HOMEWORK SET 6. CARTESIAN PRODUCTS AND MAPS

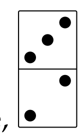
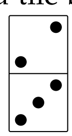
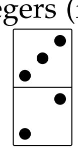
1. A database is storing information about flights. For each flight it stores the airline code, the flight number, the origin, the destination, the departure time and the arrival time. Describe an entry in that database as an element of the Cartesian product of sets.

Solution. Let A be the set of airline codes, N be the set of positive integers (flight numbers), F be the set of airport codes, H be the set of integers between 0 and 23 (including 0 and 23) and M be the set of integers between 0 and 59 (including 0 and 59).

Then a record in our database is an element of the set $A \times N \times F \times F \times H \times M \times H \times M$. Note that we use the same set F for origins and destinations; we use the Cartesian product $H \times M$ to store time.

Other solutions can also be proposed. $\square$

2. Every domino tile can be represented as a pair of integers a, b, with $0 \leq a, b \leq 6$. Is it true that $\{0, 1, 2, 3, 4, 5, 6\} \times \{0, 1, 2, 3, 4, 5, 6\}$ is the set of all domino tiles?

Solution. No. We can represent a tile by a pair of integers (for example,  corresponds to the pair $\{2, 3\}$. The problem is that  is the same as . In the Cartesian product, however, $(2, 3)$ and $(3, 2)$ are different elements, so the pair corresponding to the same tile appears twice (provided that the numbers are different). $\square$

3. Let A, B be sets and suppose that $A \times B = \emptyset$. What can we say about A and B?

Solution. Suppose that A and B are both non-empty. Then there is $a \in A$ and $b \in B$. This implies that $(a, b) \in A \times B$ and so $A \times B$ is non-empty. Thus, if both A and B are non-empty then $A \times B$ is non-empty. The contrapositive of this statement is "If $A \times B$ is empty then it is not true that both A and B are non-empty", that is, "If $A \times B = \emptyset$ then either A or B is empty". $\square$

4. Let A, A', B, B' be sets. What is the relation between $(A \times B) \setminus (A' \times B')$ and $(A \setminus A') \times (B \setminus B')$?

Solution. The set $(A \setminus A') \times (B \setminus B')$ is the set of all pairs (a, b) such that $a \in A \setminus A'$, $b \in B \setminus B'$. The set $(A \times B) \setminus (A' \times B')$ is the set of all pairs (a, b) such that $a \in A$, $b \in B$ and $(a, b) \notin A' \times B'$, that is, we only exclude pairs (a, b) with $a \in A'$ and $b \in B'$. So, $(A \setminus A') \times (B \setminus B')$ is a subset of $(A \times B) \setminus (A' \times B')$, but in general they are not equal.

Indeed, consider the following example let $A = A'$, $B' = \emptyset$. Then $(A \setminus A') \times (B \setminus B') = \emptyset$, while $A' \times B' = \emptyset$ and so $(A \times B) \setminus (A' \times B') = (A \times B) \setminus \emptyset = A \times B$.

As an exercise, take $A = B = \{1, 2, 3\}$, $A' = B' = \{1, 2\}$, find $(A \setminus A') \times (B \setminus B')$, $(A \times B) \setminus (A' \times B')$ and compare them. $\square$

5. Use Cartesian product and other set operations to describe the following sets.

a) The set of all passwords containing at least 8 and at most 10 symbols and consisting of upper- and lowercase letters

b) The set of all passwords containing at least 8 and at most 10 symbols, consisting of upper- and lowercase letters and digits and beginning with a letters

c) The set of all passwords as described in **b)** which also contain at least one digit

d) (*) The set of all passwords as described in question **c)** which contain at most three digits

Solution. We will need the following sets. Let L be the set of all upper- and lowercase letters ($L = \{A, a, B, b, \ldots, Z, z\}$). Let $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let $A = L \cup D$.

