

# A report on a basic group that was implemented and studied

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## Abstract

A topological group,  $G$  was implemented and its representations studied. It was found to have a representation into  $GL_5(R)$ . This representation was realized. This Abelian Lie was also found to have an 8-dimensional faithful unitary representation.

## 1 Basic group implementation

The construction started by considering two tuples of four numbers henceforth referred to as quartets and a multiplication law. Let us take a quartet to be  $(s, p, m, n)$ . Let  $s, p, n \in R \setminus \{0\}$ , and  $m \in R$ . We now take  $R \setminus \{0\}$  to be  $R^*$  moving forward. A law is now imposed: Given two quartets  $(s, p, m, n)$  and  $(q, r, t, u)$  then  $(s, p, m, n) \cdot (q, r, t, u) := (\frac{2}{3}sq, pr, m + (1 + t), nu)$ . To see that this construction was indeed a group, closure, inverse and identity were checked. The group is found in its most basic form to be composed of three different groups. Let's call them  $G_1, G_2, G_3$ .  $G_1$  has underlying set  $R^*$  and has the operation  $s.q := \frac{2}{3}sq$ .  $G_2$  is the multiplicative group over the real numbers excluding 0. This takes care of the second and fourth elements of the quartet, and  $G_3$  is has underlying set  $R$  with  $m.t = m + (1 + t)$ . These three groups are can be easily shown to be indeed groups [httd]. The group is then  $G_1 \times G_2 \times G_3 \times G_2$ . It has identity  $(e_1, e_2, e_3, e_2) = (\frac{3}{2}, 1, -1, 1)$  and inverse  $(\frac{9}{4s}, \frac{1}{p}, -m - 2, \frac{1}{n})$ . This group is an Abelian Lie group as seen directly from its construction.

## 2 Group Representations

It was suggested that a left regular representation exists for this group. This followed from the fact that locally compact Hausdorff topological groups have finite-dimensional representations [httb]. A representation into  $GL_5(R)$  was

suggested to exist [httb] and found.

$$\begin{pmatrix} \frac{2}{3}q & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1+t \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s \\ p \\ m \\ n \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}qs \\ pr \\ 1+m+t \\ nu \\ 1 \end{pmatrix}$$

It was also suggested [htta] that this group would have a faithful unitary representation on the sequence space  $l^P(G)$ . This follows from the fact that  $G_1$  and  $G_2$  have finite dimensional unitary representations and thus so does their product.  $\rho_1 \times \rho_2 : G_1 \times G_2 \rightarrow U(n_1) \times U(n_2) \hookrightarrow U(n_1 + n_2)$ .  $G_3$  is isomorphic to  $R$  and has a faithful unitary representation [htta] too.

### 3 Algebra

This group is an abelian Lie algebra following from the fact that  $G_1, G_2, G_3$  are abelian groups. The product of the groups  $G_i$  is an abelian Lie group. The tangent space is gotten by considering the derivative of the map,  $g := e^{ut}$  at the identity. This amounts to  $u \cdot e^{ut}|_{t=0}$  which makes the tangent space of the group a copy of  $R^* \times R^* \times R \times R^*$  [httc]. This leads to trivial Lie brackets and structure constants.

### 4 Conclusion

The abstract group  $R^4 \times C_2^3$  was realized and studied. A representation into  $GL_5$  was found, and it was also shown that this group has an 8-dimensional faithful unitary representation. The Lie Algebra of the group was constructed and found to have vanishing lie brackets and structure constants.

### Acknowledgements

I learned about the very basics groups by asking very simple questions on mathematics stack exchange. I want to acknowledge and thank every single person I interacted with while constructing a basic group, and asking silly questions.

### References

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