Homework 4, High-resolution shock-capturing methods

Kevin Mead

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1 Shallow water with non-horizontal bottom

The shallow water model of HW2 is now extended to a non-horizontal bottom "bathymetry" $B(x)$.

$$\begin{pmatrix} h_t \\ hu_t \end{pmatrix} + \begin{pmatrix} h_x u + h u_x \\ huu_x + gh(h_x + B_x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

a) Show that still water ($u = 0$) must have, as it should, a horizontal water level.

If $u = 0$ for all space and time, then u_t and u_x are also zero. Equation (1) becomes

$$\begin{pmatrix} h_t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ gh(h_x + B_x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This means that $h_t = 0$ and $h_x = -B_x$. We know that $B(x)_t = 0$ because it isn't a function of time. The total height, $D(x,t)$, is $D(x,t) = h(x,t) + B(x)$. A horizontal water level means that $D(x,t) =$ constant and $D(x,t)_x = D(x,t)_t = 0$. Equation (2) shows that derivatives with respect to x and t are zero and therefore showing that the total height $D(x,t)$ is constant.

$$\begin{aligned} D(x,t)_x &= h_x + B_x = h_x + (-h_x) = 0 \\ D(x,t)_t &= h_t + B_t = 0 + 0 = 0 \end{aligned} \tag{2}$$

b) Write the equation in conservation form for h and $m = hu$.

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ f_2(h,m) \end{pmatrix}_x = \begin{pmatrix} 0 \\ s(h,m,x) \end{pmatrix} \tag{3}$$

The first equation in Equation (3) is proved using the product rule, $m_x = h_x u + h u_x$. The second equation needs to be written out explicitly and solved for the unknowns. Substituting $m_t = h_t u + h u_t$ into Equation (3)

$$(h_t u + h u_t) + [huu_x + gh(h_x + B_x)] = h_t u$$

and substituting $u = m/h$ and $h_t = -m_x$

$$m_t + [h\left(\frac{m}{h}\right)\left(\frac{m}{h}\right)_x + gh(h_x + B_x)] = (-m_x)\left(\frac{m}{h}\right)$$

and bringing x and t derivatives together and putting derivatives other than h or m to the source function

$$m_t + [\underbrace{m\left(\frac{m}{h}\right)_x + m_x\left(\frac{m}{h}\right)}_{(m^2/h)_x} + \underbrace{ghh_x}_{(gh^2/2)_x}] = -ghB_x$$

$$m_t + (\underbrace{\frac{m^2}{h} + \frac{gh^2}{2}}_{f_2(h,m)})_x = \underbrace{-ghB_x}_{s(h,m,x)}$$

Equation (3) written in conservation form becomes,

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ \frac{m^2}{h} + \frac{gh^2}{2} \end{pmatrix}_x = \begin{pmatrix} 0 \\ -ghB_x \end{pmatrix}$$

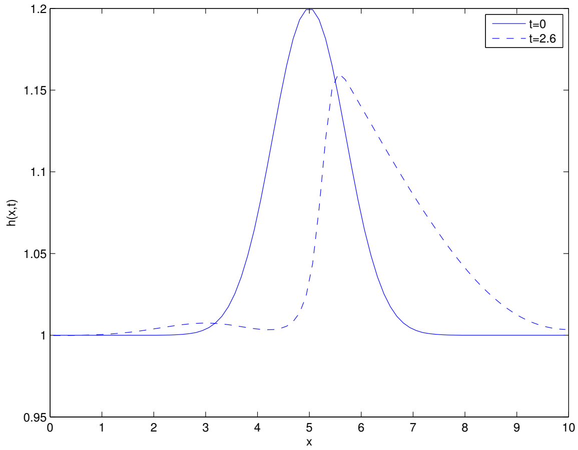
2 First order Roe scheme

1. (a) Explain how to choose m(x,0), to make a single pulse (not two!) based on the linearized problem.

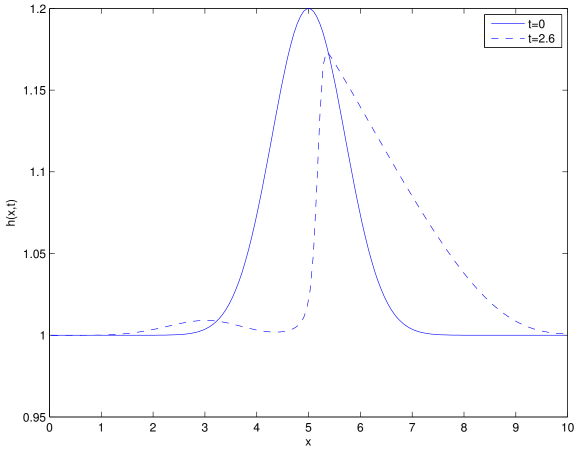
   You can make it a single pulse by giving the wave an initial speed proportional to its height in one direction. Since $m = hv$, we can set m to hc, where the characteristic speed is $c = \sqrt{gH}$, and $H = 0$ since we want a linearized approximation.

   (b) Use this initial data and run the Roe solver with n=80, 160, 320 and the CFL number as large as possible without instability, until the wave has been fully reflected at L. Plot the wave shapes.

