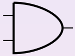
Logic circuits

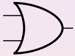
Logic circuits take one or more inputs which have T or F (0 or 1) value and produce one or more outputs with T or F (0 or 1) value.

Their building blocks are:

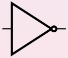
- AND gate



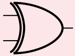
- OR gate



- NOT gate

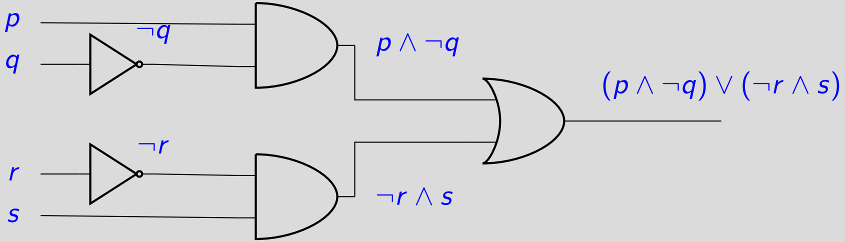


- XOR gate



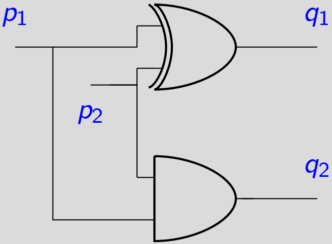
(The input is on the left, the output is on the right)

Example



What is the output of this circuit if $p = r = T$ and $q = s = F$?
Since $\neg q = T$, the upper input of the last gate (OR) is T and so the output is T regardless of what happens with the other input of that gate (which, by the way, is F).

Example



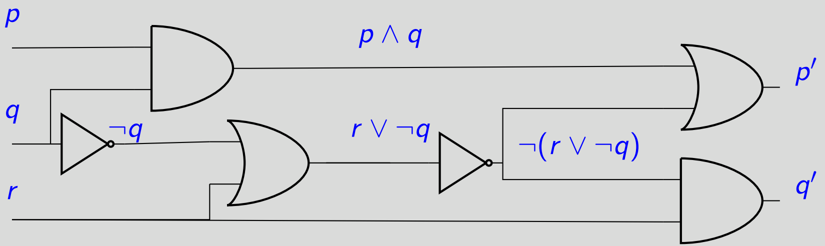
Inputs: p_1, p_2, outputs: q_1, q_2

p_1	p_2	q_2	q_1	
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	0	2

So, this circuit adds p_1 and p_2 - if we regard them as numbers

Exercise. Draw a circuit that can add two numbers between 0 and 3

Example



Thus, $p' = (p \wedge q) \vee (\neg(r \vee \neg q))$, $q' = r \wedge (\neg(r \vee \neg q))$.
What are the possible values of p' if $q' = T$? What are the possible values of q' if $p' = F$?

If $q' = T$ then $r = T$ and $\neg(r \vee \neg q) = T$. Then $p' = T$.

If $p' = F$ then $p \wedge q = F$ and $\neg(r \vee \neg q) = F$ that is $r \vee \neg q = T$. Thus, $q' = T$ if $r = T$ and $q' = F$ if $r = F$. If $r = T$ then $r \vee \neg q = T$ for any value of q. If $r = F$ then $r \vee \neg q = T$ if and only if $\neg q = T$ or $q = F$. But then $p \wedge q = F$ and so this situation never occurs. So, if $p' = F$ then $q' = T$.

System specifications and their consistence

Example

Are these system specifications consistent? "Whenever the system software is being upgraded, users cannot access the file system. If users can access the file system, then they can save new files. If users cannot save new files, then the system software is not being upgraded."

Let

- u: "The system software is being upgraded",
- f: "The users can access the file system",
- s: "The users can save new files".

Then the above specification is

$$(u \to \neg f) \wedge (f \to s) \wedge (\neg s \to \neg u). \qquad (\star)$$

We need to find if it is possible to assign T and F values to u, f and s in such a way that the truth value of $(\star)$ is T. In that case we say that the specifications are consistent.

We can replace $\neg s \to \neg u$ with its contrapositive $u \to s$ since $\neg s \to \neg u \equiv u \to s$. Then our specification becomes $(u \to \neg f) \wedge (u \to s) \wedge (f \to s)$.

