

# Homework 4, High-resolution shock-capturing methods

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## 1 Shallow water with non-horizontal bottom

The shallow water model of HW2 is now extended to a non-horizontal bottom “bathymetry”  $B(x)$ .

$$\begin{pmatrix} h_t \\ hu_t \end{pmatrix} + \begin{pmatrix} h_x u + hu_x \\ hu u_x + gh(h_x + B_x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1)$$

a) Show that still water ( $u = 0$ ) must have, as it should, a horizontal water level.

If  $u = 0$  for all space and time, then  $u_t$  and  $u_x$  are also zero. Equation (1) becomes

$$\begin{pmatrix} h_t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ gh(h_x + B_x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This means that  $h_t = 0$  and  $h_x = -B_x$ . We know that  $B(x)_t = 0$  because it isn't a function of time. The total height,  $D(x, t)$ , is  $D(x, t) = h(x, t) + B(x)$ . A horizontal water level means that  $D(x, t) = \text{constant}$  and  $D(x, t)_x = D(x, t)_t = 0$ . Equation (2) shows that derivatives with respect to  $x$  and  $t$  are zero and therefore showing that the total height  $D(x, t)$  is constant.

$$\begin{aligned} D(x, t)_x &= h_x + B_x = h_x + (-h_x) = 0 \\ D(x, t)_t &= h_t + B_t = 0 + 0 = 0 \end{aligned} \quad (2)$$

b) Write the equation in conservation form for  $h$  and  $m = hu$ .

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ f_2(h, m) \end{pmatrix}_x = \begin{pmatrix} 0 \\ s(h, m, x) \end{pmatrix} \quad (3)$$

The first equation in Equation (3) is proved using the product rule,  $m_x = h_x u + hu_x$ . The second equation needs to be written out explicitly and solved for the unknowns. Substituting  $m_t = h_t u + hu_t$  into Equation (3)

$$(h_t u + hu_t) + [hu u_x + gh(h_x + B_x)] = h_t u$$

and substituting  $u = m/h$  and  $h_t = -m_x$

$$m_t + [h \left( \frac{m}{h} \right)_x + gh(h_x + B_x)] = (-m_x) \left( \frac{m}{h} \right)$$

and bringing  $x$  and  $t$  derivatives together and putting derivatives other than  $h$  or  $m$  to the source function

$$\begin{aligned} m_t + \underbrace{\left[ m \left( \frac{m}{h} \right)_x + m_x \left( \frac{m}{h} \right) \right]}_{(m^2/h)_x} + \underbrace{ghh_x}_{(gh^2/2)_x} &= -ghB_x \\ m_t + \underbrace{\left( \frac{m^2}{h} + \frac{gh^2}{2} \right)_x}_{f_2(h, m)} &= \underbrace{-ghB_x}_{s(h, m, x)} \end{aligned}$$

Equation (3) written in conservation form becomes,

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ \frac{m^2}{h} + \frac{gh^2}{2} \end{pmatrix}_x = \begin{pmatrix} 0 \\ \frac{gh^2}{2} \end{pmatrix}_x$$