HOMEWORK SET 3. PREDICATES AND QUANTIFIERS

1. Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.

a) $\exists x P(x)$
b) $\forall x P(x)$
c) $\exists x \neg P(x)$
d) $\forall x \neg P(x)$

Answers. **a)** Some student spends more than five hours every weekday in class
b) All students spend more than five hours every weekday in class
c) Some student spends no more than five hours every week day in class
d) All students spend no more than five hours every week day in class

$\square$

2. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

a) $\forall x(C(x) \rightarrow F(x))$
b) $\forall x(C(x) \wedge F(x))$
c) $\exists x(C(x) \rightarrow F(x))$
d) $\exists x(C(x) \wedge F(x))$

Answers. **a)** All comedians are funny
b) Every person is a comedian and is funny
c) Some people are not comedians or are funny
d) There is a person who is a comedian and is funny. $\square$

3. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

a) $\forall n(n^2 \geq 0)$
b) $\exists n(n^2 = 2)$
c) $\forall n(n^2 \geq n)$
d) $\exists n(n^2 < 0)$

Answers. **a)** True
b) False
c) True
d) False $\square$

4. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ **b)** $\forall x P(x)$ **c)** $\exists x \neg P(x)$ **d)** $\forall x \neg P(x)$ **e)** $\neg \exists x P(x)$ **f)** $\neg \forall x P(x)$

Answers. **a)** $P(1) \vee P(2) \vee P(3) \vee P(4)$
b) $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$
c) $\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$
d) $\neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4)$
e) $\neg(P(1) \vee P(2) \vee P(3) \vee P(4)) \equiv \neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4)$
f) $\neg(P(1) \wedge P(2) \wedge P(3) \wedge P(4)) \equiv \neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$ $\square$

5. Express each of these statements using logical operators, predicates, and quantifiers.

a) Some propositions are tautologies.
b) The negation of a contradiction is a tautology.
c) The disjunction of two contingencies can be a tautology.
d) The conjunction of two tautologies is a tautology

Solution. Let $C(p)$ be "p is a contradiction" and $T(p)$ be "p is a tautology" (the domain is "propositions").

a) This is $\exists p T(p)$.
b) This can be rephrased as "If p is a contradition, then $\neg p$ is a tautology", which is supposed to hold for all p. Thus, we can write this as $\forall p(C(p) \to T(\neg p))$.
c) To make this answer shorter we introduce $N(p)$ which is "p is a contingency". This function is redundant since $N(p) \equiv \neg(C(p) \vee T(p))$. We are stating that there exist two contingencies, p and q, such that $p \vee q$ is a tautology, that is $\exists p \exists q(N(p) \wedge N(q) \wedge T(p \vee q))$. If we want to get rid of N, the answer becomes $\exists p \exists q(\neg C(p) \wedge \neg T(p) \wedge \neg C(q) \wedge \neg T(q) \wedge T(p \vee q))$.
d) Similarly to the previous one, we obtain $\forall p \forall q(T(p) \wedge T(q) \to T(p \wedge q))$. $\square$

Remark. If we want to be really strict about the usage of predicates we will not be allowed to use the fact that logical operators can be applied in our domain. Then we will have to introduce more predicates, say $O(p, q)$ which reads "q is the negation of q", $A(p, q, r)$ which is "$r \equiv p \wedge q$" and $D(p, q, r)$ which is "$r \equiv p \vee q$". Then the second statement becomes $\forall p \forall q(C(p) \wedge O(p, q)) \to T(q)$, the third statement becomes $\exists p \exists q \exists r(N(p) \wedge N(q) \wedge D(p, q, r) \wedge T(r)$, and the last statement becomes $\forall p \forall q \forall r(T(p) \wedge T(q) \wedge A(p, q, r)) \to T(r)$.

6. Show that $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ are logically equivalent. Are $\exists x(P(x) \wedge Q(x))$ and $\exists x P(x) \wedge \exists x Q(x)$ logically equivalent?

Solution. It is sufficient to prove that if $\exists x(P(x) \vee Q(x))$ is true (false) then $\exists x P(x) \vee \exists x Q(x)$ is true (false).

If $\exists x(P(x) \vee Q(x))$ is true then there is a in the domain such that at least one of $P(a)$ and $Q(a)$ is true. But then either $\exists x P(x)$ or $\exists x Q(x)$ is true.

If $\exists x(P(x) \vee Q(x))$ is false then for all a in the domain both $P(a)$ and $Q(a)$ are false. But then $\exists x P(x)$ and $\exists x Q(x)$ are both false and therefore their disjunction is also false.

