Theorem 1. *The replace-add implementation of PV satisfies the PV property*

Proof. Assume each PE executes with only a finite delay between instructions. Expanding the invocations of P and V, we obtain the following code for each PE, where TDR, $Temp_1$, and $Temp_2$ are local variables.

```
comment: Initially S = 1

1  cycle {
2     repeat TDR ← false
3        Temp1 ← S − 1
4        if Temp1 ≥ 0 then {
5           Temp2 ← RepAdd(S, −1)
6           if Temp2 ≥ 0  then
7              TDR ← true
8           else RepAdd(S, 1) }
9     until TDR
10    critical  section
11    RepAdd(S, 1) }
```

Continuing with our proof, we make the following

Claim. *One may assume, without loss of generality, that, during each time step, exactly one PE executes one line of PVTest.*

Proof. By the serialization principle we may assume that during each time step exactly one PE executes exactly one machine instruction. That is, we can assume a serial execution order $I_1, I_2, ...$ where each I_j represents execution of one machine instruction by one PE; of course, the instructions executed by each PE are in the sequence determined by the *PVTest* code. Note that we can interchange any two consecutive instructions I_j and I_{j+1} executed by two different PEs provided that at most one of them references a public variable. Thus, since each line of *PVTest* contains at most one reference to S, there exists a sequence of interchanges that yields an execution order in which each line *PVTest* is executed indivisibly. This proves our claim. ☐

To describe the state of a PE at time T, we introduce terminology for specifying each PE's location counter and the value of the shared variable S. For $1 \leq j \leq 11$, PE$_i$ is said to be *at* j if the next instruction to be executed by PE$_i$ is line j of the program above. We write $L_i(T) = j$ to indicate that, at time T, PE$_i$ is *at* j. For $j = 4, 6$, and 9, the three decision points of the program, we distinguish to subcases: PE$_i$ is at $j+$ (respectively, $atj-$) if PE$_i$ is at j and the condition to be tested in line j is true (respectively, false). The value of S at time T is denoted $S(T)$ and equals $1+\sum c_i(T)$, where 1 is the initial value of S and $c_i(T)$ is the cumulative contribution to S caused by actions of PE$_i$ completed before time T. This latter quantity equals

$$n_i(T, 8) + n_i(T, 11) - n_i(T, 5)$$

where $n_i(T, j)$ is the number of executions of line j by PE$_i$ completed before time T.
A simple analysis of *PVTest* as a sequential program yields

Proposition 1. *For all times T and PE indices i, $c_i(T)$ is 0 or -1. Specifically, $c_i(T)$ = -1 if and only if $L_i(T)$ = 6, 7, 8, 9+, 10, or 11.*

Corollary 1. *At any time T, -#PE < S(T) ≤ 1.*

We now prove the easy half of the theorem, namely, that mutual exclusion is guaranteed. We call a PE *critical at time T* if $L_i(T)$ = 6+, 7, 9+, 10, or 11 and define $N(T)$ to be the number of such PEs.

Proposition 2. *At any time T, N(T) ≤ 1.*

Proof. If not, there exists a time t_0 such that $N(t_0) = 1$ and $N(t_0+1) = 2$. Thus, for some processor PE$_i$, $L_i(t_0) = 5$ and $L_i(t_0 + 1) = 6+$. But, by Proposition 1, any critical PE contributes -1 to S, and thus $S(t_0) < 1$. This contradicts the statement that PE$_i$ makes a transition from 5 to 6+ at time t_0. ☐

Corollary 2. *At any time T, at most one PE can be executing its critical section.*

To complete the proof of our theorem, we must show that some time after any reachable state is established, some PE$_i$ will enter the critical section. The key to this is to verify the following:

Lemma 2. *For any time T there exists a time T' ≥ T such that S(T') = 1.*

Proof. Suppose $S(t) < 1$ for $t \geq T$. Then, after time T, no PE can make the transition from 3 to state 4+ or from state 5 to state 6+. Therefore, the flow graph for *PVTest* becomes

$$\begin{tikzcd}
6+ \arrow[r] & 7 \arrow[r] & 9+ \arrow[r] & 10 \arrow[r] & 11 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4- \arrow[r] & 9- \arrow[lll, bend right] \\
 & & & & & 1 \arrow[u] & & & \\
 & & & & & 4+ \arrow[r] & 5 \arrow[r] & 6- \arrow[r] & 8 \arrow[uu]
\end{tikzcd}$$

Hence, there exists $T' \geq T$ such that at time T' all PEs are at 2, 3, 4-, or 9-, and thus, by Proposition 1, $c_i(T') = 0$ for all i. But this implies that $S(T') = 1$, as desired. ☐

Finally, if T' is as in the lemma, it is easy to see that after time T' the first PE to execute line 5 becomes critical and then enters its critical section. ☐