Arguments

Definition

An *argument* is an assertion that a given set of propositions $p_1, \ldots, p_n$, called *premises*, yields (has as a consequence) another proposition q, called the *conclusion*. Such an argument is denoted by $p_1, \ldots, p_n \vdash q$.
We say that an argument $p_1, \ldots, p_n \vdash q$ is *valid* if q is true whenever all the $p_1, \ldots, p_n$ are true.
An argument which is not valid is called a *fallacy*.

Another notation

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

To save space we will also write $p_1, \ldots, p_n \therefore q$.

Example

Consider the following argument.

Premises:	p_1	"You have a valid password"
	p_2	"If you have a valid password, you can access the system"
Conclusion:	q	"You can access the system"

This argument is *valid* because $p_2 \equiv p_1 \to q$. So, if p_1 is true, then for p_2 to be true we need q to be true.

This *rule of inference* is called *modus ponens* (method that asserts) or *the law of detachment* (because we detach q from the conditional statement $p \to q$).

The argument $p, p \to q \vdash q$ ("p and if p to q, therefore q") is valid

Example

Consider the argument:

Premises:	p	"You can access the system"
	$q \to p$	"If you have a valid password, you can access the system"
Conclusion:	q	"You have a valid password"

Suppose that p is true. Then $q \to p$ is true regardless of the truth value of q. Thus, this argument is a *fallacy*: the validity of premises does not force the conclusion to be true

Example

Is the argument $(p \vee q) \to r, \neg r \therefore q \to r$ valid?

Suppose that both premises are true. Then $\neg r$ is true and so r is false.

Since $(p \vee q) \to r$ is true, we conclude that $p \vee q$ must be false.

This happens if only if both p and q are false.

So, if both premises are true then q and r are both false which means that $q \to r$ is true. The argument is valid.

We can also use truth tables:

p	q	r	$p \vee q$	$(p \vee q) \to r$	$\neg r$	$q \to r$
T	T	T	T	T	F	T
T	T	F	T	F	T	F
T	F	T	T	T	F	T
T	F	F	T	F	T	T
F	T	T	T	T	F	T
F	T	F	T	F	T	F
F	F	T	F	T	F	T
F	F	F	F	T	T	T

Theorem

The argument $p_1, \ldots, p_n \vdash q$ *is valid if and only if* $(p_1 \wedge \cdots \wedge p_n) \rightarrow q$ *is a tautology.*

We explain how this works for only two statements p_1 and p_2.

$$(p_1 \wedge p_2) \rightarrow q \equiv T$$
$$\neg(p_1 \wedge p_2) \vee q \equiv T$$
$$\neg p_1 \vee \neg p_2 \vee q \equiv T \qquad \text{by De Morgan's law}$$

To illustrate how this theorem is used we show that *modus ponens* is a valid argument. By the Theorem, to show that it is valid it is enough to show that $(p \wedge (p \to q)) \to q$ is a tautology.

$$
\begin{aligned}
(p \wedge (p \to q)) \to q &\equiv \neg(p \wedge (p \to q)) \vee q && \text{since } x \to y \equiv \neg x \vee y \\
&\equiv \neg(p \wedge (\neg p \vee q)) \vee q && \text{since } p \to q \equiv \neg p \vee q \\
&\equiv \neg((p \wedge \neg p) \vee (p \wedge q)) \vee q && \text{by Distributivity} \\
&\equiv \neg(F \vee (p \wedge q)) \vee q && p \wedge \neg p \equiv F \\
&\equiv \neg(p \wedge q) \vee q && F \vee x \equiv x \\
&\equiv \neg p \vee \neg q \vee q && \text{by De Morgan's law} \\
&\equiv \neg p \vee T && \neg q \vee q \text{ is a tautology} \\
&\equiv T && \text{by Dominance}
\end{aligned}
$$

Some standard rules of inference

Modus ponens (method that affirms)	$p,\ p \to q \vdash q$
Modus tollens (method that denies)	$\neg q,\ p \to q \vdash \neg p$
Hypothetical syllogism	$(p \to q),\ (q \to r) \vdash p \to r$
Disjunctive syllogism	$(p \vee q),\ \neg p \vdash q$
Resolution	$p \vee q,\ \neg p \vee r \vdash q \vee r$
Addition	$p \vdash p \vee q$
Simplification	$p \wedge q \vdash p$

Example

Syllogism (or more formally Hypothetical syllogism) states "It follows from p implies q and q implies r that p implies r".
Why is this argument valid?

Suppose that both premises $p \to q$ and $q \to r$ are true.

- If p is true then q is true (otherwise $p \to q$ would be false).
  But then also r is true, for the same reason. Then $p \to r$ is true.
- If p is false then $p \to r$ is true regardless of the value of r.

