Chap. 5-7

1. A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let X denote the number of hoses being used on the self-service island at a particular time, and let Y denote the number of hoses on the full-service island in use at that time. The joint pmf of X and Y appears in the following table:

$p(x,y)$		y = 0	1	2	$P_X(x)$
x	0	0.10	0.10	0.02	0.22
	1	0.08	0.18	0.06	0.32
	2	0.08	0.14	0.24	0.46
$P_Y(y)$		0.26	0.42	0.32	1

(a) Find the pmf's of X and Y.

(b) Are X and Y independent?

$p(x,y) = P_X(x)\,P_Y(y) \quad \forall (x,y)$

$0.10 = p(0,0) \neq P_X(0)\,P_Y(0) = 0.22 \times 0.26$

no.

(c) What is the probability that $X = 1$ **and** $Y = 1$?

$P(1,1) = 0.18$

(d) What is the probability that $X = 1$ **or** $Y = 1$?

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(X=1 \text{ or } Y=1) = P(X=1) + P(Y=1) - P(X=1 \text{ and } Y=1)$

$= P_X(1) + P_Y(1) - P(1,1) = 0.32 + 0.42 - 0.18$

(e) What is the conditional probability $P(X = 1|Y = 1)$?

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$P(X=1|Y=1) = \dfrac{P(X=1 \text{ and } Y=1)}{P_1(Y=1)} = \dfrac{P(1,1)}{P_Y(1)}$

2. Tom and Jerry have agreed to meet between 3:00 pm and 4:00 pm for coffee. Let X = Tom's arrival time and Y = Jerry's arrival time. Suppose X and Y are independent with each uniformly distributed on the interval $[3, 4]$.

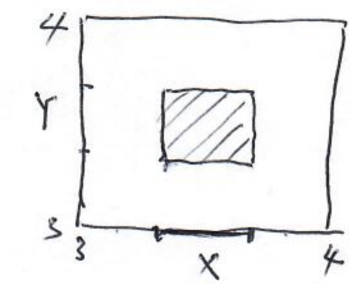
(a) What is the joint pdf of (X, Y)?

$$f_X(x) = \begin{cases} 1 & 3 \le x \le 4 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 1 & 3 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x, y) = \begin{cases} 1 & 3 \le x, y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

(b) What is the probability that they both arrive between 3:20 pm and 3:40 pm?

$$P(3\tfrac{1}{3} \le X, Y \le 3\tfrac{2}{3})$$
$$= P(3\tfrac{1}{3} \le X \le 3\tfrac{2}{3})\, P(3\tfrac{1}{3} \le Y \le 3\tfrac{2}{3})$$
$$= \frac{1}{3} \cdot \frac{1}{3}$$
$$= \frac{1}{9}$$



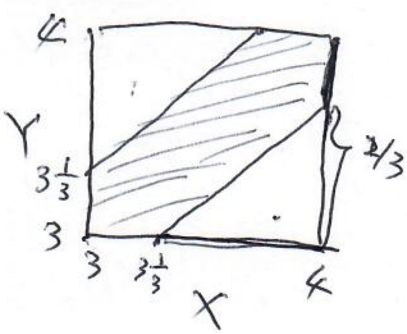
$$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

(c) If the first one to arrive will wait only 20 min before leaving, what is the probability that they will have coffee together?

$$|X - Y| \le \frac{1}{3}$$

$$X - \frac{1}{3} \le Y \le X + \frac{1}{3}$$

$$\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$



$$P(|X - Y| \le \tfrac{1}{3}) = 1 - \frac{4}{9} = \frac{5}{9}$$

3. A surveyor wishes to lay out a square region with each side having length μ. However, because of measurement error, he instead lays out a rectangle in which the north-south sides both have length X and the east-west sides both have length Y. Suppose that X and Y are independent and that each follows an exponential distribution with mean μ.

