HOMEWORK SET 4. MATHEMATICAL INDUCTION. SETS AND SET OPERATIONS

1. Prove that the product of two consecutive integers is even.

Solution. Observe first that if a and b are integers and a is even then ab is even. Indeed, since a is even, $a = 2m$ for some integer m and so $ab = 2(mb)$. Thus, ab is an integer multiple of 2 and hence is even.

Let k be an integer. If k is even then $k(k+1)$ is even by the above. Otherwise, k is odd and hence can be written as $2m+1$ for some integer m. Then $k+1 = 2m+2 = 2(m+1)$ is even. Then again $k(k+1)$ is a product of an even integer and an integer and hence is even. $\square$

2. Prove that every odd integer can be written as a sum of two consecutive integers.

Solution. Let n be odd. Then $n = 2k+1$ for some integer k and so $n = k + (k+1)$. Thus, n is a sum of two consecutive integers. $\square$

3. Define a sequence f_n, $n \geq 0$ by $f_0 = 1$ and $f_n = 1 + \frac{1}{f_{n-1}}$, $n \geq 1$. Prove that $f_n = \dfrac{F_{n+2}}{F_{n+1}}$ for all $n \in \mathbb{N}$ where F_n is the nth Fibonacci number ($F_0 = 0$, $F_1 = 1$ etc)

Remark. The sequence f_n provides approximations for the so called *golden ratio*, $(1+\sqrt{5})/2$. For example, f_{100} approximates the golden ratio with the error less than 10^{-41}.

Solution. The induction base holds since $f_0 = 1$ and $F_2 = 1 = F_1$. For the inductive step we have

$$\begin{aligned} f_{n+1} &= 1 + \frac{1}{f_n} \\ &= 1 + \frac{F_{n+1}}{F_{n+2}} \\ &= \frac{F_{n+1} + F_{n+2}}{F_{n+2}} \\ &= \frac{F_{n+3}}{F_{n+2}}. \end{aligned}$$

Here we used the recursive definition of Fibonacci numbers: $F_{n+1} + F_{n+2} = F_{n+3}$. $\square$

4. Let n be an integer. Use induction to show that $2n < 2^n - 1$ for $n \geq 3$ and that $n^2 < 2^n$ for all $n \geq 5$.

Solution. The induction base is clear: $2 \cdot 3 = 6 < 7 = 2^3 - 1$ and $5^2 = 25 < 32 = 2^5$.

For the inductive step, we have, for $n \geq 3$,

$$\begin{aligned} 2(n+1) &= 2n + 2 \\ &< (2^n - 1) + 2 \\ &= 2^n + 1 \\ &< 2^n + 2^n - 1 \\ &= 2^{n+1} - 1. \end{aligned}$$

Then we also have, for $n \geq 5$, $(n+1)^2 = n^2 + 2n + 1 < 2^n + 2^n = 2^{n+1}$ (here we used the first inequality which yields $2n + 1 < 2^n$). $\square$

5. Use induction to prove that $1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all non-negative integers n.

Solution. For $n = 0$ the sum and the expression in the right hand side are both zero.

For the inductive step, assume that the equality holds for n. Then

$$\begin{aligned}
1^2 + 2^2 + \cdots + (n+1)^2 &= 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 \\
&= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 && \text{by the induction hypothesis} \\
&= \frac{1}{6}(n+1)(n(2n+1) + 6(n+1)) \\
&= \frac{1}{6}(n+1)(2n^2 + 7n + 6) \\
&= \frac{1}{6}(n+1)(n+2)(2n+3).
\end{aligned}$$

Thus, the equality holds for $n + 1$. This proves the inductive step. □

6. For each of the following sets, determine whether 2 is its element. Do the same for $\{2\}$.

a) $\{x \in \mathbb{Z} : x > 1\}$
b) $\{x \in \mathbb{Z} \mid x = y^2 \text{ for some } y \in \mathbb{Z}\}$
c) $\{2, \{2\}\}$
d) $\{\{2\}, \{\{2\}\}\}$
e) $\{\{2\}, \{2, \{2\}\}\}$
f) $\{\{\{2\}\}\}$

