What is a set?

Definition (Intuitive)

A set is an unordered collection of objects (called its *elements* or *members*).
A set is said to *contain* its elements.
If S is a set then for any object s the statement "s is an element of S" is a proposition. If this proposition is true we write $s \in S$. If it is false we write $s \notin S$.

Example

All integers form a set (denoted $\mathbb{Z}$). All rational numbers form a set (denoted $\mathbb{Q}$). $5 \in \mathbb{Z}$ is true, while $\frac{1}{2} \in \mathbb{Z}$ is false.

We can define a set by listing its elements.

Example

- Let $S = \{1, 3, 5, 7, 9\} = \{1, 5, 3, 7, 9\} = \{1, 9, 3, 7, 5\}$ etc. Then $1, 3, 5, 7, 9 \in S$ and $2 \notin S$.
- Let V be the set of vowels and let C be the set of consonants of the English alphabet. Then $V = \{a, e, i, o, u, y\}$ and $C = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$. Note that $y \in V$ and $y \in C$.
- Let $S = \{\heartsuit, \clubsuit, \diamondsuit\}$. Then $\heartsuit \in S$ but, for example, $\spadesuit \notin S$.

We can also define a set by a property. The standard notation is $\{x \in U : P(x)\}$ where U is some other set (sometimes called the universe) and P is a predicate.

Example

- The set of all non-negative integers: $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$
- The set of all even integers: $\{n \in \mathbb{Z} : n \text{ is even}\}$.
- The set of all integers between 1 and $1\,000$: $\{n \in \mathbb{Z} : 1 \leq n \leq 1\,000\}$.
- The interval $[0, 1]$ (the set of all real numbers x such that $0 \leq x \leq 1$)

The empty set $\emptyset$ is the set which contains no elements.

The singleton set $\{\emptyset\}$ is the special set with just one element (the empty set).

Example

$\emptyset \neq \{\emptyset\}$; the first set has no element, the second one has one element.
$\emptyset \in \{\emptyset\}$
$0 \notin \{\emptyset\}$
$\{\emptyset, \emptyset\} = \{\emptyset\}$.

Subsets

Definition

Let A and B be sets. We say that A is a *subset* of B and write $A \subset B$ if $\forall a \in A$, $a \in B$.
Thus, if a is an element of A then a is an element of B.
We say that A and B are equal if they have the same elements, that is $A \subset B$ and $B \subset A$. We write $A = B$.

Example

- Let $A = \{1, 3, 5\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Then $A \subset B$ and $A \neq B$ (denoted $A \subsetneq B$).
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.
- The set of vowels of English alphabet is a subset of the set of letters.
- The interval $[2, 3]$ is a subset of the open interval $(0, 4)$.
- Let $A = \{1, 2, 5\}$ and $B = \{2, 3, 6\}$. Then $A \not\subset B$ and $B \not\subset A$.

Let $P(x, S)$ be the statement $x \in S$ (here x is any object and S is a set). Then $A \subset B$ means $\forall x, P(x, A) \to P(x, B)$.

Sets $\{1, 2, 2, 3, 1, 2, 3, 1, 3, 2\}$, $\{3, 2, 1\}$ and $\{1, 2, 3\}$ are equal

To show that A is a subset of B we need to show that every element of A is also an element of B

To show that A is not a subset of B it is enough to find one element of A which is not an element of B. Thus, $\mathbb{Q}$ is not a subset of $\mathbb{Z}$ because $\frac{1}{2}$ is a rational number but is not an integer.

Theorem

For any set S, $\emptyset \subset S$ and $S \subset S$.

Proof.

To show that $\emptyset \subset S$, we need to show that $\forall x, P(x) \to Q(x)$ where $P(x)$ is $x \in \emptyset$ and $Q(x)$ is $x \in S$.
But, since $P(x)$ is always false, $P(x) \to Q(x)$ is always true.
To show that $S \subset S$ we need to show that $\forall x, Q(x) \to Q(x)$.
But $p \to p$ is a tautology. □

Theorem

Let A, B and C be sets. If $A \subset B$ and $B \subset C$ then $A \subset C$.

Proof.

Let $P(x)$ be the statement $x \in A$, $Q(x)$ be the statement $x \in B$ and $R(x)$ be the statement $x \in C$.
We are given that $\forall x, P(x) \to Q(x)$ and $\forall x, Q(x) \to R(x)$.
For any a in our universe, we have $P(a) \to Q(a)$, $Q(a) \to R(a)$.
Therefore, $P(a) \to R(a)$. Thus, $\forall x, P(x) \to R(x)$, that is, $A \subset C$. $\square$

Set operations

A way to understand set operations is to compare them with logical operators. First, we need to introduce the universal set - that is, the set $\mathbb{U}$ which contains (has as its subsets) all sets we work with.