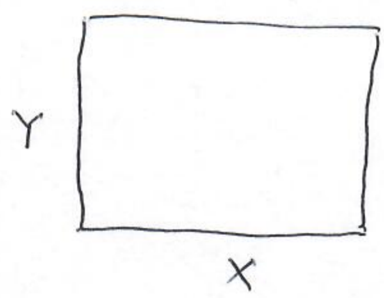
(a) What is the expected area of the resulting rectangle?

$$\text{Area} = XY$$

$$0 \overset{\text{ind}}{=} Cov(X, Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = E(X)E(Y) = \mu^2$$

$$E(XY) = \iint xy\, f(x,y)\, dx\, dy$$



(b) If X and Y are not independent but instead equal to each other, what is the expected area of the resulting square?

$$\text{Area} = X^2$$

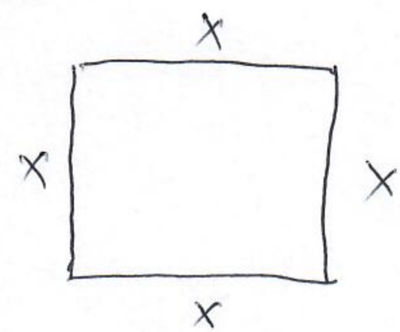
$$E(X^2) = \int x^2 f_X(x)\, dx$$

$$V(X) = E(X^2) - \{E(X)\}^2$$

$$E(X^2) = \{E(X)\}^2 + V(X)$$

$$= \mu^2 + \mu^2$$

$$= 2\mu^2$$



4. Let $X_1, \ldots, X_n$ be a random sample from some distribution with mean μ and variance σ^2, and let $\overline{X}$ be the sample mean (i.e., $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$).

(a) What is the sampling mean of $\overline{X}$?

$$E(\bar{X}) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} n\mu = \mu$$

(b) What is the sampling variance of $\overline{X}$?

$$V(\bar{X}) = \frac{1}{n^2} \sum_{i=1}^{n} V(X_i) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}.$$

(c) What is the sampling mean of $\overline{X}^2$?

$$V(Y) = E(Y^2) - \{E(Y)\}^2 \Rightarrow E(Y^2) = \{E(Y)\}^2 + V(Y)$$

$$E(\bar{X}^2) = \{E(\bar{X})\}^2 + V(\bar{X}) = \mu^2 + \frac{\sigma^2}{n}$$

(d) In which of the following cases is $\overline{X}$ approximately normally distributed?

$n > 30$

i. $n = 100$ and each X_i follows a uniform distribution ✓

ii. $n = 100$ and each X_i follows an exponential distribution ✓

iii. $n = 15$ and each X_i follows a normal distribution ✓

iv. $n = 15$ and each X_i follows an exponential distribution ✗

5. Let X be a binomial variable based on n independent Bernoulli trials with success probability p, and let $\widehat{p} = X/n$ be the sample proportion of success.

(a) What is the sampling mean of $\widehat{p}$?

$$E(\hat{p}) = \frac{E(X)}{n} = \frac{np}{n} = p$$

(b) What is the sampling variance of $\widehat{p}$?

$$V(\hat{p}) = \frac{V(X)}{n^2} = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

(c) In which of the following cases is $\widehat{p}$ approximately normally distributed?

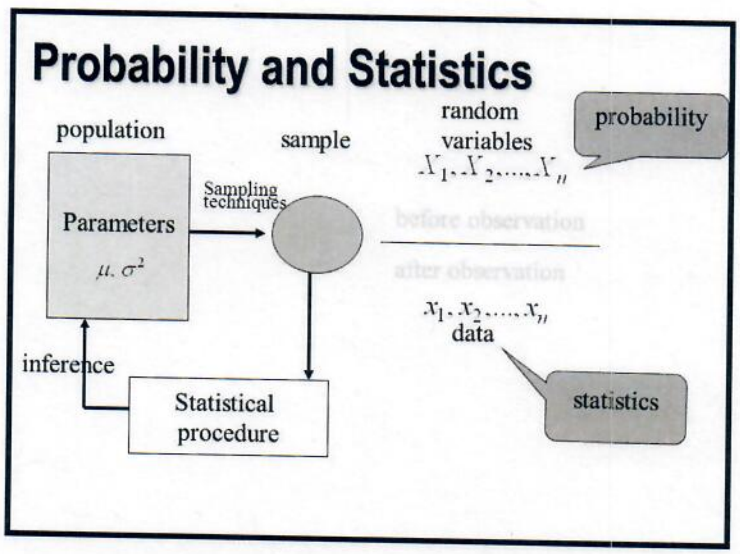
$np > 5, \quad n(1-p) > 5$

i. $n = 100$ and $p = 0.5$ $\quad np = 50, \quad n(1-p) = 50$ ✓

ii. $n = 100$ and $p = 0.01$ $\quad np = 1$ ✗

iii. $n = 50$ and $p = 0.3$ $\quad np = 15, \quad n(1-p) = 35$ ✓

iv. $n = 50$ and $p = 0.95$ $\quad np = 47.5 \quad n(1-p) = 2.5$ ✗


Probability and Statistics
population
Parameters
μ, σ^2
Sampling techniques
sample
random variables
$X_1, X_2, \ldots, X_n$
probability
before observation
after observation
$x_1, x_2, \ldots, x_n$
data
inference
Statistical procedure
statistics

6. Let $X_1, \ldots, X_n$ be a random sample from some distribution with mean μ and variance σ^2. Let $\overline{X}$ be the sample mean and s^2 the sample variance.

