Propositions, Functions, and Predicates

What is a proposition?

- A proposition is a claim made about the state of a world
  - 4 is even
  - 2 is the only even prime number
  - All odd numbers are prime

```
def isEven (n: ℕ) : Prop := n % 2 = 0

def fourIsEven := isEven 4
#check fourIsEven

def isPrime(n: ℕ): Prop := ¬(∃(m: ℕ), m > 1 ∧ m ≠ n ∧ n % m = 0)

def twoOnlyEvenPrime := ∀(n: ℕ), (isEven n) ∧ (isPrime n) → (n = 2)
#check twoOnlyEvenPrime

def allOddNumbersPrime := ∀(n: ℕ), ¬(isEven n) → (isPrime n)
#check allOddNumbersPrime
```

Propositions don’t have to be true

Proposition components

- True and False: most basic propositions
- Conjunction: whether two or more propositions are all true
- Disjunction: whether any of two or more propositions are true
- Implication: whether one proposition is true whenever another proposition is true
- Negation: whether a proposition is false
- Universal quantification: whether a proposition is true for all instances of a particular type
- Existential quantification: whether a proposition is true for some instance of a particular type

Negations are actually a bit more subtle than this

What is a function?

- A function is a mapping from one or more inputs to an output
  - The mapping must be unique
    - The square root function can return +2 or -2, but it must be consistently defined
    - Technically, it could also return a *pair* of numbers, but that pair would be the output
- Exercise:
  - Describe the function that takes an x input and returns a y input such that the function draws a circle
    - There is no such function!
- Exercise: Give some examples of interesting functions from nat to bool

```
def positive (n: nat) : bool :=
    if n > 0 then tt else ff
def uint32 (n: nat) : bool :=
    if n >= 0 ∧ n < 2^32 then tt else ff
```

Lambda expressions

- Lambda expressions are basically anonymous functions:

```
λ n : nat,
    (if n > 0 then tt else ff : bool)
```

- Many ways to create the λ symbol, but please don't use `\lamda` 🙄

```
#check λ n : nat, (if n > 0 then tt else ff : bool)
#check λ n,(if n > 0 then tt else ff)

#check λ n, n > 0
```

Alternate Formulations

- There are multiple ways to write a function
- The best way depends on taste, but also sometimes on context

```
def positive (n: nat) : bool :=
    if n > 0 then tt else ff

def positive' : nat → bool :=
    λ(n : nat), (if n > 0 then tt else ff : bool)

def positive’'' := λ n, if n > 0 then tt else ff

def positive''': ℕ → bool :=
begin
  exact λ n, if n > 0 then tt else ff
end
```

Exercise

- Exercise: Give some examples of interesting functions from ℕ to ℕ

```
def double (n: ℕ) := 2 * n

#check double
#check double 3
#reduce double 3

def square (n: ℕ) := n * n

#check square
#reduce square 3
```

Functions as types

- Recall that a function from $\mathbb{N}$ to $\mathbb{N}$ is of type $\mathbb{N} \to \mathbb{N}$
- If a function has a type, can it be an argument to another function?
- Can a function be the return value of another function?
- Discuss

Function as return value

- Consider the following function:

```
def add(x: ℕ)(y: ℕ) := x + y
#reduce add 3 4 – 7
#check add 3
```

- If add takes two arguments, what happens when we only give it one?

```
def add3(y: ℕ) := 3 + y
```

- Does format matter?

```
def add'(x y: ℕ) := x + y
#check add’ 3
```

This is an example of currying, one of the most common ways to return a value that is a function

Examples of functions as arguments

```
def compose (f: ℕ → ℕ) (g: ℕ → ℕ) (x: ℕ) : ℕ :=
  f (g x)

#check compose
#reduce compose double double 3
#reduce compose square double 3
#reduce square (double 3)

def do_twice (f : ℕ → ℕ) (x: ℕ) : ℕ := f (f x)

#check do_twice
#reduce do_twice square 3
```

