1 Lesson 15 Additional Exercise 3

You have two coins. One coin is a fair coin with a 0.5 probability of landing on heads. The other coin is a biased coin with a 0.25 probability of landing on heads. You pick one of these two coins at random, and begin flipping until you get 5 heads. It takes you 12 flips in order to get your 5 heads. What is the probability that the coin you picked was the fair coin? What is the probability you picked the biased coin?

2 Answer

This is a classic problem that can be solved using Bayes' theorem. We define the following events:

- F: The event that the fair coin was picked,
- B: The event that the biased coin was picked,
- $D = 12$: The event that it took exactly 12 flips to get 5 heads.

We are interested in finding the posterior probabilities:

$$P(F \mid D = 12) \quad \text{and} \quad P(B \mid D = 12)$$

According to Bayes' theorem:

$$P(F \mid D = 12) = \frac{P(D = 12 \mid F)P(F)}{P(D = 12)}$$

$$P(B \mid D = 12) = \frac{P(D = 12 \mid B)P(B)}{P(D = 12)}$$

2.1 Prior Probabilities

Since the coins were picked randomly, the prior probabilities are:

$$P(F) = P(B) = 0.5$$

2.2 Likelihoods

Next, we calculate the likelihoods $P(D = 12 \mid F)$ and $P(D = 12 \mid B)$, which are the probabilities of getting exactly 5 heads in 12 flips using the fair coin and the biased coin, respectively. We use the negative binomial distribution to model this.

The PMF of the negative binomial distribution is:

$$P(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

where $r = 5$ is the number of heads (successes), and $x = 12$ is the total number of flips.

2.3 Fair Coin Likelihood ($p = 0.5$)

:

$$P(D = 12 \mid F) = \binom{11}{4}(0.5)^5(0.5)^7 = \binom{11}{4}(0.5)^{12}$$

$$P(D = 12 \mid F) = \frac{330}{4096} \approx 0.08057$$

2.4 Biased Coin Likelihood ($p = 0.25$)

:

$$P(D = 12 \mid B) = \binom{11}{4}(0.25)^5(0.75)^7 = \frac{330 \times (0.75)^7}{1024}$$

$$P(D = 12 \mid B) \approx 0.04302$$

2.5 Total Probability $P(D = 12)$

The total probability is given by:

$$P(D = 12) = P(D = 12 \mid F)P(F) + P(D = 12 \mid B)P(B)$$

$$P(D = 12) = 0.08057 \times 0.5 + 0.04302 \times 0.5 = 0.040285 + 0.02151 = 0.0618$$

2.6 Posterior Probabilities

Using Bayes' theorem, we now calculate the posterior probabilities:

$$P(F \mid D = 12) = \frac{P(D = 12 \mid F)P(F)}{P(D = 12)} = \frac{0.08057 \times 0.5}{0.0618} \approx 0.652$$

$$P(B \mid D = 12) = \frac{P(D = 12 \mid B)P(B)}{P(D = 12)} = \frac{0.04302 \times 0.5}{0.0618} \approx 0.348$$

2.7 Conclusion

- The probability that the coin picked was the **fair coin** is approximately 0.652 or 65.2%.
- The probability that the coin picked was the **biased coin** is approximately 0.348 or 34.8%.