We claim that $\exists x(P(x) \wedge Q(x))$ and $\exists xP(x) \wedge \exists xQ(x)$ are not logically equivalent. Indeed, for the second to be true it suffices to have a in the domain such that $P(a)$ is true and b in the domain such that $Q(b)$ is true. For $\exists x(P(x) \wedge Q(x))$ to be true we need c in the domain such that *both* $P(c)$ and $Q(c)$ are true. □

7. (Lewis Carroll). What conclusion can be deduced from the following premises?

- No ducks are willing to waltz.
- No officers ever decline to waltz.
- All my poultry are ducks.

Solution. Let the domain be "my poultry". Let $W(x)$ be "x is willing to waltz", $O(x)$ be "x is an officer" and $D(x)$ be "x is a duck". Then the premises are $\forall x(D(x) \to \neg W(x))$, $\forall x(O(x) \to W(x))$ and $\forall xD(x))$.

The law of detachment yield $\forall xD(x)), \forall x(D(x) \to \neg W(x)) \therefore \forall x \neg W(x)$. Then modus tollens gives $\forall x(O(x) \to W(x)), \forall x \neg W(x) \therefore \forall x \neg O(x)$. Thus, the conclusion is $\forall x \neg O(x)$ which is equivalent to $\neg(\exists xO(x))$. Thus, "there are no officers among my poultry". □

8. Let $I(x)$ be the statement "x has an Internet connection" and $C(x, y)$ be the statement "x and y have chatted over the Internet," where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.

a) Someone in your class does not have an Internet connection.
b) Not everyone in your class has an Internet connection.
c) Everyone except one student in your class has an Internet connection.
d) Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.
e) Someone in your class has an Internet connection but has not chatted with anyone else in your class.
f) There are two students in your class who have not chatted with each other over the Internet.
g) There is a student in your class who has chatted with everyone in your class over the Internet.
h) There are at least two students in your class who have not chatted with the same person in your class.
i) There are two students in the class who between them have chatted with everyone else in the class.

Answers. **a)** $\exists x \neg I(x)$
b) $\neg(\forall xI(x)) \equiv \exists x \neg I(x)$
c) $\exists ! x \neg I(x)$
d) $\forall x(I(x) \to \exists y((y \neq x) \wedge C(x, y)) \equiv \forall x \exists (y \neq x)(I(x) \to C(x, y))$
e) $\exists x(I(x) \wedge (\forall (y \neq x) \neg C(x, y)) \equiv \exists x \forall (y \neq x)(I(x) \wedge \neg C(x, y))$
f) $\exists x \exists (y \neq x) \neg C(x, y)$
g) $\exists x \forall y C(x, y)$
h) $\exists x \exists (y \neq x) \forall z \neg (C(x, z) \wedge C(y, z))$

i) $\exists x \exists (y \neq x) \forall z C(x,z) \vee C(y,z)$

In some cases we need to specify that $y \neq x$. That is, we restrict the domain for the second quantifier to "all students in the class except x". For example, when we claim that there are at least two students such that... we need to make sure that x and y do not coincide. However, in the last question one may argue whether $y \neq x$ is necessary. □

9. Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) When there is less than 30 megabytes free on the hard disk, a warning message is sent to all users.

b) No directories in the file system can be opened and no files can be closed when system errors have been detected.

c) The file system cannot be backed up if there is a user currently logged on.

d) Video on demand can be delivered when there are at least 8 megabytes of memory available and the connection speed is at least 56 kilobits per second

Solution. **a)** Let $F(m)$ be "there is less than m MB free on the hard drive", and let $W(u)$ be "send a warning message to u" Then the specification is $F(30) \to \forall u W(u)$.

b) Let $E(e)$ be "the system error e is detected", $D(d)$ be "the directory d can be opened" and $F(f)$ be "the file f can be closed". Then the specification is $\forall e(E(e) \to \forall d \neg D(d) \wedge \forall f \neg F(f)$. Since $\forall d \neg D(d) \equiv \neg(\exists d D(d))$ and similarly $\forall f \neg F(f) \equiv \neg(\exists f F(f))$, we can replace $\forall d \neg D(d) \wedge \forall f \neg F(f)$ by $\neg(\exists d D(d)) \wedge \neg(\exists f F(f)) \equiv \neg(\exists d D(d) \vee \exists f F(f))$. Replacing the conditional statement by its contrapositive we can also write

$$\forall e \exists d \exists f (D(d) \vee F(f) \to \neg E(e)).$$

c) Let f be "the file system can be backed up" and $L(u)$ be "user u is logged on". Then the specification is $(\exists u L(u) \to \neg f$.

d) Let $A(m)$ be "at least m MB of memory are available" and $S(k)$ be "the connection speed is at least k Kbps". Let v be "video on demand can be delivered". Then the specification is $A(8) \wedge S(56) \to v$. □