The set operation corresponding to $\wedge$ is called *intersection*.

Definition

Let A, B be sets. Then the *intersection* of A and B, denoted $A \cap B$ is the set $\{x \in \mathbb{U} : (x \in A) \wedge (x \in B)\}$.
If $A \cap B = \emptyset$, we say that A and B are *disjoint*.

Example

- Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5, 6\}$. Then $A \cap B = \{1, 3\}$.
- Let A be the set of even integers and B be the set of odd integers. Then $A \cap B = \emptyset$.
- Let $A = \{a, \{a, b\}, \{a, b, c\}\}$ and $B = \{a, b\}$. Then $A \cap B = \{a\}$.

Theorem

Let A, B be sets. Then $A \cap B \subset A$ and $A \cap B \subset B$. Moreover, if $A \subset B$ then $A \cap B = A$.

Proof.

Let $x \in A \cap B$. Then in particular $x \in A$ (the truth of $x \in A \wedge x \in B$ implies that $x \in A$ is true) and so $A \cap B \subset A$.
To prove that $A \cap B = A$ if $A \subset B$ we also need to show that $A \subset B$ implies that $A \subset A \cap B$. But if $A \subset B$, $x \in A$ implies that $x \in B$. So, $x \in A$ implies $x \in A$ and $x \in B$. $\square$

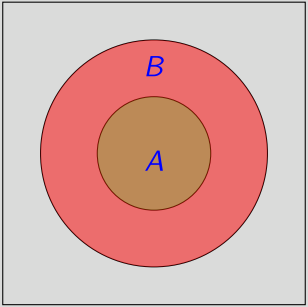
Thus, $\mathbb{Z} \cap \mathbb{N} = \mathbb{N}$ and $A \cap \emptyset = \emptyset$ for any set A.

Venn diagrams

Venn diagrams are useful for visualizing set operations. In these diagrams $\mathbb{U}$ is shown as a rectangle and sets are shown as circles inside the rectangle.

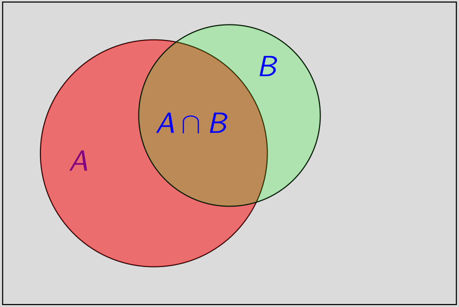
Example

The following Venn diagram illustrates the statement $A \subset B$. The set A is shown as a circle inside the other circle representing the set B.



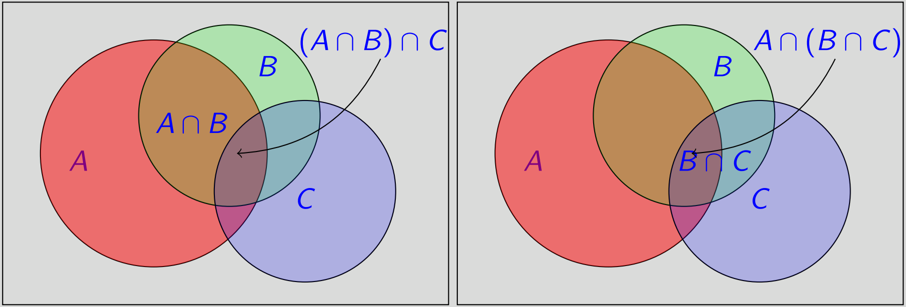
Example

The following Venn diagram shows the intersection of two sets.



Example

Like conjunction, intersection is associative: if A, B, C are sets then $A \cap (B \cap C) = (A \cap B) \cap C$. This is illustrated by the following diagram



Theorem

For any sets A, B and C, $A \cap (B \cap C) = A \cap (B \cap C)$.

Proof.

We claim that $A \cap (B \cap C)$ and $(A \cap B) \cap C$ are both equal to the set $S = \{x \in \mathbb{U} : (x \in A) \wedge (x \in B) \wedge (x \in C)\}$.
Indeed, suppose that $x \in S$. Then $x \in A$ and $x \in B$, whence $x \in A \cap B$, and $x \in C$; thus, $x \in (A \cap B) \cap C$. Likewise, $x \in B$ and $x \in C$, hence $x \in B \cap C$, and $x \in A$; thus, $x \in A \cap (B \cap C)$. So, $S \subset (A \cap B) \cap C$ and $S \subset A \cap (B \cap C)$.
Suppose that $x \in A \cap (B \cap C)$. Then $x \in A$ and $x \in B \cap C$; the latter implies that $x \in B$ and $x \in C$. Thus, $x \in S$. Likewise, $x \in (A \cap B) \cap C$ implies that $x \in A \cap B$, hence $x \in A$ and $x \in B$, and $x \in C$. Therefore, $x \in S$.
Thus, $(A \cap B) \cap C = S = A \cap (B \cap C)$. $\square$

A different proof.

