Generals transcript

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1 Introduction

Examiners: Assaf Naor (chair), Charlie Fefferman, Ian Zemke

I arrived 10 minutes early. We debated whether mathematicians were fit to serve as jurors.

Test was 13:10 - 15:10; delayed because (IZ) didn't show up. (IZ) eventually arrived at 14:30 to deliver the algebra portion (he forgot about my generals lmao).

Notation: $A \lesssim B$ means $A \leq CB$ for some absolute constant C, and define $A \gtrsim B$ similarly.

2 Complex Analysis

1. (CF) State Riemann mapping theorem.

2. (CF) What can you say about regions with a hole?

3. (CF) When are two annuli conformally equivalent? Prove it.

   I said that a friend told me the answer and that he suggested a combinatorial proof. (CF) didn't buy it and told me to do a complex analytic proof. Basically, use the exponential and logarithmic map to convert the problem to biholomorphically mapping the strip $\mathrm{Re}(z) \in (0, a)$ to $\mathrm{Re}(z) \in (0, b)$, such that $f(z + 2\pi i) = f(z) \pm 2\pi n(z)i \quad (*)$, where $n(z) \in \mathbb{Z}$. By biholomorphism, we have $n(z) = \pm 1$, and by continuity, $n(z)$ is a constant. Both regions are now simply connected, so you know the set of maps between them by Riemann mapping theorem. (Subquestion: construct explicit conformal map between strip and upper half plane. I stumbled on this until (CF) told me to consider e^z acting on the strip $\mathrm{Im}(z) \in (0, \pi)$.) And using the automorphism group of the unit disk, the only maps satisfying condition (*) are $f(z) = z$ and $f(z) = a - z$.

4. (AN) What is the automorphism group of the unit disk?

   Gave and proved using Schwarz lemma. I stumbled on describing the group structure until AN gave me the hint to consider automorphism group of upper half plane, which is Mobius transforms $\frac{az+b}{cz+d}$, parametrized by $\mathrm{SL}_2(\mathbb{C})$.

3 Real analysis

(CF) Let's declare victory over complex analysis. Let's turn to real since (IZ) isn't here yet?

1. (CF) What is a (Lebesgue) density point? What can you say about the density points of a measurable set?

   a.e. point of E is density point using Lebesgue differentiation on characteristic function of E.

2. (CF) Sketch a proof of Lebesgue differentiation theorem.

   I said to approximate f by compactly supported functions and apply the maximal function.

3. (CF) Define the maximal function. What can you say about it?

   L^1 to weak L^1, L^p to L^p.

4. (AN) How do you prove L^p to L^p bounded?

   Use Marcinkiewicz interpolation. (I butchered the pronounciation.)

5. (AN) But Mf is not linear....

   But it is sublinear!

4 Harmonic analysis

1. (CF) Define Calderon-Zygmund operator. What can you say about them?

   L^1 to weak L^1, $L^p \to L^p$ bounded for $p \in (1, \infty)$

2. (CF) Give examples of CZ operators.

   Hilbert transform.

3. (CF) Why is Hilbert transform useful?

   Given holomorphic f on upper half plane, the real part and imaginary part of f restricted to $\mathrm{Im}(z) = 0$ are related by Hilbert transform.

4. (CF) Any other CZ operators?

   Riesz transform.

5. (CF) Why is Riesz transform useful?

   Bound the Hessian $\frac{\partial u}{\partial x_i \partial x_j}$ of solution to $\Delta u = f$ (it is $R_i R_j f$ which is L^p bounded operator)

6. (AN) Prove L^p boundedness of CZ-operators for $1 < p < \infty$.

   Gave the standard proof using CZ-decomposition $f = g+b$, where $|g| \lesssim \alpha$ and b is localized to the union of small balls outside $E = \{x : |Mf(x)| > \alpha\}$.

5 Metric geometry

1. (AN) What can you say about Euclidean sections of ℓ_1^n?

   Since ℓ_1 has cotype 2, we have the bound $k(X) \gtrsim n$.

2. (AN) What is "..." theorem?

