A report on a basic group that was implemented and studied

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Abstract

A topological group, G was implemented and its representations studied. It was found to have a representation into $GL_5(R)$. This representation was realized. This Abelian Lie was also found to have an 8-dimensional faithful unitary representation.

1 Basic group implementation

The construction started by considering two tuples of four numbers henceforth referred to as quartets and a multiplication law. Let us take a quartet to be (s,p,m,n). Let s,p,n $\in R \setminus \{0\}$, and and m $\in R$. We now take $R \setminus \{0\}$ to be R^* moving forward. A law is now imposed: Given two quartets (s, p, m, n) and (q, r, t, u) then $(s, p, m, n).(q, r, t, u) := (\frac{2}{3}sq, pr, m + (1 + t), nu)$. To see that this construction was indeed a group, closure, inverse and identity were checked. The group is found in its most basic form to be composed of three different groups. Let's call them G_1,G_2,G_3. G_1 has underling set R^* and has the operation $s.q := \frac{2}{3}sq$. G_2 is the multiplicative group over the real numbers excluding 0. This takes care of the second and fourth elements of the quartet, and G_3 is has underlying set R with $m.t = m+(1+t)$. These three groups are can be easily shown to be indeed groups [httc]. The group is then $G_1 \times G_2 \times G_3 \times G_2$ It has identity $(e_1, e_2, e_3, e_2) = (\frac{3}{2}, 1, -1, 1)$ and inverse $(\frac{9}{4s}, \frac{1}{p}, -m-2, \frac{1}{n})$. This group is an Abelian Lie group as seen directly from its construction.

2 Group Representations

It was suggested that a left regular representation exists for this group. This followed from the fact that locally compact Hausdorff topological groups have finite-dimensional representations [httb]. A representation into $GL_5(R)$ was suggested to exist [httb] and found.

$$\begin{pmatrix} \frac{2}{3}q & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1+t \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s \\ p \\ m \\ n \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}qs \\ pr \\ 1+m+t \\ nu \\ 1 \end{pmatrix}$$

It was also suggested [htta] that this group would have a faithful unitary representation on the sequence space $l^P(G)$. This follows from the fact that G_1 and G_2 have finite dimensional unitary representations and thus so does their product. $\rho_1 \times \rho_2 : G_1 \times G_2 \to U(n_1) \times U(n_2) \hookrightarrow U(n_1 + n_2)$. G_3 is isomorphic to R and has a faithful unitary representation[htta] too.

3 Algebra

This group is an abelian Lie algebra following from the fact that G_1, G_2, G_3 are abelian groups. The product of the groups G_i is an abelian Lie group. The tangent space of the group at the identity is a copy of $mathbbR$. This makes the Lie brackets and structure constants trivial.

4 Conclusion

The abstract group $R^4 \times C_2^3$ was realized and studied. A representation into GL_5 was found, and it was also shown that this group has an 8-dimensional faithful unitary representation.

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References

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