Stable Fractional Matchings with Non-Unit Capacities and Fixed Matching Costs

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Abstract

In this project, we study a variant of the stable matching problem called the stable fractional matching problem, where agents can be fractionally matched with other agents and so may share agents with others. We introduce two novel extensions to this setting. The first extension allows agents to have non-unit capacities. We generalize a definition of fractional stability to fit this setting and propose a novel continuous variant of the deferred-acceptance algorithm for finding stable matchings, and then show several theoretical properties of the matchings attained by our algorithm. We also propose an algorithm for finding a stable fractional matching that maximizes social welfare.

Our second extension introduces the idea of an incurred fixed cost for being matched with each distinct agent. We formulate a novel definition of stability that makes sense for this framework and obtain negative results on how a deferred-acceptance style algorithm fails in this setting.

1 Introduction

In the standard stable matching problem, we have two sets of agents, commonly referred to as men and women[1], whom we wish to match with each other. The men and women submit preference lists over each other, and we wish to find a matching between men and women that is "stable", where there does not exist a man-woman pair who would rather be paired with each other than their assigned partners. This problem has been widely studied and applied to settings such as medical residency and school choice [5].

In the stable fractional matching problem, the traditional setting is extended in two ways. First, cardinal preferences are used instead of just ordinal preferences to capture exact agent utilities. Second, fractional pairings are allowed, so that agents may be matched with fractions of other agents and be matched with multiple different agents, and the utility gained from each matching is weighted by the fraction matched.

We motivate this fractional matching problem with a time-sharing example. Consider a set of employees and supervisors who wish to be matched with each other. Each employee and supervisor has preferences over each other, and they also have some amount of time that they can allocate to be matched with each other. This time can be split up into fractions such that a single employee can split up their available time between multiple supervisors, and a single supervisor can split up their available time between multiple employees. We also find motivation in the farmers and traders problem, where farmers wish to sell their crops to traders and traders want to buy crops from farmers. The crops (like time) are divisible, and so it is not necessary for a farmer to sell all of their crops to a single trader, thus

[1] While this heterosexual convention is antiquated (and also does not make real-world sense when we introduce the idea of fractional matchings), we adopt this notation to stay consistent with the convention in the field.

suggesting a fractional matching setting.

In our project, we introduce two novel extensions to the stable fractional matching problem. First, we eliminate the underlying assumption that all agents have unit capacity. For example, we wish to consider the situation where a man may only wish to be matched with 0.5 women, or where a woman may wish to be matched with 3.2 men (again, "men" and "women" are notational conventions used to describe the two types of agents and should not be interpreted literally in the context of fractional matchings). Second, we introduce the idea of a fixed cost for being matched with each unique agent. For example, a woman who is (fractionally) matched with 3 different men would have to pay the fixed cost 3 times.

The non-unit capacity extension fits well with our motivating examples. In the time-sharing example, non-unit capacities imply that different employees and supervisors may be available for different amounts of times, which makes sense since some employees may wish to work shorter hours and some supervisors may require more working hours. Similarly in the farmers and traders example, farmers and traders may have different amounts of supply and demand to be satisfied.

Meanwhile, the fixed cost extension captures one-time costs that are accumulated when the matchings to occur. For example, in both the time-sharing and the crop-trading examples, if two agents match with each other, then there is a commute/travel cost that is fixed regardless of the amount they are matched. Thus, by studying these two extensions of the stable marriage problem, we hope to more realistically model a wider range of real-world matching markets.

2 Related Work

Caragiannis et al. [1] and Chen et al. [2] introduced and studied the original stable fractional matchings problem. Caragiannis et al. focused specifically on the two-party marriage setting with cardinal preferences. They introduced their own definition of stability for the fractional domain, but it does not apply well to our non-unit capacity setting. Thus, we formulate our own definition of stability which is different from the one they use in their work. Furthermore, most of their work is devoted towards showing algorithmic complexity results and approximation results, specifically in finding stable solutions that maximize social welfare.

