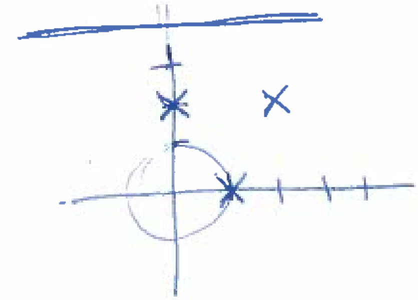
$$D = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} * \\ 4 \end{pmatrix} \right\} \qquad \Theta = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \sigma_1^2 \\ \sigma_2^2 \end{bmatrix} \qquad \Theta^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad x_4 = \begin{pmatrix} x_{41} \\ 4 \end{pmatrix}$$



$$Q(\theta, \theta^{old}) = \int_z \ln p(x, z \mid \theta)\, p(z \mid x, \theta^{old})$$

E-STEP

$$p(z \mid x, \theta^{old}) = \frac{p(z, x \mid \theta^{old})}{p(x \mid \theta^{old})}$$

$$= \frac{p(z, x \mid \theta^{old})}{\int p(z, x \mid \theta^{old})\, dz}$$

$$= \frac{p(x_{41}, x_{42} \mid \theta^{old})}{\int p(x_{41}', x_{42} \mid \theta^{old})\, dx_{41}'}$$

E-STEP

$$p(z \mid X, \theta^{old}) = \frac{\frac{1}{2\pi \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^{1/2}} e^{-\left[\frac{1}{2}x_{41}^2 + \frac{4^2}{2}\right]}}{\int \frac{1}{2\pi \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^{1/2}} e^{-\left(\frac{1}{2}x_{41}^2 + \frac{4^2}{2}\right)} dx_{41}}$$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_{41}^2} \cancel{\frac{1}{\sqrt{2\pi}} e^{-\frac{4^2}{2}}}}{\cancel{\frac{1}{\sqrt{2\pi}} e^{-\frac{4^2}{2}}} \underbrace{\cancel{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_{41}^2} dx_{41}}}_{1}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_{41}^2}$$

note $e^{-\left[\frac{1}{2}x_{41}^2 + \frac{4^2}{2}\right]}$

is $e^{-\frac{1}{2}\begin{bmatrix} x_{41} & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_{41} \\ 4 \end{bmatrix}}$

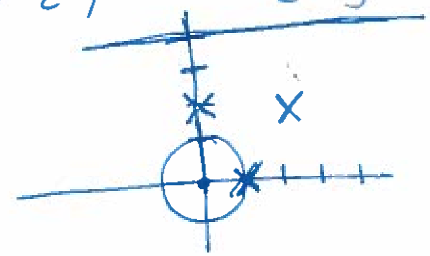
$= e^{-\frac{1}{2}\begin{bmatrix} x_{41} & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{41} \\ 4 \end{bmatrix}}$

$= e^{-\frac{1}{2}x_{41}^2 + \frac{4^2}{2}}$

and $\begin{bmatrix} x_{41} & 4 \end{bmatrix} = [x_{41} - 0 \quad 4-$

$$D = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} * \\ 4 \end{pmatrix} \right\}$$

$$\theta = \begin{Bmatrix} \mu_1 \\ \mu_2 \\ \sigma_1^2 \\ \sigma_2^2 \end{Bmatrix} \qquad \theta^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad x_4 = \begin{pmatrix} x_{41} \\ 4 \end{pmatrix}$$



$$Q(\theta, \theta^{old}) = \int_z \ln p(X, Z \mid \theta)\, p(Z \mid X, \theta^{old})$$

$$= \int_{-\infty}^{\infty} \ln p(X, Z \mid \theta) \frac{1}{\sqrt{2\pi}} e^{-\frac{x_{41}^2}{2}} dx_{41}$$

$$= \sum_{k=1}^{3} \ln p(x_k \mid \theta) \cancelto{1}{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_{41}^2}{2}} dx_{41}} + \int_{-\infty}^{\infty} \ln p\left(\begin{matrix} x_{41} \\ 4 \end{matrix} \middle| \theta\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x_{41}^2}{2}} dx_{41}$$