A password is a sequence of symbols; that is, it is important in which order we list these symbols (for example, AxY is not the same as YAx).

a) A password consisting of k letters is a sequence of k elements of L. Thus, a password consisting of 8 letters is an element of L^8, a password consisting of 9 letters is an element of L^9 etc. So, the answer to the first question is $L^8 \cup L^9 \cup L^{10}$.

b) If a password begins with a letter and the remaining symbols can be digits or letters then it is an element of $L \times A^{k-1}$ where k is the length of the password. Thus, the answer is

$$(L \times A^7) \cup (L \times A^8) \cup (L \times A^9).$$

c) We want to exclude from consideration all passwords which contain no digits. The set of such passwords was described in part **a)**. Thus, the set we are after is

$$((L \times A^7) \setminus L^8) \cup ((L \times A^8) \setminus L^9) \cup ((L \times A^9) \setminus L^{10}).$$

Note that $(L \times A^7) \setminus L^8$ is *NOT* the same as $L \times (A \setminus L)^7 = L \times D^7$.

d) We want to exclude from consideration all passwords which contain more than 3 digits. For example, for passwords of length 8 beginning with the letter we will have to exclude $L \times R$ where R is the union of all the sets of the form

$$X_1 \times X_2 \times X_3 \times X_4 \times X_5 \times X_6 \times X_7 \times X_8$$

where 3 of the X_i are equal to D and the remaining are equal to A.

□

6. Consider these maps from the set of students in a discrete mathematics class. Under what conditions is the map injective if it assigns to a student his or her

a) mobile phone number.

b) student identification number.

c) final grade in the class.

d) home town.

Solution. The first function is likely to be injective. It is not injective only if there are two students sharing a mobile phone.

The second function is aways injective. Indeed, the student ID number is supposed to identify the student uniquely, so two different students cannot have the same SID number

The 3rd function is very unlikely to be injective (later we will see that it automatically fails to be injective if there are more students in the class than possible grades). For it to be injective, we need each student in the class to have a different grade.

The last function is injective only if no two students in class have the same home town, which is not likely to happen, especially in a university located in a big city.

□

7. Recall that $\mathbb{N}$ is the set of all non-negative integers. Give an example of a map $f : \mathbb{N} \to \mathbb{N}$ which is

a) injective but not surjective
b) surjective but not injective
c) bijective but not equal to $\mathrm{id}_{\mathbb{N}}$
d) neither injective nor surjective

Solution. **a)** Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n+1$, $n \in \mathbb{N}$. It is injective because if $f(n) = f(m)$ then $n + 1 = m + 1$ and so $n = m$. However, it is not surjective since $f(0) > 0$ for all $n \in \mathbb{N}$ and so $0 \notin \operatorname{im} f$.

Another possibility is to define f by $f(n) = 2n$. This function is manifestly injective, but its image is the set of all non-negative even numbers.

b) Let $f : \mathbb{N} \to \mathbb{N}$ be defined by

$$f(n) = \begin{cases} n-1, & n > 0 \\ 0, & n = 0. \end{cases}$$

Then f is surjective, since every non-negative integer m can be written as $m = (m+1) - 1 = f(m+1)$. It is not injective because $f(1) = f(0)$.

c) Define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} n+1, & n \text{ even} \\ n-1, & n \text{ odd} \end{cases}$$

We claim that f is bijective but not $\mathrm{id}_{\mathbb{N}}$. First, note that f changes the parity of an integer: it sends an even integer to an odd one and vise versa. So, if $f(m) = f(n)$ then m and n are both even or both odd. If m, n are both even then $f(m) = m+1$ and $f(n) = n+1$ and so $f(m) = f(n)$ implies that $m + 1 = n + 1$ hence $m = n$. Likewise, if m and n are both odd then $f(m) = m - 1$ and $f(n) = n - 1$ and so $f(n) = f(m)$ implies that $m - 1 = n - 1$ and again $m = n$. We proved that f is injective.

If m is even then $m + 1$ is odd and $m = (m + 1) - 1 = f(m + 1)$. If m is odd then $m - 1$ is even (and is non-negative, since $m \geq 1$) and so $m = (m - 1) + 1 = f(m - 1)$. In both cases we conclude that $m = f(n)$ for some n and so f is surjective.

