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CS-225: Discrete Structures in CS
Homework 6
Canvas Problems #1-5

1.

$e_k = 5e_{k-1} + 3$ for all integers $k \geq 2$ recurrence relation
$e_1 = 2$ initial condition

$e_2 = 5(e_1) + 3 = 5(2) + 3 = 5^1 \cdot 2 + 5^0 \cdot 3$

$e_3 = 5(e_2) + 3 = 5(5^1 \cdot 2 + 5^0 \cdot 3) + 3 = 5^2 \cdot 2 + 5^1 \cdot 3 + 5^0 \cdot 3$

$e_4 = 5(e_3) + 3 = 5(5^2 \cdot 2 + 5^1 \cdot 3 + 5^0 \cdot 3) = 5^3 \cdot 2 + 5^2 \cdot 3 + 5^1 \cdot 3 + 5^0 \cdot 3$

$e_5 = 5(e_4) + 3 = 5(5^3 \cdot 2 + 5^2 \cdot 3 + 5^1 \cdot 3 + 5^0 \cdot 3) = 5^4 \cdot 2 + 5^3 \cdot 3 + 5^2 \cdot 3 + 5^1 \cdot 3 + 5^0 \cdot 3$

Guess:

$e_n = 5^{n-1} \cdot 2 + 5^{n-2} \cdot 3 + 5^{n-3} \cdot 3 + \ldots + 5^2 \cdot 3 + 5^1 \cdot 3 + 5^0 \cdot 3$

$e_n = 5^{n-1} \cdot 2 + 3(5^{n-2} + 5^{n-3} + \ldots + 5^2 + 5^1 + 5^0)$

$= 5^{n-1} \cdot 2 + 3 \cdot \sum_{i=0}^{n-2} 5i$

$= 2 \cdot 5^{n-1} + 3 \cdot \frac{5^{n-2+1}-1}{5-1}$ by formula $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$

$= \frac{4 \cdot 2 \cdot 5^{n-1}}{4} + \frac{3(5^{n-1}-1)}{4} = \frac{8 \cdot 5^{n-1} + 3 \cdot 5^{n-1} - 3}{4} = \frac{11 \cdot 5^{n-1} - 3}{4}$ (ans)

2. <u>Proof by mathematical induction:</u>

Let $e_1, e_2, e_3 \ldots$ be the sequence defined by $e_1 = 2$ and $e_k = 5e_{k-1} + 3$ for all integers $k \geq 2$, and let the property P(n) be the definition

$e_n = \frac{11 \cdot 5^{n-1} - 3}{4}$ for each integer $n \geq 1$. We must prove that for each integer $n \geq 1$, P(n) is true.

<u>Basis step:</u>
We have $P(1) = e_1 = \frac{11 \cdot 5^{1-1} - 3}{4} = \frac{11 \cdot 1 - 3}{4} = \frac{8}{4} = 2$
Also the initial condition gives $e_1 = 2$. Therefore, P(1) is true.
<u>Inductive hypothesis</u>: Let m be any integer with $m \geq 1$ and suppose P(m) is true.

$P(m) \equiv e_m = \dfrac{11 \cdot 5^{m-1} - 3}{4}$

<u>Inductive step:</u> We will show that for all integers m ≥ 1, if P(m) is true, then P(m + 1) is true. We must show that $P(m+1) \equiv e_{m+1} = \dfrac{11 \cdot 5^{m-1+1} - 3}{4}$

The left hand side of P(m+1) is:

$e_{m+1} = 5e_{(m-1)+1} + 3$ by recursive definition of the sequence
$= 5e_m + 3$

$= 5 \cdot \dfrac{11 \cdot 5^{m-1} - 3}{4} + \dfrac{3 \cdot 4}{4}$ substitution from inductive hypothesis and rewriting 3 as a fraction

$= \dfrac{5 \cdot 11 \cdot 5^{m-1} - 15 + 12}{4}$ by expressing 5 · (-3) as -15

$= \dfrac{11 \cdot 5^{1} \cdot 5^{m-1} - 3}{4}$ by property of exponents

$= \dfrac{11 \cdot 5^{m-1+1} - 3}{4}$

Which is the right hand side of P(m+1). Hence the property is true for n = m + 1. Since both the basis step and inductive step has been proved, P(n) is true for all integers n ≥ 1.

3.
$t_k = t_{k-1} + 7k + 2$ for all integers k ≥ 1 recurrence relation
$t_0 = 0$ initial condition

$t_1 = t_0 + 7(1) + 2 = 7 \cdot 1 + 2$

$t_2 = t_1 + 7(2) + 2 = (7 \cdot 1 + 2) + (7 \cdot 2 + 2)$

$t_3 = t_2 + 7(3) + 2 = (7 \cdot 1 + 2) + (7 \cdot 2 + 2) + (7 \cdot 3 + 2)$

$t_4 = t_3 + 7(4) + 2 = (7 \cdot 1 + 2) + (7 \cdot 2 + 2) + (7 \cdot 3 + 2) + (7 \cdot 4 + 2)$

Guess:

$t_n = (7 \cdot 1 + 2) + (7 \cdot 2 + 2) + (7 \cdot 3 + 2) + \dots (7 \cdot n + 2)$

$= (7 \cdot 1 + 7 \cdot 2 + 7 \cdot 3 + 7 \cdot 4 \dots + 7 \cdot n) + (2 + 2 + 2 + 2 \dots$for however many n times)

$= 7(1 + 2 + 3 + 4 \dots + n) + 2n$

$= \quad 7 \cdot \frac{n(n+1)}{2} + 2n$ by formula $\sum_{k=1}^{n} k \;\Big|\; \frac{n(n+1)}{2}$

$= \frac{7n(n+1)}{2} + 2n$

$= \frac{7n(n+1)+4n}{2}$

$= \frac{n(7(n+1)+4)}{2}$

$= \frac{n(7n+7+4)}{2}$

$= \frac{n(7n+11)}{2} = t_n = \frac{7n^2+11n}{2}$ for all $n \geq 1$ (ans)

4.

Let S be the set of all strings of a's and b's where all strings contain exactly one a -

1. Base: a $\in$ S
II. Recursion: If u $\in$ S, then
   a. bu $\in$ S
   b. ub $\in$ S

III. Restriction: There are no elements of S other than those obtained from the base and recursion of S.

5.

Let S be the set of all strings of a's and b's where all strings are odd lengths -

I. Base: a $\in$ S and b $\in$ S
II. Recursion: If u $\in$ S, then
   a. aau $\in$ S
   b. bbu $\in$ S
   c. bau $\in$ S
   d. abu $\in$ S

III. Restriction: There are no elements of S other than those obtained from the base and recursion of S.