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Interactivity of Visual Mathematical Representations: Factors Affecting Learning and Cognitive Processes

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Computer-based mathematical cognitive tools (MCTs) are a category of external aids intended to support and enhance learning and cognitive processes of learners. MCTs often contain interactive visual mathematical representations (VMRs), where VMRs are graphical representations that encode properties and relationships of mathematical concepts. In these tools, interaction enables learners to perform epistemic actions on VMRs to explore and learn mathematical concepts. Interactivity of VMRs refers to the feel, form, properties, and quality of this interaction. As such, interactivity of VMRs can influence how and what learners learn. A number of factors affect learners' cognitive processes while interacting with VMRs. Researchers from several disciplines have attempted to characterize interactivity and the multiplicity of factors that affect it. However, as many of these characterizations and factors are inapplicable to VMR-based MCTs, understanding of the factors that affect learning and cognitive processes can help in the analysis of interactive VMRs. This article draws on research from various disciplines to identify and describe the applicability of 12 interactivity factors that affect learning and cognitive processes of learners who use VMR-based MCTs. Collectively, the factors can then serve as a descriptive and conceptual framework to help in the design and evaluation of MCTs and to allow designers to discuss and substantiate their design choices of interactive VMRs.

The cognitive perspective of learning primarily deals with how the mind interacts with external information and how it processes the information internally (Eysenck & Keane, 1990; Thagard, 1996). In this perspective, learning involves a set of internal cognitive processes such as perceiving, attending, recalling, reasoning, interpreting, evaluating, sense-making, decision-making, and understanding (Ormrod, 1995).

External cognitive aids and artifacts, such as representations and tools, greatly influence and affect learning and cognitive processes (Zhang & Norman, 1994; Glasgow, Narayanan, & Chandrasekaran, 1995; Hutchins, 1995; Scaife & Rogers, 1996; Arcavi & Hadas, 2000; National Council of Teachers of Mathematics [NCTM], 2000; Jonassen, 2003). Computer-based cognitive tools are a category of external aids that can support cognitive, knowledge-based tasks, that is, support performing epistemic actions on information. These tools can support epistemic activities by enhancing, amplifying, transforming, and guiding cognitive processes of learners (Norman, 1993; Pea, 1993; Card, MacKinlay, & Shneiderman, 1999; Lajoie, 2000; Beynon, Nehaniv, & Dautenhahn, 2001; Jonassen, 2003). By sharing and distributing the cognitive processing load and activity of learners, these tools can transform the way cognitive tasks are performed (Kieran, Boileau, & Garancon, 1996; Lajoie, 2000; de Léon, 2002; Yerushalmy, 2004). Cognitive tools can act as partners in cognitive activities of learners (Salomon, Perkins, & Globerson, 1991), and their interface can act as an augmentative prosthetic support for the perceptive and cognitive capabilities of learners (Stojanov & Stojanoski, 2001). To support epistemic activities of learners, many cognitive tools provide structural, logical, and visuospatial formalisms of ideas and concepts in the form of interactive representations (Jonassen & Carr, 2000).

Mathematical cognitive tools (MCTs) are a subset of cognitive tools that allow learners to investigate and explore mathematical information (Sedig, 2004). Similar to other forms of epistemic activities, much of mathematical learning and thinking also involves working with and elaborating on representations in order to understand them or use them for problem solving (NCTM, 2000; Pimm, 1995; Cuoco & Curcio, 2001). Visual mathematical representations (VMRs) constitute an important subset of mathematical representations. VMRs are used extensively to support the investigation and exploration of mathematical ideas – both in static, noninteractive media such as books, as well as in interactive media such as MCTs (English, 1997; Duval, 1999; Whiteley, 2002; Sedig & Sumner, in press). VMRs are graphical representations that encode causal, functional, structural, logical, and semantic properties and relationships of mathematical structures, objects, concepts, problems, patterns, and ideas (Hitt, 2002; Sedig & Sumner, in press). Some examples of VMRs include 2D and 3D visualizations of geometric structures, patterns, graphs, and diagrams.

MCTs incorporating interactive VMRs that can effectively support learning and cognitive processes of learners are difficult to design. To design epistemologically sound MCTs, designers of these tools need to consider several interrelated issues, such as the pedagogical goals of the tool, the kinds of VMRs used, the possible techniques for interacting with VMRs, and the degree and types of interactivity or cognitive support (Sedig, Klawe, & Westrom, 2001; Gadanidis, Sedig, & Liang, 2004; Sedig, 2004; Sedig & Sumner, in press). Designers of MCTs need to make critical decisions about how much cognitive support is appropriate for effective learning. A few studies indicate that there are interactivity factors that affect learners' cognitive processes and overall learning (Jackson, Krajcik, & Soloway, 1998; Sedig et al., 2001; Sedig, Rowhani, Morey, & Liang, 2003; Travaglini, 2003; Chai, 2003). For example, Sedig et al. (2001) conducted a study in which a few interactivity factors of 2D transformation geometry VMRs were operationalized differently. They found that these factors significantly affected children's processing of the information and their degree of learning.[1] Therefore, characterizing the interactivity factors that affect learners' cognitive processes and learning is an educational imperative, as it can help designers of MCTs to design and analyze these tools in a systematic fashion.

Part of the challenge of analysis and design of MCTs is that there is a lack of a proper conceptual framework and a common language describing and characterizing the interactivity of VMRs. Such a framework can help designers clarify their thinking and provide a vocabulary for them to express and substantiate their design decisions. Research in visual reasoning, cognitive technologies, information visualization, and human-computer interaction design has identified a number of factors that affect cognition. Many of these factors apply to interactive VMRs. Unfortunately, they are discussed and reported in different research domains and are scattered across different bodies of literature. Often, the characterizations and descriptions of these factors are bound to the domain in which they are discussed, making it difficult to apply them to the design of interactive VMRs (Sedig & Sumner, in press). Knowing what factors affect learning and cognitive processes during interaction with VMRs and developing a language to describe these factors can be a step in the direction of creating a conceptual framework for the design and analysis of VMR-based MCTs.

This article is part of a larger research plan aimed at developing a systematic framework to provide guidelines for the design and evaluation of MCTs. Earlier research has developed a framework that categorizes and characterizes a set of interactions by which learners can explore and investigate VMRs (Sedig & Sumner, in press). However, this framework looks only at interaction with VMRs, and does not address their interactivity. It does not address how different interactivity factors can affect the quality of interaction, and hence, learning and cognitive processes of learners. This article aims to extend the earlier research. The purpose of this article is to discuss interactivity factors that affect learning and cognitive processes of learners who use VMR-based MCTs. Collectively, the factors can then serve as a framework to help the design and evaluation of such tools. To achieve its purpose, this article draws together research from mathematics learning, visual reasoning, cognitive technologies, information visualization, and human-computer interaction design. Before discussing the factors, conceptual and terminological issues with regard to interaction and interactivity are presented next.

Interaction and Interactivity

Interaction and interactivity are essential features of MCTs. VMRs by themselves are noninteractive. This means that much of their semantic and relational properties are hidden and latent. In order to reason and learn with static, noninteractive VMRs, learners need to analyze, evaluate, process, and elaborate on them (Ormrod, 1995; Peterson, 1996). In the context of MCTs, interaction can allow learners to perform "epistemic actions" on VMRs to adapt the visual information according to their needs (Kirsh & Maglio, 1994; Neth & Payne, 2002; Schwan, 2002). Interaction can act as an epistemic extension of static representations. It extends the communicative power of VMRs by adding a temporal dimension to them, making them dynamic and allowing their latent meanings to become visible. It allows learners to customize the "what" and "how" of the presentation of visual information. Interacting with VMRs allows learners to perform numerous cognitive activities, such as visualizing, analyzing, interpreting, modeling, and organizing. Allowing learners to interact with VMRs leads to some general benefits, such as: supporting dialectic reasoning with and through VMRs; providing opportunistic experimentation and exploration of hypothetical "what if" queries; making mental manipulation of concepts easier; facilitating the acquisition of qualitative insight into and understanding of the nature of VMRs; and coordinating learners' internal mental models with external VMRs (Sedig & Sumner, in press). Sedig and Sumner have identified and characterized 12 different micro-level, task-based interactions for performing cognitive tasks involving VMRs: animating, annotating, chunking, composing, cutting, filtering, fragmenting, probing, rearranging, repicturing, scoping, and searching.

