HOMEWORK SET 1. SOLUTIONS OF SELECTED PROBLEMS.

1. Let p and q be the propositions "The election is decided" and "The votes have been counted", respectively. Express each of the following compound propositions as an English sentence.

a) $\neg p$	**c)** $\neg p \wedge q$	**e)** $\neg q \to \neg p$	**g)** $p \leftrightarrow q$
b) $p \vee q$	**d)** $q \to p$	**f)** $\neg p \to \neg q$	**h)** $\neg q \vee (\neg p \wedge q)$

Answers. **a)** The election is not decided;
b) The election is decided or the votes have been counted;
c) The election is not decided and (but) the votes have been counted;
d) If the votes have been counted then the election is decided;
e) If the votes have not been counted then the election is not decided;
f) If the election is not decided then the votes have not been counted;
g) The election is decided if and only if the votes have been counted;
h) The votes have not been counted or the election is not decided and the votes have been counted.

□

2. Let p and q be the propositions
p : You drive over 65 miles per hour.
q : You get a speeding ticket.
Write the following propositions using p and q and operations $\neg$, $\wedge$ and $\vee$:
a) You drive over 65 miles per hour, but you do not get a speeding ticket.
b) You will get a speeding ticket if you drive over 65 miles per hour.
c) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.
d) Driving over 65 miles per hour is sufficient for getting a speeding ticket.
e) You get a speeding ticket, but you do not drive over 65 miles per hour.

Answers. **a)** $p \wedge \neg q$;
b) $p \to q$
c) $\neg p \to \neg q$;
d) $p \to q$;
e) $q \wedge \neg p$.

□

3. State the converse, contrapositive, and inverse of each of these conditional statements.
a) If it snows today, I will ski tomorrow.
b) I come to class whenever there is going to be a quiz.
c) A positive integer is a prime only if it has no divisors other than 1 and itself.

Answers. The order is: converse, contrapositive, inverse.

a) If I ski tomorrow then it snows today; If I do not ski tomorrow it does not snow today; If it does not snow today, I won't ski tomorrow.

b) If I come to class then there is going to be a quiz; If I do not come to class then there is no quiz; If there is no quiz then I do not come to class.

c) A positive integer that has no divisors other than 1 and itself is a prime; A positive integer that has divisors other than 1 and itself is not a prime; A positive integer that is not a prime has divisors other than 1 and itself.

□

4. Construct a truth table for each of these compound propositions.

a) $p \oplus (p \vee q)$
b) $(p \wedge q) \rightarrow (p \vee q)$
c) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$
d) $(p \vee q) \rightarrow (p \oplus q)$
e) $(p \vee q) \oplus (p \wedge q)$
f) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
g) $(p \oplus q) \wedge (p \oplus \neg q)$
h) $(p \vee q) \wedge \neg r$

Answers.

p	q	(a)	(b)	(c)	(d)	(e)	(g)
T	T	F	T	F	F	F	F
T	F	F	T	F	T	T	F
F	T	T	T	F	T	T	F
F	F	F	T	T	T	F	F

p	q	r	(f)	(h)
T	T	T	F	F
T	T	F	T	T
T	F	T	T	F
T	F	F	F	T
F	T	T	F	F
F	T	F	T	T
F	F	T	T	F
F	F	F	F	F

□

5. Evaluate each of the following expressions.

a) $11000 \wedge (01011 \vee 11011)$
b) $(01111 \wedge 10101) \vee 01000$
c) $(01010 \oplus 11011) \oplus 01000$
d) $(11011 \vee 01010) \wedge (10001 \vee 11011)$

Answers. **a)** 11000
b) 01101
c) 11001
d) 11011

□

6. Are these system specifications consistent? "If the file system is not locked, then new messages will be queued. If the file system is not locked, then the system is functioning normally, and conversely. If new messages are not queued, then they will be sent to the message buffer. If the file system is not locked, then new messages will be sent to the message buffer. New messages will not be sent to the message buffer."

Solution. Let f be "the file system is locked", n be "the system is functioning normally", q be "new messages are queued", b be "new messages are sent to the message buffer". Then the specification is

$$(\neg f \to q) \wedge (\neg f \leftrightarrow n) \wedge (\neg q \to b) \wedge (\neg f \to b) \wedge \neg b.$$

For this to be true we must have $\neg b$ true that is, b is false. Then $\neg f \to b$ and $\neg q \to b$ can only be true if $\neg f$ and $\neg q$ are false, that is, f and q are true. $\neg f \to q$ is then true, and n is false. So, the specifications are consistent in the following way: "The file system is locked. The system is not function normally. New messages are queued and are not sent to the message buffer". □

7. An island is populated by knights (who always tell the truth) and knaves (who always lie). You encounter two people, A and B. Determine, if possible, what A and B are if they address you in the ways described. If you cannot determine what these two people are, can you draw any conclusions?

a) A says "At least one of us is a knave" and B says nothing.
b) A says "We are both knights" and B says "A is a knave."
c) A says "I am a knave or B is a knight" and B says nothing.
d) A says "We are both knaves" and B says nothing.

