	0.408246204927	0.408253005124	1.66570968953e-05
0.8	0.402370647493	0.402377010395	1.58135330773e-05
0.9	0.406623610277	0.406629470977	1.44130841299e-05
1.0	0.419169104046	0.419174435538	1.27191921361e-05
1.1	0.438503947953	0.438508749523	1.09498900288e-05
1.2	0.463397441612	0.463401730193	9.25464964861e-06
1.3	0.492841972768	0.492845776566	7.71808968941e-06
1.4	0.526012578282	0.526015932134	6.37599327566e-06
1.5	0.56223383546	0.562236777503	5.2327744184e-06
1.6	0.600952754973	0.600955324301	4.2754238162e-06
1.7	0.64171658745	0.641718822516	3.48294778679e-06
1.8	0.684154653059	0.684156590621	2.83205260436e-06
1.9	0.72796346482	0.727965139295	2.30021749034e-06
2.0	0.772894548611	0.772895991712	1.86713802392e-06

4th order Runge-Kutta Method, Step size (h=0.05):

t	*x*	**Exact Solution**	**Relative Error**
0	1.0	1.0	0.0
0.05	0.906046875	0.906046772545	1.13079207134e-07
0.10	0.823413626758	0.823413441347	2.25172830288e-07
0.15	0.751023027501	0.751022775852	3.35075490418e-07
0.20	0.687900361147	0.687900057545	4.41346545691e-07
0.25	0.633163668029	0.633163324641	5.42337845355e-07
0.30	0.58601491797	0.586014545118	6.3625188795e-07
0.35	0.545732023339	0.545731629739	7.21233208355e-07
0.40	0.511661612168	0.511661205147	7.95490265961e-07
0.45	0.483212489	0.483212074676	8.57437898139e-07
0.50	0.459849718016	0.459849301464	9.05843107368e-07
0.55	0.441089269225	0.441088854623	9.399520245e-07
0.60	0.426493174142	0.42649276489	9.59575687946e-07
0.65	0.415665142458	0.415664741293	9.65117954092e-07
0.70	0.408246595839	0.408246204927	9.57539460949e-07
0.75	0.403913079162	0.403912700186	9.38264150029e-07
0.80	0.402371013267	0.402370647493	9.09045528404e-07
0.85	0.403354756717	0.403354405066	8.7181559348e-07
0.90	0.406623947181	0.406623610277	8.28539084456e-07
0.95	0.411961095807	0.411960774028	7.81090568683e-07
1.0	0.419169410527	0.419169104046	7.31164271487e-07
1.05	0.428070826498	0.428070535316	6.80219002859e-07
1.10	0.438504223971	0.438503947953	6.2945479453e-07
1.15	0.450323815758	0.450323554654	5.79814559604e-07
1.20	0.463397688141	0.463397441612	5.32003054503e-07
1.25	0.477606480643	0.47760624828	4.86515965037e-07
1.30	0.492842191429	0.492841972768	4.4367332604e-07
1.35	0.509007096387	0.509006890925	4.03653137565e-07
1.40	0.526012771077	0.526012578282	3.66522553709e-07
1.45	0.543779205749	0.543779025071	3.32265247422e-07
1.50	0.562234004582	0.56223383546	3.00804425446e-07
1.55	0.581311661121	0.581311502992	2.72021538830e-07
1.60	0.60095290267	0.600952754973	2.45771055889e-07
1.65	0.621104097069	0.621103959252	2.21891824392e-07
1.70	0.641716715932	0.64171658745	2.00215585716e-07
1.75	0.662746848952	0.662746729278	1.80573182744e-07
1.80	0.684154764439	0.684154653059	1.62798945791e-07
1.85	0.705904511668	0.705904408088	1.46733668173e-07
1.90	0.727963561076	0.72796346482	1.32226507986e-07
1.95	0.750302478695	0.750302389307	1.19136095997e-07
2.0	0.772894631567	0.772894548611	1.07331056988e-07


4th order Runge-Kutta Method
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
Step size 0.2
Step size 0.1
Step size 0.05


Relative Error - 4th order Runge-Kutta
0.00035
0.00030
0.00025
0.00020
0.00015
0.00010
0.00005
0.00000
0.0
0.5
1.0
1.5
2.0
Step size 0.2
Step size 0.1
Step size 0.05

Proof: We wish to show that $x(t) = \frac{5}{4}e^{-2t} + \frac{1}{4}(2t - 1)$ is the exact solution to $\frac{dx}{dt} = t - 2x$ with the initial conditions $t_0 = 0,\ x_0 = 1$.