   ??? not Klinchine? said I had no idea.

   I didn't quite catch the name, it sounded like Klinchine or Krivine. The theorem was about finding "nicely complemented" subspaces of ℓ_1^n. I thought it was Krivine's theorem but don't see any similarity between it and the problem....

3. (AN) Define cotype.

4. (AN) Prove the bound $k(X) \gtrsim n$ for cotype 2.

   By John's theorem, find Euclidean norm $|\cdot|$ such that $\frac{1}{\sqrt{n}}|x| \le \|x\| \le |x|$. By Dvorestky-Rogers, find ONB $\{e_1, e_2, \cdots, e_n\}$ such that $\|e_i\| \ge \frac{1}{4}$ for $1 \le i \le \frac{n}{2}$. Let M_r be the median of the function $r : S^{n-1} \to \mathbb{R}$ defined by $r(x) = \|x\|$. Then by concentration of measure on S^{n-1}, $k(X) \gtrsim n \cdot M_r^2$ (take a random k-dimensional subspace and see what you get). Finally, M_r is close to the average $\int_{S^{n-1}} \|x\| dx$. By symmetry, it is $\int_{S^{n-1}} \mathbb{E}_\varepsilon \| \sum_{i=1}^n e_i x_i \varepsilon_i \| dx$, where $\varepsilon_i \in \{-1, 1\}^n$ each with probability $\frac{1}{2}$. By triangle inequality, it is $\ge \int_{S^{n-1}} \mathbb{E}_\varepsilon \| \sum_{i=1}^{n/2} e_i x_i \varepsilon_i \| dx$. Now use cotype (+ Kahane's inequality and $\|e_i\| \ge \frac{1}{4}$ for $1 \le i \le \frac{n}{2}$) and it becomes $\ge \int_{S^{n-1}} \left(\sum_{i=1}^{n/2} |x_i|^2 \right)^{1/2} dx$, which is $\ge c$.

   (AN) asked me to state and prove Kahane's inequality, which I stumbled on.

5. (AN) State and prove Bourgain's embedding theorem.

   Gave the probabilistic proof suggested to me by Alan Chang.

(CF) runs to get (IZ), who admits to forgetting about my generals :(

6 Algebra

1. (IZ) Classify finitely-generated modules over PID.

2. (IZ) State and prove Jordan canonical form.

3. (IZ) Classify chain complexes over fields.

   ??????

4. (IZ) What else have you studied?

   Representation theory of finite groups, Sylow theorems, finite groups, Galois theory... (like, what I'm supposed to???)

5. (IZ) Sorry, I don't know any of these topics.

   (to myself) Bruh

6. (AN) How about some Galois theory. What can you say about polynomials that are solvable by radicals?

   If $f \in \mathbb{Q}[x]$, then f is solvable by radicals iff $\mathrm{Gal}(f)$ is solvable.

7. (AN) What if the field is not $\mathbb{Q}$?

   Separable?

8. (AN) Define nilpotent group.

   Uh... I know nilpotent ring...

9. (CF) How about a different topic. What are the elementary symmetric polynomials?

10. (CF) Are there any types of symmetric polynomials besides the elementary ones?

    No: every symmetric polynomial is a polynomial in the elementary symmetric ones.

11. (CF) What is the relation of roots of a polynomial to its coefficients?

    Vieta's formulas

12. (CF) Let's solve a cubic equation. Say $z^3+az+b=0$ has roots x_1, x_2, x_3. Consider $(x_1+\omega x_2+\omega^2 x_3)^3$ and $(x_1+\omega^2 x_2+\omega x_3)^3$. What can you say about their sum and product?

    Both symmetric in x_1, x_2, x_3, hence expressible as polynomials in terms of a, b.

13. (CF) How can you solve a cubic?

    Use the quadratic formula to solve for $(x_1+\omega x_2+\omega^2 x_3)^3$ and $(x_1+\omega^2 x_2+\omega x_3)^3$. Take cube roots and the rest is easy.

I was asked to leave the room. After a few min, they congratulated me and said I passed!