Meanwhile, Chen et al. take a more general approach and study both the fractional marriage market and the fractional roommates market. They introduce three different types of stability, and we modify their definition of "ordinal stability" to satisfy the non-unit capacity domain. They study the relationships between their types of stability and show several structural and algorithmic runtime properties regarding the stable fractional matching problem. However, they assume unit capacities throughout their work, which we introduce as a novel extension.

Finally, Mnich and Schlotter [4], as well as Gharote et al. [3], studied additional extensions and constraints of the original (integral) stable matching problem. Mnich and Schlotter explored the idea of introducing quotas, where rural hospitals get lower quotas and popular hospitals get upper quotas. Our extension differs as we do not have a strict quota but rather assign a numerical penalty for each unique match. Meanwhile, Gharote et al. focus on optimizing over additional objectives (such as maximizing social welfare, fairness/equity) over the set of stable matchings, which is more analogous to what Caragiannis et al. worked on. As far as we are aware, while stable fractional matchings and stable matchings under non-unit capacities have been independent studied, there has not yet been a study which combines the two settings, nor anything that explores fixed costs in the matching domain.

3 Definitions

Let $M := \{m_1, \ldots, m_n\}$ be the set of n men, and $W := \{w_1 \ldots, w_n\}$ be the set of n women. Let the **capacity** $C(m)$ or $C(w)$ of a man m or woman w be the number of units the agent has available to match with other agents. We can always ensure the same number of men and women by adding dummy agents with capacity 0. While prior work has always assumed that the capacities of all agents are always 1, in our first extension we allow each capacity to be any non-negative real number. Let $U(m, w)$ be the value derived by man m from being matched with each unit (of capacity) of woman w, and let $V(m, w)$ be the value derived by woman w from being matched with each unit of man m. Define the preference operator $\succ_m$ by $w \succ_m w'$ if and only if $U(m, w) > U(m, w')$.

In the first part of this report where we initially introduce the non-unit capacity extension, we focus on the ordinal preferences of the agents. By guaranteeing ordinal stability, we also ensure that any cardinal preferences that follow that ordinal ranking are also stable. In the second part of this report, we consider cardinal preferences in order to consider social welfare and the fixed costs extension.

Define a fractional matching $\mu : M \times W \to \mathbb{R}_{\geq 0}$ as an assignment of non-negative weights to all man-woman pairs such that

$$\sum_{w \in W} \mu(m, w) \leq C(m) \qquad \forall m \in M$$

and

$$\sum_{m \in M} \mu(m, w) \leq C(w) \qquad \forall w \in W$$

The **utility** a man m derives from a fractional matching μ is defined as $u_m(\mu) := \sum_{w \in W} U(m, w)\mu(m, w)$. Similarly, the utility a woman m derives from from μ is $v_w(\mu) := \sum_{m \in M} V(m, w)\mu(m, w)$.

We would like to find a "stable" fractional matching for this setting with non-unit capacities. However, we must first consider what a stable fractional matching would look like. We propose the following definition of stability that requires only ordinal preferences and generalizes to non-unit capacities.

Definition 1 (Stability)**.** *Given a fractional matching μ, a man-woman pair (m, w) is said to be a **blocking pair** if for some $m' \in M$ and $w' \in W$, we have $w \succ_m w'$, $m \succ_w m'$, $\mu(m, w') > 0$, and $\mu(m', w) > 0$. A fractional matching μ is **stable** if there are no blocking pairs and there does not exist a man-woman pair (m, w) where both m and w have unused capacity remaining.*

Our definition is generalized from the definition of ordinal stability that Chen et al. [2] propose and strong stability that Caragiannis et al. [1] briefly mention in their appendix. At a high level, a blocking pair (m, w) exists when m and w each prefer each other more than they prefer some partner they are currently fractionally matched with. In this case, m and w would have an incentive to reduce the fraction that they are matched with less-preferred agents and match more with each other instead, creating instability.

4 A Matching Algorithm for Non-Unit Capacities

The first question we address is: does a fractional stable matching always exist in the setting with non-unit capacities? We answer in the affirmative by proposing a modified deferred-acceptance algorithm and showing that it outputs such a matching.

