Propositional functions

Example

$2 \leq 4$ (2 is less than or equal to 4) that is “$2 < 4$” $\vee$ “$2 = 4$”
Now, suppose that we want to analyze $x \leq 4$. This is *NOT* a proposition, since it can be true (if say $x = 0$) or false (if say $x = 5$).
However, for each specific value of x it *IS* a proposition.
This is an example of a *propositional function*. x is its variable and “≤ 4” is a *predicate*. If we denote the predicate by P then the propositional function is denoted by $P(x)$.
Thus, $P(1)$ is $1 \leq 4$ (which is true) and $P(8)$ is $8 \leq 4$ (which is false).
Moreover, we can now consider two propositional functions: $P_1(x)$ and $P_2(x)$ where P_1 is “< 4” and P_2 is “$= 4$”. Then $P(x) = P_1(x) \vee P_2(x)$.

Example

- "Computer C is under a DOS (Denial-of-Service) attack" is a propositional function $A(C)$ with predicate A ("is under a DOS attack") and variable C identifying the computer (for example, C can be the assigned name or the IP address).
- "$x + y < 2$" is a propositional function $P(x, y)$ with predicate P ("< 2") and variables x and y. Then $P(1, 0)$ and $P(0, 1)$ are both true and $P(1, 1)$ is false.
- Let $P(x, y)$ be "$x + y < 2$" and $Q(x, y)$ be "$x + y > 1$". Then $P(x, y) \wedge Q(x, y)$ is "$1 < x + y < 2$". If x and y are integers then it is always false; if x and y are rational then $P(1, 1/2)$ is true and $P(1, 1)$ is false. We can also think about this function as $(P \wedge Q)(x, y)$.
- Let $A(c, n)$ be the statement "Computer c is connected to the network n" (for example, we have three Wi-Fi networks available on campus, UCRWPA, Mobilenet and Eduroam). The predicate is "is connected to the network" and the variables are c (some unique identifier of the computer) and n (a unique identifier of the network).

Example

- Let $A(c, n, r)$ be the statement "Computer c is connected to the network n via the router r". This is an example of a propositional function depending on three variables.
- Let $P(x, y, z)$ be $x + y \leq z$. Then the truth value of $P(1, 2, 3)$ is T and the truth value of $P(3, 2, 1)$ is F.

A propositional function can depend on any number of variables, just like any other kind of function.

The value of a propositional function for some chosen x in its domain is a *proposition*, not T or F.

However, by abuse of notation, we often write $P(1, 2, 3) = T$ as a shortcut for "the truth value of $P(1, 2, 3)$ is T".

Preconditions and postconditions

Example

Consider the following pseudo-code

```
temp:=x
x:=y
y:=temp
```

This pseudo-code implements a (very simple) algorithm for interchanging the values of variables x and y. How do we know that it actually works? Let $P(x, y)$ be "$x = a$ and $y = b$" and let $Q(x, y)$ be "$x = b$ and $y = a$". We want to make sure that if $P(x, y)$ is true then $Q(x, y)$ is true. That means that our algorithm works.

Suppose that $P(x, y)$ is true. Then $x = a$ is true and $y = b$ is true. After the first assignment, $temp = x$ is true which means that $temp = a$ is true. After the second assignment $x = y$ is true which means that $x = b$ is true. After the last assignment $y = temp$ is true which means $y = a$ is true. Thus, $Q(x, y)$ is true.

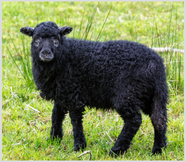
Quantifiers

We can make a proposition out of a propositional function by assigning values to its variables.

Another way to produce propositions is to use *quantifiers*. In English we use words such as *all*, *some*, *any*, *none*, *many*, *few* as quantifiers (like in “All dogs have four legs”, “Some horses are bay”)

Example

An astronomer, an engineer and a mathematician are on a train in Scotland. The astronomer looks out of the window, sees a black sheep grazing in a meadow, and remarks, "How odd. *All* the sheep in Scotland are black!" "No, no, no!" protests the engineer. "Only *some* Scottish sheep are black." The mathematician rolls his eyes and says, "In Scotland, there is at least one meadow, where there is at least one sheep, at least one side of which appears to be black from here some of the time."



The universal quantifier

Definition

The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain of discourse" (or simply "in the domain"). The notation $\forall x\, P(x)$ denotes the universal quantification of $P(x)$.
$\forall$ is called the *universal quantifier.*
We read $\forall x P(x)$ as "for all x, $P(x)$" or "for every x,$P(x)$." A value of x for which $P(x)$ is false is called a *counterexample* of $\forall x P(x)$.

