HOMEWORK SET 2. EQUIVALENCES AND ARGUMENTS

1. Show that each of the following propositions is a tautology. You can use truth tables or properties of logical operators.

a) $(p \wedge q) \to p$	**c)** $\neg p \to (p \to q)$	**e)** $\neg(p \to q) \to p$
b) $p \to (p \vee q)$	**d)** $(p \wedge q) \to (p \to q)$	**f)** $\neg(p \to q) \to \neg q$

Solution. **a)** If p is true than $(p \wedge q) \to p$ is true regardless of the value of q since $r \to T$ is true for all values of r. If p is false then $p \wedge q$ is also false and so $(p \wedge q) \to p$ is true.

Another solution: $(p \wedge q) \to p \equiv \neg(p \wedge q) \vee p \equiv \neg p \vee \neg q \neg p \equiv T \vee \neg q \equiv T$.

b) If p is false then the conditional statement is true regardless of the value of q. If p is true then $p \vee q$ is also true and so the conditional statement is true. Alternatively, we have $p \to (p \vee q) \equiv \neg p \vee (p \vee q) \equiv T \vee q \equiv T$.

c) if p is true then $\neg p \to (p \to q)$ is true regardless of the value of q. If p is false then $p \to q$ is true regardless of the value of q and so $\neg p \to (p \to q)$ is true. Alternatively, $\neg p \to (p \to q) \equiv p \vee (p \to q) \equiv p \vee (\neg p \vee q) \equiv T \vee q \equiv T$.

d) If p is false then $p \wedge q$ is false and so $(p \wedge q) \to (p \to q)$ is true. If p is true then $(p \wedge q) \to (p \to q)$ is true unless $p \to q$ is false and q is true. But if q is true then $p \to q$ is true. Alternatively, $(p \wedge q) \to (p \to q) \equiv \neg(p \wedge q) \vee (\neg p \vee q) \equiv (\neg p \vee \neg q) \vee \neg p \vee q \equiv \neg p \vee \neg q \vee q \equiv \neg p \vee T \equiv T$. □

2. Determine whether $(\neg p \wedge (p \to q)) \to \neg q$ is a tautology. What about $(\neg q \wedge (p \to q)) \to \neg p$?

Solution. For the first implication to be false $\neg q$ must be false (that is q is true) and $\neg p \wedge (p \to q)$ must be true. Since q is true, $p \to q$ is automatically true so $\neg p \wedge (p \to q)$ is true provided that p is false. Thus, if q is true and p is false the first implication is false. Therefore, it is not a tautology.

Similarly, if the second implication is false we must have p true and $\neg q \wedge (p \to q)$ true, which forces $\neg q$ and $p \to q$ to be true. But $\neg q$ is true means that q is false and then $p \to q$ becomes $T \to F$ which is false. Thus, the second implication is never false. Therefore, it is a tautology. □

3. Determine whether $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$ are logically equivalent. What about $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$?

Solution. We have

$$\begin{aligned}(p \wedge q) \to r &\equiv \neg(p \wedge q) \vee r \\ &\equiv (\neg p \vee \neg q) \vee r.\end{aligned}$$

On the other hand,

$$\begin{aligned}(p \to r) \wedge (q \to r) &\equiv (\neg p \vee r) \wedge (\neg q \vee r)\\ &\equiv (\neg p \wedge \neg q) \vee r.\end{aligned}$$

If r is false then the first one is equivalent to $\neg p \vee \neg q$ and the second is equivalent to $\neg p \wedge \neg q$. These are clearly not equivalent.