(a) What are unbiased estimators of μ and σ^2?

$$E(\bar{X}) = \mu$$
$$E(S^2) = \sigma^2$$

(b) What is the standard error of $\overline{X}$?

$$V(\bar{X}) = \frac{\sigma^2}{n}$$
$$se(\bar{X}) = \sqrt{V(\bar{X})} = \frac{\sigma}{\sqrt{n}}$$

(c) How do you estimate the standard error of $\overline{X}$?

$$\frac{S}{\sqrt{n}}$$

(d) How do you obtain a confidence interval for μ? How do you interpret it?

$$\bar{X} \pm z_{\alpha/2}\, S/\sqrt{n}$$

Before data collection: $P(\bar{X} - z_{\alpha/2} S/\sqrt{n} < \mu < \bar{X} + z_{\alpha/2} S/\sqrt{n}) = 1-\alpha$

After " : $1-\alpha$ confident $\bar{x} - z_{\alpha/2} S/\sqrt{n} < \mu < \bar{x} + z_{\alpha/2} S/\sqrt{n}$

(e) How do you choose a sample size to produce a confidence interval for μ with desired width? To estimate μ within some distance?

$$n = \left(\frac{2 z_{\alpha/2}\, \sigma}{w}\right)^2$$

S

7. Let X be a binomial variable based on n independent Bernoulli trials with success probability p, and let $\widehat{p} = X/n$ be the sample proportion of success.

(a) Is $\widehat{p}$ unbiased for p?

$E(\hat{p}) = p.$

(b) What is the standard error of $\widehat{p}$?

$$V(\hat{p}) = \frac{p(1-p)}{n}$$

$$se(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

(c) How do you estimate the standard error of $\widehat{p}$?

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

(d) How do you obtain a confidence interval for p? How do you interpret it?

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

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(e) How do you choose a sample size to produce a confidence interval for p with desired width? To estimate p within some distance?

$$n = \left(\frac{2 z_{\alpha/2}}{w}\right)^2 \hat{p}(1-\hat{p}) \leq \left(\frac{z_{\alpha/2}}{w}\right)^2$$

8. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2.

(a) What are the method of moment estimators of μ and σ^2? Are they unbiased?

$$\hat{\mu} = \bar{X}, \quad \text{unbiased.}$$

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{n-1}{n} S^2$$

$$E(\hat{\sigma^2}) = \frac{n-1}{n} E(S^2) = \frac{n-1}{n}\sigma^2, \quad \text{biased.}$$

(b) What are the maximum likelihood estimators of μ and σ^2? Are they unbiased?

$$\hat{\mu} = \bar{X}, \quad \text{unbiased}$$

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2, \quad \text{biased.}$$

9. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameters (n, p). What is the method of moments estimator of p? What is the maximum likelihood estimator of p? Are they unbiased?

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}, \quad \text{unbiased.}$$

10. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with mean λ.

(a) What is the method of moment estimator of λ? Is it unbiased?

$$E(X) = \lambda \overset{set}{=} \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\Rightarrow \hat{\lambda} = \bar{X}$$

$$E(\hat{\lambda}) = E(\bar{X}) = E(X_i) = \lambda, \quad \text{unbiased}$$

(b) What is the maximum likelihood estimator of λ? Is it unbiased?

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$L = \prod_{i=1}^{n} p(x_i;\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\prod_{i=1}^{n}(x_i!)}$$

$$\ln(L) = -n\lambda + (\Sigma x_i)\ln(\lambda) - \sum_{i=1}^{n}\ln(x_i!)$$

$$\frac{d\ln(L)}{d\lambda} = -n + \frac{\Sigma x_i}{\lambda} \overset{set}{=} 0$$

$$\Rightarrow \hat{\lambda} = \frac{\Sigma X_i}{n} = \bar{X}, \quad \text{unbiased.}$$