First we can rewrite $\frac{dx}{dt} = t - 2x$ as $\frac{dx}{dt} + 2x = t$.

Then we let

$$\rho = e^{\int 2dt} = e^{2t}$$

Next we multiply by ρ:

$$(e^{2t})\frac{dx}{dt} + (e^{2t})2x = (e^{2t})t$$

We know by construction:

$$\frac{d}{dt}[e^{2t}x] = e^{2t}t$$

$$e^{2t}x = \int e^{2t}tdt$$

We integrate:

$$e^{2t}x = \frac{t}{2}e^{2t} - \frac{1}{2}\int e^{2t}dt$$

$$= \frac{t}{2}e^{2t} - \frac{1}{4}e^{2t} + C$$

Finally we have:

$$x(t) = \frac{t}{2} - \frac{1}{4} + Ce^{-2t}$$

We can now use our initial value conditions to solve for the constant, C:

$$t_0 = 0 \text{ and } x_0 = 1$$

$$1 = \frac{0}{2} - \frac{1}{4} + Ce^{0}$$

Therefore $C = \frac{5}{4}$

So we have, as we wished to prove:

$$x(t) = \frac{5}{4}e^{-2t} + \frac{t}{2} - \frac{1}{4}$$

$= \frac{5}{4}e^{-2t} + \frac{1}{4}(2t - 1)$

Problem 2:
We will now approximate numerical solutions for a different differential equation:

$$\frac{d^2x}{dt^2} + 4\pi^2 x = 0$$

over the interval $[0, 1.2]$ with initial conditions $t_0 = 0, x_0 = 1, x'_0 = 0$. We will use Euler's method with step sizes h=0.2, h=0.1, and h=0.05.

P2-Euler's Method, Step size (h = 0.2):

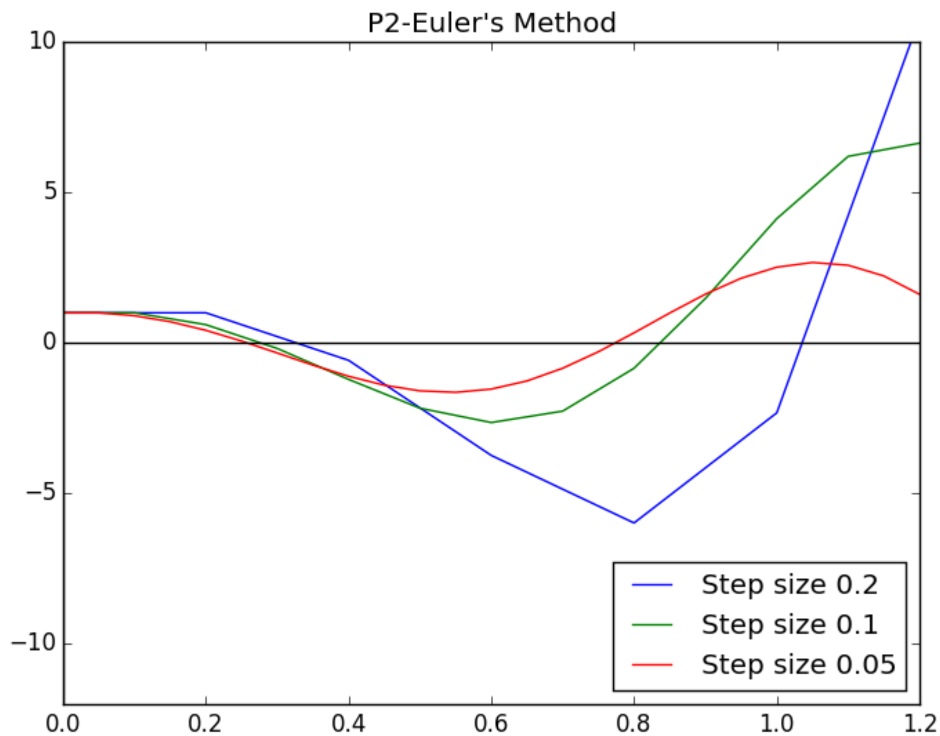
t	Exact Solution	*x*	Relative Error
0	1.0	1.0	0.0
0.2	0.309016994375	1.0	2.2360679775
0.4	-0.809016994375	-0.579136704174	0.284147665375
0.6	-0.809016994375	-3.73741011252	3.61969295887
0.8	0.309016994375	-5.98114749458	20.3553998759
1.0	1.0	-2.32300338939	3.32300338939
1.2	0.309016994375	10.7801902576	33.8854284838