Interaction and interactivity are closely related, yet they are distinct concepts. They have different connotations and meanings in different contexts (Norman & Draper, 1986; Laurillard, 1993; Kirsh, 1997; Sims, 1999, 2000; Yacci, 2000; Otero, Rogers, & du Boulay, 2001; Preece, Rogers, & Sharp, 2002; Roussou, 2004; Shneiderman & Plaisant, 2004). In the context of this article, interaction refers to a learner communicating with one or more VMRs through a human-computer interface (Sedig & Sumner, in press). Interaction with a VMR takes place in the time-space continuum and has two implications: (a) the learner acting upon a VMR and (b) the VMR responding or reacting in some form for the learner to interpret. Interactivity refers to the feel, form, properties, quality, and dimensions of interaction (Svanæs, 1999; Burgoon et al., 2000). Both, interaction and interactivity implicitly suggest how and what a learner learns. As such, a VMR can be interactive, but depending on its interactivity, interaction with it may require different amounts of cognitive effort, engage and support different thought processes, facilitate different degrees and depths of reasoning and learning, and allow different types of interchange between the learner and the content (Steuer, 1992; Laurillard, 1993; Golightly, 1996; Kirsh, 1997; Sims, 1999; Burgoon et al., 2000; Sedig et al., 2001; Preece et al., 2002). For instance, learners may interact with a VMR; however, interacting with the VMR directly or indirectly, through another intermediary representation, can affect the learners' attentive processes and degree of learning (Sedig et al., 2001). Similarly, whether interaction is scaffolded or not can influence learners' decision-making processes (Kirsh, 1997). As such, an understanding of the factors that affect learning and cognitive processes can help in the analysis and evaluation of interactive VMRs. The benefit of interactive VMRs may not be fully harnessed if designers of MCTs do not consider these interactivity factors in their designs.

Researchers from numerous disciplines have attempted to characterize interactivity and the multiplicity of factors that affect it. For instance, human-computer interaction researchers often discuss interactivity in terms of efficiency, affect, helpfulness, learnability, and usability of interactive tools (Teoa, Oha, Liua, & Weib, 2003; Shneiderman & Plaisant, 2004). Media and communication scientists have also investigated interactivity (Steuer, 1992; Jensen, 1998; McMillan & Downes, 2000; Gustavsen & Tilley, 2003; Laine, 2003; Liu, 2003; Liu & Shrum, 2003; Bhatt, 2004;), but they emphasize the communication medium and exchange of information. Many such characterizations are not easily transferable and applicable to interactive VMRs. One of the primary reasons for this is that such characterizations are not concerned with cognitive tools for learning, but rather, with productivity tools and commercial websites whose goals are different. Another reason is that the previously mentioned research does not deal with visual representations and reasoning.

This article is an attempt to create a formal framework for analyzing interactivity of VMRs. A number of factors affect the interactivity of VMRs. The interactivity factors presented in this article provide a descriptive framework to allow designers of MCTs to discuss and support their design choices. The framework presents 12 interactivity factors. Before presenting an indepth discussion of these factors, they are briefly defined below:

1. *Affordance:* Provision of interface cues to advertise possible interactions.
2. *Cognitive offloading:* Provision of interactions that can shoulder the load of some cognitive processes.
3. *Constraints:* Restriction of possible interactions.
4. *Distance:* Degree of difficulty in understanding how to act upon a VMR and interpret its responses.

5. *Epistemic appropriateness:* Suitability and harmony of interactions in supporting learning.
6. *Feedback:* Exchange of information and the direction of communication during interaction between learners and the VMR.
7. *Flexibility:* Range and availability of interactive choices and options available to learners.
8. *Flow:* Duration of interaction with the VMR in time and its effect on learners' perception of the relationship between cause and effect.
9. *Focus:* Locus of learners' attention during interaction with the VMR.
10. *Involvement:* Learners' engagement with and contribution to the information content of a VMR allowed by the available interactions.
11. *Scaffolding:* Provision of interactions to cognitively support and augment learners' reasoning and understanding of embedded concepts.
12. *Transition:* Communication of visual changes to the VMR.

These factors are nonorthogonal: that is, they are closely related and can interact with one another. Their labeling and characterization, however, help in the analysis, design, and evaluation of VMR-based MCTs.

The remainder of this article is organized as follows. The next section discusses the 12 interactivity factors. A number of examples taken from several MCTs are used to clarify and demonstrate the applicability of each factor. The last section of the article presents a summary of what has been discussed and suggests directions for future research in this area.

FACTORS AFFECTING LEARNING AND COGNITIVE PROCESSES

Affordance

The affordance factor deals with advertising the interaction possibilities of the VMR. That is, affordance refers to the interface cues that help learners know what can be done with the VMR and how (Norman, 1999; Svanæs, 1999; Preece et al., 2002). The affordance factor affects learners' perceptive and attentive processes by facilitating awareness of possible interactions. Interaction possibilities can be advertised by such means as color change in visual elements or cursor change resulting from mouse-over actions. For instance, an MCT, Archimedean Kaleidoscope (Morey & Sedig, 2004b), allows learners to rotate 3D geometric solids. Figure 1 shows two snapshots of a solid being rotated. As seen in the figure, to facilitate awareness of the availability and form of this interaction with the VMR, the MCT provides two cues: a circle with a bi-directional arrow on its circumference, and the flashing of the arrow when learners move the mouse over the circle. The bi-directional arrow is intended to perceptually advertise the existence of the rotation interaction, and the flashing arrow is intended to draw attention to when and how it can be used.

Figure 2 shows a VMR whose interaction possibilities are not advertised properly. As shown in the figure, learners can interact with a traditional Euclidean coordinate plane (a) by stretching it to an infinite hyperbolic plane (d). Learners can drag the mouse in different directions to stretch the plane. However, learners are not provided with any visual cues that advertise this interaction, and, as a result, they may have difficulty perceiving both its existence and usage. Generally, VMRs should clearly communicate their affordances to learners, making it easy for them to perceive and attend to the interactions that are possible.

Cognitive Offloading

The cognitive offloading factor is related to the interactive abilities and features of the VMR upon which learners' cognitive processes can be offloaded (Larkin & Simon, 1987; Scaife & Rogers, 1996; Card, MacKinlay, & Shneiderman, 1999; Preece et al., 2002; Ainsworth & Loizou, 2003; Rogers & Brignull, 2003). That is, as a mental partner, the VMR contains interactive features to enable it to shoulder some cognitive activities for the learner. Cognitive offloading[2] has generally been discussed in terms of the role of external representations in reducing the amount of cognitive effort required to solve informationally equivalent problems. However, since interaction in effect can extend the communicative power and epistemic utility of external representations (Sedig & Sumner, in press), cognitive offloading also applies to the role that interaction plays in bearing some of the mental load of the learner (Yamamoto, Nakakoji, & Aoki, 2002; Nakakoji & Yamamoto, 2003). Through dynamic externalization of cognitive activities,