More compose examples

```
def compose (g f : ℕ → ℕ) : ℕ → ℕ :=
  λ k : ℕ, g (f k)

def compose' : (ℕ → ℕ) → (ℕ → ℕ) → (ℕ → ℕ) :=
  λ g : ℕ → ℕ,
    λ f : ℕ → ℕ,
      λ k : ℕ,
        g(f(k))

def square : ℕ → ℕ := λ k, k^2
def increment : ℕ → ℕ := λ k, k + 1
```

```
def square_then_increment : ℕ → ℕ :=
  (compose increment square)

def increment_then_square : ℕ → ℕ :=
  (compose square increment)

#eval square_then_increment 5
#eval increment_then_square 5
```

Alternate do_twice representations

```
def do_twice (f : ℕ → ℕ) (x: ℕ) : ℕ := f (f x)

def do_twice' : (ℕ → ℕ) → ℕ → ℕ := λ f x, f (f x)

def do_twice'' : (ℕ → ℕ) → ℕ → ℕ :=
  λ f : (ℕ → ℕ),
    λ (x : ℕ), f (f x)

theorem dt_eq_dt : do_twice = do_twice'' := rfl
```

What do these types mean? (1 of 2)

- `ℕ → ℕ`
  - Function that takes a natural number as an argument and returns a natural number
- `ℕ → (ℕ → ℕ)`
  - Function that takes a natural number as an argument and returns a function that takes a natural number as an argument and returns a natural number
  - E.g.,

```
def constNatFun (n: ℕ): ℕ → ℕ := λ k, n
def constNatFun' (n: ℕ) (k: ℕ): ℕ := n
```

- `ℕ → ℕ → ℕ`
  - Same as above

What do these types mean? (2 of 2)

- `(ℕ → ℕ) → ℕ`
  - Function that takes a function (that maps naturals to naturals) as an argument and returns a natural number — hard to imagine a useful example

```
def unuseful(f: ℕ → ℕ): ℕ := 3
```

- `(ℕ → ℕ) → (ℕ → ℕ)`
  - Function that takes a function as an argument and returns another function
  - E.g., `do_twice`
- `(ℕ → ℕ) → (ℕ → ℕ) → (ℕ → ℕ)`
  - Function that takes a function as an argument and returns a function that takes a function as an argument and returns another argument
  - E.g., `compose`

Inception

```
def do_twice' (f: (ℕ → ℕ) → ℕ → ℕ) (x: (ℕ → ℕ)) : (ℕ →
ℕ) := f (f x)

#check do_twice'
#check do_twice' do_twice
#eval (do_twice' do_twice) square 2
```