Let $x \in \mathbb{U}$, $S \subset \mathbb{U}$ and let $P(x, S)$ be the statement "$x \in S$". Then

$$\begin{aligned} P(x, A \cap (B \cap C)) &\equiv P(x, A) \wedge P(x, B \cap C) \\ &\equiv P(x, A) \wedge (P(x, B) \wedge P(x, C)) \\ &\equiv (P(x, A) \wedge P(x, B)) \wedge P(x, C) \qquad \text{since } \wedge \text{ is associative} \\ &\equiv P(x, A \cap B) \wedge P(x, C) \\ &\equiv P(x, (A \cap B) \cap C). \end{aligned}$$

This proves that $x \in (A \cap B) \cap C$ if and only if $x \in A \cap (B \cap C)$. Thus, $A \cap (B \cap C) = (A \cap B) \cap C$. $\square$

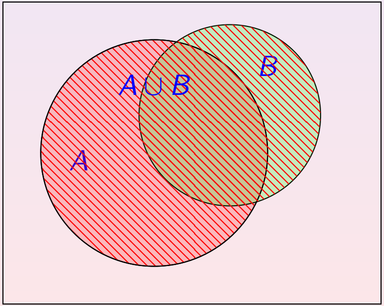
This proof can be written in a "set theoretic" way, that is, without using the formal logic. Let $x \in A \cap (B \cap C)$. Then $x \in B \cap C$, hence $x \in B$ and $x \in C$, and also $x \in A$. This implies that $x \in A \cap B$; since $x \in C$ we conclude that $x \in (A \cap B) \cap C$. Thus, $A \cap (B \cap C) \subset (A \cap B) \cap C$. The opposite inclusion is proven similarly.

Our next operation corresponds to $\vee$ in logic.

Definition

Let A, B be sets. Then the union of A and B, denoted $A \cup B$, is the set $\{x \in \mathbb{U} : (x \in A) \vee (x \in B)\}$.

The corresponding Venn diagram is



Example

- Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$
- Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5, 6\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$
- Let $A = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$, $B = \{\{b\}, \{a, c\}, \{a, b\}, \{a, b, c\}\}$. Then $A \cup B = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$.

Theorem

Let A and B be sets. Then A and B are subsets of $A \cup B$. Moreover, if $A \subset B$ then $A \cup B = B$.

Theorem

Let A, B and C be sets. Then $(A \cup B) \cup C = A \cup (B \cup C)$ and

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \qquad (A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Exercise. Draw Venn diagrams illustrating the above statements.

Proof.

The argument for associativity of $\cup$ is similar to that for associativity of $\cap$.

Suppose that $x \in (A \cup B) \cap C$. Then $x \in C$ and either $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cap C$. If $x \in B$ then $x \in B \cap C$. Thus, either $x \in A \cap C$ or $x \in B \cap C$. Therefore, $x \in (A \cap C) \cup (B \cap C)$ and so $(A \cup B) \cap C \subset (A \cap C) \cup (B \cap C)$.

Suppose that $x \in (A \cap C) \cup (B \cap C)$. Then either $x \in (A \cap C)$, which means that $x \in A$ and $x \in C$, or $x \in (B \cap C)$, with means that $x \in B$ and $x \in C$. Thus, in both cases $x \in C$, and also $x \in A$ or $x \in B$. Thus, $x \in (A \cup B) \cap C$ and so $(A \cap C) \cup (B \cap C) \subset (A \cup B) \cap C$.

We proved that $(A \cup B) \cap C \subset (A \cap C) \cup (B \cap C)$ and $(A \cap C) \cup (B \cap C) \subset (A \cup B) \cap C$. Therefore, $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$.