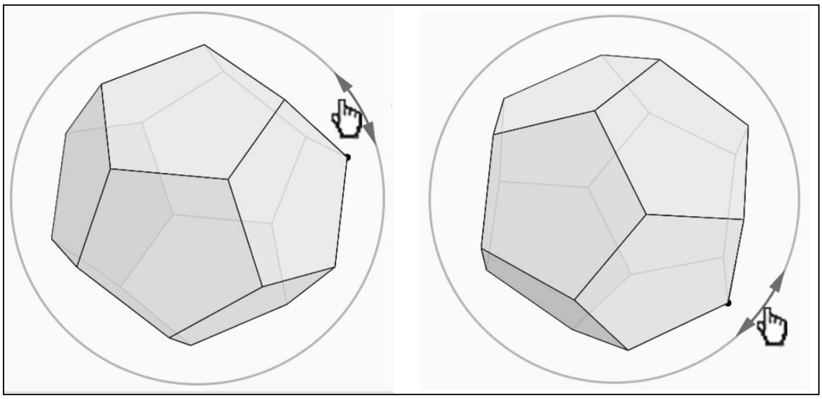

Figure 1. An example of advertising interaction affordance

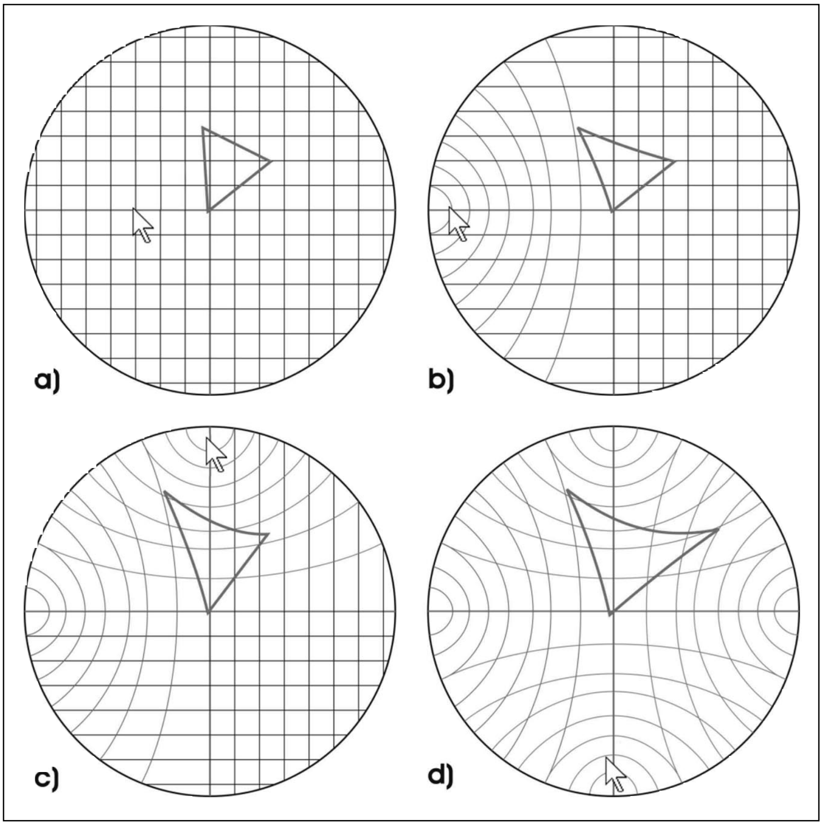

Figure 2. An example of not advertising interaction affordance

interaction with VMRs can aid learners' perception, goals, plans, memory, and reasoning. For instance, given snapshots of two static, noninteractive geometric solids, learners often find it difficult to mentally visualize and reason about how one geometric solid metamorphoses into the other. By providing morphing as an interaction technique, an MCT, PARSE (Sedig et al., 2003), makes visualizing and reasoning much easier for learners. A study of PARSE showed that learners used interactive morphing to explore the process involved in the transformation of the solids (Figure 3). By visually externalizing the intermediary stages of the metamorphic process, this interaction technique allowed some of the learners' cognitive load involved in reasoning about and visualizing of the transformation to be offloaded onto the interactive VMR.

It is not always desirable to provide maximum cognitive offloading (Rogers & Scaife, 1998). Maximum cognitive offloading is often desirable

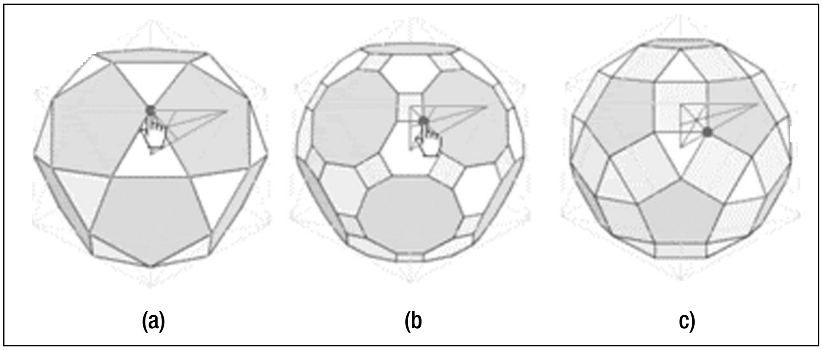

Figure 3. Snapshots of solids generated during interactive morphing

for productivity tools whose goal is to use interactions and representations that reduce the amount of conscious thinking of users, thereby making productivity tasks easy to perform. However, in the case of MCTs, cognitive offloading is not necessarily desirable with respect to VMRs (Sedig et al., 2001). Cognitive offloading, in these cases, is to facilitate understanding, rather than minimize cognitive effort. For instance, although sometimes it is important to provide animation as an interaction, one of the drawbacks of animation is that it may increase the visual explicitness of VMRs, resulting in the reduction of mental effort and reasoning on the part of learners due to overconfidence in the amount of knowledge obtained from the animation, and hence inducing shallowness of learning (Jones & Scaife, 2000).

Constraints

The constraints factor deals with restrictions in the possible interactive operations that can be performed with the VMR (Trudel & Payne, 1995; Jones & Scaife, 2000; Preece et al., 2002). This factor refers to how interaction can focus, canalize, and direct learners' cognitive processes – that is, to attract attention towards the VMR's salient elements or features, to guide thinking and reasoning while interacting with it, and to narrow and direct the course of goal formulation while exploring it. Trudel and Payne (1995) suggested that exploratory learning can be significantly improved when learners' interactions and explorations are constrained and restricted.

Interaction constraints can be based on other constraints, such as relational, boundary, geometric, logical, or algebraic constraints. Figure 4 shows an example of an MCT designed to support learners' exploration of the relationship of side length, volume, and surface area of 3D geometric shapes (Gadanidis, Sedig, & Liang, 2004). The interactive representations in the tool complement and support one another. Several types of constraints are used to guide learners' thinking and exploration. For instance, by interacting

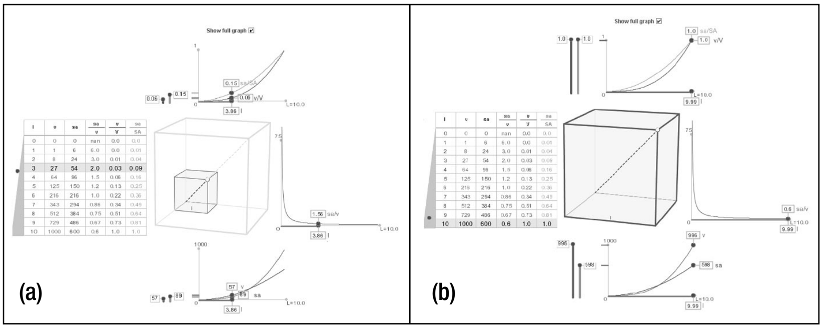

Figure 4. An example of an MCT that combines multiple interaction constraints

with the control on the inner cube, learners can resize the cube and observe the rate of change of its volume and surface area in the lower graph. This dynamical linking of multiple interactive representations is an example of relational constraining, where interacting with any one of these representations causes simultaneous change in all the other ones. In the same MCT, boundary constraining limits the maximum size of the inner cube to correspond to that of the outer one; geometric constraining restricts learners from rotating the cube; logical constraining guides the growth or shrinking of the cube only along a diagonal line; and, algebraic constraining delimits the length of the side of the inner cube to range from zero to ten.

Distance

The distance factor is related to the executive and interpretive gulfs that exist between the VMR and learners (Hutchins, Hollan, & Norman, 1986; Norman, 1991; Sedig et al., 2001). That is, distance can be conceptualized in terms of two gulfs between the VMR and learners: gulf of execution and gulf of evaluation. The former signifies the difficulty that learners experience in understanding how to act upon the VMR, and the latter signifies the difficulty that learners experience in interpreting the responses or state of the VMR. The distance learners must cross to overcome the gulfs of execution and evaluation affects their reasoning, amount of mental effort and elaboration, and hence the learners' depth of learning (Sedig et al., 2001). If there is too much or too little distance between the VMR and learners, investigation and exploration of the VMR might not yield the desirable learning outcomes.

In the context of learner-VMR interaction, four types of distance can be discussed: semantic, articulatory, conceptual, and presentation (Hutchins et al., 1986; Strothotte, 1998; Sedig et al., 2001). Semantic distance refers to the level of matching between what learners intend to do with the VMR, and how interactions support the expression of learners' intentions (Hutchins et al., 1986). That is, how much structure is provided by the MCT and how much by learners. For instance, in a tool called Super Tangrams, learners can interact with a visual representation of the concept of rotation to transform 2D geometric shapes (Figure 5). The MCT can provide structure in the form of a ghost image of where the shape will move to bridge the gulf of execution and match the learners' expectations of what to anticipate from the action. If the ghost image did not exist, the semantic distance would be increased, as the learners would have to mentally elaborate to determine how to act upon the VMR (for an indepth discussion of this example, see Sedig et al., 2001). Articulatory distance refers to the difficulty in the form of input and output expressions (Hutchins et al., 1986). For instance, in the case of rotating a 2D geometric shape, rotation can be expressed by directly pointing to the VMR and turning it, or it can be expressed by typing a command, such as ROTATE (center[1, 3], angle[-225°]), to perform the same action. Articulatory distance is shorter in the first case than in the second.

Conceptual distance refers to the distance between learners' current conceptual model of how to act upon the VMR and the model provided by the MCT. For instance, in the previous example of rotation, learners may know how to use rotation to transform a shape. However, the tool may only have reflection as an available interaction. In this case, even though any transformation may be achieved using composite reflection, learners may not have the conceptual model to act upon the VMR using this interaction. In the latter case, the conceptual distance is greater. Presentation distance is related to an aspect of interest of a VMR, such as size and/or perspective, and how, through interaction, learners can adjust the presentation of the VMR to investigate it more closely. The VMR's visual presentation refers to its spatial display. Interactions which allow adjusting the VMR's

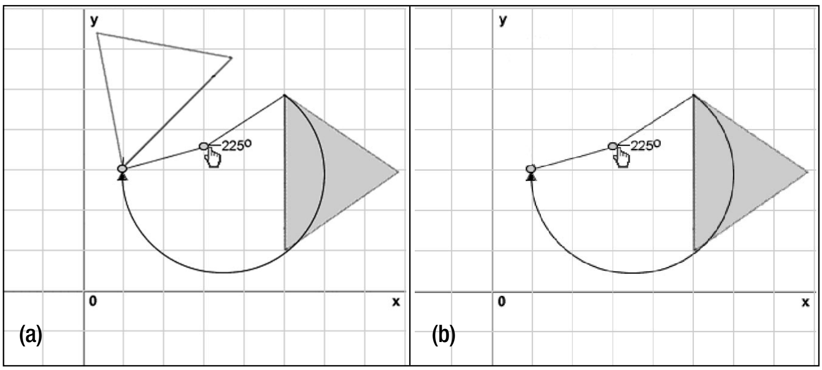

Figure 5. Example of increasing semantic distance from (a) when rotating a shape (shaded triangle), learners are provided with a ghost image, to (b) the ghost image is now unavailable for learners

presentation according to learners' cognitive preferences can shorten this distance. Figure 6 shows an example of a VMR providing several interactions for adjusting its visual presentation. Learners can use either magnification to increase the VMR's viewing size (6b), rotation to change its viewing angle (6c), and/or stretching to distort its perspective presentation (6d).