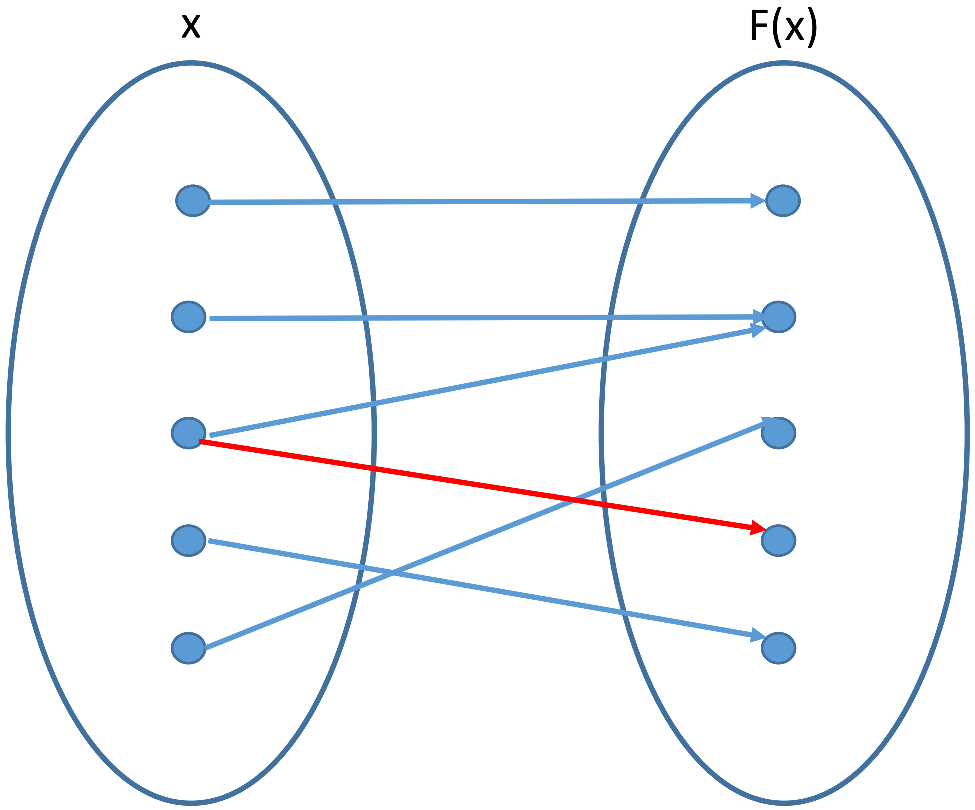
- Why not #reduce?

Important Properties about Functions

- Single-valued (always true of functions)
- Total (all domain values are valid)
- Partial (some domain values are not valid)
- Injective (total and no output is duplicated)
- Surjective (every element of the codomain/range is a valid output of the function)
- Bijective (injective and surjective)
- Extensionality