$f \neq \mathrm{id}_{\mathbb{N}}$ because $f(0) = 1 \neq 0$.

d) Define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} n, & n \text{ is odd} \\ n+1, & n \text{ is even} \end{cases}$$

It is not injective since $f(1) = 1 = f(0)$. It is not surjective since only even numbers appear in its image.

$\square$

8. For each map $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined below determine whether it is surjective.

a) $f(m, n) = 2m - n$;
b) $f(m, n) = m^2 + n^2$;
c) $f(m, n) = m + n + 1$;
d) $f(m, n) = |m| - |n|$;
e) $f(m, n) = m^2 - 4$.

Solution. **a)** This map is surjective because we have $x = f(0, -x)$ for any $x \in \mathbb{Z}$.

b) This map is not surjective because a sum of squares of two integers can never be a negative number. Even if we regard f as a map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ it is still not surjective since we proved that a sum of two perfect squares cannot have remainder 3 when divided by 4; since there are integers which have remainder 3 when divided by 4, we conclude that the map is not surjective.

c) This map is surjective since $x = f(x-1, 0) = f(0, x-1)$ for any $x \in \mathbb{Z}$.
d) This map is surjective. Indeed, let $x \in \mathbb{Z}$. If $x > 0$ then $x = f(x, 0)$. If $x < 0$ then $x = f(0, -x)$.
e) This map is not surjective. If it was surjective, we would have $1 = f(m, n) = m^2 - 4$ for some $m, n \in \mathbb{Z}$ and so $m^2 = 1 + 4 = 5$. But 5 is not a perfect square.

$\square$

9. Let $f : A \to B$ be a map. Let S and T be subsets of A. Recall that $f(S) = \{f(s) : s \in S\} \subset B$. Show that

a) $f(S \cup T) = f(S) \cup f(T)$
b) $f(S \cap T) \subset f(S) \cap f(T)$
c) Give an example of f, S and T such that $f(S \cap T) \neq f(S) \cap f(T)$
d) Show that if f is injective then the inclusion in part (b) is an equality.

Solution. **a)** We have

$$\begin{aligned} x \in f(S \cup T) &\iff x = f(y),\ y \in S \cup T \\ &\iff (x = f(y),\ y \in S) \vee (x = f(z),\ z \in T) \\ &\iff (x \in f(S)) \vee (x \in f(T)) \\ &\iff x \in f(S) \cup f(T). \end{aligned}$$

b) Let $x \in f(S \cap T)$. Then $x = f(y)$ where $y \in S \cap T$. Thus, $x \in f(S)$ and $x \in f(T)$, that is, $x \in f(S) \cap f(T)$.
c) Let S be the set of even integers and T be the set of odd integers. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 0$, $x \in \mathbb{Z}$. Then $f(S \cap T) = f(\emptyset) = \emptyset$, while $f(S) \cap f(T) = \{0\} \neq \emptyset$.
d) Suppose that f is injective and let $x \in f(S) \cap f(T)$. Then $x = f(s)$ for some $s \in S$ and $x = f(t)$ for some $t \in T$. Thus, $f(s) = f(t)$. Since f is injective this forces $s = t$. But then $s = t \in S \cap T$ and so $x \in f(S \cap T)$.

$\square$

10. Let $f : A \to B$ be a map. Recall that for a subset X of B we denote $f^{-1}(X) = \{a \in A : f(a) \in X\}$. Let S, T be subsets of B.

a) Show that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$
b) Show that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

Solution. **a)** Let $x \in f^{-1}(S \cup T)$. This happens if and only if $f(x) \in S \cup T$ that is, if and only if $f(x) \in S$ or $f(x) \in T$, that is, if and only if $x \in f^{-1}(S) \cup f^{-1}(T)$.
b) Let $x \in f^{-1}(S \cap T)$. This happens if and only if $f(x) \in S \cap T$ that is, if and only if $f(x) \in S$ and $f(x) \in T$, that is, if and only if $x \in f^{-1}(S) \cap f^{-1}(T)$.

$\square$