Distance determines how much cognitive load learners have to bear and how much mental effort they need to exert to explore, make sense of, or use

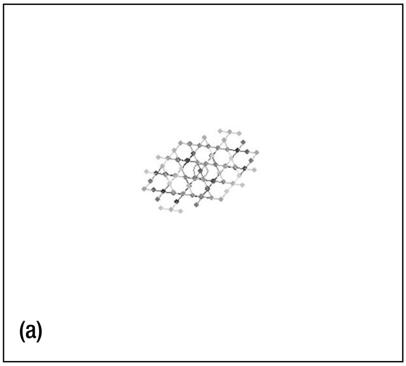

Figure 6a. Normal view of a lattice structure

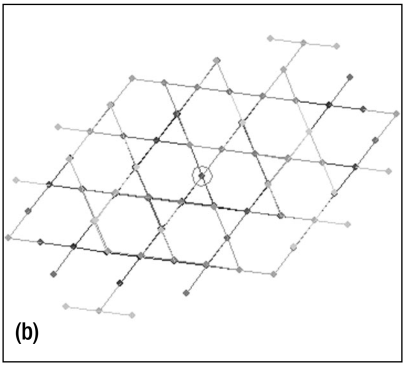

Figure 6b. Applying magnification to the VMR in 6a to increase its viewing size

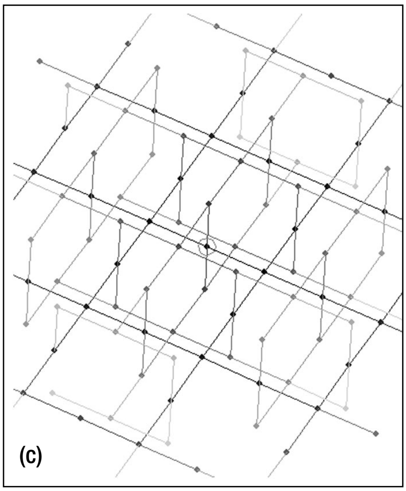

Figure 6c. Applying rotation to the VMR in 6b to change its viewing angle

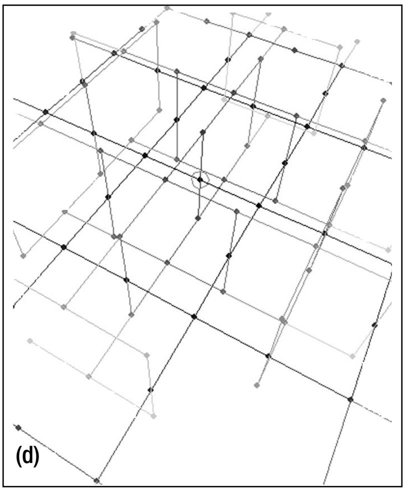

Figure 6d. Applying stretching to the VMR in 6c to change its viewing perspective

a given VMR (see Sedig et al., 2001). It can be reduced in two ways: either the interactive VMR is designed so that it matches the meanings, forms, conceptual models, and presentation needs of learners, or learners, through exertion of mental effort, try to bridge the gulfs and understand the interactive language, the form of articulation, and the conceptual presentation of the VMR. However, as Sedig et al. demonstrated, it is not always advantageous to reduce distance. When operationalized properly, increasing the different types of distance can render the VMR a more effective partner in cognitive activities by engaging learners in reflective thinking and deeper reasoning.

Epistemic Appropriateness

The epistemic appropriateness factor is concerned with the suitability of the interaction techniques used in supporting learners in performing a cognitive task.[3] That is, given some cognitive task involving VMRs, this factor deals with whether the interaction used in an MCT fits the task and has a harmonious effect on learning and cognitive processes, such as learners' perception, goals, plans, memory, reasoning, and understanding.

Selecting an appropriate interaction that epistemologically fits a cognitive task is important, as some interactions can support and enhance performance of cognitive tasks, while others can have undesirable effects (Svendsen, 1991; Golightly, 1996; Holst, 1996; Sedig et al., 2001). For instance, Sedig et al. (2001) conducted a study in which they compared the appropriateness of different interaction styles in supporting the cognitive task of learning transformation geometry concepts. In this study, learners interacted with visual representations of these concepts. The study showed that while one style significantly supported learners in their understanding of the concepts, another style was unsuitable. The latter style not only limited learners' reasoning and understanding of these mathematical concepts, but also amplified their misunderstanding and naïve notions of these concepts.

Appropriateness also deals with the optimal number of interaction techniques used and how they work in harmony to support learners' cognitive activities (Yamamoto et al., 2002; Beaudouin-Lafon, 2004). Superfluous and inharmonious interactions can put unnecessary demand on learners' memory, and needlessly add to their attentional load (see de Léon, 2002 for an elaboration of how tools that provide superfluous actions can have negative cognitive effects). Yet another aspect of appropriateness deals with propagating the effects of an interaction across a collection of dynamically linked VMRs in a harmonious manner (van der Meij & de Jong, 2003; Gadanidis et al., 2004). In such cases, the number of different interaction techniques is not as important as how the effect of an interaction is propagated. Examples of MCTs in which appropriateness exists with respect to either the number of interactions used or the distribution of its effects are illustrated next.

An MCT can provide a number of interaction techniques that work together to support learners in performing coordinated and integrated cognitive tasks (Morey & Sedig, 2004a; Sedig & Sumner, in press). In this type of appropriateness, although each individual interaction independently supports a particular cognitive task, all interactions collectively support more sophisticated cognitive tasks and learning processes (Morey & Sedig, 2004a). For instance, an MCT, PARSE (Sedig et al., 2003), provides learners with several interaction techniques to help in their exploration and learning of a subset of geometric solids (Figure 7).[4] In a study of PARSE, it was observed that learners would use these different interactions to engage in different forms of mental activities (e.g., varying their reasoning styles) to be able to coordinate their investigation of the shapes.

An MCT can contain a collection of dynamically linked VMRs. In such tools, learners' understanding depends on exploring the harmony and relationships between these connected VMRs. For instance, Gadanidis et al. (2004) presented an MCT that used several dynamically-interconnected VMRs to allow learners to explore the relationships among the edge length, surface area, and volume of similar rectangular prisms (see Figure 4 in the constraints section). In this tool, interacting with any one VMR has an effect on all the other VMRs, causing simultaneous change in them. Thus, the effect of interaction is appropriately propagated across all the VMRs, allowing learners to explore relationships among these VMRs.

Feedback

The feedback factor is concerned with the exchange of information and the direction of communication between learners and the VMR. That is, if interaction with a VMR is regarded as a communication "loop" (Kirsh, 1997; Yacci, 2000), then feedback is the mechanism by which this loop is created between learners and the VMR, allowing them to "converse" with each other (Pérez-Quiñones & Sibert, 1996). Feedback allows learners to know the effect of their actions, the VMR sending back information about what action has been received and what has been accomplished (Preece et al., 2002). It also suggests that there is an exchange of direction in communication in which both learners and the VMR interchange roles when sending and receiving information.

Providing feedback during interaction is as important as the interaction itself (Gibbons & Fairweather, 1998). Feedback can supply information for learners to correct erroneous, inaccurate knowledge, and to learn to self-monitor their own performance (Gibbons & Fairweather, 1998; Corbett & Anderson, 2001). Feedback can be used to challenge, revise, and rectify learners' knowledge structures. Moreover, feedback can play a crucial role in learners' attitude and motivation towards learning (Gibbons & Fairweather, 1998). As such, this factor affects learners' interpretative, decision-making, self-monitoring, motivational, and evaluative processes.

