HOMEWORK SET 8. CARDINALITY AND COUNTING

1. Assuming that no person has more than 1,000,000 hairs on their head and that the population of Los Angeles is 4,041,707, show that there are at least five people in Los Angeles with the same number of hairs on their heads.

Solution. We have $N = 4,041,707$ of "pigeons", to be distributed among $k = 1,000,000$ boxes (or $1,000,001$ if we assume that there exist completely bald people). Since $4 < N/k < 5$, we conclude that there are at least 5 "pigeons" in one of the boxes.

We can also conclude (although that was not asked in the problem) that there is some number of hairs which does not occur in more than 4 people. □

2. At UCR, there are 15 classrooms which have at least 100 seats. Their total capacity is 3,292 seats.

a) Show that there is a classroom with at least 220 seats and a classroom with at most 219 seats.

b) Given that the largest classroom has 570 seats and the second largest classroom has no more than 416 seats, what is the maximal possible number of classrooms which have exactly 100 seats?

Solution. We need to distribute $N = 3,292$ seats among $k = 15$ "boxes" (classrooms). Since $219 < N/k < 220$ there is a classroom with at least 220 seats and a classroom with at most 219 seats.

To answer the second question, note that all large classrooms put together, except the largest one, contain $3,292 - 570 = 2,722$ seats. Suppose that k classrooms contain exactly 100 seats. Clearly $k < 14$ (we know that $k \leq 14$ and $14 \cdot 100 = 1,400 < 2,722$). If we distribute $2,722 - 100k$ seats among $14-k$ "boxes" we must not have a "box" containing more than 416 items. So, we must have $(2,722 - 100k)/(14 - k) \leq 416$. This means that $2,722 - 100k \leq 416(14 - k)$ or $316k \leq 3102$. Thus, $k \leq 1551/158$. Since k is an integer, the maximal possible number of such rooms is 9.

In fact, if there are 10 rooms with exactly 100 seats, then we need to distribute $2722 - 1000 = 1722$ among 4 classrooms which contain more than 100 and at most 416 seats. By the generalized pigeonhole principle we must have a room with at least 430 seats, so this is impossible. □

3. Find the minimal number of cables required to connect eight computers to four printers to guarantee that for every choice of four of the eight computers, these four computers can directly access four different printers. Can you find the answer for an arbitrary number of computers and printers?

Solution. We claim that each printer must be connected to at least 5 computers. Indeed, if there is a printer connected to 4 computers or less, we can find a set of $8 - 4 = 4$ computers none of which is connected to that printer. This means that we will have 4 computers directly connected to 3 printers only, which contradicts the requirements. Thus, the minimal number of connections per printer is 5 and so the minimal total number of connections is $4 \cdot 5 = 20$.

This can be implemented by connecting 4 computers to 4 different printers and then connecting the remaining 4 computers to all 4 printers.

In general, we claim that $P(C - P + 1)$ cables are needed so that any P computers are directly connected to P different printers. Indeed, each printer must be connected to at least $C - P + 1$ computers. If there is a printer which is connected to computers from a set Y with $|Y| \leq C - P$ then the complement of Y has cardinality $C - |Y| \geq C - (C - P) = P$. Thus we can find a set X of P computers (a subset of the complement of Y) which are not connected to that printer. Thus, computers from the set X will be directly connected to at most $P-1$ printers. This contradicts the requirements.

Thus, the minimal number of cables is $P(C-P+1)$. It can be attained by attaching first P computers to P different printers and then attaching the remaining $C-P$ computers to all printers. This gives $P+(C-P)P=(C-P+1)P$ connections. $\square$

4. What is the minimum number of domestic (=citizens or permanent residents of the United States) students who must be enrolled in a university to guarantee that there are at least 70 who come from the same state?

Solution. There are 51 "boxes", corresponding to each of the 50 states and the District of Columbia. We need to find minimal N such that $70 \geq N/51 > 69$; the pigeonhole principle will then imply that at least one "box" contains at least 70 "objects". Then $N = 51 \cdot 69 + 1 = 3,520$ (if we do not include D.C., the minimal number will be $50 \cdot 69 + 1 = 3,451$). $\square$

5. How many ways are there for six men and four women to be seated at a round table so that no two women sit next to each other?

Solution. If seats are not numbered, we first position the men (in $5!$ ways, since we can choose one of the man and always place him first) and then we have 6 spaces between them where the four women can be seated. Thus, the total number is $5! \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 43200$. If seats are numbered we need to multiply this number by 10 to take into account all possible rotations. $\square$

6. How many permutations of the string $abcdefgh$ contain

a) The string cd
b) The string dc
c) The string ade
d) The strings ac and eh?
e) The strings fed and cab?
f) The strings bed and cab?

