Kevin Sekuj

CS-225: Discrete Structures in CS

Homework 5, Part 2

Exercise Set 5.4: Problem: #3, #7, #8.a

Extra Canvas Problem

Set 5.4

#3.

Let the property P(n) be the statement “c_n is even” for each integer n ≥ 0. Let c_0, c_1, c_2 be the sequence defined by $c_0 = 2$, $c_1 = 2$, and $c_2 = 6$.

Basis step:

P(0), P(1), and P(2). We are given $c_0 = 2$, $c_1 = 2$, and $c_2 = 6$, and 2 and 6 are all even integers. Thus, basis cases P(0), P(1), and P(2) are true.

Inductive hypothesis:

Let k be any integer with k ≥ 2, suppose that P(i) is true for each integer i with 0 ≤ i ≤ k.

c_i is even for each integer i with 0 ≤ i ≤ k.

Inductive step:

We must show that P(k+1) is true. That means we must show that c_{k+1} is even

$c_k = 3c_{k-3}$ thus

$c_{k+1} = 3c_{k-2}$

By the inductive hypothesis, k-2 is definitively ≤ k. Thus, c_{k-2} is even and can be expressed as 2r, as an even integer is defined as twice some integer.

= 3(2r)

Let s = (2r). Then

= 2s.

Therefore, c_{k+1} can be expressed as twice some integer, which means c_{k+1} is even. This proves that P(k+1) is true [*since both the basis step and inductive step have been proved, P(n) is true for a ll integers n ≥ 0*].

#7.

Let g_0, g_1, g_2 be the sequence defined by $g_0 = 2$, $g_1 = 2$, and $g_k = 3g_{k-1} – 2g_{k-2}$ for each integer $k \geq 3$. Let the property P(n) be the statement "$g_n = 2^n + 1$" for each integer $n \geq 1$, which we will prove by strong induction.

<u>Basis step:</u>

$P(1) = g_1 = 2^1 + 1 = 3$

$P(2) = g_2 = 2^2 + 1 = 5$

Also, we are given the values of g_1 and g_2 by the sequence definition. Thus, basis cases P(1), and P(2) are true.

<u>Inductive hypothesis:</u>

Let k be any integer with $k \geq 2$, suppose that P(i) is true for each integer i with $0 \leq i \leq k$.

$g_i = 2^i + 1$ for all integers i from through k

<u>Inductive step:</u>

We must show that P(k+1) is true. That is, we must show that $g_{k+1} = 2^{k+1} + 1$.

$g_k = 3g_{k-1} – 2g_{k-2}$ thus

$g_{k+1} = 3g_{k-1+1} – 2g_{k-2+1}$

$= 3g_k – 2g_{k-1}$

$= 3(2^k + 1) – 2(2^{k-1} + 1)$ by inductive hypothesis

$= 3 \cdot 2^k + 3 – 2 \cdot 2^{k-1} – 2$ by distributive property

$= 3 \cdot 2^k + 3 – 2 \cdot 2^k \cdot 2^{-1} – 2$ by exponent laws

$= 3 \cdot 2^k + 3 – (2 \cdot 2^k \cdot ½) – 2$ as $2^{-1} = ½$

$= 3 \cdot 2^k + 3 – (2^k \cdot 1) – 2$ by arithmetic

$= 3 \cdot 2^k - 2^k + 1$ " "

$= (2^k + 2^k + 2^k) – 2^k +1$ by expressing $3 \cdot 2^k$ alternatively

$= 2^k + 2^k +1$

$= 2 \cdot 2^k + 1$ by exponent laws

$= 2^{k+1} + 1$

Therefore, P(k+1) is true. Also, since the basis step and inductive step have been proven, the statement P(n): "$g_n = 2^n + 1$" for each integer $n \geq 1$ is true.

#8.a.

Let the property P(n) be the statement " $h_n \leq 3^n$ " for each integer $n \geq 0$. Let h_0, h_1, h_2 be the sequence defined by $h_0 = 1$, $h_1 = 2$, and $h_2 = 3$.

<u>Basis step:</u>

Using $h_n \leq 3^n$

P(0): $h_0 = 1 \leq 1 = 3^0$

P(1): $h_1 = 2 \leq 3 = 3^1$

P(2): $h_3 = 3 \leq 9 = 3^2$

Also, we are given the values of h_0, h_1, and h_2 by the sequence definition. Thus, basis cases P(0), P(1), and P(2) are true.

<u>Inductive hypothesis:</u>

Let P(0), P(1), P(2) to P(k) be true. Suppose that P(i) is true for each integer i = 0, 1, 2 . . .k where $k \geq 2$.

$h_i \leq 3^i$ for all integers i through k

<u>Inductive step:</u>

We must show that P(k+1) is true

$h_{k+1} = 3^{k+1}$

$h_{k+1} = h_{k+1-1} + h_{k+1-2} + h_{k+1-3}$ by sequence of $h_{\mathbf{k}} = h_{k-1} + h_{k-2} + h_{k-3}$

$= h_k + h_{k-1} + h_{k-2}$

$\leq 3^k + 3^{k-1} + 3^{k-2}$ by inductive hypothesis

$\leq 3^k + 3^k + 3^k$ by 3^k, 3^{k-1}, 3^{k-2} each $\leq 3^k$

$= 3 \cdot 3^k = 3^{k+1}$

Therefore, P(k+1) is true. Also, since the basis step and inductive step have been proven, the statement P(n): "P(n) be the statement " $h_n \leq 3^n$" for each integer $n \geq 0$ is true.

Extra question on Canvas

a.

Basis step:

P(12) 12 = 3(4)

P(13) 13 = 2(4) + 1(5)

P(14) 14 = 1(4) + 2(5)

P(15) 15 = 3(5)

b.

Let k be any integer with $k \geq 15$. Suppose P(i) is true for each i with $12 \leq i \leq k$.

c.

In the inductive step, we must prove that P(k+1) is true. That is, form P(k+1) cents of postage using just 4 cent and 5 cent stamps.

d.

P(k-3) formed by 4 cent and 5 cent stamps is obtained from the inductive hypothesis. Since we have the sequence for P(12), we simply need to add one more 4 cent coin to that, thus

$P_{(k+1)} = P_{(k-3)} + 4$	by sequence P(12) + one 4 cent coin
$= (4a + 5b) + 4$	by inductive hypothesis where $p_i = 4a + 5b$
$= 4a + 4 + 5b$	
$= 4(a+1) + 5b$	by algebra

Therefore, P(k+1) is true. Also, since the basis step and inductive step have been proven, the given statement is true.