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CS-225: Discrete Structures in CS

Homework 5, Part 1

Exercise Set 5.2: Problem #(12, 16)

Exercise Set 5.3: Problem #(12, 18, 26)

Exercise Set 5.2

12.

Let the property P(n) be the equation

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}\, for\, every\, integer\, n\geq 1$$

Basis step: We must show that P(1) is true.

The left hand side of P(1) =

$$\frac{1}{1\cdot(1+1)}=\frac{1}{1+1}=\frac{1}{2}$$

The right hand side of P(1) =

$$\frac{n}{n+1}=\frac{1}{1+1}=\frac{1}{2}$$

Since the LHS and RHS of both equal ½, P(1) is true.

Inductive hypothesis: For all integers k ≥ 1, suppose P(k) is true. That is,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{(k(k+1))}=\frac{k}{k+1} \quad (*)$$

Inductive step: We will show that for all integers k ≥ 1, if P(k) is true, then P(k+1) is true. That is, we must show that:

$$p(k+1)\equiv\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{(k+1)((k+1)+1)}=\frac{k+1}{(k+1)+1}$$

Left hand side of P(k+1)

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{(k+1)((k+1)+1)}$$

$$= \quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{((k+1)+1)}+\frac{1}{(k+1)((k+1)+1)}$$ making next-to-last term explicit

$= \quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)}$ substitution from inductive hypothesis (*), and by algebra

$= \quad \frac{(k)(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)}$ multiplied both sides by (k+2) for common denominator

$= \frac{(k)(k+2)+1}{(k+1)(k+2)}$ combined terms by addition

$= \frac{k^2+2k+1}{(k+1)(k+2)}$ distributive property

$= \frac{(k+1)^2}{(k+1)(k+2)}$ by algebra

$= \frac{k+1}{k+2}=\frac{k+1}{(k+1)+1}$ canceling common factors and expressing (k+2) as (k+1) + 1

This equals the right hand side of P(k+1). Thus the property is true for n = k + 1. Since the basis and inductive step have been proven, P(n) is true for all integers n ≥ 1.

16.

Let the property P(n) be the equation

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})\ldots(1-\frac{1}{n^2})=\frac{n+1}{2n}\ \textit{for every integer}\ n\geq 2.$$

<u>Basis step</u>: We must show that P(2) is true.

Left hand side of P(2) =

$$(1-\frac{1}{n^2})=(1-\frac{1}{2^2})=1-\frac{1}{4}=\frac{3}{4}$$

Right and side of P(2) =

$$\frac{n+1}{2n}=\frac{2+1}{2(2)}=\frac{3}{4}$$

Since the LHS and RHS both equal ¾, P(2) is true.

<u>Inductive hypothesis</u>: For all integers k ≥ 2, suppose P(k) is true. That is,

P(k) = $\quad (1-\frac{1}{2^2})(1-\frac{1}{3^2})\ldots(1-\frac{1}{k^2})=\frac{k+1}{2k}$ (*)

<u>Inductive step:</u> We will show that for all integers k ≥ 2, if P(k) is true, then P(k+1) is true. That is, we must show that

P(k+1) = $\quad (1-\frac{1}{2^2})(1-\frac{1}{3^2})\ldots(1-\frac{1}{(k+1)^2})=\frac{(k+1)+1}{2(k+1)}$

Left hand side of P(k+1):

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})...(1-\frac{1}{(k+1)^2})$$

$$= (1-\frac{1}{2^2})(1-\frac{1}{3^2})...(1-\frac{1}{k^2})(1-\frac{1}{(k+1)^2})$$ writing the (k+1)th term separately from first k terms

$$= (\frac{k+1}{2k})(1-\frac{1}{(k+1)^2})$$ by substitution from inductive hypothesis (*)

$$= \frac{k+1}{2k}-\frac{k+1}{(k+1)^2(2k)}$$ distributive property

$$= \frac{k+1}{2k}-\frac{1}{(k+1)(2k)}$$ by algebra

$$= \frac{k+1}{2k}-\frac{k+1}{(k+1)^2(2k)}$$ by multiplying left side by (k+1) for common denominator

$$= \frac{(k+1)^2}{(2k)(k+1)}-\frac{1}{(2k)(k+1)}$$ by subtraction

$$= \frac{k^2+2k+1-1}{2k(k+1)}$$ by evaluating $(k+1)^2$

$$= \frac{k^2+2k}{(2k)(k+1)}$$ canceling out like terms

$$= \frac{k+2}{2(k+1)}=\frac{(k+1)+1}{2(k+1)}$$ canceling out common factors and expressing k+2 as (k+1)+1

This equals the right-hand side of P(k+1). Hence the property is true for n = k + 1. [*Since the basis and inductive step have been proven, the statement is true for all integers n ≥ 2.*]

Exercise Set 5.3

12.