Example

- Let $P(x)$ be $x^2 \geq 0$. Then $\forall x\, P(x)$ is true for all integer x (and also for all rational x and even real x).
- Let $Q(x)$ be $x < 2$. Then $\forall x\, Q(x)$ is false if the domain is "all integer x" or "all real x" since 3 is a counterexample.

Example

- Let $R(x)$ be the statement "odd integers are prime numbers" (more formally, "if x is an odd integer then x is a prime number"). If the domain is "all positive integers", $\forall x\, R(x)$ is false since 9 is a counterexample ($9 = 3 \times 3$ is not a prime number). However, if the domain is "all positive integers greater than 1 and less than 9" then $\forall x\, R(x)$ is true since the only odd integers in that domain - 3, 5 and 7 - are prime numbers.

- The truth value of $\forall x\, P(x)$ depends on the domain
- If the domain is empty, $\forall x\, P(x)$ is true (since there is no x for which $P(x)$ is false).
- A single counterexample is enough to establish that $\forall x\, P(x)$ is false. Usually we cannot establish that $\forall x\, P(x)$ is true by just checking it for some collection of x (it is only possible if there are very few x in the domain).
- If the domain is a *finite list* (say, $x_1, x_2, \ldots, x_n$) then $\forall x\, P(x)$ is equivalent to $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$. In our previous example, if the domain was $\{2, 3, 4, 5, 6, 7, 8\}$ then $\forall x\, R(x)$ becomes $R(2) \wedge R(3) \wedge \cdots \wedge R(8)$ or even $R(3) \wedge R(5) \wedge R(7)$.

Example

- Consider $\forall x\,(x^2 \geq x)$ if x is a real number. This inequality holds unless $0 < x < 1$ (for example, $\left(\frac{1}{3}\right)^2 < \frac{1}{3}$). Thus, this statement is false. However, if x is an integer then the inequality always holds since there are no integers x satisfying $0 < x < 1$.
- Let $N(c)$ be "Computer c is connected to the Internet" where c is a computer on campus. Then $\forall c, N(c)$ means "All computers on campus are connected to the Internet". To check the validity of this statement we have to verify that each and every computer on campus is indeed connected to the Internet.

The existential quantifier

Often we want to state that it is possible to find at least one object with a certain property ("there is at least one meadow in Scotland...")

Definition

The *existential quantification* of $P(x)$ is the proposition "There exists x in the domain such that $P(x)$." We use the notation $\exists x\, P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

Like with $\forall$, we always need to specify the domain when we use $\exists$. Otherwise it has no meaning.

Other forms of $\exists$: "There is at least one x such that $P(x)$", "There is an x such that $P(x)$" or "For some x, $P(x)$".

Example

- $\exists x(x > 3)$ if x is an integer is true since $4 > 3$. $\exists x(x^2 < 0)$ if x is an integer is false since $x^2 \geq 0$ for all integer x.
- $\exists x(x = x + 1)$ is false if x is an integer (or a real number)
- Let $U(c)$ be "The operating system on c is being upgraded" where the domain is "all computers on campus". Then $\exists c, U(c)$ means "There is a computer on campus whose operating system is being upgraded".
- Let $B(s)$ be "s is a black sheep" there the domain is "all sheep in Scotland". Then $\exists s, B(s)$ means "There is at least one black sheep in Scotland".

If we can list all x in the domain (say, as $x_1, x_2, \ldots, x_n$) then $\exists x, P(x)$ is the same as

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n).$$

Example

What is the truth value of $\exists x, P(x)$ if $P(x)$ is "$x^2 > 11$" and the domain is $x = 1, 2, 3, 4$?
We have $P(1) = F$, $P(2) = F$, $P(3) = F$ and $P(4) = T$. Then $P(1) \vee P(2) \vee P(3) \vee P(4) = T$ and so $\exists x, P(x)$ is true.

Uniqueness quantifier

Definition

The notation $\exists! x, P(x)$ means that there exists precisely one x in the domain such that $P(x)$ is true.

Example

$\exists! x(2x + 3 = 0)$ where x is a real number means that the equation $2x + 3 = 0$ has a unique solution.

Relationship between $\forall$ and $\exists$

What is the meaning of "$\forall x, P(x)$ is false"? It means that there is at least one x such that $P(x)$ is false, which can be stated as "$\exists x, \neg P(x)$".

What is the meaning of "$\exists x, P(x)$ is false"? That for all x, $P(x)$ is false, which is $\forall x, \neg P(x)$.

Example

- Let $P(s)$ be "s has taken a course in calculus" where the domain is "all students in our class". Then $\forall s, P(s)$ means "All students in our class have taken a course in calculus". This is false if there is a student in our class who has not taken a course in calculus.
- Let $H(x)$ be "x is honest" where the domain is "all politicians". Then $\exists x, H(x)$ means "There is an honest politician". This is false if all politicians are dishonest.