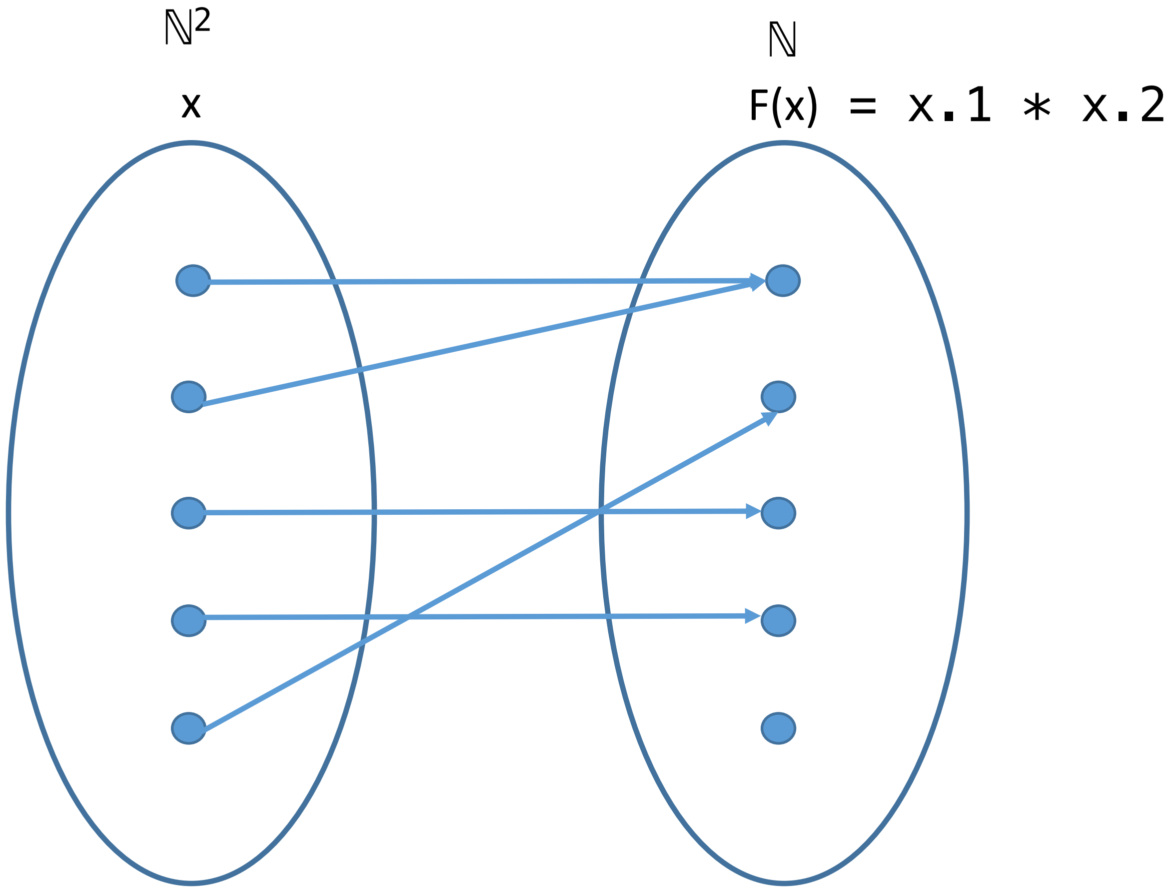
Single-valued

- The value of x uniquely determines the value of F(x)