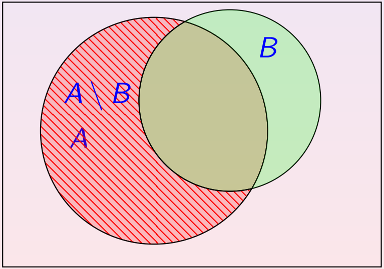
The remaining equality is proved similarly (Exercise). $\square$

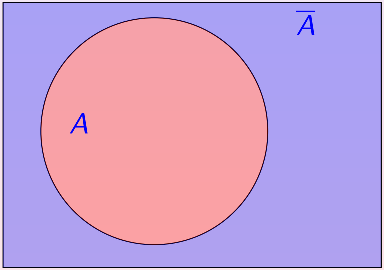
Difference and complements

Definition

Let A, B be sets. Then the *difference* of A and B (denoted $A \setminus B$) is the set $\{x \in A : x \notin B\}$.
The set $\mathbb{U} \setminus A$ is called *the complement* of A and is denoted A^C or $\overline{A}$.

The corresponding Venn diagrams are





Complement plays the role of $\neg$ in logic: $x \in \overline{A}$ if and only if $\neg(x \in A)$ is true.

Example

- If $A = \{1, 2, 3, 4, 6\}$, $B = \{2, 3, 5, 7\}$ then $A \setminus B = \{1, 4, 6\}$ and $B \setminus A = \{5, 7\}$.
- If $\mathbb{U}$ is the set of all integers and A is the set of even integers then $\overline{A}$ is the set of odd integers.
- If $\mathbb{U}$ is the set of all positive integers and A is the set of perfect squares (that is, integers n such that $n = k^2$ for some integer k) then $\overline{A}$ is the set of integers which are not perfect squares. For example, $2, 3, 5, 6, 7 \in \overline{A}$ and $4, 9 \notin \overline{A}$.

Clearly, $A \setminus B \subset A$ but $A \setminus B \not\subset B$. Moreover, $(A \setminus B) \cap B = \emptyset$.

Theorem

Let A, B be sets. Then $A \setminus B = A \cap \overline{B}$, $A \cap \overline{A} = \emptyset$ and $A \cup \overline{A} = \mathbb{U}$.

Proof.

Let $x \in \mathbb{U}$. Then $x \in A \setminus B$ if and only if $x \in A$ and $x \notin B$, that is, $x \in A$ and $x \in \overline{B}$, which is equivalent to saying that $x \in A \cap \overline{B}$. Thus, $A \setminus B = A \cap \overline{B}$.
Take $B = A$. Then $A \cap \overline{A} = A \setminus A = \emptyset$, since $A \setminus A$ is the set of all elements $x \in \mathbb{U}$ such that $x \in A$ and $x \notin A$.
For the last equality, observe that for any $x \in \mathbb{U}$, $(x \in A) \vee (x \notin A)$ is true, that is, $\mathbb{U} \subset A \cup \overline{A}$. Since $A \cup \overline{A} \subset \mathbb{U}$, we obtain the desired equality of sets. $\square$

Theorem

Let A, B be sets. Then $A \cap (B \setminus A) = \emptyset$ and $A \cup (B \setminus A) = A \cup B$. In particular, if $A \subset B$ then $A \cup (B \setminus A) = B$.

Proof.

Since $x \in B \setminus A$ implies that $x \notin A$, $(x \in (B \setminus A)) \wedge (x \in A)$ is false for all $x \in \mathbb{U}$ which means that $A \cap (B \setminus A) = \emptyset$.
A similar argument can be used to prove the second equality. Alternatively,

$$\begin{aligned} A \cup (B \setminus A) &= A \cup (B \cap \overline{A}) \\ &= (A \cup B) \cap (A \cup \overline{A}) \\ &= (A \cup B) \cap \mathbb{U} \\ &= A \cup B. \end{aligned}$$

The second statement follows from the first since $A \subset B$ implies that $A \cup B = B$. $\square$

De Morgan's laws for sets

Theorem

Let A and B be sets. Then $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof.

As the name suggests, the De Morgan's laws for sets follows from De Morgan's laws for propositions. For example, let $P(x)$ be "$x \in A$" and $Q(x)$ be "x in B".

Then $x \in A \cap B$ is $P(x) \wedge Q(x)$.

Then $x \in \overline{A \cap B}$ is $\neg(P(x) \wedge Q(x)) \equiv \neg P(x) \vee \neg Q(x)$ by De Morgan's laws.

The last statement reads $(x \notin A) \vee (x \notin B)$ which is $x \in \overline{A} \cup \overline{B}$.

We established that

$$(x \in \overline{A \cap B}) \equiv (x \in \overline{A} \cup \overline{B}). \qquad \square$$

Symmetric difference

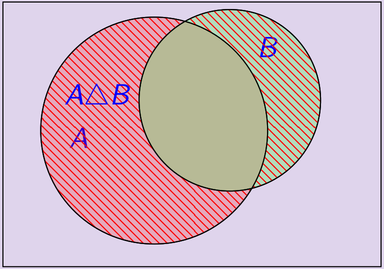
Another set operation, which is the analog of *XOR*, is the symmetric difference.