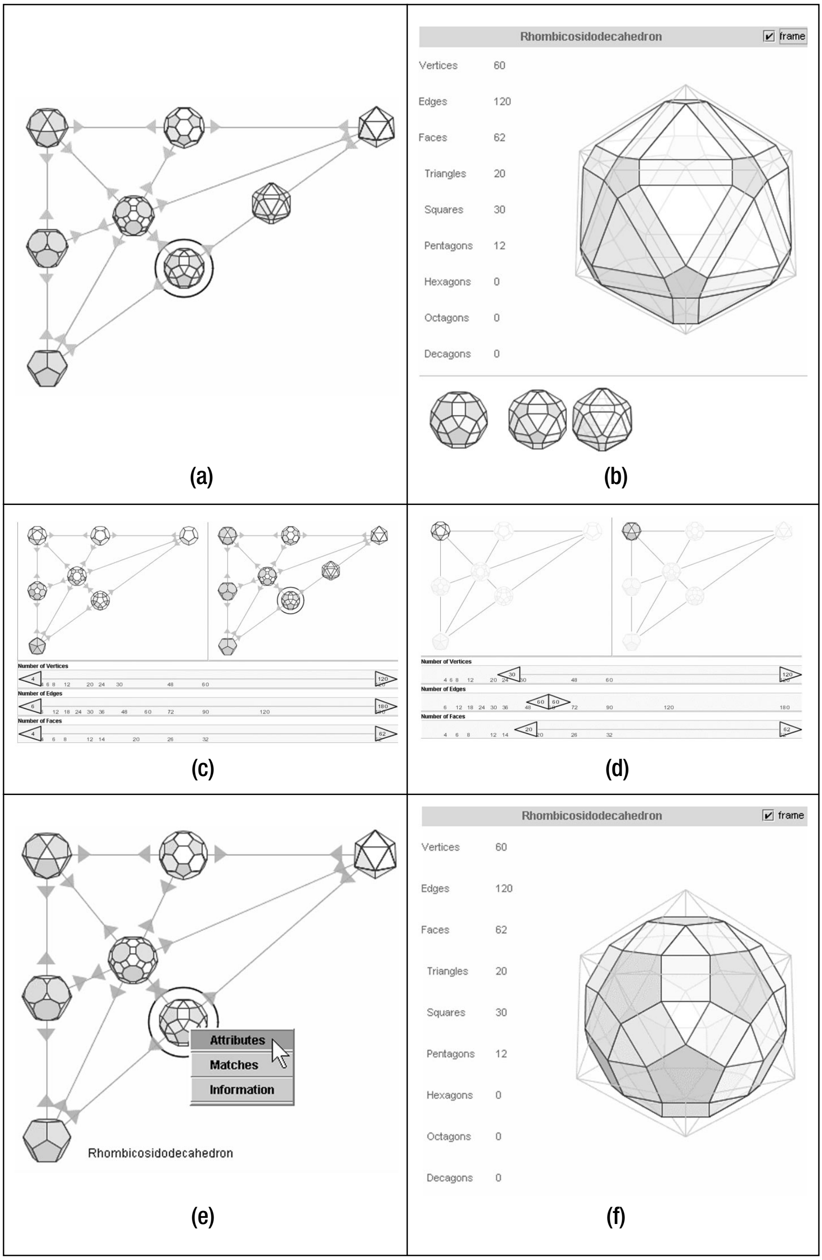

Figure 7. Some possible interactions provided in PARSE: (a, b) *interactive animated morphing*; (c, d) *searching*, and (e, f) *probing*

During interaction with a VMR, feedback can occur before and/or after learners have committed an action. In the former case, the VMR can communicate to learners what will be done if they commit a particular action; that is, the VMR initiates the conversation by communicating to learners the action's potential effects. In the latter case, once the action has been committed, the VMR informs learners of its actual effects. Figure 8 shows an example of an MCT, PARSE (Sedig et al., 2003), that provides both before and after feedback. In this tool, learners can explore the relationships among different solids. When the mouse is moved over a triangle, it is highlighted and the cursor is changed to a hand (Figure 8a). At this stage, although the VMR has advertised its interactivity through the highlighted triangle and the change in the cursor, the potential effect of a mouse-click action is not indicated, and thus it is hidden from learners. To make this more explicit, PARSE provides additional feedback by displaying the text 'Click to augment vertices.' This additional feedback allows learners to assess the potential effect of the action prior to its execution—that is, before feedback. On a mouse-click, the VMR shows the actual effect of the action by dynamically displaying the transformation of the shape from one form to another—that is, after feedback (Figure 8b).

Feedback can be immediate, delayed, or requested (Alessi & Trollip, 2001; Corbett & Anderson, 2001). In immediate feedback, the effect of interaction is communicated without any time lag. In delayed feedback, there is a temporal gap between interaction and discovering its effect. In requested feedback, or on-demand feedback, learners do not receive any feedback unless they ask for it. Providing a particular type of feedback is dependent on the instructional goals and cognitive processes that the MCT intends to support. Immediate feedback is corrective in nature because it is intended to guide, facilitate interpretation, and rectify learners' actions; as such, it may be preferred by or may

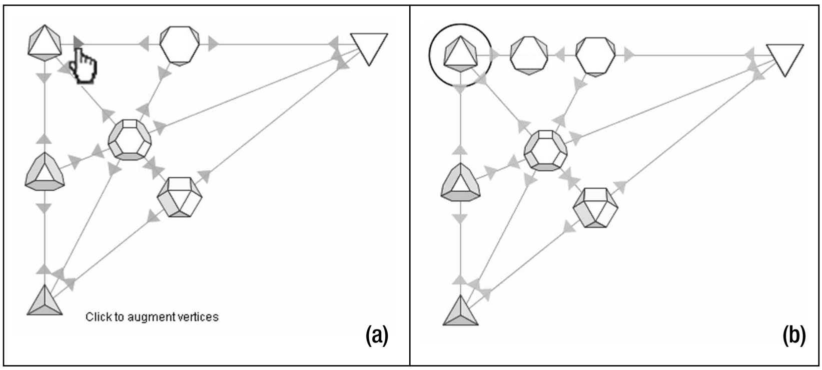

Figure 8. An example of before feedback (a), and after feedback (b)

be more suitable for novice learners (Alessi & Trollip, 2001). However, as learners gain further understanding, immediate feedback can sometimes hinder the development of their meta-cognitive, self-monitoring processes, and can, more generally, disrupt their cognitive processes while executing a task (Corbett & Anderson, 2001). In this context, delayed feedback may be preferable over immediate feedback because it forces learners to engage in reflective cognition to gradually advance in the learning process, and can facilitate retentive processes (ibid.). For instance, immediate and requested feedback are provided in TileLand, an MCT intended to assist children in learning how to describe tiling patterns by writing procedural code (Sedig, Morey, & Chu, 2002; Travaglini, 2003). In preliminary stages of the learning process, to acquaint children with the environment, immediate feedback is provided, allowing children to observe the effect of each command on the created tilling pattern without delay (Figure 9a). At a later stage, requested feedback is provided so that children do not see the outcome of each command unless they ask for it (Figure 9b, c). This temporal lag is intended to encourage children to mentally visualize the sequence of steps required in constructing a pattern, and to mentally detect errors before committing to the execution of the whole body of code that they have written.

Flexibility

The flexibility factor is related to the range of interactive choices and options that the VMR offers to learners (Milheim & Martin, 1991; Kristof & Satran, 1995; Steuer, 1995; Kettanurak, Ramamurthy, & Haseman, 2001). That is, the VMR may allow learners to have some control over and freedom in: deciding the speed and pace of interaction (e.g., speed of animation of the VMR), selecting interactions that suit their needs and preferences (e.g., animating the VMR rather than manually morphing it – see Sedig et al., 2003), determining the presentation of interaction (e.g., stacked rather than distributed – see the transition factor), adjusting the amount and type of feedback (e.g., immediate feedback rather than delayed – see the feedback factor), customizing the perceptual characteristics of the VMR (e.g., color, brightness, viewing perspective, level of detail, and size – see the distance factor), and so on. An example of flexibility of an interactive VMR can be found in an MCT, TileLand (Sedig et al., 2002; Travaglini, 2003), which allows children to choose between immediate or requested feedback to support their learning of sequential, procedural programming skills by constructing tiling patterns (see the feedback section). To increase the level of challenge and children's mastery of the subject, children can change the type of feedback, from immediate to requested. In the latter mode, there is a temporal disconnect between devising a solution for building a tiling pattern and testing it.