Answers. **a)** Yes; No
b) No (2 is not a square of an integer); No
c) Yes; Yes
d) No; Yes
e) No; Yes
f) No; No

□

7. For each of the following statements determine whether it is true or false.

a) $0 \in \emptyset$
b) $\emptyset \in \{0\}$
c) $\{0\} \subset \emptyset$
d) $\emptyset \subset \{0\}$
e) $\{0\} \in \{0\}$
f) $\{0\} \subset \{0\}$
g) $\{\emptyset\} \subset \{\emptyset\}$
h) $\emptyset \in \{\emptyset\}$
i) $\emptyset \in \{\emptyset, \{\emptyset\}\}$
j) $\{\emptyset\} \in \{\emptyset\}$
k) $\{\emptyset\} \in \{\{\emptyset\}\}$
l) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
m) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
n) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Answers. **a)** False
b) False
c) False
d) True ($\emptyset$ is a subset of every set)
e) False
f) True
g) True
h) True
i) True
j) False
k) True
l) True
m) True
n) True

□

8. Suppose that A is the set of sophomores at UCR and B is the set of students in discrete mathematics at UCR (we can take $\mathbb{U}$ to be the set of all undergraduate students at UCR). Use set operations to express each of the following sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in UCR
b) the set of sophomores at UCR who are not taking discrete mathematics
c) the set of students at UCR who either are sophomores or are taking discrete mathematics
d) the set of students at UCR who either are not sophomores or are not taking discrete mathematics
e) the set of students at UCR who are not sophomores and take discrete mathematics.

Answers.

a) $A \cap B$
b) $A \setminus B$
c) $A \cup B$
d) $\overline{A} \cup \overline{B}$
e) $\overline{A} \cap B = B \setminus A$

$\square$

9. Let $A = \{a, c, e, g, i, k\}$, $B = \{a, b, c, d, e, f, g\}$ and $C = \{e, f, g, h, i, j, k\}$. Find

a) $A \cap B \cap C$
b) $A \cup B \cup C$
c) $(A \cup B) \cap C$.
d) $(A \cap B) \cup C$
e) $(A \setminus B) \cup (B \setminus C)$
f) $(A \cup B) \setminus C$

Answers. **a)** $\{e, g\}$
b) $\{a, b, c, d, e, f, g, h, i, j, k\}$
c) $\{e, f, g, i, k\}$
d) $\{a, c, e, f, g, h, i, j, k\}$
e) $\{a, b, c, d, i, k\}$
f) $\{a, b, c, d\}$

$\square$

10. Let A, B, C be sets. Show that
a) $A \setminus B = A \cap \overline{B}$
b) $A \setminus \emptyset = A$ and $\emptyset \setminus A = \emptyset$
c) $(B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A$
d) $(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$
Draw the corresponding Venn diagrams.

Solution. **a)** Let p be the statement "$x \in A$" and q be the statement "$x \in B$". Then $x \in A \setminus B$ is $p \wedge \neg q$ which is exactly the same as $x \in A \cap \overline{B}$.

Alternatively, suppose that $x \in A \setminus B$. Then $x \in A$ and $x \notin B$ which means that $x \in A \cap \overline{B}$. On the other hand, if $x \in A \cap \overline{B}$ then $x \in A$ and $x \notin B$ which means that $x \in A \setminus B$.

b) By the previous part, $A \setminus \emptyset = A \cap \overline{\emptyset} = A \cap \mathbb{U} = A$. Similarly, $\emptyset \setminus A = \emptyset \cap \overline{A} = \emptyset$.