Our algorithm has men proposing to women with women temporarily accepting fractional proposals until the algorithm terminates. Unlike in the traditional stable marriage DA algorithm, these proposals also come with a capacity indicating how much a man would like to match with a woman. Going back to the example with farmers and traders, this corresponds to a trader indicating the volume of crops they're willing to purchase from the farmer they're proposing to.

1. While there exist men with unmatched capacity and who have not yet been rejected by all women:

   (a) Arbitrarily pick one such man m and have him propose to his most preferred woman w whom he hasn't proposed to yet. Let $c_m^{unmatched}$ be all of m's unmatched capacity at this point in the algorithm's execution. m proposes to w with capacity $c_m^{unmatched}$.

   (b) Let $c_w^{matched}$ be w's matched capacity at this point in the algorithm's execution. Woman w handles the proposal from m by first considering all $c_w^{matched} + c_m^{unmatched}$ units currently available to her and rejecting the "worst" $c_w^{matched} + c_m^{unmatched} - C(w)$ units corresponding to her least preferred proposals. Note that if $c_w^{matched} = C(w)$ and m is less preferred than all other men whom w is currently fractionally matched to, she simply rejects m's proposal.

There is some intuition from chemistry for what happens when a woman receives a proposal (Figure 1). Imagine a proposal's capacity is some quantity of liquid with density corresponding to how much the woman prefers the man, where her most preferred men have the most dense proposals. The woman pours the proposal liquid into a graduated cylinder with max volume corresponding to her capacity. If the woman's cylinder exceeds her capacity, then proposals spill out the top, starting with the least dense liquids corresponding to proposals from the least preferred men. The fractions of proposals that spill out are rejected.

Proof. Now we show that the matching μ output by this fractional DA algorithm

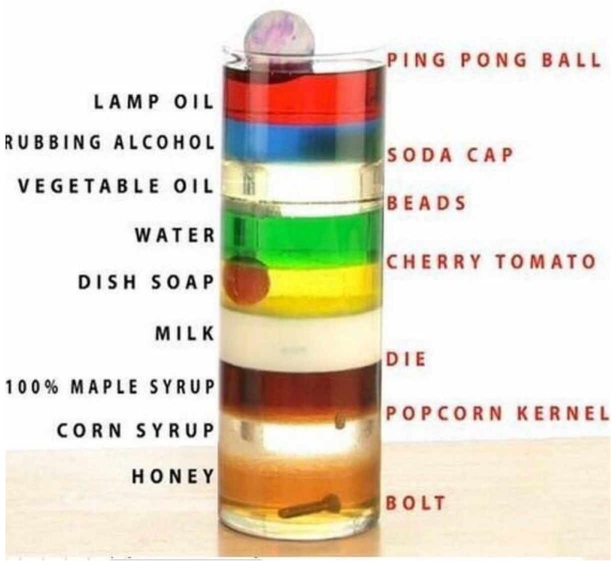

Figure 1: Intuition from chemistry, where the more dense liquids (e.g. honey and syrup) correspond to very preferred men and the volume of the cylinder corresponds to the woman's capacity.

is stable. Assume for the sake of contradiction that there exists some blocking pair (m, w), where there exists some m' and w' such that $w \succ_m w'$, $m \succ_w m'$, $\mu(m, w') > 0$, and $\mu(m', w) > 0$.

By the definition of our algorithm, m proposed to w before w', so since $\mu(m, w') > 0$ some fraction of the proposal from m to w was not accepted, either because w reached her capacity or w later rejected some fraction of the proposal by m.

In the first case where w reached her capacity with the proposal by m, this implies that after the proposal by m, w was no longer fractionally matched to any less preferred man. Thus, when m' proposes to w, all of the fractional proposals held by w are of men that w prefers at least as much as she prefers m, w must reject all of the proposal from m', and we get a contradiction to $\mu(m', w) > 0$.

In the other case where w later rejected some fraction of the proposal by m, this would only happen if w was trying to make room for some more preferred proposal AND w already rejected all fractional proposals by men less preferred than m (taking the chemistry intuition, the less preferred proposals have already "spilled out the top"). In this case as well we reach the state where all of the fractional proposals held by w are of men whom w prefers at least as much as she prefers m, so w will never end up with a fraction of m'. Thus, we again get a contradiction.