- If u is F then the first two conditionals are true regarless of values of f and s. We only need to exclude the case when f is T and s is F.
- If u is T then $u \to s$ is true if and only if s is T and $u \to \neg f$ is true if and only if f is F. The conditional $f \to s$ is true if f is F and s is T.

Thus, $(\star)$ is true in the following cases:

- $u = T$, $f = F$, $s = T$: the system software is being upgraded; the users cannot access the file system but can save new files;
- $u = F$, $f = T$, $s = T$: the system software is not being upgraded; the users can access the file system and can save new files;
- $u = F$, $f = F$, $s = T$: the system software is not being upgraded; the users cannot access the file system but can save new files;
- $u = F$, $f = F$, $s = F$: the system software is not being upgraded; the users cannot access the file system and cannot save new files;

In particular, the specifications are consistent.

Example

Are these system specifications consistent? "The system is in multiuser state if and only if it is operating normally. If the system is operating normally, the kernel is functioning. The kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode. The system is not in interrupt mode."

Let

m: "The system is in multiuser state"

n: "The system is operating normally"

k: "The kernel is functioning"

i: "The system is in interrupt mode"

The specification is: $(m \leftrightarrow n) \wedge (n \rightarrow k) \wedge (\neg k \vee i) \wedge (\neg m \rightarrow i) \wedge (\neg i)$.

This proposition is clearly false if $i = T$. Thus, we must have $i = F$ which forces $m = T$ (and so $n = T$) and $k = F$. But then $n \rightarrow k$ is false. Thus, the specification is not consistent.

Contradictions and tautologies

Definition

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*.
A compound proposition that is always false is called a *contradiction*.
A compound proposition that is neither a tautology nor a contradiction is called a *contingency*

Example

Let p be a proposition. Then $p \wedge \neg p$ and is a contradiction and $p \vee \neg p$ is a tautology.

Definition

We say that p and q are *logically equivalent* and write $p \equiv q$ if $p \leftrightarrow q$ is a tautology.

As we already discussed, two compound propositions are logically equivalent if and only if they have the same truth tables.

Theorem

Let p, q, r be propositions. Then

1° $p \equiv p$

2° $p \equiv q$ *implies that* $q \equiv p$

3° $p \equiv q$, $q \equiv r$ *implies that* $p \equiv r$.

1° is obvious since $p \leftrightarrow p$ is a tautology.

2° is obvious since $p \leftrightarrow q$ has the same truth table as $q \leftrightarrow p$.

Suppose that $p \leftrightarrow q$ and $q \leftrightarrow r$ are both tautologies. If p is true then, since $p \leftrightarrow q$ is a tautology, q is true. Then, since $q \leftrightarrow r$ is a tautology, r is also true. Thus, $p \leftrightarrow r$ is true.

Likewise, if p is false then q is false and then r is false. Therefore, $p \leftrightarrow r$ is true.

Thus, $p \leftrightarrow r$ is a tautology or $p \equiv r$.

Important equivalences

Identity laws	$p \wedge T \equiv p$ $p \vee F \equiv p$
Dominance laws	$p \wedge F \equiv F$ $p \vee T \equiv T$
Idempotent laws	$p \wedge p \equiv p$ $p \vee p \equiv p$
Double negation law	$\neg(\neg p) \equiv p$
Commutativity	$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$
Associativity	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ $p \vee (q \vee r) \equiv (p \vee q) \vee r$
Distributivity	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example

First De Morgan's law: $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

How to check logical equivalence without truth tables

The statement $p \equiv q$ means that p is true if and only if q is true and p is false if and only if q is false.

Let a be "p is true" and b be "q is true". Then the contrapositive of $a \to b$ is $\neg b \to \neg a$ (that is, "if q is false then p is false") and the contrapositive of $b \to a$ is "if p is false then q is false".

So, in the table

If p is true then q is true	If q is true then p is true
If p is false then q is false	If q is false then p is false

it is enough to check *any two statements NOT connected by an arrow*.

Associativity of $\vee$ and $\wedge$ allows us to define

$$p_1 \vee p_2 \vee p_3 \vee \cdots \vee p_n, \qquad p_1 \wedge p_2 \wedge \cdots \wedge p_n,$$

for any collection of propositions $p_1, \ldots, p_n$. The truth value of the first is F if all of the $p_1, \ldots, p_n$ are false and T otherwise. The truth value of the second is T if all of the $p_1, \ldots, p_n$ are true and F otherwise.