c) Using the first part we can write

$$\begin{aligned}(B \setminus A) \cup (C \setminus A) &= (B \cap \overline{A}) \cup (C \cap \overline{A}) \\ &= (B \cup C) \cap \overline{A} \\ &= (B \cup C) \setminus A.\end{aligned}$$

d) Using the first part we obtain

$$\begin{aligned}(A \setminus C) \setminus (B \setminus C) &= (A \cap \overline{C}) \cap \overline{B \cap \overline{C}} \\ &= (A \cap \overline{C}) \cap (\overline{B} \cup C) \\ &= (A \cap (\overline{B} \cup C)) \cap \overline{C} \\ &= ((A \cap \overline{B}) \cup (A \cap C)) \cap \overline{C} \\ &= ((A \setminus B) \setminus C) \cup (A \cap C \cap \overline{C}) \\ &= ((A \setminus B) \setminus C) \cup (A \cap \emptyset) \\ &= ((A \setminus B) \setminus C) \cup \emptyset \\ &= (A \setminus B) \setminus C.\end{aligned}$$

$\square$

11. What can we conclude about sets A and B if

a) $A \cup B = A$
b) $A \cap B = A$
c) $A \setminus B = A$
d) $A \setminus B = B \setminus A$
e) $(A \setminus B) \cup (B \setminus A) = A$?

Solution. **a)** We always have $A \subset A \cup B$. The equality means that also $A \cup B \subset A$. This implies that $B \subset A$. Indeed, if $x \in B$ then $x \in A \cup B$ and so $x \in A$.

b) We always have $A \cap B \subset A$. The equality means that also $A \subset A \cap B$. Thus, if $x \in A$ then $x \in B$ that is $A \subset B$

c) $A \setminus B = A \cap \overline{B} = A$. By the previous part this means that $A \subset \overline{B}$.

d) Let $x \in A \setminus B$. This implies that $x \in A$ and $x \notin B$. Since $A \setminus B = B \setminus A$, we also have that $x \in B \setminus A$, which means that $x \in B$ and $x \notin A$. Therefore, $A = B = \emptyset$.

e) We can analyze this equality using logic. Let $P(x)$ be $x \in A$ and $Q(x)$ be $x \in B$. Then $A \triangle B = A$ means that $\forall x(P(x) \oplus Q(x) \leftrightarrow P(x))$. But if $Q(x)$ is true for some x then $P(x) \oplus Q(x)$ and $P(x)$ have the opposite truth values. So $\exists x Q(x)$ is false that is $B = \emptyset$.

We can also argue using elements. Suppose that there exits $x \in B$. If $x \notin A$ then $x \in A \triangle B$. Since $A \triangle B = A$ this leads to the conclusion that $x \in A$, which is a contradiction. Thus, if $x \in B$ then $x \in A$. Then $x \notin A \triangle B = A$. So, this is also impossible. Thus, our assumption $x \in B$ is always false and so $B = \emptyset$. □

12. Prove the De Morgan's law for sets $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by showing that each side is a subset of the other side.

Solution. Suppose that $x \in \overline{A \cap B}$. Then $x \notin A \cap B$ which means that $x \notin A$ or $x \notin B$. Therefore, $x \in \overline{A} \cup \overline{B}$.

Conversely, if $x \in \overline{A} \cup \overline{B}$ then $x \notin A$ or $x \notin B$. Then $x \notin A \cap B$. □

13. Is it true that the set difference is associative (that is, if A, B, C are sets then $(A \setminus B) \setminus C = A \setminus (B \setminus C)$)? What about the symmetric difference?

Solution. The first set is $A \cap \overline{B} \cap \overline{C}$. The second set is $A \cap \overline{B \cap \overline{C}} = A \cap (\overline{B} \cup C)$. For this to be always true we need $\overline{B} \cap \overline{C}$ to be equal to $\overline{B} \cup C$. But this is manifestly false.

The easiest way to study the associativity of $\triangle$ is to determine whether *XOR* is associative. For, note that $F \oplus p \equiv p$ while $T \oplus p \equiv \neg p$. Thus, if $p = T$ then $(p \oplus q) \oplus r \equiv \neg q \oplus r$, while $p \oplus (q \oplus r) \equiv \neg(q \oplus r) \equiv \neg q \oplus r$. If $p = F$ then $(p \oplus q) \oplus r \equiv q \oplus r \equiv p \oplus (q \oplus r)$. Thus, $\oplus$ is associative and so $\triangle$ is associative. □