$\square$

5 Man-Optimal Matching

We now define a man-optimal (and symmetrically woman-optimal) matching and show that our extension of the deferred acceptance algorithm with men proposing results in a man-optimal matching. We begin with some definitions. Let $B_m(w)$ denote the set of women w' that m weakly prefers over w, i.e. $B_m(w) := \{w' \in W : w' \succeq_m w\}$. Define $\mu(m, \succeq w) := \sum_{w' \in B_m(w)} \mu(m, w')$. That is, $\mu(m, \succeq w)$ is the sum of the fractions of women whom m is matched with whom he weakly prefers over w. Then we say that a man m weakly prefers a matching μ over μ' if and only if $\mu(m, \succeq w) \geq \mu'(m, \succeq w)$ for all $w \in W$. A man-optimal stable fractional matching is a stable fractional matching that every man weakly prefers over any other stable fractional matching (and symmetrically for woman-optimal).

While we show an approximation result for man-optimality, we were not able to prove that the output of our fractional DA algorithm is an exact man-optimal fractional matching, or that a man-optimal matching must exist in general. The proof ideas from the traditional stable marriage rely heavily on two points that do not apply in the fractional setting:

- In the stable setting, no man is ever rejected by a woman that he's matched to in some stable matching. This doesn't work in the fractional setting and we can easily construct an example where there must always be a man that is fractionally rejected by a woman (consider one woman with medium capacity and two men; one which is more preferred but who has low capacity and one who is less preferred but has high capacity).

- In the stable setting, a man only proposes to a woman if he has been rejected by all more preferred women. This is simply not true in the fractional setting; in particular, if one man has enormous capacity he will inevitably propose to all women.

Showing that an exact man-optimal matching always exists and is output by our algorithm would require a completely different approach, and in our attempts we've unfortunately been unsuccessful.

5.1 δ-Man Optimal

While we do not show exact man-optimality, we can still show that our DA algorithm outputs a matching that is arbitrarily close to man-optimal by relating to a discretized version where we discretize our agents' capacities and enforce a minimum matching proportion. We first define approximate versions of fractional stability and man-optimality.

Definition 2 (ϵ-Stability)**.** *Given a fractional matching μ, a man-woman pair (m, w) is said to be an ϵ-blocking pair if for some $m' \in M$ and $w' \in W$, we have $w \succ_m w'$, $m \succ_w m'$, $\mu(m, w') > \epsilon$, and $\mu(m', w) > \epsilon$. A fractional matching μ is ϵ-stable if there are no ϵ-blocking pairs and there does not exist a man-woman pair (m, w) where both m and w have unused capacity of over ϵ remaining.*

Correspondingly, we say that a man m δ-weakly prefers a matching μ over μ' if and only if $\mu(m, \succeq w) \geq \mu'(m, \succeq w) - \delta$ for all $w \in W$. A δ-man-optimal stable fractional matching is a stable fractional matching that every man δ-weakly prefers over any other stable fractional matching (and symmetrically for woman-optimal). We now prove that our continuous man-proposing DA algorithm attains a stable fractional matching that is δ-man-optimal for any $\delta > 0$.

Fix some small $\epsilon > 0$. Consider some agent m with capacity $C(m)$. Let k be the largest integer such that $k\epsilon \leq C(m)$. For each such agent m, we replace m with k identical clones $m_1, \ldots, m_k$, each with capacity ϵ and identical preferences as m. We leave the remaining $C(m) - k\epsilon$ capacity of the original m unmatched, which is fine as it is contained within the error of ϵ-stability.

We can then run the standard discrete man-proposing DA algorithm on this modified setting to obtain a man-optimal stable matching under the discrete definitions of man-optimal and stability. Let the matching resulting from our continuous man-optimal DA algorithm be μ, and the matching resulting from our discretized algorithm be μ^*. We briefly argue that for all m and w, $|\mu(m,w) - \mu^*(m,w)| \leq n\epsilon$.