The symmetric difference of two sets A and B is the set $(A \cup B) \setminus (A \cap B) = \{x \in \mathbb{U} : (x \in A) \oplus (x \in B)\}$. This set is often denoted by $A \triangle B$; sometimes $A \oplus B$ or $A \ominus B$ is used.

Example

- Let $A = \{1, 3, 4, 6, 8, 11\}$, $B = \{2, 4, 6, 7, 9, 11\}$. Then $A \triangle B = \{1, 2, 3, 7, 8, 9\}$.
- Let A be the set of all even integers and B be the set of perfect squares. Then $A \triangle B$ is the set of all odd perfect squares and all even integers which are not perfect squares. For example, $2, 6, 9, 25 \in A \triangle B$ but $3, 4, 7 \notin A \triangle B$.

Venn diagram of the symmetric difference



Theorem

Let A, B be sets. Then $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Proof based on formal logic.

Let $P(x)$ be "$x \in A$" and $Q(x)$ be "$x \in B$".
Then $x \in A \triangle B$ is

$$\begin{aligned}
P(x) \oplus Q(x) &\equiv \neg(P(x) \leftrightarrow Q(x)) \\
&\equiv \neg((P(x) \to Q(x)) \wedge (Q(x) \to P(x))) \\
&\equiv \neg((\neg P(x) \vee Q(x)) \wedge (\neg Q(x) \vee P(x))) \\
&\equiv (\neg(\neg P(x) \vee Q(x))) \vee \neg(\neg Q(x) \vee P(x)) \\
&\equiv (P(x) \wedge \neg Q(x)) \vee (Q(x) \wedge \neg P(x)).
\end{aligned}$$

Since $P(x) \wedge \neg Q(x)$ is "$x \in A \setminus B$" and similarly $Q(x) \wedge \neg P(x)$ means "$x \in B \setminus A$", the last assertion is $x \in (A \setminus B) \cup (B \setminus A)$. $\square$

Proof based on set operations.

Since $A \setminus B = A \cap \overline{B}$ we have

$$
\begin{aligned}
(A \setminus B) \cup (B \setminus A) &= (A \cap \overline{B}) \cup (B \cap \overline{A}) \\
&= ((A \cap \overline{B}) \cup B) \cap ((A \cap \overline{B}) \cup \overline{A}) \\
&= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A}) \\
&= ((A \cup B) \cap \mathbb{U}) \cap (\mathbb{U} \cap (\overline{B} \cup \overline{A})) \\
&= (A \cup B) \cap (\overline{A} \cup \overline{B}) \\
&= (A \cup B) \cap \overline{A \cap B} \\
&= (A \cup B) \setminus (A \cap B) \\
&= A \triangle B.
\end{aligned}
$$

$\square$

Comparison of set operations and logical operators

Negation, $\neg$	Complement, $\overline{}$
Conjunction, *AND*, $\wedge$	Intersection, $\cap$
Disjunction, *OR*, $\vee$	Union, $\cup$
Exclusive or, *XOR*, $\oplus$	Symmetric difference, $\triangle$
Conditional statement, $\rightarrow$	Subset, $\subset$
Biconditional statement, $\leftrightarrow$	Equality of sets
True	Universal set
False	Empty set, $\emptyset$

Algebra of logic vs algebra of set operations

Identity laws	$p \wedge T \equiv p$ $p \vee F \equiv p$	$A \cap \mathbb{U} = A$ $A \cup \emptyset = A$
Dominance laws	$p \wedge F \equiv F$ $p \vee T \equiv T$	$A \cap \emptyset = \emptyset$ $A \cup \mathbb{U} = \mathbb{U}$
Idempotent laws	$p \wedge p \equiv p$ $p \vee p \equiv p$	$A \cap A = A$ $A \cup A = A$
Double negation law	$\neg(\neg p) \equiv p$	$\overline{\overline{A}} = A$
Commutativity	$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	$A \cap B = B \cap A$ $A \cup B = B \cup A$

Associativity in logic: $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ and $p \vee (q \vee r) \equiv (p \vee q) \vee r$.

For sets: $(A \cap B) \cap C = A \cap (B \cap C)$ and, respectively, $(A \cup B) \cup C = A \cup (B \cup C)$.

Distributivity: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (respectively $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$)

For sets: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (respectively, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$).

De Morgan's laws $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (respectively, $\neg(p \vee q) \equiv \neg p \wedge \neg q$)
For sets $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (respectively, $\overline{A \cup B} = \overline{A} \cap \overline{B}$).