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CS-225: Discrete Structures in CS

Homework 3. Part 2

Exercise Set #4.7: Problem #13, #22, #28, #29

Exercise Set #4.8: Problem #18(a), (b)

Exercise Set #4.7

13.

Negation: There exists an irrational number and any nonzero rational number whose product is rational.

Supposition: Suppose not [*We must take the negation of the statement and suppose it to be true*]. That is, suppose there exists an irrational number *s* and a nonzero rational number *r* whose product is rational [*We must deduce a contradiction*].

Goal: Deduce the contradiction and show the product is rational.

Deduction:

By the definition of a rational number where *r* is rational, let

$r = \frac{a}{b}$ for some integers *a* and *b* where $b \neq 0$ definition of a rational number

Since the product of *r* and *s* is rational, let

$r \cdot s = \frac{c}{d}$ for some integers *c* and *d* where $d \neq 0$

Then

$\frac{a}{b} \cdot s = \frac{c}{d}$ by substitution

Then

$s = \frac{cb}{da}$ by algebra and division

Since the products of integers are integers, and $d \neq 0$ and $a \neq 0$ by the zero-product property, *cb* and *da* are integers. Thus, the product of *r* and *s* is rational.

Then, *s* must be a rational number by the definition of a rational, as it is expressed as a quotient of integers with a nonzero denominator. However, this is contradictory to the original supposition which states *s* is an irrational number, as *s* cannot be both rational and irrational.

Conclusion: The contradiction of the statement is incorrect; thus, the original statement is true {*as was to be shown*}.

#22.

a. Contradiction

Supposition: Suppose not [*We must take the negation of the statement and suppose it to be true*]. That is, suppose there exists a real number *r* such that r^2 is irrational and *r* is rational [*We must deduce a contradiction*].

Goal: We need to deduce that this supposition leads to a contradiction.

Deduction:

Let

$r = \frac{p}{q}$ definition of a rational, for some integers *p* and *q* where q ≠ 0.

Then

$r^2 = \frac{p^2}{q^2}$ by substitution and algebra

$\frac{p^2}{q^2}$ are integers as the products of integers are integers, thus, r^2 is rational by the definition of a rational number, as it is expressed as the quotient of two integers with a non-zero denominator.

Conclusion: This contradicts the supposition where r^2 is irrational, hence the original statement is true{*as was to be shown*}.

b. Contraposition

Supposition: For every real number *r*, if *r* is not irrational then r^2 is not irrational {*we must show that r^2 is rational*}.

Goal: We would need to show that r^2 is not irrational (in other words, that r^2 is rational).

Deduction:

Let

$r = \frac{p}{q}$ by the definition of a rational, for some integers *p* and *q* where q ≠ 0.

Then

$r^2 = \frac{p^2}{q^2}$ by substitution and algebra

As in part a, $\frac{p^2}{q^2}$ are integers as the products of integers are integers, thus, r^2 is rational by the definition of a rational number, as it is expressed as the quotient of two integers with a non-zero denominator, this proves the contrapositive.

Conclusion: The contraposition is proven, which means the original statement is also true{*as was to be shown*}.

#28.

Proof by contradiction.

Supposition: Suppose not {*We must take the negation of the statement and suppose it to be true*}. That is, suppose that there exists some integers a, b, and c such that $a|b$ and $a\nmid c$ and $a|(b + c)$ {*We must deduce a contradiction}.*

Goal: Deduce the contradiction of this supposition.

Deduction:

By the definition of divisibility, there exists integers k and i so that

$b = ak$ for some integer k and

$b + c = ai$ for some integer i

Then,

$ak + c = ai$ by substitution

$c = ai - ak$ by algebra

$c = a(i - k)$ by factoring out a

Let $m = i - k$. Then m is an integer because i and k are integers and because the difference of integers are integers.

Hence,

$c = am$ where m is some integer. By the definition of divisibility, this can be written as $a|c$, which contradicts the supposition.

Conclusion: Therefore, the supposition is false and the original statement is true.

#29.

Proof by contraposition: For all integers m and n, if m and n are both not even and m and n are both not odd, then $m + n$ is not even.

Supposition: Suppose that m and n are any [*particular but arbitrarily chosen]* integers such that m is even and n is odd.

Goal: We are trying to show that m + n is not even, in other words, odd.

Deduction: By the definition of even and odd integers, let

$m = 2k$ where k is some integer

$n = 2l + 1$ where *l is some integer*

Then,

$m + n = 2k + 1 + 2l$ by substitution

$= 2k + 2l + 1$ by associative property of addition

$m + n = 2(k + l) + 1$ by factoring

Since k is some integer and i is some integer, and the sum of integers are integers, as well as integers are closed under addition operations, then by definition of odd integers, $m + n$ is odd.

Conclusion: The contraposition of the statement is true. Hence, the statement is true. If we consider the other case where m is odd and n is even, we can also show that $m + n$ is odd.

Exercise Set 4.8

#18.

a. **Proof by contradiction**

Supposition: Suppose not. That is, suppose there is some {*particular but arbitrarily chosen*} integer a such that if a^3 is even then a is odd {*we have to deduce the contradiction*}.

Goal: Deduce the contradiction.

Deduction:

By the definition of odd, let

$a = 2k + 1$ for some integer k definition of odd integer

Thus

$a^3 = (2k + 1)^3$ by substitution

$= (2k + 1)(2k + 1)^2$ by factoring out $(2k + 1)$

$= (2k + 1)(4k^2 + 4k + 1)$ $(a + b)^2 = a^2 + 2ab + b^2$

$= 8k^3 + 12k^2 + 6k + 1$ distributive property and addition

$a^3 = 2(4k^3 + 6k^2 + 3k) + 1$ factor out 2

Let $t = 4k^3 + 6k^2 + 3k$ where t is an integer, because the products and sums of integers are integers. Thus $a^3 = 2t + 1$. Hence, a^3 is odd by the definition of an odd integer where it is twice some number plus 1.

Conclusion: We have deduced that a^3 is odd when a is odd, which is a contradiction. Therefore, the original statement is true{*as was to be shown*}.

b. Proof by contradiction

Supposition: Suppose not. That is, suppose $\sqrt[3]{2}$ is rational {*We need to deduce the contradiction*}. By the definition of a rational number,

$\sqrt[3]{2} = \frac{a}{b}$ for some integers a and b where $b \neq 0$ definition of a rational number

Then

$2 = \frac{a^3}{b^3}$ cubing each side

$= 2b^3 = a^3$ by algebra

Since the products of integers are integers, b^3 is an integer, and as a^3 is twice some integer, a is even by the definition of even integers.

Thus,

$a^3 = (2k)^3$ for some integer k

$= 8k^3$ by algebra

$= 2(4k^3)$ by factoring out 2

Since $a^3 = 2b^3$, this means that $b^3 = 4k^3 = 2(2k^3)$. Since the products of integers are integers, k^3 is an integer and thus $2k^3$ is an integer. Thus, b^3 can be written as twice some integer k^3, meaning b^3 is even.

However, this proves that 2 is a common factor of both a and b, which is a contradiction, as it was implied initially with $\sqrt[3]{2} = \frac{a}{b}$ that neither a nor b have common factors. Thus, we have deduced a contradiction.

Conclusion: We have deduced that $\sqrt[3]{2}$ is rational is a contradiction, thus the original statement is true.