Solution. We denote by a the proposition of A and by b the proposition of B (if any).

a) If a is true then A is a knight (because A tells the truth) and B is a knave (because a is true, one of them must be a knave, and we already know that A isn't a knave).
If a is false then A is a knave. But then a is true (because one of them is a knave), which is a contradiction. Thus, the only solution is the one we already found.

b) If a is true then both B and A are knights and b is true and false at the same time (b must be true because B is a knight, but since B is calling another knight a knave b is also false). Thus, a is false and A is a knave. Then b is true and so B is a knight.

c) If a is false then A is not a knave and B is not a knight. But since A is not a knave, A is a knight and so a cannot be false. This is a contradiction and so a is true. Since a is true and the proposition "I am a knave" is false we conclude that "B is a knight" is true. Thus, both A and B are knights.

d) a is actually a compound statement $a_1 \wedge a_2$ (where a_1 is "A is a knave" and a_2 is "B is a knave"). Clearly, a cannot be true because if a is true then a_1 is true but then A cannot be telling the truth and so a must be false. Since a is false, A is a knave and so a_1 is true. But then for a to be false a_2 must be false, which means that B is a knight. □

8. An island is populated by knights who always tell the truth, knaves who always lie, and normals who can either lie or tell the truth. You encounter three people, A, B, and C. You know one of these people is a knight, one is a knave, and one is a normal. Each of the three people knows the type of person each of other two is. Determine, if possible, whether there is a unique solution and who the knave, knight, and normal are. When there is no unique solution, list all possible solutions or state that there are no solutions.

a) A says "C is the knave," B says, "A is the knight," and C says "I am the normal."
b) A says "I am the knight," B says "I am the knave," and C says "B is the knight."'
c) Each says "I am the knave."
d) A says "I am the knight," B says "A is telling the truth," and C says "I am the normal."

Solution. We denote the propositions made by A, B and C by a, b and c, respectively.

a) If a is true then C is the knave and so A and B are both not knaves. If A is a normal then B is lying and so is not a knight. This is a contradiction. So A is a knight and B is a normal.

If a is false then b is also false. Thus, A and B are both lying, so C must be the knight. But then C is lying which is a contradiction. Thus, the only solution is: A is a knight, B is a normal and C is a knave.

b) Note that b is automatically false because a knave cannot make the proposition "I am a knave". Then c is also false, since B is not a knight. It follows that B is a normal (otherwise b is not false) and C is a knave. This means that A is a knight which agrees with a being true. We found the only possible solution.

c) This is impossible since one of them is a knave, but a knave cannot state "I am a knave".

d) If a is true then b is also true and so A is a knight and B is a normal. Then C is a knave which agrees with c being false.

If a is false then also b is false which means that both A and B are not knights. But then C must be a knight. This is a contradiction since then c is false and true at the same time. $\square$

9. The police have three suspects for the murder of Mr. Cooper: Mr. Smith, Mr. Jones, and Mr. Williams. All suspects declare that they did not kill Cooper. Smith also states that Cooper was a friend of Jones and that Williams disliked him. Jones also states that he did not know Cooper and that he was out of town the day Cooper was killed. Williams also states that he saw both Smith and Jones with Cooper the day of the killing and that either Smith or Jones must have killed him. Can you determine who the murderer was if one of the three men is guilty, the two innocent men are telling the truth, but the statements of the guilty man may or may not be true?

Solution. The statement of Jones contradicts the statement of Smith (because Jones claims he did not know Cooper and Smith claims that Cooper was a friend of Jones) and the statement of Williams (Jones claims that he was out of town when Cooper was killed and Williams claims that he sow Jones with Cooper that day). So, if Jones is telling the truth then both Smith and Williams are lying. But since we are given that only one of the three is guilty and only the guilty one can lie it follows that Jones cannot be telling the truth. Thus, Jones is lying and hence is the murdered.

It remains to check that everything else is consistent (because it can happen that the puzzle does not have a solution). The first statement of Smith being true agrees with the first statement of Jones being false; the second statement of Smith does not contradict any statements of others. The first statement of Williams being true agrees with the second statement of Jones being false. The second statement of Williams is true since Jones is the murderer. $\square$