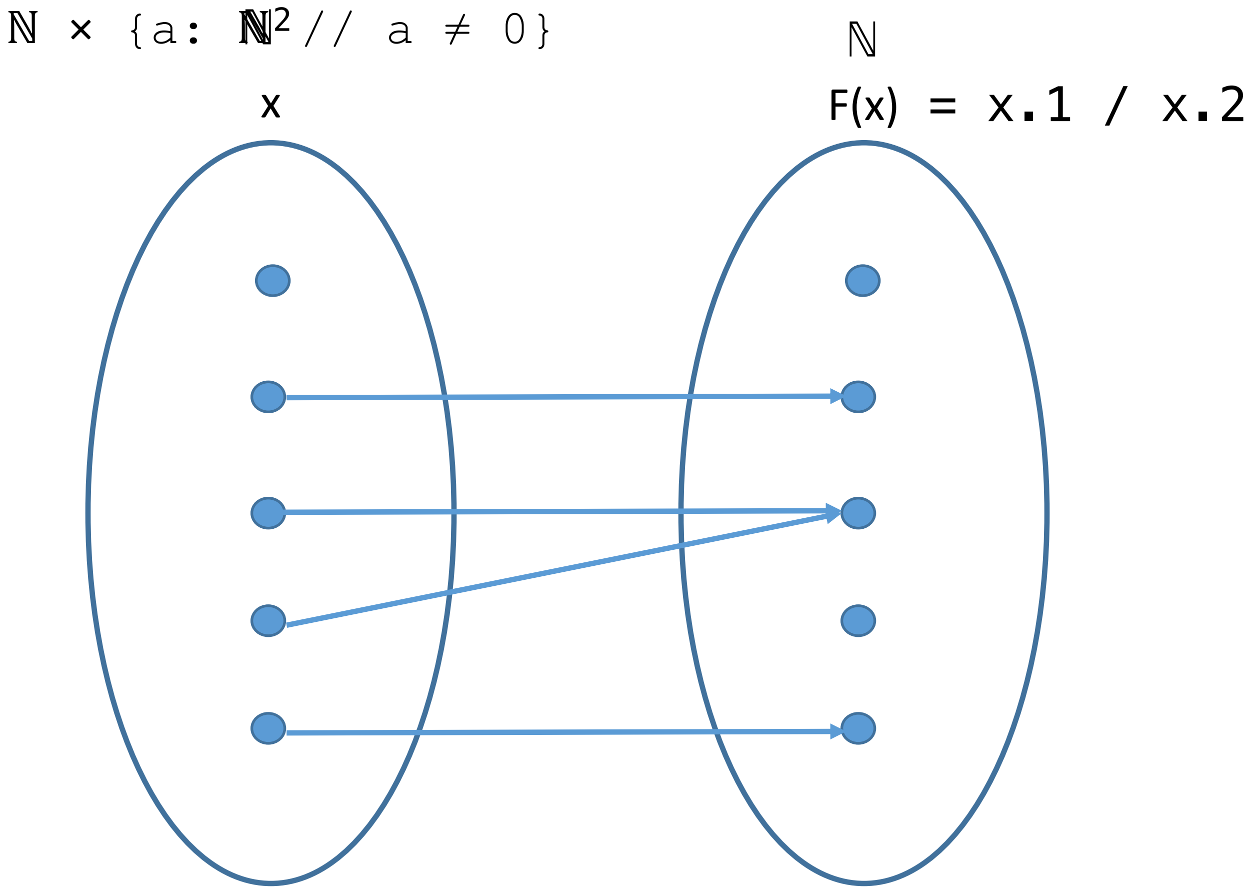
Total

- All possible values for x yield a valid result for F(x)



Total
Partial

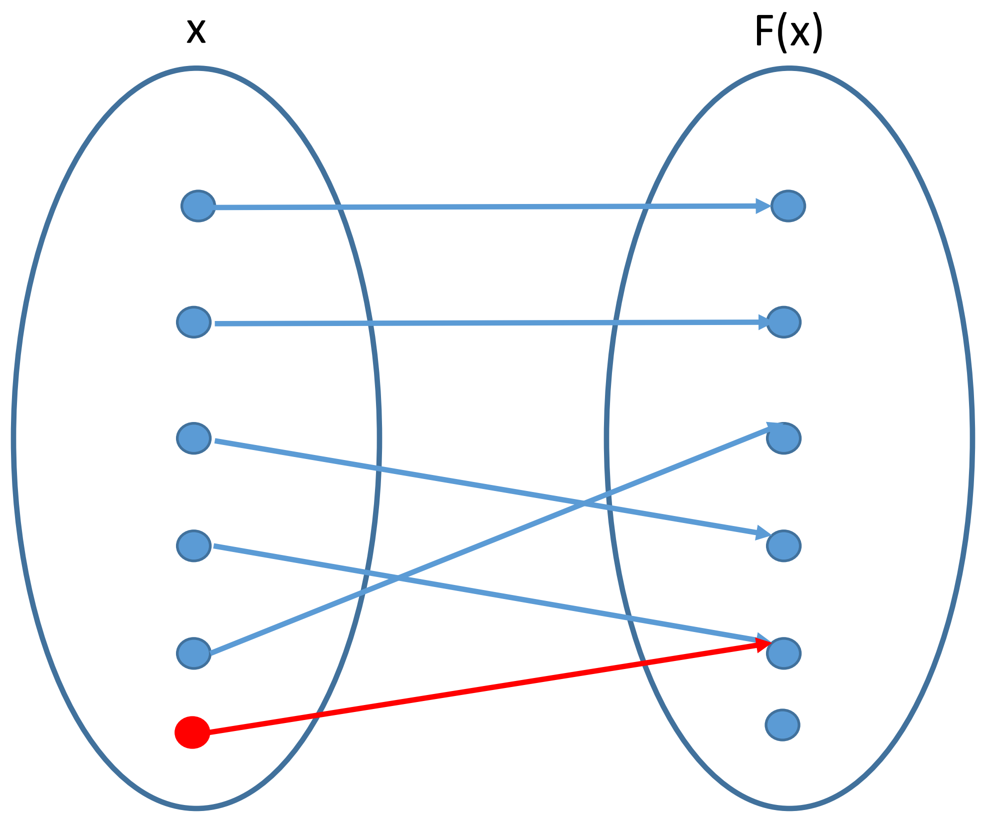
- There are some values of x (e.g., 0) that do not yield a valid value for F(x)
- Is subtraction over the naturals total or partial?