Each time a man proposes to a woman, he can accumulate at most ϵ additional discrepancy between the continuous setting and the discretized setting of his match proportion with the woman since the leftover unused capacity is at most ϵ for each agent. Since each man proposes to at most n women, his match proportion discrepancy with each woman can be at most $n\epsilon$.

Finally, we recall our definition of man-optimal. Since μ^* is man-optimal in the discrete setting, we have $\mu^*(m, \succeq w) \geq \mu'(m, \succeq w) - n\epsilon$ for all stable μ' in the fractional setting. From the definition of $\mu(m, \succeq w)$, we also have that $|\mu(m, \succeq w) - \mu^*(m, \succeq w)| \leq n^2\epsilon$. Thus, we have $\mu(m, \succeq w) \geq \mu'(m, \succeq w) - n\epsilon - n^2\epsilon$ for all stable μ', and so μ is δ-man-optimal where $\delta = n\epsilon + n^2\epsilon$. By making ϵ and δ arbitrarily small, we show that our algorithm produces a matching that is arbitrarily close to man optimal.

5.2 Man Optimal is Woman Pessimal

We can show that a man optimal matching, if it exists, is necessarily female pessimal in the sense that every woman weakly prefers any other stable matching.

Proof. Let μ^M be the male optimal matching and assume there exists μ such that some woman w does not weakly prefer μ over μ^M. Then by the definition of not preferring μ, there must exist m such that $\mu^M(w, \succeq m) > \mu(w, \succeq m)$.

Since $\mu^M(w, \succeq m) > \mu(w, \succeq m)$ and since the capacity $C(w)$ is the same in either matching, we know there must exist some $m' \in M \cup \{\emptyset\}$ such that $m' \prec_w m$ and $\mu^M(w, m') < \mu(w, m')$. Similarly, by the definition of man optimal, we have $\mu^M(m, \succeq w) \geq \mu(m, \succeq w)$ and there must exist $w' \in W \cup \{\emptyset\}$ such that $w' \prec_m w$ and $\mu^M(m, w') < \mu(m, w')$.

Now we can see that $m \succ_w m'$, $w \succ_m w'$, $\mu(m, w') > 0$, and $\mu(m', w) > 0$, so by definition (m, w) are a blocking pair in μ and we get a contradiction. Thus, for every μ, every woman weakly prefers μ over μ^M and μ^M is the woman pessimal matching.

It's worth noting that we only require our definition of man optimal to give that if w does not prefer μ over μ^M then there must exist m such that $\mu^M(w, \succeq m) > \mu(w, \succeq m)$. If we have to weaken our definition of man optimal in order to show that our fractional DA algorithm always outputs a man optimal matching, as long as it satisfies this one condition then these subsequent results still hold.

$\square$

6 Rural Hospital Theorem

While prior literature did not consider the Rural Hospital theorem at all in the fractional setting, we extend the Rural Hospital theorem to our fractional and non-unit capacity setting.

For some stable matching μ, let $|\mu(m)|$ be the matched capacity of man m under μ, not including matching to $\emptyset$ corresponding to having unmatched capacity (resp. $|\mu(w)|$). We say that $|\mu(M)| \supseteq |\mu'(M)|$ if $\forall m \in M$, we have $|\mu(m)| \geq |\mu'(m)|$ (resp. $|\mu(W)| \supseteq |\mu'(W)|$). If $|\mu(M)| \supseteq |\mu'(M)|$ and $|\mu(M)| \subseteq |\mu'(M)|$ we simply write $|\mu(M)| = |\mu'(M)|$.

The Rural Hospital theorem for stable fractional matchings states that for any stable matchings μ and μ', $|\mu(M)| = |\mu'(M)|$ and $|\mu(W)| = \mu'(W)|$, so in all stable matchings all players are matched with the same capacity.

Proof. Let μ^M be the man optimal matching that we assume to exist. First we claim that for any stable μ, we have $|\mu^M(M)| \supseteq |\mu(M)|$. Towards a contradiction, assume there exists some m where $|\mu^M(m)| < |\mu(m)|$. Let w_0 be the woman that m prefers the least. By the definition of $|\mu(m)|$, we have $\mu^M(m, \succeq w_0) < \mu(m, \succeq w_0)$. This contradicts the definition of man optimal, so $|\mu^M(M)| \supseteq |\mu(M)|$ for all μ. By a symmetric argument using the definition of woman pessimal, $|\mu^M(W)| \subseteq |\mu(W)|$.