Example

Deduce second De Morgan's law $\neg(p \vee q) \equiv \neg p \wedge \neg q$ from the first De Morgan's law.

Let $p' = \neg p$, $q' = \neg q$. Then by the first de Morgan's law

$$\neg(p' \wedge q') \equiv \neg p' \vee \neg q'$$

Apply $\neg$ to both sides

$$\neg(\neg(p' \wedge q')) \equiv \neg(\neg p' \vee \neg q').$$

The left hand side is equivalent to $p' \wedge q' \equiv \neg p \wedge \neg q$ by the Double Negation law.

The right hand side is equivalent to $\neg(p \vee q)$, again by the Double Negation law.

Thus

$$\neg p \wedge \neg q \equiv p' \wedge q' \equiv \neg(\neg p' \vee \neg q') \equiv \neg(p \vee q).$$

This is precisely what we wanted to establish.

Example

Distributivity $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Suppose that $p \wedge (q \vee r)$ is true. Then p is true and $q \vee r$ is true. In particular, at least one of q, r is true. Then either $p \wedge q$ is true or $p \wedge r$ is true and so $(p \wedge q) \vee (p \wedge r)$ is true.

Suppose that $p \wedge (q \vee r)$ is false. Then either p is false or $q \vee r$ is false. If p is false then $p \wedge q$ and $p \wedge r$ are both false. If $q \vee r$ is false then q and r are both false and so $p \wedge q$ and $p \wedge r$ are again false. We conclude that $(p \wedge q) \vee (p \wedge r)$ is false.

Some other equivalences

$$p \to q \equiv \neg p \vee q$$

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\neg p \vee q) \wedge (\neg q \vee p)$$

$$p \oplus q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$$

De Morgan's Law and the above equivalences imply that $\neg$, $\wedge$ as well as $\neg$, $\vee$ are *fundamental systems of logical operators*. This means that every other logical operator can be expressed in terms of $\neg$, $\wedge$ or $\neg$, $\vee$.

Indeed, we deduce from $\neg(p \wedge q) \equiv \neg p \vee \neg q$ that $p \vee q \equiv \neg(\neg p \wedge \neg q)$.

Example

Is it true that $(p \to q) \to r \equiv p \to (q \to r)$?

$$
\begin{aligned}
(p \to q) \to r &\equiv (\neg p \vee q) \to r && \text{because } p \to q \equiv \neg p \vee q \\
&\equiv \neg(\neg p \vee q) \vee r \\
&\equiv (\neg(\neg p) \wedge \neg q) \vee r && \text{by De Morgan's law} \\
&\equiv (p \wedge \neg q) \vee r && \text{by Double Negation law}
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
p \to (q \to r) &\equiv p \to (\neg q \vee r) \\
&\equiv \neg p \vee (\neg q \vee r) \\
&\equiv (\neg p \vee \neg q) \vee r && \text{by Associativity} \\
&\equiv \neg(p \wedge q) \vee r && \text{by De Morgan's law}
\end{aligned}
$$

Both propositions are true if $r = T$ by Dominance law. If $r = F$ we can ignore r using the Identity law.

So we need to find out whether $\neg(p \wedge q)$ is equivalent to $p \wedge \neg q$.

This is clearly not the case since $p \wedge \neg q = T$ if and only if $p = T$, $q = F$, while $\neg(p \wedge q) = T$ if and only if $p = q = F$.

Conclusion

$(p \to q) \to r$ is not equivalent to $p \to (q \to r)$.

Another method: the proposition $p \to (q \to r)$ is false if and only if $p = T$ and $q \to r$ is false which happens if and only if $p = q = T$ and $r = F$.

On the other hand, the proposition $(p \to q) \to r$ is false if and only if $r = F$ and $p \to q$ is true. In particular, $(p \to q) \to r$ is false if $p = q = F$, $r = F$. But for these truth values of logical variables p, q, r the proposition $p \to (q \to r)$ is true.

Thus, we found values of p, q, r for which $p \to (q \to r)$ and $(p \to q) \to r$ have different truth values. Therefore, these two statements are not logically equivalent.