Example

Consider the statement "All students in this class have taken a course in calculus". We already saw that it is easy to turn it into a conditional function with the domain "All students in this class". But this might be too restrictive. Another way to make this work is to introduce two functions: $S(x)$ which is "x is a student in this class" and $C(x)$: "x has taken a course in calculus". The domain now can be "all students at UCR" or even "all people". Our proposition can be expressed as $\forall x, S(x) \to C(x)$.

$\forall x(S(x) \wedge C(x))$ is wrong, since it says that all people are students in this class and have studied calculus.

Bound and free variables

Definition

A variable which is quantified is called *bound*. A variable which is not bound is called *free*.

For example, if $Q(x, y)$ is a propositional function then in the expression $\forall x, Q(x, y)$, x is a bound variable and y is a free variable.

Nested quantifiers

If a propositional function has more than one variable we can use more than one quantifier.

Example

- Let x, y be integers. Then $\forall x, \exists y, (x + y = 0)$ means that for every integer x, we can find another integer y such that their sum is zero
- Let x, y, z be integers. Then $\forall x \forall y \forall z, ((x + y) + z = x + (y + z))$ means that the addition of integers is associative
- Consider the statement "No one has climbed every mountain in the Himalayas".
  Let $C(x, m)$ be the statement "x climbed m" where the domain for x is "all people" and the domain for m is "all mountains". Let $H(m)$ be the statement "m is in the Himalayas", the domain for m being "all mountains". Then our statement becomes $\forall x \exists m (H(m) \wedge \neg C(x, m))$.

Example

Let $a \leq b$ be real numbers and let f be a function defined on the interval (a, b) (except, may be, at $x = x_0$) which takes real values. Do you recognize the following statement?

$\forall(\epsilon > 0)\exists(\delta > 0)\forall x((a < x < b) \wedge (0 < |x - x_0| < \delta) \rightarrow f(x) > \epsilon)$

This is the definition of "the limit of f as x approached x_0 is $+\infty$"

Equivalence in predicate calculus

Definition

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example

We claim that $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$ if the same domain is used.

Indeed, $\forall x(P(x) \wedge Q(x))$ is true if and only if for all a in the domain $P(a) \wedge Q(a)$ is true which happens if and only if both $P(a)$ and $Q(a)$ are true. This means $\forall x P(x)$ and $\forall x Q(x)$ are both true.

$\forall x(P(x) \wedge Q(x))$ is false if and only if there is a in the domain such that $P(a) \wedge Q(a)$ is false. Thus, one of $\forall x P(x)$, $\forall x Q(x)$ is false.

De Morgan's law for quantifiers

Theorem

$\neg(\forall x P(x)) \equiv \exists x \neg P(x)$
$\neg(\exists x P(x)) \equiv \forall x \neg P(x)$

We will explain why the first equivalence holds.

- Suppose that $\exists x \neg P(x)$ is true.
  Then for some a in the domain $\neg P(a)$ is true that is $P(a)$ is false.
  But then $\forall x P(x)$ is false (a is a counterexample) which means that $\neg(\forall x P(x))$ is true.
- Suppose that $\exists x \neg P(x)$ is false.
  Then for all a in the domain $\neg P(a)$ is false
  Thus, $P(a)$ is true for all a in the domain, that is $\forall x P(x)$ is true.
  This means that $\neg(\forall x P(x))$ is false

The second equivalence can be deduced from the first.

Let $Q = \neg P$. Then we have $\neg(\forall x, Q(x)) \equiv \exists x, \neg Q(x)$.

Applying $\neg$ to both sides we obtain $\forall x, Q(x) \equiv \neg(\exists x, \neg Q(x)$.

Thus, $\forall x, \neg P(x) \equiv \neg(\exists x, \neg(\neg P(x))) \equiv \neg(\exists x, P(x))$.

Rules of inference for quantified statements

Universal instantiation	$\forall x, P(x) \therefore P(c)$
Universal generalization	$P(c)$ for arbitrary $c \therefore \forall x, P(x)$
Existential instantiation	$\exists x, P(x) \therefore P(c)$ for some c
Existential generalization	$P(c)$ for some $c \therefore \exists x, P(x)$

Example

Let the premises be "All humans are mortal", "Socrates is a human". The domain is "All humans". The propositional function is $M(x)$: "x is mortal". Since $\forall x, M(x)$, universal instantiation implies $M(\text{Socrates})$, that is, "Socrates is mortal".

Example

Is the following argument valid: "Every computer science major takes a course in discrete mathematics. Jane takes a course in discrete mathematics, therefore, Jane is a computer science major".