$$= \sum_{k=1}^{3} \ln p(x_k \mid \theta) + \int_{-\infty}^{\infty} \ln \left[\frac{1}{2\pi \begin{vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{vmatrix}^{1/2}} e^{-\frac{(x_{41}-\mu_1)^2}{2\sigma_1^2} - \frac{(4-\mu_2)^2}{2\sigma_2^2}} \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{x_{41}^2}{2}} dx_{41}$$

$$= \sum_{k=1}^{3} \ln p(x_k \mid \theta) + \int_{-\infty}^{\infty} \left[\ln\left(\frac{1}{2\pi \sigma_1 \sigma_2}\right) - \frac{(x_{41}-\mu_1)^2}{2\sigma_1^2} - \frac{(4-\mu_2)^2}{2\sigma_2^2} \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{x_{41}^2}{2}} dx_{41}$$

$$= \sum_{k=1}^{3} \ln p(x_k \mid \theta) + \left[\ln\left(\frac{1}{2\pi \sigma_1 \sigma_2}\right) - \frac{(4-\mu_2)^2}{2\sigma_2^2} \right] \cancelto{1}{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_{41}^2}{2}} dx_{41}} + \int_{-\infty}^{\infty} -\frac{(x_{41}-\mu_1)^2}{2\sigma_1^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_{41}^2}{2}} dx_{41}$$

$$= \sum_{k=1}^{3} \ln p(x_k|\theta) + \ln\left(\frac{1}{2\pi\sigma_1\sigma_2}\right) - \frac{(4-\mu_2)^2}{2\sigma_2^2} + \int_{-\infty}^{\infty} -\frac{x_{41}^2}{2\sigma_1^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x_{41}^2}{2}}dx_{41} + \int_{-\infty}^{\infty}\frac{2x_{41}\mu_1}{2\sigma_1^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x_{41}^2}{2}}dx_{41} - \int_{-\infty}^{\infty}\frac{\mu_1^2}{2\sigma_1^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x_{41}^2}{2}}dx_{41}$$

$$= \sum_{k=1}^{3} \ln p(x_k|\theta) + \ln\left(\frac{1}{2\pi\sigma_1\sigma_2}\right) - \frac{(4-\mu_2)^2}{2\sigma_2^2} - \frac{1}{2\sigma_1^2}\underbrace{\int x_{41}^2\frac{1}{\sqrt{2\pi}}e^{-\frac{x_{41}^2}{2}}dx_{41}}_{\text{Variance}=1} + \frac{2\mu_1}{2\sigma_1^2}\underbrace{\int_{-\infty}^{\infty} x_{41}\frac{1}{\sqrt{2\pi}}e^{-\frac{x_{41}^2}{2}}dx_{41}}_{\text{mean}=0} - \frac{\mu_1^2}{2\sigma_1^2}\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x_{41}^2}{2}}dx_{41}}_{=1}$$

$$= \sum_{k=1}^{3} \ln p(x_k|\theta) + \ln\left(\frac{1}{2\pi\sigma_1\sigma_2}\right) - \frac{(4-\mu_2)^2}{2\sigma_2^2} - \frac{1}{2\sigma_1^2} - \frac{\mu_1^2}{2\sigma_1^2}$$

$$= \sum_{k=1}^{3} \ln \frac{1}{2\pi\sigma_1\sigma_2} - \left[\frac{(x_{k1}-\mu_1)^2}{2\sigma_1^2} + \frac{(x_{k2}-\mu_2)^2}{2\sigma_2^2}\right] - \frac{(4-\mu_2)^2}{2\sigma_2^2} - \frac{1}{2\sigma_1^2} - \frac{\mu_1^2}{2\sigma_1^2} + \ln\left(\frac{1}{2\pi\sigma_1\sigma_2}\right)$$