Injective

- No output is duplicated
- Inverse is valid function

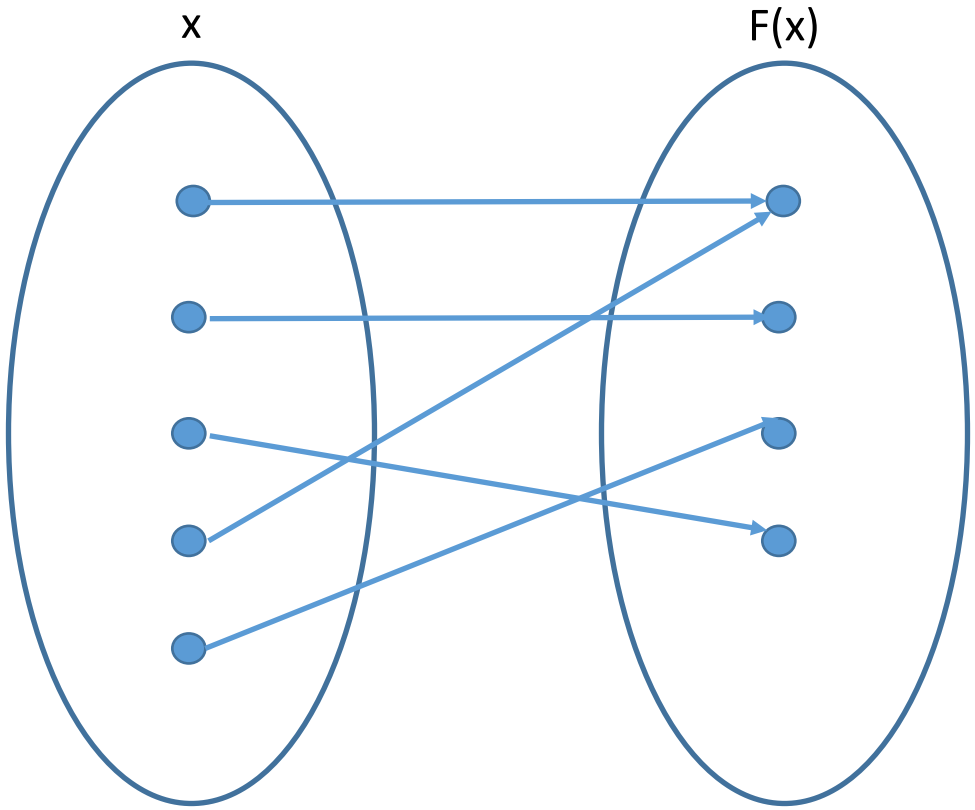
```
def injective(d: Type)(c: Type)
               (F: d → c): Prop :=
  ∀(x y: d),
    ((F x) = (F y)) → (x = y)
```