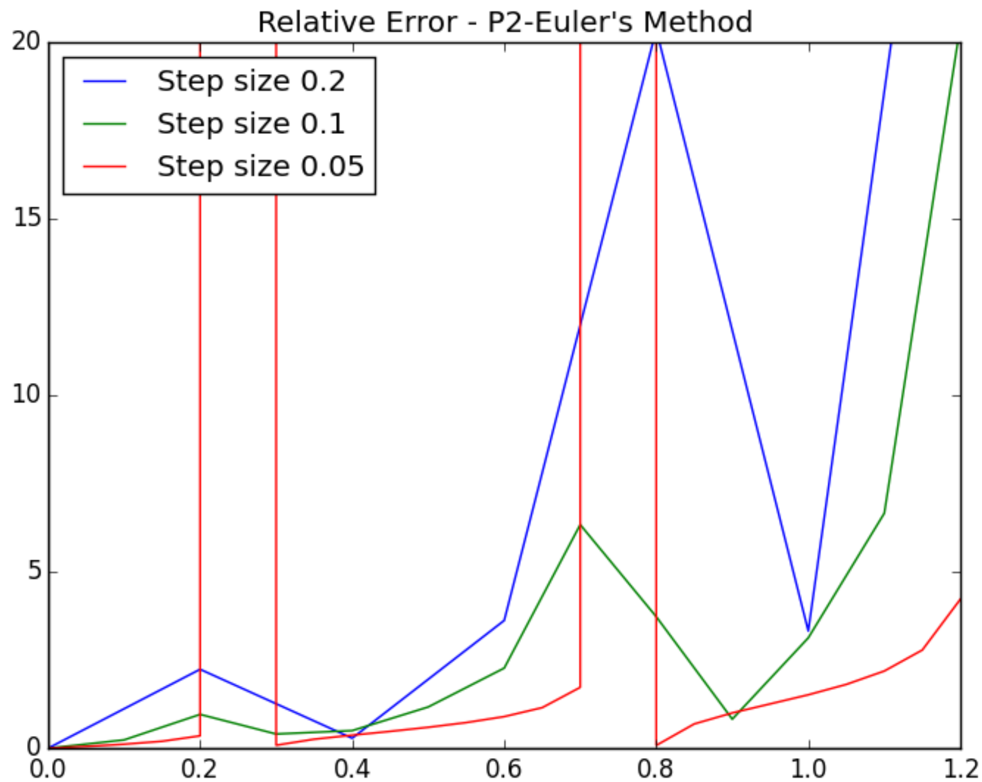
P2-Euler's Method, Step size (h = 0.1):

t	Exact Solution	*x*	Relative Error
0	1.0	1.0	0.0
0.1	0.809016994375	1.0	0.2360679775
0.2	0.309016994375	0.605215823956	0.958519547382
0.3	-0.309016994375	-0.184352528131	0.403422687145
0.4	-0.809016994375	-1.21285051061	0.499165677656
0.5	-1.0	-2.16856903216	1.16856903216
0.6	-0.809016994375	-2.64547336423	2.26998491085
0.7	-0.309016994375	-2.26626095773	6.33377451398
0.8	0.309016994375	-0.842657528898	3.72689704526
0.9	0.809016994375	1.47562986484	0.823978822566
1.0	1.0	4.1265851168	3.1265851168
1.1	0.809016994375	6.19498504843	6.65742263946
1.2	0.309016994375	6.63427447485	20.468963182

