

Theorem 1. *The replace-add implementation of PV satisfies the PV property*

Proof. Assume each PE executes with only a finite delay between instructions. Expanding the invocations of P and V , we obtain the following code for each PE, where TDR , $Temp_1$, and $Temp_2$ are local variables.

comment: Initially $S = 1$

```

1  cycle {
2    repeat  $TDR \leftarrow false$ 
3       $Temp_1 \leftarrow S - 1$ 
4      if  $Temp_1 \geq 0$  then {
5         $Temp_2 \leftarrow RepAdd(S, -1)$ 
6        if  $Temp_2 \geq 0$  then
7           $TDR \leftarrow true$ 
8        else  $RepAdd(S, 1)$  }
9    until  $TDR$ 
10   critical section
11    $RepAdd(S, 1)$  }
```

Continuing with our proof, we make the following

Claim. *One may assume, without loss of generality, that, during each time step, exactly one PE executes one line of PVTest.*

Proof. By the serialization principle we may assume that during each time step exactly one PE executes exactly one machine instruction. That is, we can assume a serial execution order I_1, I_2, \dots where each I_j represents execution of one machine instruction by one PE; of course, the instructions executed by each PE are in the sequence determined by the $PVTest$ code. Note that we can interchange any two consecutive instructions I_j and I_{j+1} executed by two different PEs provided that at most one of them references a public variable. Thus, since each line of $PVTest$ contains at most one reference to S , there exists a sequence of interchanges that yields an execution order in which each line $PVTest$ is executed indivisibly. This proves our claim. \square

To describe the state of a PE at time T , we introduce terminology for specifying each PE's location counter and the value of the shared variable S . For $1 \leq j \leq 11$, PE_i is said to be *at j* if the next instruction to be executed by PE_i is line j of the program above. We write $L_i(T) = j$ to indicate that, at time T , PE_i is at j . For $j = 4, 6$, and 9 , the three decision points of the program, we distinguish to subcases: PE_i is at $j+$ (respectively, $atj-$) if PE_i is at j and the condition to be tested in line j is true (respectively, false). The value of S at time T is denoted $S(T)$ and equals $1 + \sum c_i(T)$, where 1 is the initial value of S and $c_i(T)$ is the cumulative contribution to S caused by actions of PE_i completed before time T . This latter quantity equals

$$n_i(T, 8) + n_i(T, 11) - n_i(T, 5)$$

where $n_i(T, j)$ is the number of executions of line j by PE_i completed before time T .

A simple analysis of $PVTest$ as a sequential program yields

Proposition 1. *For all times T and PE indices i , $c_i(T)$ is 0 or -1. Specifically, $c_i(T) = -1$ if and only if $L_i(T) = 6, 7, 8, 9+, 10$, or 11 .*

Corollary 1. *At any time T , $-\#PE < S(T) \leq 1$.*

We now prove the easy half of the theorem, namely, that mutual exclusion is guaranteed. We call a PE *critical at time T* if $L_i(T) = 6+, 7, 9+, 10$, or 11 and define $N(T)$ to be the number of such PEs.

Proposition 2. *At any time T , $N(T) \leq 1$.*

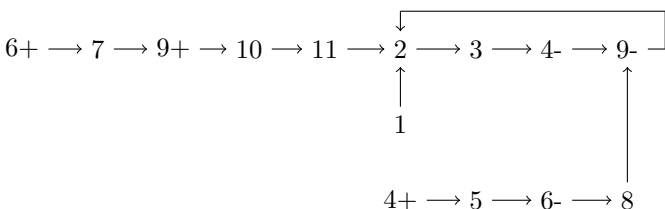
Proof. If not, there exists a time t_0 such that $N(t_0) = 1$ and $N(t_0+1) = 2$. Thus, for some processor PE_i , $L_i(t_0) = 5$ and $L_i(t_0+1) = 6+$. But, by Proposition 1, any critical PE contributes -1 to S , and thus $S(t_0) < 1$. This contradicts the statement that PE_i makes a transition from 5 to 6+ at time t_0 . \square

Corollary 2. *At any time T , at most one PE can be executing its critical section.*

To complete the proof of our theorem, we must show that some time after any reachable state is established, some PE_i will enter the critical section. The key to this is to verify the following:

Lemma 2. *For any time T there exists a time $T' \geq T$ such that $S(T') = 1$.*

Proof. Suppose $S(t) < 1$ for $t \geq T$. Then, after time T , no PE can make the transition from 3 to state 4+ or from state 5 to state 6+. Therefore, the flow graph for $PVTest$ becomes



Hence, there exists $T' \geq T$ such that at time T' all PEs are at 2, 3, 4-, or 9-, and thus, by Proposition 1, $c_i(T') = 0$ for all i . But this implies that $S(T') = 1$, as desired. \square

Finally, if T' is as in the lemma, it is easy to see that after time T' the first PE to execute line 5 becomes critical and then enters its critical section. \square