These relations give us the inequalities

$$\forall m, |\mu^M(m)| \geq |\mu(m)|$$
$$\forall w, |\mu^M(w)| \leq |\mu(w)|$$

so summing over everything gives

$$\sum_m |\mu^M(m)| \geq \sum_m |\mu(m)|$$
$$\sum_w |\mu(w)| \geq \sum_w |\mu^M(w)|$$

By the fact that a fractional matching is symmetric, we know that

$$\sum_m |\mu(m)| = \sum_w |\mu(w)|$$
$$\sum_m |\mu^M(m)| = \sum_w |\mu^M(w)|$$

and thus

$$\sum_m |\mu^M(m)| = \sum_m |\mu(m)|$$
$$\sum_w |\mu^M(w)| = \sum_w |\mu(w)|$$

Now assume there is some m such that $|\mu^M(m)| > |\mu(m)|$. Then clearly there must exist some m' where $|\mu^M(m')| < |\mu(m')|$ which is a contradiction, so for all m we have $|\mu^M(m)| = |\mu(m)|$ and thus $|\mu^M(M)| = |\mu(M)|$. The same argument works for the women, and this gives us the Rural Hospital theorem for stable fractional matchings. □

7 Optimal Stable Fractional Matching

Now that we know that stable fractional matchings always exist and that we can find them, we redirect our attention to the cardinal preferences of agents. We ask: are some stable fractional matchings better for overall social welfare? To address this question, we define the idea of social welfare as:

Definition 3 (Social Welfare)**.** *Given a fractional matching μ, the **social welfare** of μ is defined as*

$$S(\mu) := \sum_{m\in M} u_m(\mu) + \sum_{w\in W} v_w(\mu) = \sum_{m\in M}\sum_{w\in W} (U(m,w) + V(m,w))\mu(m,w)$$

Social welfare is a natural objective to optimize over, as it is the sum of utilities of all agents. However, we still want to preserve stability in our matching since a matching with high welfare but that is unstable may still face unraveling from the instability, which is undesirable. We thus define an optimal stable fractional matching as a matching which has highest social welfare among all stable fractional matchings.

We can find an optimal stable fractional matching in exponential time using mixed-integer programming by setting our objective to maximize social welfare and our constraints to ensure a stable fractional matching. Let K be some large constant which is at least as large as all agent capacities. Then we solve the following MIP:

$$
\begin{aligned}
&\text{maximize} \sum_{m \in M} u_m + \sum_{w \in W} v_w \\
&\text{subject to} \\
\mu(m, w') + K y(m, m', w, w') &\leq K && \forall m, m' \in M, w, w' \in W \text{ such that } w \succ_m w', m \succ_w m' \\
\mu(m', w) + K(1 - y(m, m', w, w')) &\leq K && \forall m, m' \in M, w, w' \in W \text{ such that } w \succ_m w', m \succ_w m' \\
\sum_{w \in W} U(m, w)\mu(m, w) &= u_m && \forall m \in M \\
\sum_{m \in M} V(m, w)\mu(m, w) &= v_w && \forall w \in W \\
\sum_{w \in W} \mu(m, w) &\leq C(m) && \forall m \in M \\
\sum_{m \in M} \mu(m, w) &\leq C(w) && \forall w \in W \\
\mu(m, w) &\geq 0 && \forall m \in M, w \in W \\
y(m, m', w, w') &\in \{0, 1\} && \forall m, m' \in M, w, w' \in W
\end{aligned}
$$

where we get $O(2^{n^4})$ linear programs for each possible assignment of $y(m, m', w, w')$ for each tuple (m, m', w, w'). The objective of this MIP directly aims to maximize the social welfare. The constraints ensure that the solution is stable and valid. The first two constraints ensure there are no blocking pairs by guaranteeing that, for every m, m', w, w' such that $w \succ_m w', m \succ_w m'$, either $\mu(m, w') = 0$ or $\mu(m', w) = 0$, and so m and w are not a blocking pair. The remaining constraints verify that the matching is valid and that agent utilities are correctly computed.