P2-Euler's Method, Step size (h = 0.05):

t	Exact Solution	*x*	Relative Error
0	1.0	1.0	0.0
0.05	0.951056516295	1.0	0.0514622242383
0.10	0.809016994375	0.901303955989	0.114072957992
0.15	0.587785252292	0.703911867967	0.19756639899
0.20	0.309016994375	0.417564645038	0.351267576344
0.25	6.12323399574e-17	0.0617441054081	1.00835776407e+15
0.30	-0.309016994375	-0.335288412806	0.0850160959081
0.35	-0.587785252292	-0.738414829965	0.256266344017
0.40	-0.809016994375	-1.10844960718	0.370119064104
0.45	-0.951056516295	-1.40560576183	0.477941360739
0.50	-1.0	-1.59336232527	0.593362325274
0.55	-0.951056516295	-1.64239116058	0.726912262778
0.60	-0.809016994375	-1.53416143771	0.896327825473
0.65	-0.587785252292	-1.26383420458	1.15016317549
0.70	-0.309016994375	-0.84209130666	1.72506471161
0.75	7.04481399828e-16	-0.295612972468	4.19617852991e+14
0.80	0.309016994375	0.333976442388	0.0807704704504
0.85	0.587785252292	0.992741688184	0.688953039078
0.90	0.809016994375	1.61854478032	1.00063137311
0.95	0.951056516295	2.14636819512	1.25682507647
1.0	1.0	2.51444764304	1.51444764304
1.05	0.951056516295	2.67068904111	1.80812863941
1.10	0.809016994375	2.57876440394	2.18752810122
1.15	0.587785252292	2.22325332363	2.78242447383
1.20	0.309016994375	1.61322839821	4.22051675984