Flexibility of interaction is especially important in the constructivist model of learning in which autonomy of action is promoted (Alessi & Trol-

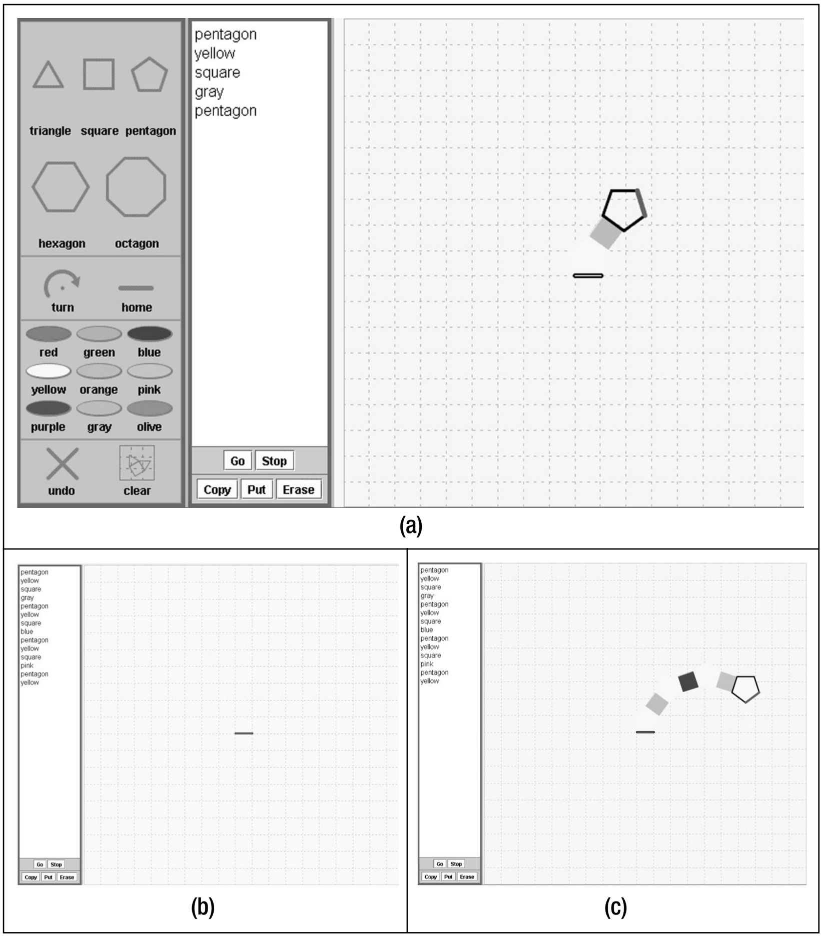

Figure 9. Example of (a) immediate feedback: Tiling pattern is updated each time children enter a command, and (b) requested feedback: Children can request feedback by clicking on the Run button to display the tiling pattern as shown in (c)

lip, 2001). The flexibility factor affects cognitive self-determination, an important condition in regulation of learning (Ormrod, 1995). For instance, in a study of PARSE (see Figure 7), an MCT that provides a range of interaction choices, learners would choose different interaction techniques to engage in different forms of mental activities, such as varying their reasoning style, switching from structural to process-based thinking, and solving dissimilar tasks (Sedig et al., 2003).

Flow

The flow factor is concerned with the duration of interaction with the VMR in time (Sedig & Sumner, in press). Both action (cause) and reaction (effect) parts of interaction occur in time. The flow factor influences learners' causal reasoning – that is, their perception of the relationship between their actions and the effect of those actions – as well as their temporal reasoning (Sedig et al., 2005). Interaction flow can happen in two ways: continuous or discrete. Continuous flow is analog-like in behavior, where cause and/or effect occur in a fluid, uninterrupted manner over a span of time. Discrete flow is digital-like in behavior, where cause and/or effect occur in a distinct, separate manner at an instance in time. Given continuous and discrete flow, learners can perceive interaction in four ways: (a) continuous action, discrete reaction; (b) continuous action, continuous reaction; (c) discrete action, continuous reaction; and (d) discrete action, discrete reaction. Figure 4 shows an MCT that provides continuous-continuous interaction. In this example, the inner cube can be resized by dragging its edge along the diagonal line (continuous action). The effect of this action is dynamic transformation of the cube in a fluid manner (continuous reaction), enabling learners to visualize the intermediary snapshots or stages of change over a span of time (Gadanidis et al., 2004). An example of discrete-continuous interaction is when learners click on a button to animate a VMR, where the clicking action is discrete and the animation response is continuous (Figure 8).

Two widely used forms of flow are continuous-continuous (or simply, "continuous") and discrete-discrete (or simply, "discrete"). In continuous interaction, because cause and effect take place simultaneously and smoothly over time, learners are in control of the communication and flux of information. They can stop and freeze action in time to observe and reason about intermediary reactions or responses from the VMR. However, in discrete interaction, interactions happen one at a time. Upon committing an action, learners lose control of communication until a snapshot response appears. Then, they can commit the next action. This latter form of flow is often not conducive to dynamic, exploratory interaction with VMRs, as incremental reversal of action is not easy (for dynamic exploration, see Shneiderman, 1994; Spence, 2001).[5] To compensate for this, learners often need to resort to undoing their actions, which may result in more premeditated actions.

Focus

The focus factor refers to the locus of interaction – that is, the center of attention of learners while interacting with the VMR (Sedig & Morey, 2004). Given a VMR, learners can explore it directly or indirectly. In the former case, they interact with the elements inside the VMR itself. In the latter case, they interact with the VMR through another intermediary – either an interface control (e.g., button or menu) or another VMR. In direct interaction, learners focus on the VMR and interact with it without any other intermediaries. In indirect interaction, learners attend to the intermediary representation to act on the VMR. An example of direct interaction is seen in Figure 3, where learners morph the geometric solid by interacting with the VMR itself. A simple example of indirect interaction is seen in Figure 1, where learners rotate the geometric solid by interacting with the double-headed arrows.

In the early days of computing, the prevalent interaction technique was conversational and command-based. Linguistic commands were used to indirectly interact with interface objects. As direct manipulation (Shneiderman, 1988; Preece et al., 2002; Shneiderman & Plaisant, 2004), as an interaction technique, gained more popularity, it became the prevalent technique used both in productivity tools as well as in educational tools. The main goal of directness of interaction was to minimize learners' cognitive load by allowing them to engage with the onscreen representations directly (Hutchins et al., 1986). However, some researchers began to notice that in some learning situations direct manipulation involved little reflective thought compared to indirect command-based interactions (Svendsen, 1991; Golightly, 1996; Holst, 1996). For instance, in solving the 8-puzzle game[6], changing the focus of interaction from direct to indirect (moving puzzle pieces using adjacent buttons rather than pointing at the pieces and moving them directly) resulted in learners paying more attention to each move and using a "look-ahead" strategy, as opposed to the ones who used the direct manipulation version of the activity (Golightly, 1996). The latter group's strategy involved less planning and was based mainly on "trial-and-error" since recovering from errors was easy. Therefore, "directness" of interaction has been cited as the main source of the problem since learners may perceive the interaction to be easy. However, through an empirical study of children using different interface styles, Sedig et al. (2001) found that the problem is not with the directness of interaction, but rather the representations or structures to which the learners are attending.

Direct and indirect interaction can serve different purposes. For instance, in a study (Sedig et al., 2005), learners were given the two VMRs in Figure 10 with which to interact: a geometric solid (a), and a diagrammatic map of transitional relationships among a set of solids (b). Learners could directly interact with the control point on the enlarged solid (a) to morph the solid. Learners could also directly interact with the map (b) by clicking on the transitional arrows. As learners interact with the enlarged solid, they indirectly navigate (i.e., move on) the map; and as they interact with the map, they indirectly interact with the solid on the left, causing it to morph. During the study, learners would interact with and focus on the map to reason about the transitions, and they would interact with and focus on the geometric solid to reason about its structure. The study found that having access to both direct and indirect interaction helped learners develop a deeper understanding of the structures and their relationships.

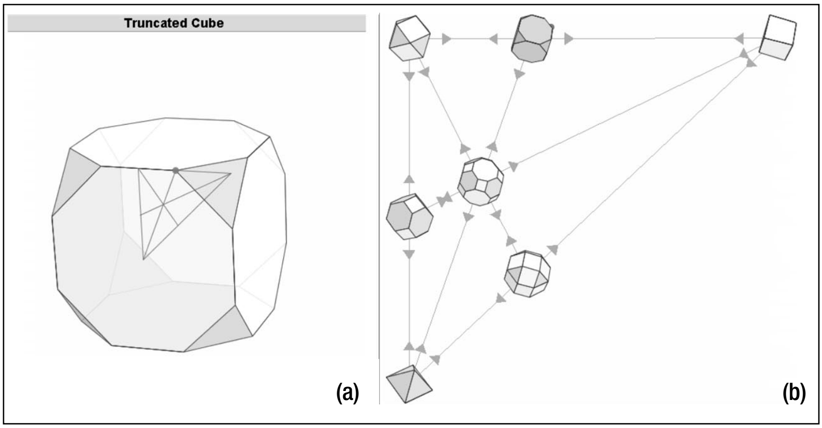

Figure 10. Direct interaction with the map (a), resulting in indirect interaction with the solid (b), and vice versa

Involvement

The involvement factor is concerned with how interaction allows learners to engage with and contribute to the information content of the VMR. Involvement is a continuum spanning from purely observational interaction all the way to constructionist interaction (Harel & Papert, 1991; Hannafin, 1992; Jonassen, Peck, & Wilson, 1999). Between these two poles of the continuum, interaction can provide learners with annotative capabilities, enabling them to contribute to the information content of the VMR. The involvement factor can have an overall effect on learners' learning strategy and cognitive processes.