For every integer n ≥ 0, let P(n) be the statement that,

$$7^n - 2^n \textit{ is divisible by } 5$$

Basis step: To establish that P(0), we must show that $7^0 - 2^0$ is divisible by 5.

$7^0 - 2^0 = 1 - 1 = 0$, and 0 is divisible by 5. Hence, P(0) is true.

Inductive hypothesis: For all integers k ≥ 0, suppose P(k) is true. That is,

P(k) = $7^k - 2^k$ is divisible by 5, thus

$7^k - 2^k = 5 \cdot q$ . . . . for some $q \in \mathbb{Z}$ (*)

Inductive step: We will show that for all integers $k \geq 0$, if P(k) is true then P(k+1) is true. That is, we must show $P(k+1) = 7^{k+1} - 2^{k+1}$ is divisible by 5

Now

$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k$ by laws of exponents

$= (7^k \cdot 5) + (7^k \cdot 2) - (2^k \cdot 2)$ by algebra

$= (7^k \cdot 5) + 2(7^k - 2^k)$ factoring out 2

$= 7^k \cdot 5 + 2 \cdot 5p$ from inductive hypothesis (*)

$= 5(7^k + 2p)$ factoring out 5

Let $q = (7^k + 2p)$. Then q is an integer because the sum of integers are integers. Therefore,

$7^k - 2^k = 5q$ where by the definition of divisibility, $7^k - 2^k$ is divisible by 5. Therefore, P(k+1) holds, and the property is true for n = k + 1. [*Since both the basis step and inductive step have been proved, P(n) is true for all integers n ≥ 0.*]

18.

For every integer $n \geq 2$, let P(n) be the inequality

$5^n + 9 < 6^n$

Basis step: To establish P(2), we must show that $5^2 + 9 < 6^2$

Now, $5^2 + 9 = 34$ and $6^2 = 36$ and $34 < 36$. Hence, P(2) is true.

Inductive hypothesis: For every integer $k \geq 2$, suppose P(k) is true. That is,

$P(k) = 5^k + 9 < 6^k$ (*)

Inductive step: We will show that for all integers $k \geq 2$, if P(k) is true then P(k+1) is true. That is, we must show that

$P(k+1) = 5^{k+1} + 9 < 6^{k+1}$

Now

$5^{k+1} + 9 = 5 \cdot 5^k + 9$ by exponent laws

$< 5 \cdot (6^k - 9) + 9$ by inductive hypothesis, but rearranging it from $5^k + 9 < 6^k$ to $5^k < 6^{k-} 9$

$= 5 \cdot 6^k - 36$ by distributive property and simplifying

$= 5 \cdot 6^k - 36$ because $-36 < 0$

So, $5 \cdot 6^k < 6 \cdot 6^k$ by exponent laws for 6^{k+1}

The quantity $5 \cdot 6^k$ is definitively less than $6 \cdot 6^k$ or 6^{k+1}. Therefore, P(k+1) holds. Hence, the property is true for n = k + 1. {*Since both the basis step and inductive step have been proven, P(n) is true for all integers n ≥ 2.*]

26.

For every integer n ≥ 0, let P(n) be the statement $c_n = 3^{2^n}$

Basis step: We must establish that the base case P(0) is true

P(0) = $c_0 = 3^{2^0} = 3$

Also, it is given that $c_0 = 3$. Thus, P(0) is true.

Inductive hypothesis: For all integers k ≥ 0, suppose P(k) is true. That is,

$$c_k = 3^{2^k}$$

Inductive step: We will show that for all integers k ≥ 0, if P(k) is true, then P(k+1) is true. That is, we must show that

$$c_{k+1} = 3^{2^{k+1}}$$

By the given sequence definition where $c_k = (c_{k-1})^2$ for each integer $n \geq 0$, then

$(c_{(k+1)-1})^2 = (c_k)^2$ by sequence definition

= $(3^{2^k})^2$ by inductive hypothesis

= $3^{2 \cdot 2^k}$ by exponent rules

= $3^{2^{k+1}}$ by expressing the exponent $2 \cdot 2^k$ as $2^{2^{k+1}}$

Thus P(k+1) holds, and the property is true for n = k + 1. {*Since both the basis step and inductive step have been proven, P(n) is true for all integers n ≥ 0.*]