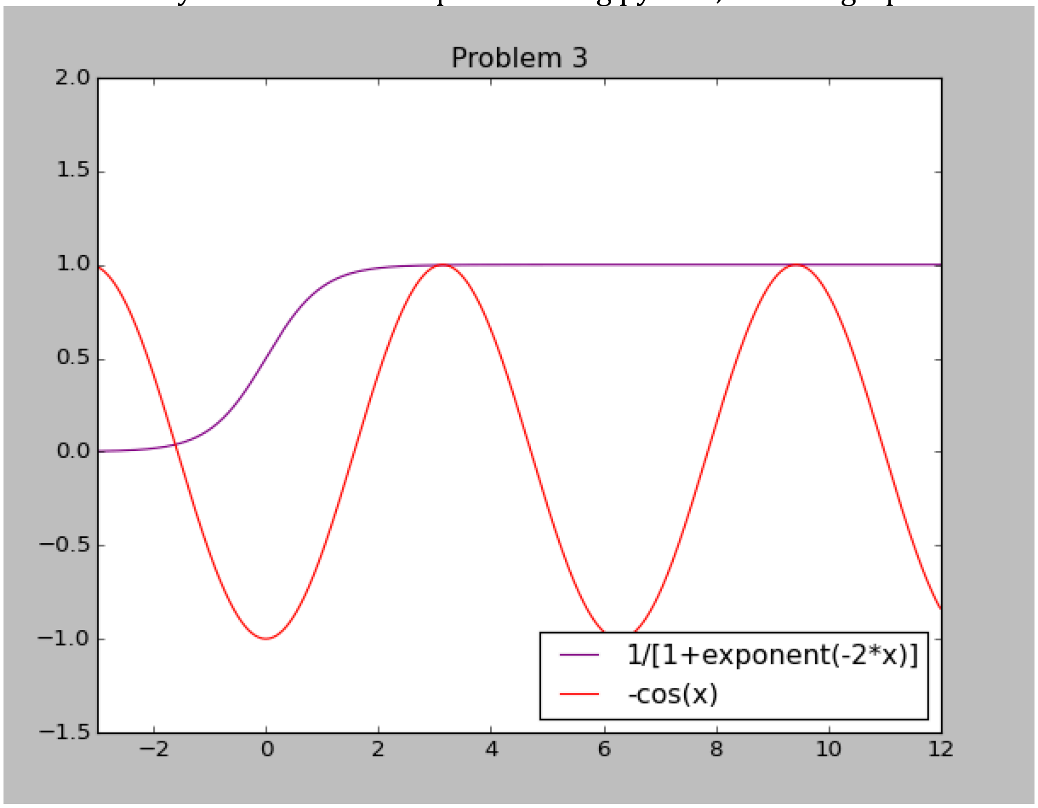

P2-Euler's Method
10
5
0
−5
−10
0.0
0.2
0.4
0.6
0.8
1.0
1.2
Step size 0.2
Step size 0.1
Step size 0.05


Relative Error - P2-Euler's Method
Step size 0.2
Step size 0.1
Step size 0.05
20
15
10
5
0
0.0
0.2
0.4
0.6
0.8
1.0
1.2

Problem 3:
We wish to find the two smallest positive roots of the following equation:

$$\frac{1}{1+e^{-2x}} = -cosx$$

with accuracy to seven decimal places. Using python, we first graph the functions:



From this graph we ascertain that the smallest roots are around x=-2 and x=3. Note also a root nearest x=9. This is greater than the other two roots, so we will not use it in our estimation. We will find the actual smallest roots using Newton's method. We will use these estimations as the initial values. The code attached as 3.py includes code written to perform Newton's method.

The root nearest x=-2 was calculated as x=-1.6092791099426. Using Newton's method, this required 3 iterations.

The root nearest x=3 was calculated as x=3.076421138060. This required 4 iterations.

Problem 4:
We have a system:

$$3x^2 - y^2 = 0$$
$$3xy^2 - x^2 = 1$$

with a solution that is not too far from the point (1, 1). We will find this solution using the two-dimensional Newton's method, and using the one-dimensional Newton's method with substitution. We run attached code 4.py

```
Two-Dimentional Newton's Method:

x(0) = [[1], [1]]
When X = 1, Y = 1
The solution is: [[2], [1]]

x(1) = [[0.6000000000000001], [0.8]]
When X = 0.6, Y = 0.8
The solution is: [[0.44000000000000017], [-
0.20799999999999974]]

x(2) = [[0.5453333333333333], [0.952]]
When X = 0.545333333333, Y = 0.952
The solution is: [[-0.014138666666666522],
[0.18532489955555542]]

x(3) = [[0.5074952925958794], [0.8795500336346556]]
When X = 0.507495292596, Y = 0.879550033635
The solution is: [[-0.0009538456457918176], [-
0.0797438186795465]]

...

x(18) = [[0.5207858853660806], [0.9020276133187929]]
When X = 0.520785885366, Y = 0.902027613319
The solution is: [[-4.440892098500626e-16],
[3.29178479585579e-07]]

x(19) = [[0.5207858100494335], [0.9020274828665331]]
When X = 0.520785810049, Y = 0.902027482867
The solution is: [[2.220446049250313e-16], [-
1.439088492816154e-07]]

x(20) = [[0.52078584297605], [0.902027539897106]]
When X = 0.520785842976, Y = 0.902027539897
The solution is: [[-2.220446049250313e-16],
[6.291347087739041e-08]]
```

```
One-Dimentional Newton's Method

Initial Guesse is:1.0

Keep Guessing:
0.72
0.573399592253
0.52580207075
0.520837435227
0.520785838496

So the Final Solution:
x = 0.520785838496
y = 0.902027532138
```