1 Lesson 13 Additional Exercise 1

In the carnival game chuck-a-luck, three dice are rolled. You can make a bet on a particular number (1, 2, 3, 4, 5, 6) showing up. The payout is 1 to 1 if that number shows on (exactly) one die, 2 to 1 if it shows on two dice, and 3 to 1 if it shows up on all three. (You lose your initial stake if your number does not show on any of the dice.) If you make a $1 bet on the number three, what is the distribution of the amount you win? (Hint: The random variable is not binomial but very closely related to a binomial. You can should be able to write the p.m.f. as a table.)

2 Answer

Let X represent the number of dice that show the number 3 in a single roll of three dice. X can take values 0, 1, 2, or 3. The probability that a die shows 3 is $p = \frac{1}{6}$, and the probability that a die does not show 3 is $1 - p = \frac{5}{6}$. Since there are three independent dice rolls, $X \sim \text{Binomial}(3, \frac{1}{6})$.

Step 1: Probabilities for each outcome

Using the binomial probability mass function:

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

where $n = 3$, $p = \frac{1}{6}$, and $1 - p = \frac{5}{6}$, we calculate the probabilities for $X = 0, 1, 2, 3$:

- For $X = 0$ (no dice show the number 3):

$$P(X = 0) = \binom{3}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3 = 1 \times 1 \times \frac{125}{216} = \frac{125}{216}$$

- For $X = 1$ (exactly one die shows the number 3):

$$P(X = 1) = \binom{3}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^2 = 3 \times \frac{1}{6} \times \frac{25}{36} = \frac{75}{216}$$

- For $X = 2$ (exactly two dice show the number 3):

$$P(X = 2) = \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^1 = 3 \times \frac{1}{36} \times \frac{5}{6} = \frac{15}{216}$$

- For $X = 3$ (all three dice show the number 3):

$$P(X = 3) = \binom{3}{3} \left(\frac{1}{6}\right)^3 = 1 \times \frac{1}{216} = \frac{1}{216}$$

Step 2: Corresponding payouts

The net gain for each outcome depends on X, the number of dice showing the number 3:

- If $X = 0$, you lose your \$1 bet, so your net gain is -1.

- If $X = 1$, you win 1 to 1, so your net gain is $1 \times 1 = 1$.

- If $X = 2$, you win 2 to 1, so your net gain is $2 \times 1 = 2$.

- If $X = 3$, you win 3 to 1, so your net gain is $3 \times 1 = 3$.

Step 3: Probability distribution of winnings

We can summarize the probability distribution of the net gain in the table below:

Number of dice showing 3 (X)	Net gain	Probability
0	-1	$\frac{125}{216} \approx 0.579$
1	1	$\frac{75}{216} \approx 0.347$
2	2	$\frac{15}{216} \approx 0.069$
3	3	$\frac{1}{216} \approx 0.005$

Conclusion

The distribution of the amount you win in the game chuck-a-luck when betting on the number 3 is described by the table above. The probabilities follow a binomial distribution, and the net gains depend on how many times the number 3 appears on the three dice.