Chen et al. show that, when preferences can have ties, the problem of finding the welfare-maximizing stable matchings under the ordinal stability definition is NP-complete in the unit capacity setting. When we set all of our capacities to 1, our setting reduces to the setting Chen et al. studied; thus, for our setting, it must also be NP-hard to find the welfare-maximizing stable matching when preferences can have ties (because the existence of a polynomial-time algorithm would result in a contradiction). The decision problem of checking whether a matching is stable and if the social welfare is at least some threshold is clearly polynomial, so the complexity of our problem is in NP and thus is NP-complete.

However, in the case where preferences cannot have ties, Chen et al. show that a polynomial-time algorithm for finding the welfare-maximizing stable matching in the unit capacity setting exists. They do so by showing that the welfare-maximizing fractional stable matching always coincides with the welfare-maximizing integral stable matching, which is solvable in polynomial time. However, we cannot use the same reasoning that they do, since in our setting the agents can have fractional capacities that prevent an integral solution from being optimal. Thus, it remains an open problem if a polynomial-time algorithm exists for this setting.

8 Fixed Matching Costs

Now consider the setting where each unique partner an agent is (fractionally) matched with incurs a fixed cost, regardless of the fraction matched. For example, in the time-sharing scenario where employees are matched with supervisors, each with some available time capacity, the fixed cost can be interpreted as the transition/setup cost for each employee switching between supervisors (and vice versa). We first consider the simpler scenario where the fixed cost experienced by all agents for being matched with each additional agent is the same cost K. Then formally, the utility a man m derives from being matched with a woman w under μ is $U(m,w)\mu(m,w) - I(\mu(m,w) > 0)K$ where I is the indicator operator. The total utility m derives under μ is $u_m(\mu) = (\sum_{w \in W} U(m,w)\mu(m,w)) - N(m)K$, where $N(m)$ is the number of distinct women whom m is positively fractionally matched with. The definition of utility for women follows analogously.

Again, we wish to study what it means for a matching to be stable in this setting setting with fixed costs. We first redefine stability so that we can incorporate cardinal preferences.

Definition 4 (Stability (Cardinal Version))**.** *Given a fractional matching μ, a man-woman pair (m,w) is said to be a blocking pair if there exists another fractional matching μ' such that:*

- $u_m(\mu') > u_m(\mu)$
- $v_w(\mu') > v_w(\mu)$
- $\mu'(m,w) > \mu(m,w)$
- $\mu'(m,w') = \mu(m,w')$ *or* $\mu'(m,w') = 0 \quad \forall w' \in W \setminus \{w\}$
- $\mu'(m',w) = \mu(m',w)$ *or* $\mu'(m',w) = 0 \quad \forall m' \in M \setminus \{m\}$

A fractional matching μ is stable if there does not exist a blocking pair and there does not exist a man who receives negative utility from some matched woman, and there does not exist a woman who receives negative utility from some matched man (i.e. would rather not be matched with that person at all).

Roughly speaking, m and w are a blocking pair if they each can gain more utility by completely dropping some of their current partners in order to increase the proportion they match with each other, while keeping their matches with their other partners the same. When considering which partners to reduce the proportion of match with, we must either drop the partner completely or not reduce the match at all. This is because if, for example, m wishes to reduces his match with w' by a little bit but not completely, then the reduction may cause w' to now gain negative utility from being matched with m, causing w' to withdraw her remaining match with m. To prevent this kind of unraveling, we assume that any reduction in a match is a complete reduction.

We wish to find an algorithm that attains a stable fractional matching in this fixed costs setting. Unfortunately, we show that a deferred-acceptance style algorithm does not work, even for integral matchings.

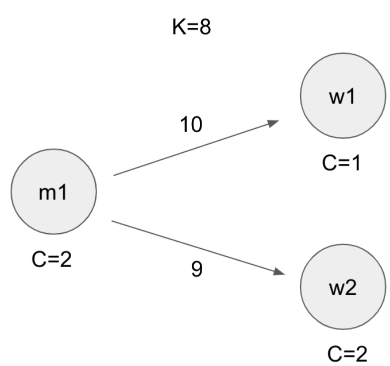

Figure 2: Counterexample for DA-style algorithms for the fixed costs setting.