For the second pair we have

$$\begin{aligned}(p \vee q) \to r &\equiv \neg(p \vee q) \vee r\\ &\equiv (\neg p \wedge \neg q) \vee r\\ &\equiv (\neg p \vee r) \wedge (\neg q \vee r)\\ &\equiv (p \to r) \wedge (q \to r).\end{aligned}$$

□

4. Verify the following equivalences.

a) $p \vee (p \wedge q) \equiv p$

b) $p \wedge (p \vee q) \equiv p$

c) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

d) $\neg p \to (q \to r) \equiv q \to (p \vee r)$

Solution. **a)** If p is true then the left hand side is true since $T \vee r \equiv T$. If p is false then also $p \wedge q$ is false and so $p \vee (p \wedge q)$ is false. Thus, the truth value of the left hand side is the same as the truth value of the right hand side.

b) We can use an argument similar to that in the previous part. Or we can use De Morgan's law: since $p \vee (p \wedge q) \equiv p$, $\neg p \equiv \neg(p \vee (p \wedge q)) \equiv \neg p \wedge \neg(p \wedge q) \equiv \neg p \wedge (\neg p \vee \neg q)$. Replacing p by $\neg p$ and q by $\neg q$ we obtain the desired equivalence.

c) We have

$$\begin{aligned}(p \to q) \wedge (p \to r) &\equiv (\neg p \vee q) \wedge (\neg p \vee r)\\ &\equiv \neg p \vee (q \wedge r)\\ &\equiv p \to (q \wedge r)\end{aligned}$$

d) We have

$$\begin{aligned}\neg p \to (q \to r) &\equiv p \vee (q \to r)\\ &\equiv p \vee (\neg q \vee r)\\ &\equiv \neg q \vee (p \vee r)\\ &\equiv q \to (p \vee r).\end{aligned}$$

□

5. Consider a specification: "If the database is opened, then the monitor is put in a closed state, if the system is not in its initial state." Find an equivalent, easier-to-understand specification that does not involve conditional statements.

Solution. Let d be "the database is opened", m be "the monitor is put in a closed state" and s be "the system is in the initial state". Then the specification reads $d \wedge \neg s \to m \equiv \neg(d \wedge \neg s) \vee m \equiv \neg d \vee s \vee m$. This can be translated as "the monitor is put in a closed state, or the system is in its initial state, or the database is not opened". □

6. Show that the argument

$$\begin{array}{l} (p \wedge t) \to (r \vee s) \\ q \to (u \wedge t) \\ u \to p \\ \neg s \\ \hline \therefore q \to r \end{array}$$

is valid.

Solution. Assuming that all premises are true we conclude that $\neg s$ is false and therefore $r \vee s \equiv r$. Thus, we can rewrite the argument as

$$(p \wedge t) \to r,\ q \to (u \wedge t),\ u \to p \therefore q \to r.$$

We claim that $u \to p \therefore (u \wedge t) \to (p \wedge t)$ is a valid argument. Indeed, suppose $(u \wedge t) \to (p \wedge t)$ is false. Then u and t are both true and $p \wedge t$ is false hence p is false. But this forces $u \to p$ to be false.

Thus, our argument can be written as

$$q \to (u \wedge t),\ (u \wedge t) \to (p \wedge t),\ (p \wedge t) \to r \therefore q \to r.$$

This is manifestly valid since it is a combination of two syllogisms.

Another approach: we can get rid of s as above and consider the argument

$$(p \wedge t) \to r,\ q \to (u \wedge t),\ u \to p \therefore q \to r.$$

Suppose that $q \to r$ is false (that is, q is true and r is false). If $q \to (u \wedge t)$ is false then we are done so assume that it is true; then $u \wedge t$ is true hence t is true and u is true. Then $(p \wedge t) \to r$ becomes $p \to F$ and $u \to p$ becomes $T \to p$. If p is T then the first one is false, while if p is F then the second one is false. We showed that if $q \to r$ is false then at least one of the premises is false. The argument is valid. □

7. Determine whether the following arguments are valid.

a) All animals in Australia are marsupials. Kangaroos live in Australia. Therefore, kangaroos are marsupials.

If a is "it lives in Australia", k is "it is a kangaroo" and m is "it is a marsupial", our argument is $a \to m,\ k \to a \therefore k \to m$. This is a syllogism and therefore is valid.

b) All marsupials live in Australia. Kangaroos are marsupials. Therefore, kangaroos live in Australia.