In observational interaction, learners are engaged in interpreting and making sense of the VMR by using interactions such as searching, probing, animating, morphing, and rotating[7]. Interactions that support such involvement engage learners in interpreting the mathematical concepts by observing, manipulating, and analyzing the VMR (Jonassen et al., 1999; Morey & Sedig, 2004a, 2004b). An example of an MCT that engages learners in observational interaction is Archimedean Kaleidoscope (Morey & Sedig, 2004b). This MCT allows learners to investigate how 3D solids can be derived from one another by truncating or augmenting their vertices and edges (Figure 11). Learners interact with given shapes by morphing and rotating them to make sense of their relationships.

In constructionist interaction, learners are engaged in creating, generating, designing, and constructing the VMR by using interactions such as composing and chunking. Interactions that support such involvement engage learners in understanding the VMR by creating and generating it (Harel & Papert, 1991;

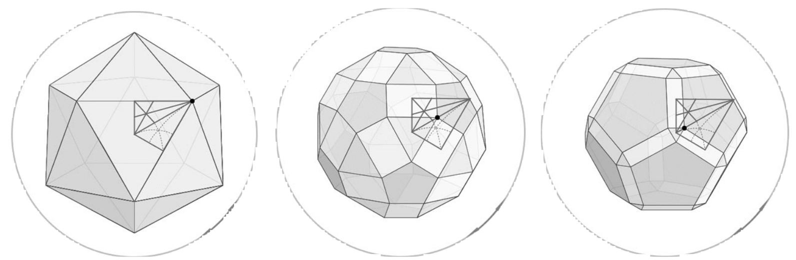

Figure 11. Snapshots of a solid being morphed during observatory interaction in Archimedean Kaleidoscope

Sedig et al., 2002; Sedig, Morey, Mercer, & Wilson, 2004). An example of an MCT that can engage learners in generative, constructionist interaction is Cabri-géomètre, which allows learners to compose geometric structures (Straesser, 2001; Holzl, 1996). Constructionist interaction has been used to promote deductive reasoning and provide means to work out geometric proofs (Hanna, 2000).

Annotative interaction strikes a balance between observational and constructionist interactions; that is, although unable to create the VMR from scratch, learners have the ability to modify the VMR by augmenting it and adding their own personal metadata to it. Learners can annotate the VMR by placing marks on it or adding notes to it. An example of an MCT that uses annotative interaction is LatticeSpace (Sedig & Sumner, in press). In this tool, while investigating the structure of a 3D lattice, learners can annotate it by adding markings to it to trace their path (Figure 12). These markings act like electronic footprints to help learners walk the lattice structure, helping learners know what parts of the VMR they have visited and where they currently are.

Scaffolding

The scaffolding factor deals with the process of providing support structures to gradually improve learners' reasoning and understanding of the VMR and its embedded mathematical concepts (Sedig et al., 2001; Reiser, 2002). That is, scaffolding refers to changing external aids (i.e., scaffolds) during the progressive stages of interaction with the VMR. The scaffolding factor affects the amount of mental load that learners bear at different stages of learning, such as planning, predicting, analyzing, and evaluating their actions (Jackson et al., 1998; Sedig et al., 2001; Fretz, Wu, Zhang, Krajcik, & Soloway, 2002; Reiser, 2002). During the scaffolding process, the scaffolds progressively fade in the course of interaction with the VMR. This progressive fading is intended to bring learners out of an automatic processing of the information and induce epistemic conflict to promote reflection and

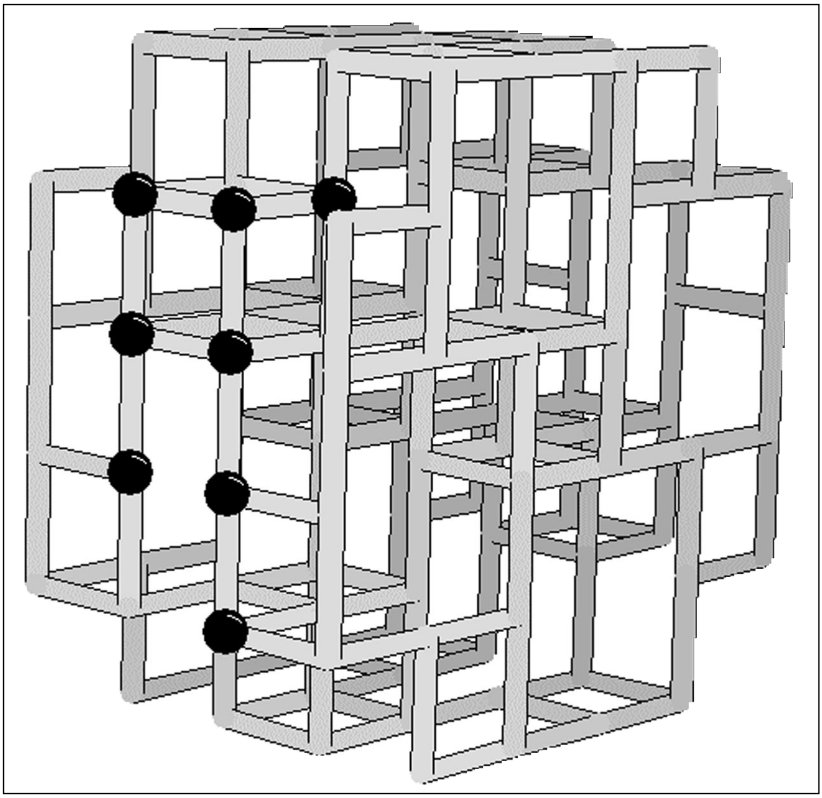

Figure 12. An example of annotative interaction: Walking (or navigating) the lattice by placing black marks on it

deeper processing of the VMR's encoded meanings (Guzdial & Kehoe, 1998; Jackson et al., 1998; Sedig et al., 2001; Loh et al., 2001; Reiser, 2002). Scaffolding can be applied to the VMR and/or to the interaction with it (Toth, 2000; Hart & Barden-Gabbei, 2002; Quintana et al., 2002).

In scaffolding of the VMR, there is a gradual change in the visual elements of the VMR during the course of interaction. For example, an MCT, Super Tangrams (Sedig et al., 2001), uses three-level scaffolding of representations of 2D transformational geometry concepts to assist learners in their understanding of these concepts (Figure 13). In this tool, to rotate an object, learners need to specify both the center and the angle of rotation by interacting with the two handles provided on the VMR. The handle at the head of the arc allows learners to set the angle of rotation, and the handle at the center of arc allows learners to adjust the position of the center of rotation. While the interaction technique remains the same in all three figures, the VMR itself is scaffolded. The first representation (Figure 13a; see also Figure 5a) provides learners with a ghost image and an arc of rotation of the object to be rotated. At this level, the inclusion of the ghost image in the representation provides explicit visual cues to support learners in their reasoning about the VMR and the geometric concepts. In the next representation (Figure 13b), the ghost image disappears, requiring learners to mentally visualize the angle of rotation. At the last level (Figure 13c), the arc of rotation is changed forcing learners to reflect more deeply to determine both the angle and the center of rotation (see Sedig et al., 2001 for a detailed explanation of this technique).

In scaffolding of interaction, there is a gradual change in the form and style of interaction. For instance, a tool for creating polygonal tiling patterns (Sedig et al., 2002; Travaglini, 2003) provides scaffolding of interaction to gradually allow children to create more complex patterns. At the most primitive level, children can run animations to observe how preconstructed tiling patterns are created using polygons. This guides them visually in understanding how a pattern is constructed. At the next level, interaction changes to direct manipulation, allowing children to click on different visual icons to construct their patterns (Figure 9a). At another level, interaction changes to linguistic commands, allowing children to issue simple sequential commands to specify how a pattern is to be created (Figure 9b). Yet, at a higher level, to create more complicated, recursive patterns, children can write subroutines. In an experimental study of the tool, scaffolding of interaction,

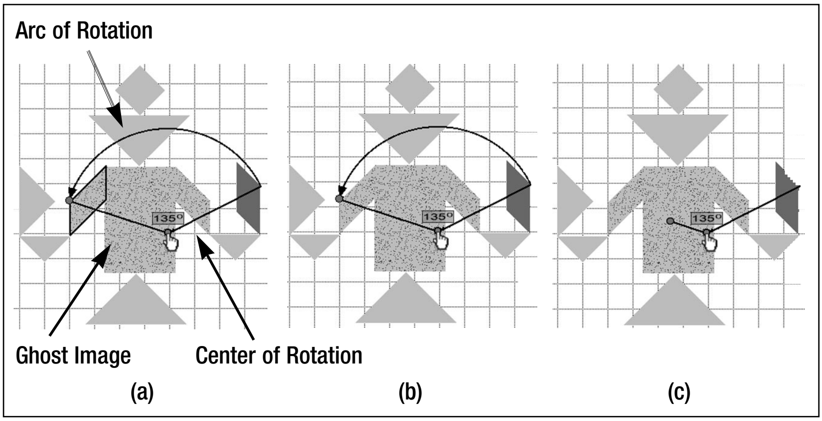

Figure 13. Gradual reduction from (a) to (c) of the level of detail in the representation of rotation while keeping the interaction style constant

from animation to linguistic routines, provided cognitive support for very young children (six-year olds) to gradually help their understanding of and engagement with tiling patterns (Travaglini, 2003).

