1 Lesson 14 Example 2

Samuel Pepys was interested in the probability of getting at least 1 six in 6 throws of a die. Previously, we saw how this probability could be calculated using the complement rule or the binomial distribution. Show how this probability can also be calculated by defining an appropriate geometric random variable.

2 Answer

Let X represent the number of rolls it takes to get the first six. The probability of rolling a six in one trial is $p = \frac{1}{6}$, and the probability of not rolling a six is $1 - p = \frac{5}{6}$.

The probability mass function (PMF) for the geometric distribution is:

$$P(X = x) = (1 - p)^{x-1}p$$

for $k = 1, 2, 3, \ldots$, where $P(X = x)$ is the probability that the first success occurs on the x-th trial (roll). We need to find the probability of getting at least 1 six within the first 6 rolls.

2.1 Compute the probability of getting a six on each roll

We calculate the probability of rolling the first six on the 1st, 2nd, 3rd, 4th, 5th, or 6th roll.

$$\begin{aligned}
P(X = 1) &= (1 - p)^0 p = 1 \times \frac{1}{6} = \frac{1}{6}, \\
P(X = 2) &= (1 - p)^1 p = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}, \\
P(X = 3) &= (1 - p)^2 p = \left(\frac{5}{6}\right)^2 \times \frac{1}{6} = \frac{25}{216}, \\
P(X = 4) &= (1 - p)^3 p = \left(\frac{5}{6}\right)^3 \times \frac{1}{6} = \frac{125}{1296}, \\
P(X = 5) &= (1 - p)^4 p = \left(\frac{5}{6}\right)^4 \times \frac{1}{6} = \frac{625}{7776}, \\
P(X = 6) &= (1 - p)^5 p = \left(\frac{5}{6}\right)^5 \times \frac{1}{6} = \frac{3125}{46656}.
\end{aligned}$$

2.2 Compute the total probability

To find the probability of getting at least 1 six within the first 6 rolls, we sum the probabilities:

$$P(\text{at least 1 six in 6 rolls}) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$$

Substituting the values:

$$P(\text{at least 1 six in 6 rolls}) = \frac{1}{6} + \frac{5}{36} + \frac{25}{216} + \frac{125}{1296} + \frac{625}{7776} + \frac{3125}{46656}$$

Let's convert these to a common denominator:

$$P(\text{at least 1 six in 6 rolls}) = \frac{7776}{46656} + \frac{6480}{46656} + \frac{5400}{46656} + \frac{4500}{46656} + \frac{3750}{46656} + \frac{3125}{46656}$$

Summing the numerators:

$$P(\text{at least 1 six in 6 rolls}) = \frac{31031}{46656} \approx 0.6651$$

2.3 Conclusion

The probability of getting at least 1 six in 6 rolls of a die, calculated using the geometric distribution, is approximately 0.6651, or 66.51%.