Chapter 3 - Ex 47

解:

a) $Q(\theta;\theta^0) = \int p(D_b|D_g;\theta^0)\ln[p(D_g,D_b|\theta)]\,dD_b$

$$= E_{D_b}[\ln p(D_g,D_b|\theta)]$$

$$= \sum_{k=1}^{3} E_{x_{kb}}[\ln p(x_{kg},x_{kb}|\theta)]$$

$$= \int_{-\infty}^{+\infty}[\ln(p(x_1|\theta))+\ln(p(x_2|\theta))+\ln(p(x_3|\theta))]\cdot p(x_{32}|x_{31}=2;\theta^0)\,dx_{32}$$

$$= \int_{-\infty}^{+\infty}\ln(p(x_1|\theta))\cdot p(x_{32}|x_{31}=2;\theta^0)\,dx_{32}$$

$$+\int_{-\infty}^{+\infty}\ln(p(x_2|\theta))\cdot p(x_{32}|x_{31}=2;\theta^0)\,dx_{32}$$

$$+\int_{-\infty}^{+\infty}\ln(p(x_3|\theta))\cdot p(x_{32}|x_{31}=2;\theta^0)\,dx_{32}$$

$$= \ln(p(x_1|\theta))+\ln(p(x_2|\theta))$$

$$+\int_{-\infty}^{+\infty}\ln\left[p\left(\begin{pmatrix}x_{31}\\x_{32}\end{pmatrix}\Big|\theta\right)\right]\cdot p(x_{32}|x_{31}=2;\theta^0)\,dx_{32}$$

令 $K=\int_{-\infty}^{+\infty}\ln\left[p\left(\begin{pmatrix}x_{31}\\x_{32}\end{pmatrix}\Big|\theta\right)\right]p(x_{32}|x_{31}=2;\theta^0)\,dx_{32}$

$$=\int_{-\infty}^{+\infty}\ln\left[p\left(\begin{pmatrix}x_{31}=2\\x_{32}\end{pmatrix}\Big|\theta\right)\right]\cdot\frac{p(x_{32},x_{31}=2|\theta^0)}{p(x_{31}=2|\theta^0)}\,dx_{32}$$

又 $p(x_1)\sim\begin{cases}\frac{1}{\theta_1}e^{-\frac{x_1}{\theta_1}}, & x_1\geq 0\\ 0 & 其它\end{cases}$，且 $\theta_1^0=2$

$\therefore p(x_{31}=2|\theta^0)=\frac{1}{2e}$

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$\therefore K = 2e \int_{-\infty}^{+\infty} \ln[p(\binom{x_{31}=2}{x_{32}}|\theta)] \cdot p(x_{32}, x_{31}=2|\theta^0)\, dx_{32}$

假设每个样本第1维与第2维特征相互独立.

$\therefore p(x_{32}, x_{31}=2|\theta^0) = p(x_{32}|\theta^0) \cdot p(x_{31}=2|\theta^0)$

$\therefore K = 2e \int_{-\infty}^{+\infty} \ln[p(\binom{x_{31}=2}{x_{32}}|\theta)] \cdot p(x_{31}=2|\theta^0) \cdot p(x_{32}|\theta^0)\, dx_{32}$

$= 2e \int_0^{\theta_2} \ln(\frac{1}{\theta_1} e^{-\frac{2}{\theta_1}} \cdot \frac{1}{\theta_2}) \cdot \frac{1}{2e} \cdot \frac{1}{4}\, dx_{32}$

$= \frac{1}{4} \int_0^{\theta_2} \ln(\frac{1}{\theta_1\theta_2} \cdot e^{-\frac{2}{\theta_1}})\, dx_{32}$

$= \frac{1}{4}\theta_2 \ln(\frac{1}{\theta_1\theta_2} \cdot e^{-\frac{2}{\theta_1}})$

$\therefore Q(\theta;\theta^0) = \ln[p(x_1|\theta)] + \ln[p(x_2|\theta)] + \frac{1}{4}\theta_2 \cdot \ln(\frac{1}{\theta_1\theta_2} e^{-\frac{2}{\theta_1}})$

$= \ln(\frac{1}{\theta_1} e^{-\frac{1}{\theta_1}} \cdot \frac{1}{\theta_2}) + \ln(\frac{1}{\theta_1} e^{-\frac{3}{\theta_1}} \cdot \frac{1}{\theta_2}) + \frac{1}{4}\theta_2 \ln(\frac{1}{\theta_1\theta_2} e^{-\frac{2}{\theta_1}})$

$= \ln(\frac{1}{\theta_1^2\theta_2^2} \cdot e^{(-\frac{1}{\theta_1}-\frac{3}{\theta_1})}) + \frac{1}{4}\theta_2 \ln(\frac{1}{\theta_1\theta_2} e^{-\frac{2}{\theta_1}})$

$= \ln e^{-\frac{4}{\theta_1}} - \ln(\theta_1^2\theta_2^2) + \frac{1}{4}\theta_2 [\ln e^{-\frac{2}{\theta_1}} - \ln(\theta_1\theta_2)]$

$= -\frac{4}{\theta_1} - 2\ln(\theta_1\theta_2) + (-\frac{1}{2})\frac{\theta_2}{\theta_1} - \frac{1}{4}\theta_2 \ln(\theta_1\theta_2)$

$= -\frac{8+\theta_2}{2\theta_1} - (2+\frac{\theta_2}{4}) \ln(\theta_1\theta_2)$

b) $\frac{\partial Q(\theta;\theta^0)}{\partial \theta_1} = (-\frac{8+\theta_2}{2}) \cdot (-\frac{1}{\theta_1^2}) - (2+\frac{\theta_2}{4}) \frac{1}{\theta_1\theta_2} \cdot \theta_2$

$= \frac{8+\theta_2}{2\theta_1^2} - (2+\frac{\theta_2}{4}) \cdot \frac{1}{\theta_1} = 0$

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$$\Rightarrow \quad (8+\theta_2) - 2\theta_1\left(2+\frac{\theta_2}{4}\right) = 0 \quad ①$$

又 $$\frac{\partial Q(\theta;\theta^0)}{\partial \theta_2} = -\frac{1}{2\theta_1} - \frac{1}{4}\ln(\theta_1\theta_2) - \left(2+\frac{\theta_2}{4}\right)\cdot\frac{1}{\theta_1\theta_2}\cdot\theta_1$$

$$= -\frac{1}{2\theta_1} - \frac{1}{4}\ln(\theta_1\theta_2) - \left(\frac{2}{\theta_2}+\frac{1}{4}\right)$$

$$= -\frac{\theta_2}{2} - \frac{\theta_1\theta_2}{4}\ln(\theta_1\theta_2) - 2\theta_1 - \frac{\theta_1\theta_2}{4} = 0 \quad ②$$

结合 ①，② 可算出 使 $Q(\theta;\theta^0)$ 最大化

的 $\theta^1 = \begin{pmatrix}\theta_1^1\\ \theta_2^1\end{pmatrix}$