Solution. **a)** Treat cd as a single letter. Then we are considering permutations of a string of $8-2+1=7$ letters, and there are $7!$ of them.

b) The number is the same as in the previous part. Indeed, let A be the set of permutations of $abcdefgh$ containing cd and B be the set of permutations of the same string containing dc. Then we have a function $f : A \to B$ which interchanges c and d. This function is manifestly bijective.

c) Like in previous part, we can treat ade as a single letter. Note that it does not matter to us that there is no such substring in the original string: $adebcfgh$ is also a permutation of the original string. Then we are looking for permutations of a string of $8-3+1=6$ letters, and there are $6!$ of them.

d) In this part, we treat ac as one letter and eh as another letter. Then we are counting permutations of a string of $8-2\cdot 2+2=6$ letters, and again there are $6!$ of them.

e) Similarly to the previous part, we treat fed and cab as single letters. This leaves us to permute $8-2\cdot 3+2=8-4=4$ strings and yields $4!$ permutations.

f) Observe that if a permutation of our string contains cab then it can contain bed too in one situation only: if it contains the substring $cabed$. Thus, if we treat $cabed$ as a single letter, we are left with $8-5+1=4$ letters to permute, and there are $4!=24$ permutations of them.

$\square$

7. How many bit strings of length 10 contain

a) Exactly three 0s?
b) At least three 0s?
c) At most three 0s?
d) More 0s than 1s?

Solution. Identifying the set of bit strings with the power set of the set of 10 elements, we conclude that the number of strings with exactly three 0s is the number of subsets of $10-3=7$ elements, which is $\binom{10}{7}=\binom{10}{3}=10\cdot 9\cdot 8/3!=10\cdot 3\cdot 4=120$.

Likewise, the number of strings with at least three 0s is the number of subsets with at most 7 elements which is $\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\binom{10}{3}+\binom{10}{4}+\binom{10}{5}+\binom{10}{6}+\binom{10}{7}=1+10+45+2\cdot\binom{10}{3}+2\binom{10}{4}+\binom{10}{5}$.

Since

$$\binom{10}{4}=\frac{10\cdot 9\cdot 8\cdot 7}{4!}=\frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2}=10\cdot 3\cdot 7=210$$

and

$$\binom{10}{5}=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2}=9\cdot 4\cdot 7=252$$

the number of bit strings with at least three zeros is 968.

The number of bit strings with at most three zeros is the number of subsets with at least 7 elements, which is $\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10}=\binom{10}{3}+\binom{10}{2}+\binom{10}{1}+\binom{10}{0}=120+45+10+1=176$.

If a bit string of length 10 has more zeros than 1s then its has at least 6 zeros, that is, at most 4 ones. So the number of such strings is $\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\binom{10}{3}+\binom{10}{4}=386$.

As a matter of fact, this number can be obtained without computing the sum. Since $\binom{10}{0}+\binom{10}{1}+\cdots+\binom{10}{10}=2^{10}$ and $\binom{10}{k}=\binom{10}{10-k}$, we have $2^{10}=2(\binom{10}{0}+\cdots+\binom{10}{4}))+\binom{10}{5})$, hence the sum we are interested in is equal to $(2^{10}-\binom{10}{5})/2=(1024-252)/2=386$. □

8. A club has 25 members.

a) How many ways are there to choose four members of the club to serve on an executive committee?
b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?
c) How many ways are there to choose a president, vice president, secretary and treasurer of the club if the same person can be the secretary and the treasurer?

Solution. For the first part, we are interested in the number of subsets of a set of 25 elements which contain exactly 4 elements, which is $\binom{25}{4}=25\cdot 24\cdot 23\cdot 22/4!=25\cdot 23\cdot 22=12,650$.

The second part is about counting the number of injective functions from the set $\{1,2,3,4\}$ to a set of 25 elements. This function assigns the President to 1, the Vice President to 2 and so on. There are $25\cdot 24\cdot 23\cdot 22=303,600$ such functions.

To answer the last question, note that we already found the number of ways to select four officials so that no person can hold more than one office. If the same person is the secretary and the treasurer then we are filling 3 positions and so consider the number of injective functions from $\{1,2,3\}$ to a set of 25 elements. There are $25\cdot 24\cdot 23$ of them. Thus, the total number is $25\cdot 24\cdot 23\cdot(22+1)=317,400$. □

9. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?
b) contain exactly two heads?
c) contain at most three tails?
d) contain the same number of heads and tails?

Idea of a solution. Encode the outcomes of each flipping by 0s (heads) and 1s (tails). Then each outcome is a bit sequence of length 10. $\square$

10. How many ways are there to select 12 countries in the United Nations to serve on a council if 3 are selected from a block of 45, 4 are selected from a block of 57, and the others are selected from the remaining 69 countries?

Solution. There are $\binom{45}{3}$ ways to choose 3 countries out of 45, $\binom{57}{4}$ ways to choose 4 countries out of 57 and $\binom{69}{5}$ ways to choose the remaining $12-3-4=5$ countries out of 69. So, the total number is

$$\binom{45}{3} \cdot \binom{57}{4} \cdot \binom{69}{5} = 62,994,022,035,644,700.$$

$\square$