Using the same notation as above, we write our argument as $m \to a,\ k \to m \therefore k \to a$. This is again a syllogism and therefore is valid.

c) All marsupials live in Australia. Kangaroos live in Australia. Therefore, kangaroos are marsupials.

This argument is $m \to a, k \to a \therefore k \to m$. It is a fallacy since if a is true then both $m \to a$ and $k \to a$ are true for all values of k and m. But $k \to m$ is false if k is true and m is false.

d) All marsupials live in Australia. Reindeer do not live in Australia. Therefore, reindeer are not marsupials.

Let r be "it is a reindeer". Then the argument is $m \to a$, $r \to \neg a \therefore r \to \neg m$. Since $m \to a \equiv \neg a \to \neg m$, this argument is equivalent to $r \to \neg a$, $\neg a \to \neg m \therefore r \to \neg m$ which is a syllogism (and hence valid).

e) Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.

Let d be "takes discrete mathematics", c be "computer science major". The argument is $c \to d, d \therefore c$, which is a fallacy as we already established.

f) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.

This is valid because it is an example of the detachment rule (or modus ponens).

g) If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on. If the sailing race is held, then the trophy will be awarded. The trophy was not awarded. Therefore, it rained.

Let r be "it rains", f be "it is foggy", s be "the sailing race is held", l be "the lifesaving demonstration is held" and t be "the trophy will be awarded". Then the argument is

$$\begin{array}{l} \neg r \vee \neg f \to s \wedge l \\ s \to t \\ \neg t \\ \hline \therefore r \end{array}$$

Since $s \to t, \neg t \therefore \neg s$ is valid, we can write this as

$$\begin{array}{l} \neg r \vee \neg f \to s \wedge l \\ \neg s \\ \hline \therefore r \end{array}$$

Using the contrapositive we replace the first one by $\neg(s \wedge l) \to \neg(\neg r \vee \neg f) \equiv \neg s \vee \neg l \to r \wedge f$. Thus, the argument becomes

$$\begin{array}{l} (\neg s \vee \neg l) \to (r \wedge f) \\ \neg s \\ \hline \therefore r \end{array}$$

Replacing $\neg s$ by $\neg s \vee \neg l$ (this is a valid argument) and using the law of detachment we obtain $r \wedge f \therefore r$ which is manifestly true.

Remark. If we replace "or" by "and" in the first premise (which makes more sense than the present one) the argument becomes invalid.

8. For each of these collections of premises, what relevant conclusion or conclusions can be drawn? Explain your reasoning.

a) "Every computer science major has a personal computer." "Ralph does not have a personal computer." "Ann has a personal computer."

b) "If I take the day off, it either rains or snows." "I took Wednesday off or I took Friday off." "It was sunny on Friday." "It did not snow on Wednesday."

c) "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."
d) "I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see blue rabbits."

Answers. **a)** "Ralph is not a computer science major" (by modus tollens)
b) "I took a Wednesday off and it rained on Wednesday".
c) "Dragonflies have six legs", "Spiders are not insects and eat insects".
d) "I see blue rabbits"

□

9. (Lewis Carroll) Consider the following premises:

- None of the unnoticed things, met with at sea, are mermaids.
- Things entered in the log, as met with at sea, are sure to be worth remembering.
- I have never met with anything worth remembering, when on a voyage.
- Things met with at sea, that are noticed, are sure to be recorded in the log.

Show that these premises imply the conclusion "I have never met a mermaid at sea".

Solution. The universe here is "things met with at sea". Let n be "It is noticed", m be "It is a mermaid", l be "It is entered in the log", w be "It is worth remembering". Then our argument is

$$\begin{array}{l} \neg n \to \neg m \equiv m \to n \\ l \to w \\ \neg w \\ n \to l \\ \hline \neg m \end{array}$$

It is valid because $m \to n, n \to l \therefore m \to l$ is a hypothetical syllogism, then $m \to l, l \to w \therefore m \to w$ again is a hypothetical syllogism. Thus, we are left with $m \to w, \neg w \therefore \neg m$ which is modus tollens. □