Transition

The transition factor is concerned with how, through interaction, the VMR narrates or communicates the transformations and transitional changes that happen to it (Tufte, 1990, 1997; Sedig et al., 2005).[8] This factor influences learners' navigational, spatial, and temporal reasoning processes. Transitions in the VMR can be communicated using two models: stacked or distributed.[9]

In the stacked model, transitional changes to the VMR are unveiled in one location and are superimposed over one another – that is, one representation is overlaid over the previous one and replaces it in time. Animations and movies are examples of this model of narrating visual change. This type of communication of transitional change allows learners to appreciate the temporal relationships between the visual snapshots, and get a feeling for the dynamics of change of the VMR (Jones & Scaife, 2000). In this model, learners avoid the disorienting back-and-forth movements of the eye needed to compare spatially distributed, separate images (Tufte, 1997). However, the main problem of using the stacked model to communicate transitional change is that, during learners' observation of visual changes in the VMR, they have to compare the start state with the end state in their memory, as they cannot simultaneously see both states. This is why such stacked narration should take place smoothly in time and learners should be able to control the speed of change to reduce cognitive disorientation. For instance, in Figure 14, the transitional process of morphing one solid to another using the stacked model is shown in P1. At the end of this process, due to temporal transience, only the final state of the VMR remains visible, that is, Figure 14 (P1.c).

In the distributed model, communication of the transitional changes to the VMR is not limited to one location only, but it spans within a regional space. In this model, two styles can be used to narrate the transitional changes: traced or untraced. When narration of transitions is traced, the VMR leaves some form of residual visual information, allowing learners to observe and compare changes that happen to the VMR. For instance, in Figure 14(P2), the VMR communicates changes happening to it by leaving visual snapshots that are mapped onto and distributed in different locations within a space. In this case, the VMR distorts the temporal dimension of interaction and presents visual residues as discrete, multiple, parallel images in several spatial locations. Through spatial parallelism, the traced style supports the human visual "capacity to compare and reason about multiple images that appear simultaneously within our eyespan" (Tufte, 1997, p. 80). Transitional changes to the VMR narrated using the untraced style are not captured and displayed as a

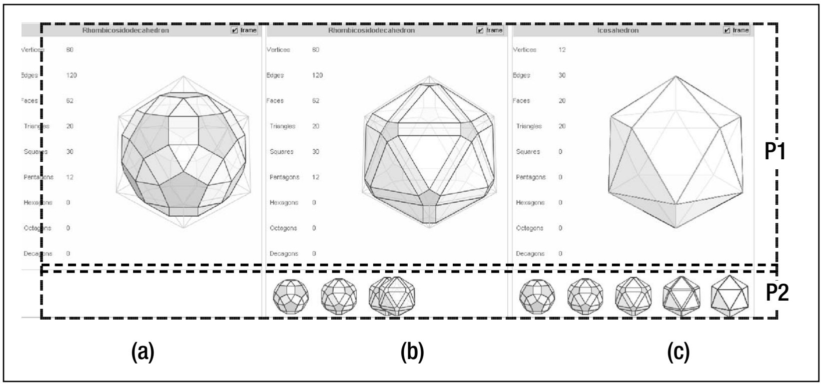

Figure 14. Snapshots of a transitional process of morphing from one solid to another communicated using both transition models: Stacked (P1) and distributed (P2)

series of visual snapshots. Figure 15 shows an example of a VMR that visually narrates how one solid morphs into another along a specific navigation path. The VMR temporally displays its transitional transformations along this path (Figure 15b), but does not leave any traces behind. Unlike the stacked model, where learners can only observe the last state of the VMR at the end of the transitional process, in this example learners can still see both the start and end states, as they always remain visible.

SUMMARY AND FUTURE WORK

VMRs are graphical notations that encode a vast array of mathematical ideas and are an essential component of MCTs. Adding interactivity to VMRs allows learners to perform epistemic actions on them and can increase their

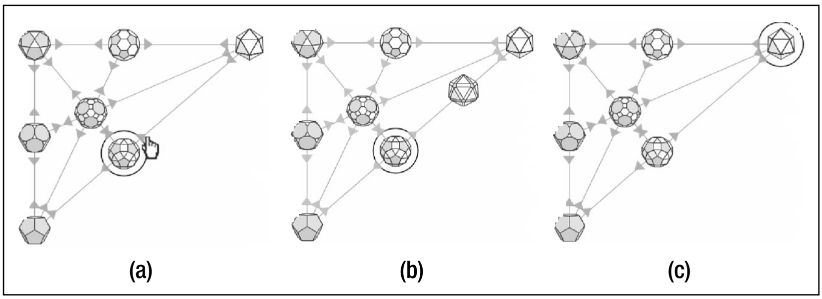

Figure 15. Snapshots of a VMR showing an untraced distributed transitional process

epistemic utility. These epistemic actions involve cognitive activities and processes such as attending, perceiving, reasoning, decision-making, interpreting, planning, and evaluating. Learning is a high-level cognitive activity that encompasses all such cognitive processes. Interaction then acts as a mediatory mechanism that can support, enhance, and/or transform the cognitive tasks that can be performed on or with static, displayed representations.

Interactivity and interaction are central features of interactive VMRs. They are closely related; yet, they are distinct concepts. Interaction with a VMR refers to acting upon it. A VMR's interactivity refers to the feel, form, properties, and quality of this interaction. A VMR can be interactive, but depending on its interactivity, epistemic actions may require different amounts of cognitive effort and engage and support different cognitive processes. As such, the quality and effectiveness of the learner-VMR interaction can be in large part determined by its interactivity. The interactions used and the interactivity of a VMR implicitly suggest how and what a learner learns. This article has presented 12 factors for analyzing the interactivity of VMRs. These 12 factors are: affordance, cognitive offloading, constraints, distance, epistemic appropriateness, feedback, flexibility, flow, focus, involvement, scaffolding, and transition. These factors have been developed by bringing together and integrating research findings from a number of research areas such as mathematics learning, visual reasoning, cognitive technologies, information visualization, and human-computer interaction design.

The conceptual framework presented in this article is descriptive. It provides a framework for the design and analysis of VMR-based MCTs. The framework helps in the design of MCTs by allowing a systematic analysis of the factors that affect the learning and cognitive processes of learners. Beyond MCTs, most of the factors in this framework can be used to analyze other tools that allow interaction with visualization of other domains of knowledge. However, as can be observed, the framework presented in this article is not prescriptive. It does not provide rules or guidelines as how or when to operationalize the different interactivity factors, given different learning tasks, VMRs, and situations. A possible line of future research is to use both the interaction framework (Sedig & Sumner, in press) and the interactivity framework to develop a more comprehensive prescriptive framework. Some elements of such a framework may include: (a) organization of VMRs according to their features, (b) analysis of the cognitive tasks involved in mathematics problem solving and learning, (c) general rules for when and how to use which interactions, (d) rules for how to effectively operationalize the interactivity factors, and (e) organization and categorization of best-practice tools and designs to be consulted (see Sedig, 2004 for a discussion of this topic). Developing such a framework is not an easy task. Much empirical evaluation of tools and interaction techniques is needed to develop, validate, and refine such a framework.

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Notes

[1] Another recent study shows how the transition factor (discussed in this article) can affect the understanding of transitional processes (Sedig, Rowhani, & Liang, 2005).

[2] Cognitive offloading is also referred to as computational offloading.

[3] The issue of appropriateness has been discussed in the context of how different representations and tools fit the needs of different tasks, users, processes, and so on (Norman, 1991, 1993; Peterson, 1996; Scaife & Rogers, 1996; de Léon, 2002). Since interaction extends the communicative power of external representations (Sedig & Sumner, in press), this factor also applies to the role that interaction plays in affecting the mental processes of learners.

[4] Some of the interactions were: (a) *interactive animated morphing* – to assist in reasoning about transitional relationships among the shapes; (b) *searching* – to help with locating shapes that have similar attributes; and (c) *probing* – to provide further information about particular shapes.

[5] Also, see Preece et al. (2002) for how immediate, rapid feedback affects quality of direct manipulation

[6] The 8-puzzle is a game consisting of a 3 x 3 square grid. Eight of the squares have numbers from 1 to 8, and one of the squares is empty. This allows for moving the other 8 squares around into different positions until the squares are arranged in an ascending order, with the last square empty.

[7] For a list of different interactions, see Sedig and Sumner (in press).

[8] See Tufte (1990, 1997) for a discussion on "Time-Space Narrative," elaborating on how transformations to representations can be narrated through different models.

[9] See also Beaudouin-Lafon (2000) for a discussion on the difference between spatial and temporal interaction instruments.