Surjective

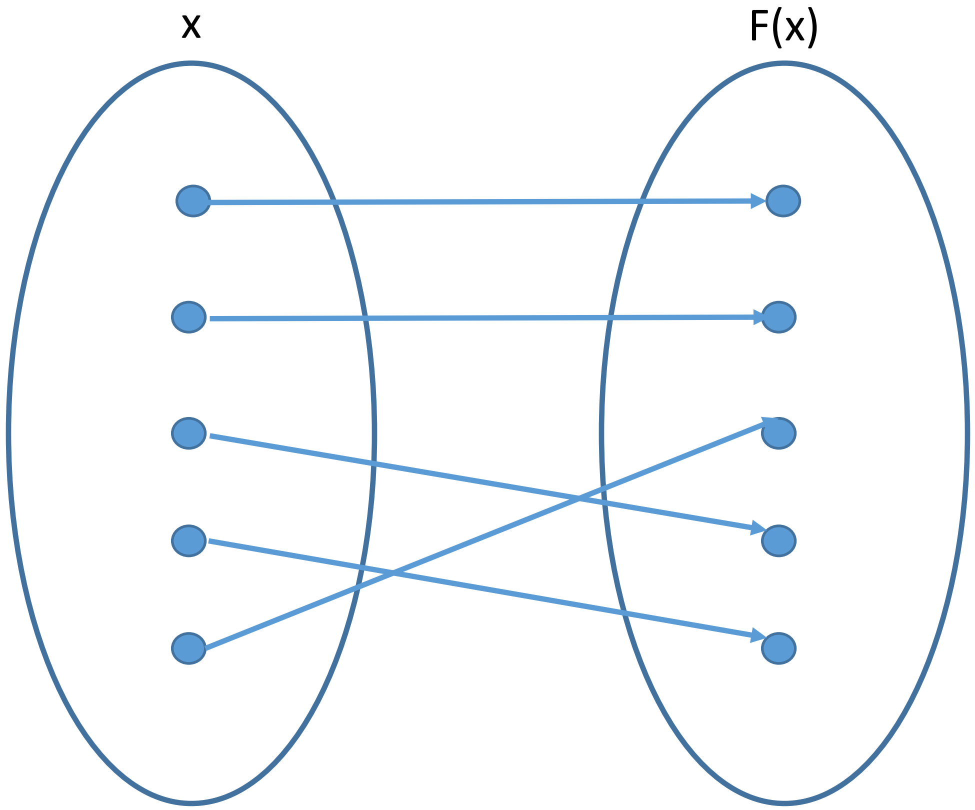
- Codomain of F(x) is covered

```
def surjective(d: Type)(c: Type)
               (F: d → c): Prop :=
  ∀(y: c),
    ∃(x: d),
      (F x) = y
```



Bijective

- Injective and surjective



Extensionality

- If f and g have the same domain, and they both map to the same values for all members in that domain, then f and g are equivalent

```
∀(S T: Type), ∀(f g: S → T), ∀(x: S),
   ((f x)=(g x)) ↔ (f = g)
```

Story time

- It’s important to get your functions correct
- Omitting an “overbar” from an equation in the Mariner 1 software (1962) caused the guidance system to interpret normal movement in the spacecraft as something that needed to be compensated for
- Mariner 1 had to be destroyed 293 seconds into its mission
- How could this have been prevented?

