Homework 3
Stability and other properties of numerical schemes
(Max. 3.0 p.)
Deadline Mon. 14 Apr. 2014

Topics: Stability by Fourier analysis and Lax-Richtmyer stability. Modified equation, dissipation and dispersion.

Purpose: To get acquainted with different techniques for analyzing numerical schemes approximating initial-boundary value problems.

Instructions: Write a report with the plots and answers to the questions posed. Make sure the plots are annotated and there is explanation for what they illustrate. Append the code to the report.

1 Stability by von Neumann analysis (0.5 p)

Consider 2π-periodic Cauchy problem for $\alpha = -1$,

$$u_t = \alpha u_{xxxx},$$
$$u(x,0) = \sin x.$$

Use a finite difference approximation with central differences in space and forward difference in time. Apply von Neumann analysis to determine how Δt and Δx should be related for the method to be stable.

2 L1 Lax-Richtmyer stability and the modified equation (1.0 p)

Consider the Lax-Friedrichs method applied to the Cauchy problem ($-\infty < x < +\infty$) for the advection equation,

$$u_t + au_x = 0$$

with initial L_1 - data $u(x,0)$.

a) Show that the method is Lax-Richtmyer stable in the L_1-norm if the CFL condition ("numerical domain of dependence includes mathematical domain of dependence") is satisfied. (0.5 p)

b) Derive the modified equation and discuss how to choose Δt and Δx to *avoid* strong damping effects. You may do this for the finite difference version, rather than the finite volume version. (0.5 p)

3 Shallow water equations, dissipation and dispersion (1.5 p)

a) Run your Lax-Friedrichs program for Homework 2 with different values of $\Delta t/\Delta x$. How large can Δt be without violating the CFL condition?
Plot solutions for different $\Delta t/\Delta x$. When is the damping of waves largest/smallest? Compare with your prediction in the previous question. (0.5 p)

b) Next, solve the problem in Homework 2 by the two-step Lax-Wendroff method, known as the McCormack scheme, see page 337 in LeVeque. (1.0 p)

i) According to section 8.6.2 the method is dispersive. Hand in plots of the solution where effects of dispersion errors can be seen. *Don't use the "magic time step"*.

ii) How is this effect changed by higher spatial resolution? Do you see any damping?

iii) Compare (dissipation/dispersion) with the solution obtained with the Lax-Friedrichs method.