$$= 4\ln\left(\frac{1}{2\pi\sigma_1\sigma_2}\right) - \frac{\mu_1^2}{2\sigma_1^2} - \frac{(2-\mu_2)^2}{2\sigma_2^2} - \frac{(1-\mu_1)^2}{2\sigma_1^2} - \frac{\mu_2^2}{2\sigma_2^2} - \frac{(2-\mu_1)^2}{\sigma_1^2} - \frac{(2-\mu_2)^2}{2\sigma_2^2} - \frac{(4-\mu_2)^2}{2\sigma_2^2} - \frac{1}{2\sigma_1^2} - \frac{\mu_1^2}{2\sigma_1^2}$$

M step - Maximize Q wrt Θ

$$\frac{\partial(Q(\theta,\theta^{old}))}{\partial \mu_1} = \frac{-2\mu_1}{2\sigma_1^2} + \frac{2(1-\mu_1)}{2\sigma_1^2} + \frac{2(2-\mu_1)}{2\sigma_1^2} - \frac{2\mu_1}{2\sigma_1^2}$$

setting $=0$ $\quad -2\mu_1 + 2 - 2\mu_1 + 4 - 2\mu_1 - 2\mu_1 = 0$

$$6 = 8\mu_1$$

$$\mu_1 = 3/4$$

$$\frac{\partial(Q(\theta,\theta^{old}))}{\partial \mu_2} = \frac{2(2-\mu_2)}{2\sigma_2^2} - \frac{2\mu_2}{2\sigma_2^2} + \frac{2(2-\mu_2)}{2\sigma_2^2} + \frac{2(4-\mu_2)}{2\sigma_2^2}$$

setting $=0$ $\quad 4 - 2\mu_2 - 2\mu_2 + 4 - 2\mu_2 + 8 - 2\mu_2 = 0$

$$16 = 8\mu_2$$

$$\mu_2 = 2$$

$$\frac{\partial(Q(\theta,\theta^{old}))}{\partial \sigma_1^2} = -2\frac{\sigma_2^2}{\sigma_1^2\sigma_2^2} + \frac{\mu_1^2}{2(\sigma_1^2)^2} + \frac{(1-\mu_1)^2}{2(\sigma_1^2)^2} + \frac{(2-\mu_1)^2}{2(\sigma_1^2)^2} + \frac{1}{2(\sigma_1^2)^2} + \frac{\mu_1^2}{2(\sigma_1^2)^2}$$

setting $=0$ $\quad -4(\sigma_1^2) = -\mu_1^2 - (1-\mu_1)^2 - (2-\mu_1)^2 - 1 - \mu_1^2$

$$\sigma_1^2 = \frac{\frac{9}{16} + \left(\frac{1}{16}\right) + \left(\frac{25}{16}\right) + \frac{16}{16} + \frac{9}{16}}{4}$$

$$= \frac{60}{64} = .9375$$

$$4\ln\left(\frac{1}{2\pi\sigma_1\sigma_2}\right) = 4\ln\left(\frac{1}{2\pi}\right) + 4\ln\frac{1}{\sigma_1\sigma_2}$$

$$= 4\ln\frac{1}{2\pi} + 4\ln\left[(\sigma_1^2\sigma_2^2)^{-1/2}\right]$$

$$= 4\ln\frac{1}{2\pi} - 2\ln(\sigma_1^2\sigma_2^2)$$

M-step

$$\frac{\partial Q(\theta;\theta^{old})}{\partial \sigma_2^2} = \frac{-2\cancel{\sigma_1^2}}{\cancel{\sigma_1^2}\sigma_2^2} + \frac{(2-\mu_2)^2}{2(\sigma_2^2)^2} + \frac{\mu_2^2}{2(\sigma_2^2)^2} + \frac{(2-\mu_2)^2}{2(\sigma_2^2)^2} + \frac{(4-\mu_2)^2}{2(\sigma_2^2)^2}$$

setting $=0$ $\quad +4\sigma_2^2 = 0 + 4 + 0 + 4$

$$\sigma_2^2 = 2$$