1 Lesson 15 Example 2

Calculate the following probabilities.

1. You toss a coin 4 times. The probability that you get (exactly) 2 heads.
2. You toss a coin until you get 2 heads. The probability that it takes (exactly) 4 tosses.

Which is larger? Explain why the answer makes intuitive sense.

2 Answer

2.1 Problem 1: Probability of Getting Exactly 2 Heads in 4 Tosses

This problem can be modeled using the **binomial distribution**, where:

- $n = 4$ is the number of trials (4 tosses),
- $p = 0.5$ is the probability of success (getting a head) on each toss,
- X is the random variable representing the number of heads obtained.

The probability mass function (PMF) for a binomial distribution is given by:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Substituting $k = 2$, $n = 4$, and $p = 0.5$, we calculate the probability of getting exactly 2 heads:

$$P(X = 2) = \binom{4}{2} (0.5)^2 (0.5)^{4-2}$$

$$P(X = 2) = \frac{4!}{2!2!} (0.5)^2 (0.5)^2 = 6 \times 0.25 \times 0.25 = 6 \times 0.0625 = 0.375$$

Thus, the probability of getting exactly 2 heads in 4 tosses is 0.375, or 37.5%.

2.2 Problem 2: Probability of Taking Exactly 4 Tosses to Get 2 Heads

This problem can be modeled using the **negative binomial distribution**, where:

- $r = 2$ is the number of successes (2 heads),
- $p = 0.5$ is the probability of success (getting a head) on each toss,
- X is the random variable representing the number of trials until 2 heads are obtained.

The probability mass function (PMF) for the negative binomial distribution is given by:

$$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Substituting $k = 4$, $r = 2$, and $p = 0.5$, we calculate the probability that it takes exactly 4 tosses to get 2 heads:

$$P(X = 4) = \binom{3}{1} (0.5)^2 (0.5)^{4-2}$$

$$P(X = 4) = 3 \times (0.5)^2 \times (0.5)^2 = 3 \times 0.25 \times 0.25 = 3 \times 0.0625 = 0.1875$$

Thus, the probability that it takes exactly 4 tosses to get 2 heads is 0.1875, or 18.75%.

2.3 Comparison

The probability of getting exactly 2 heads in 4 tosses (0.375) is higher than the probability of taking exactly 4 tosses to get 2 heads (0.1875). This makes intuitive sense because, in the first scenario, the heads can be spread across the 4 tosses more flexibly, whereas in the second scenario, the restriction that the second head must appear on the 4th toss makes it less likely. Additionally, it makes intuitive sense why the second probability is exactly half the first probability. This is because, in the binomial case, the heads can be freely distributed across the 4 tosses. Out of the 6 possible ways to place 2 heads in 4 tosses, exactly half of those ways involve the second head occurring on the 4th toss (which is what we are measuring with the negative binomial distribution),