Consider the following example. There is one man, m_1, with capacity 2. There are two women, w_1 and w_2. w_1 has capacity 1, w_2 has capacity 2. Let $u(m_1, w_1) = 10, u(m_1, w_2) = 9$, and the fixed cost $K = 8$. Now consider any man-proposing algorithm in which the men propose to their most valued women first. Under such an algorithm, m_1 would match 1 with w_1 and 1 with w_2, receiving a total utility of $10 + 9 - 2 \cdot 8 = 3$. However, if m_1 had instead matched only with w_2, then he would have received utility $9 \cdot 2 - 8 = 10$, a significant increase. Thus, m_1 and w_2 are a blocking pair, and the arrangement is unstable.

In order to accurately assign a stable matching for this example, we need to consider the capacities of the agents as well as the exact utilities gained from the matchings, not just the ordinal rankings. We can generate a capacity-weighted value function $U'(m, w) = U(m, w) \cdot \min(C(m), C(w))$. U' represents the maximum utility m can derive from being matched with w. If we apply a man-proposing deferred-acceptance algorithm using this capacity-weighted value function to our above example, then m_1 will propose to w_2 first and give the stable matching as desired. However, it is easy to see that this method breaks down quickly as agents have some of their capacity filled up by other agents, making the original capacity-weighted value function no longer accurate.

We tried several other approaches including maximizing social welfare, but this is a hard setting and unfortunately we were not able to achieve any positive results for finding a stable fractional matching. Meanwhile, we did not find any counterexamples in which a stable fractional matching did not exist.

9 Conclusion

In this work we extended existing work on stable fractional marriages to the more general setting where participants do not have unit demands. There are several natural problems that motivate work in this direction, such as farmers and traders matching up to sell crops and employees sharing work hours between multiple companies.

We define a stable fractional matching with non unit capacities and give an algorithm based on deferred acceptance that always outputs a stable matching. We also give a notion of a player preferring one matching over another, but are unfortunately unsuccessful with showing that an optimal matching, which is preferred over any other stable matching, must always exist. Instead, we outline some of the challenges and discuss why the classic proof techniques cannot be applied here, and offer an approach that guarantees approximate man optimality.

Assuming the existence of a man-optimal matching, we prove that it is woman pessimal with the same definition of preferring a matching. Using this, we can state and prove the Rural Hospital theorem for the stable fractional setting, which tells us that every player always results with the same matched capacity, no matter which matching we choose.

We also consider social welfare in a cardinal preferences setting and give an exponentially large linear program to find the optimal stable matching. We show that finding the optimal stable fractional matching is NP-complete when ties are allowed, so we expect this to be the best we can do.

However, when we restrict the problem to not allow ties in preferences, we find an open problem of whether a polynomial time algorithm exists to finding an optimal stable matching.

Finally, we attempt to define stable fractional matchings with fixed matching costs, where every distinct agent you match with incurs some flat cost to utility. This again is motivated by natural problems; taking the farmer and trader example, each trader may have some cost for visiting each farmer, so matching to more farmers does not come for free. Unfortunately we are able to show that a DA type algorithm does not work for fractional matchings with fixed costs, and the strange nature of paying an up front cost coupled with possible fractional rejection later makes the problem seem especially hard. We were unable to find an algorithm that outputs a stable matching in this setting but were also unable to find a counterexample where no stable matching exists.

9.1 Future Work

The most compelling continuation of this work is proving that an exact man-optimal stable matching always exists in the fractional setting with no fixed costs, and ideally that our fractional DA algorithm outputs it. Combining this result with the rural hospital theorem would push our understanding of fractional matchings with non unit capacities.

We also leave open the problem of finding an efficient algorithm for searching for a social welfare optimal stable fractional matching.

Finally, much more work needs to be done in the space of fractional matching with fixed costs, which seems to be a completely different type of problem from the stable marriage and variations that we're used to seeing.

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