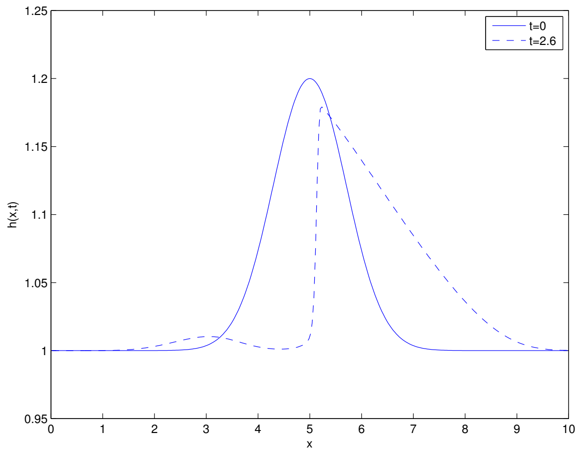
   Figure 1 shows the plots using different N values. The N value seemed to affect the sharpness of the front edge of the wave. As N increased, the sharpness increased.



(a) $N = 80$



(b) $N = 160$



(c) $N = 320$

Figure 1: These are the plots with different N values. The CFL condition was set to 80% of the value that would be ideal for a linear problem. The wave edges get sharper when N is increased

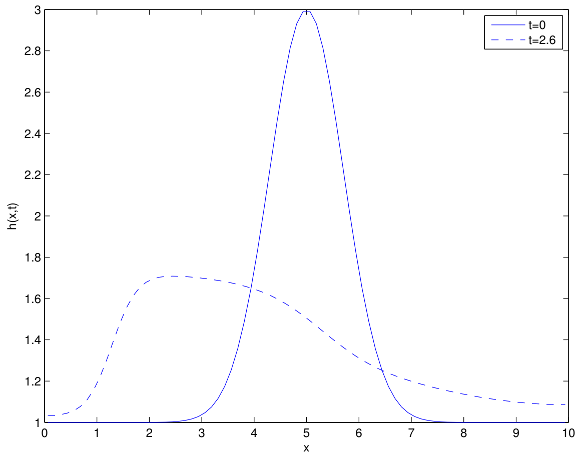
   (c) Compare with solutions obtained using your Lax-Friedrichs solver from HW2.

   The solution obtained here looks very similar to the Lax-Friedrichs solver from HW2 as the shape of the wave forms a sharp edge and looks like a triangle instead of a Gaussian. This solution isn't stable at the CFL condition and we must make the time step a little smaller. The Lax-Friedrich solution was also not stable at the CFL condition, but I was forced to decrease the time step by a much larger amount in order to see stability for a few seconds.

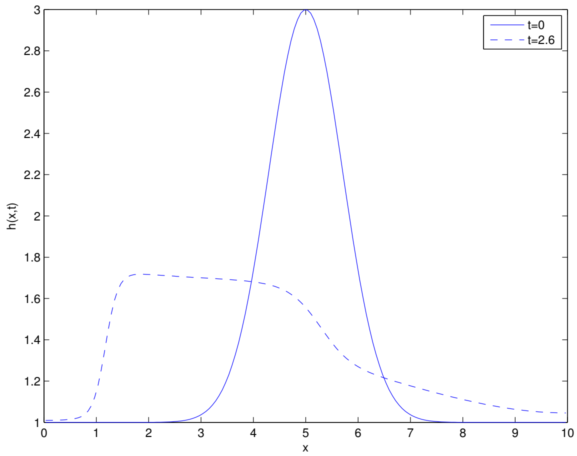
(d) Comment about order of accuracy, dissipation and dispersion.

(e) Then try also a higher pulse, say a = 2H. Comment on how well the single pulse initial data works.

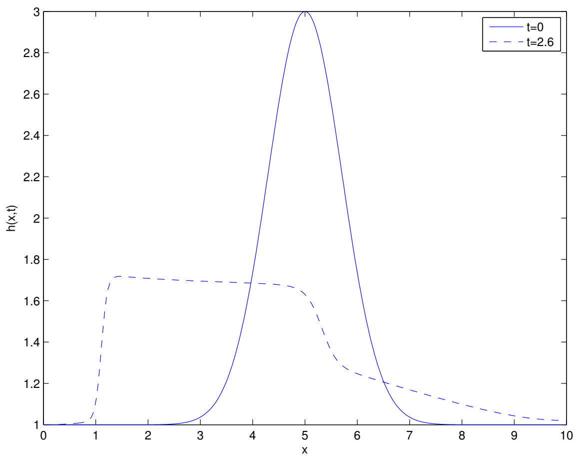
A pulse with $a > 1/5H$ gives unstable results using a time step of 80% of the CFL condition, so I had to reduce it to 40% in order to get the results shown in Figure 2. This figure is different from Figure 1 in that the height of the left moving pulse isn't as negligible as it once was. The overlap of the smaller wave is much more noticeable when a is large compared to H.



(a) $N = 80$



(b) $N = 160$



(c) $N = 320$

Figure 2: These are the plots with different N values. The CFL condition was set to 40% of the value that would be ideal for a linear problem. The wave edges get sharper when N is increased, but changing the value of a has also changed the characteristic speed and shape of the wave after it recombines with its small wave that moves leftward.