Thus, the argument "$p \to q$, $q \to r$, therefore $p \to r$" is valid

Alternatively, we can check that $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology:

$$
\begin{aligned}
&((p \to q) \wedge (q \to r)) \to (p \to r) && \\
&\quad \equiv ((\neg p \vee q) \wedge (\neg q \vee r)) \to (\neg p \vee r) && \text{because } a \to b \equiv \neg a \vee b \\
&\quad \equiv \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee \neg p \vee r && a \to b \equiv \neg a \vee b \\
&\quad \equiv (\neg(\neg p \vee q)) \vee (\neg(\neg q \vee r)) \vee \neg p \vee r && \text{De Morgan's law} \\
&\quad \equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee \neg p \vee r && \text{De Morgan's law} \\
&\quad \equiv ((p \vee \neg p) \wedge (\neg q \vee \neg p)) \vee (q \wedge \neg r) \vee r && \text{Distributivity} \\
&\quad \equiv (\neg q \vee \neg p) \vee ((q \vee r) \wedge (\neg r \vee r)) && \text{Distributivity} \\
&\quad \equiv \neg q \vee \neg p \vee q \vee r && r \vee \neg r \equiv T,\ a \wedge T \equiv a \\
&\quad \equiv (\neg q \vee q) \vee (\neg p \vee r) && \\
&\quad \equiv T \vee (\neg p \vee r) && \neg q \vee q \equiv T \\
&\quad \equiv T. && T \vee a \equiv T
\end{aligned}
$$

Example

Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Let s be "It is sunny this afternoon", g be "We go swimming", c be "we take a canoe trip" and h be "we will be home by sunset". Using simplification we can write the first premise as $\neg s$ (the proposition "It is colder than yesterday" is not used anywhere else in the argument and can be ignored), the second as $g \to s$, the third as $\neg g \to c$ and the last as $c \to h$.

Step 1 *Modus tollens* applied to premises $\neg s$, $g \to s$ yields $\neg g$.

Step 2 *The law of detachment* (*modus ponens*) applied to $\neg g$ and $\neg g \to c$ yields c.

Step 3 Applying *modus ponens* to c, $c \to h$ we obtain h.

Note that the truth table for this argument would have 16 rows.

What will happen if we replace $g \to s$ by $s \to g$ (so the premises are $\neg s, s \to g, \neg g \to c,\ c \to h$)?

First, is $\neg s,\ s \to g, \neg g \to c,\ c \to h \therefore h$ still valid? We can simply this as $\neg s,\ s \to g, \neg g \to h \therefore h$.

Suppose that h is false. If $\neg g \to h$ is false then we are done. If it is true then we must have g true. But then $s \to g$ is automatically true for all values of s. Thus, we can have that all premises are true and h is still false. This argument is no longer valid.

What conclusion can we obtain from this argument (h is not a valid conclusion).

We can work with the simplified premises $\neg s$, $s \to g$, $\neg g \to h$. If all of them are true then s is false; $s \to g$ is false for all values of g and $\neg g \to h$ true means that we cannot have g and h both false. So, the only conclusion we can draw is "We go swimming or we will be home by sunset".

The validity means that the conclusion can be obtained from the premises using the rules of logic. It says absolutely nothing as to whether the conclusion can be deduced from the premises using some other facts and premises.

As an example, consider the following argument:

If two sides of a triangle are equal, then the opposite angles are equal
Two sides of a triangle are not equal

$\therefore$ The opposite angles are not equal.

If p is "Two sides of a triangle are equal" and q is "The opposite angles are equals" then the argument is $p \to q, \neg p \therefore \neg q$.

This argument is easily seen to be a fallacy. Indeed, if $\neg p$ is true then p is false. Then $p \to q$ is true regardless of the value of q.

However, using Euclidean geometry we can deduce $\neg q$ from $\neg p$.

Example (Catch-22)

Consider the following premises:

- For a pilot to be excused from flying combat missions on the grounds of insanity, they must both be insane and have requested an evaluation.
- An insane person does not request an evaluation because they do not realize they are insane

Our universe is "combat pilots". Let e be "is excused from flying", i be "is insane" and r be "requests evaluation".

Then the first premise is $e \to (i \wedge r)$ and the second premise is $i \to \neg r \equiv \neg i \vee \neg r \equiv \neg(i \wedge r)$.

By *modus tollens*, $e \to (i \wedge r), \neg(i \wedge r) \therefore \neg e$. In other words, nobody can be excused from flying on the grounds of insanity.

Example (Lewis Carroll)

What can be deduced from the following premises:

p_1 All hummingbirds are richly colored

p_2 No large birds live on honey

p_3 Birds that do not live on honey are dull in color

Our "universe" consists of birds. Let us write down all simple statements about birds found in these premises.

q_1 It is a hummingbird

q_2 It is richly colored

q_3 It is large

q_4 It lives on honey

Then we have $p_1 \equiv q_1 \to q_2 \equiv \neg q_2 \to \neg q_1$, $p_2 \equiv q_3 \to \neg q_4 \equiv q_4 \to \neg q_3$.
$p_3 \equiv \neg q_4 \to \neg q_2 \equiv q_2 \to q_4$
We can combine these together as $q_1 \to q_2,\ q_2 \to q_4,\ q_4 \to \neg q_3 \therefore q_1 \to \neg q_3$.
That is "If it is a hummingbird then it is not large" or "All hummingbirds are small".