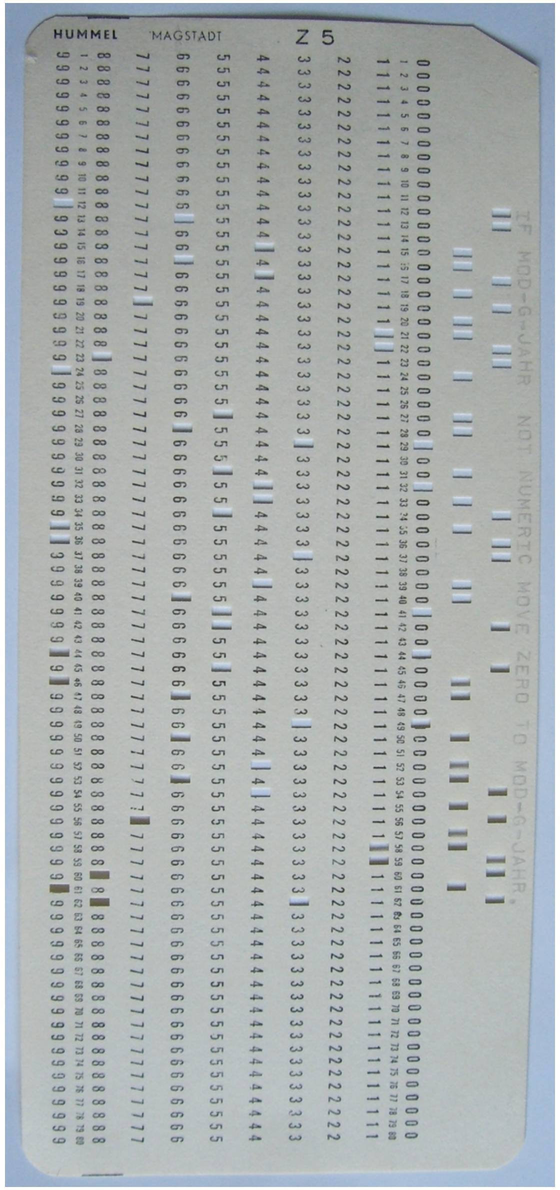


Image courtesy Wikimedia commons: https://commons.wikimedia.org/wiki/File:Punch-card-cobol.jpg

What is a predicate?

- A predicate is a *function* that returns a *proposition*
  - E.g., fromCharlottesville(p: Person): Prop := …
  - E.g., onSegment(s: Segment)(p: Point): Prop := …
- Note that in Lean, predicates have a return type of Prop and not bool
  - In other languages (e.g., PVS), this distinction is not meaningful
- Consider the example fromCharlottesville(p: Person)
  - fromCharlottesville(Maya) is the proposition that Maya is from Charlottesville
  - fromCharlottesville(Jamal) is the proposition that Jamal is from Charlottesville
  - fromCharlottesville(Bao) is the proposition that Bao is from Charlottesville
  - Not every proposition derived from a predicate will be true

Examples

- What predicates have we already seen in these slides?

```
def isEven(n: ℕ): Prop := n % 2 = 0
def isPrime(n: ℕ): Prop := ¬(∃(m: ℕ), m > 1 ∧ n % m = 0)
```

- Technically, these are not predicates:

```
def positive(n: ℕ): bool :=
    if n > 0 then tt else ff
def uint32(n: ℕ): bool :=
    if n >= 0 ∧ n < 2^32 then tt else ff
```

Exercises

- Write a predicate that takes a number *n*, and returns the proposition that *n* is positive

```
def positive(n: ℕ): Prop :=
  if n > 0 then true else false

def positive'(n: ℕ): Prop := n > 0
```

- Write a predicate that takes two numbers, *n* and *m*, and returns the proposition that *n* is evenly divisible by *m* (i.e., that *m* divides *n*)

```
def isDivisible(n m: ℕ): Prop := n % m = 0
```

Exercises

- What are other properties of natural numbers that could be expressed as predicates?
- Define a predicate that is true for every natural number (i.e., is *trivial*).
  - `def is_absorbed_by_zero(n: ℕ): Prop := n * 0 = 0`
- Define a predicate that is false for every natural number (i.e., is *unsatisfiable*)
  - `def equals_self_plus_one(n: ℕ): Prop := n = n + 1`

Inductive types

```
inductive day : Type
| Monday
| Tuesday
| Wednesday
| Thursday
| Friday
| Saturday
| Sunday

#check day.Tuesday

open day -- no longer need to day. prefix

#check Tuesday
```

Predicate for our type

```
def isWeekend : day → Prop :=
  λ d, d = Saturday ∨ d = Sunday
```

- Proof that Saturday is part of the weekend

```
theorem satIsWeekend: isWeekend Saturday :=
begin
  unfold isWeekend, -- unfold tactic
  apply or.intro_left,-- backwards reasoning
  apply rfl -- finally, equality
end
```

Relations

- Predicates can have more than one argument
  - E.g., onSegment: Segment → Point → Prop := λ (seg: Segment)(pt: Point),…
- Predicates of multiple arguments can be used to specify properties of *tuples* (e.g., pairs) of values
- Properties of tuples are called *relations*
- Properties of pairs are called *binary relations*
  - E.g., =, <, ≥, … (infix notation)
  - We could also define a predicate `areEqual` that takes two arguments
    - E.g., `areEqual 2 3` (prefix notation)

Fin