Coupon collector's problem with unlike probabilities

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Abstract

In this note we study the coupon collector's problem with unlike probabilities using majorization and a Schur concave function.

Introduction and Results

The coupon collector's problem, an important example in a course on elementary probability, can be described as follows: Suppose that there are t types of coupons, and that each day a collector randomly gets a coupon with probability p_i corresponding to the ith coupon ($1 \le i \le t$). How many days does the collector need to collect in order to have at least one of each type? There is a lot of literature about this problem (e.g. see [1, I.2 (b.11))], [2, §4], [3, Example II.3.11], [7]).

Let $X_{\mathbf{p}}$ be a random variable representing the number of days until all kinds of coupons have been collected, where $\mathbf{p} = (p_1, \dots, p_t)$ is a stochastic vector, namely

$$\sum_{i=1}^{t} p_i = 1 \quad \text{and} \quad p_i \ge 0.$$

Note that the explicit expectation form is known as

$$E(X_{\mathbf{p}}) = \sum_{q=0}^{t-1} (-1)^{t-1-q} \sum_{|J|=q} \frac{1}{1-P_J}, \tag{1}$$

where $P_J = \sum_{j \in J} p_j$ (see [2, Equation (14b)]). [7, §2] writes the distribution in more generality and with a different notation. When p_i is equal to $\frac{1}{t}$ for all $1 \le i \le t$, the expectation is well-known, that is,

$$E\left(X_{\left(\frac{1}{t},\dots,\frac{1}{t}\right)}\right) = t \sum_{i=1}^{t} \frac{1}{i} \sim t \log t.$$

Moreover the distribution is also well-known, that is,

$$P\left(X_{\left(\frac{1}{t},\dots,\frac{1}{t}\right)} \le n\right) = \sum_{k=0}^{t} (-1)^k \binom{t}{k} \left(1 - \frac{k}{t}\right)^n$$

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(see [1, IV.2 Equation (2.3)]), which is easily represented by

$$P\left(X_{\left(\frac{1}{t},\ldots,\frac{1}{t}\right)} \le n\right) = \frac{t!\left\{ {n \atop t} \right\}}{t^n}, \tag{2}$$

where $\left\{ {n \atop t} \right\}$ is Stirling numbers of the second kind (see [3, Example II.3.11]).

In this note, we show that $P(X_{\mathbf{p}} \le n)$ is monotone with respect to $\mathbf{p}$ in the sense of *majorization*. The same setting for the Birthday Problem was studied by Joag-Dev and Proschan [4].

Before stating the main theorem, we define a few terms and introduce some standard notations (see [6, 8]). For a stochastic vector $\mathbf{p} = (p_1, \ldots, p_t)$ let $p_{[j]}$ be the jth largest value of $\{p_1, \ldots, p_t\}$, that is, $p_{[1]} \ge p_{[2]} \ge \cdots \ge p_{[t]}$. A stochastic vector $\mathbf{p} = (p_1, p_2, \ldots, p_t)$ is *majorized* by a stochastic vector $\mathbf{q} = (q_1, q_2, \ldots, q_t)$, or $\mathbf{p} \prec \mathbf{q}$, if

$$\sum_{i=1}^{k} p_{[i]} \le \sum_{j=1}^{k} q_{[j]}$$

for all $1 \le k \le t-1$. By definition it is easy to see that $\left(\frac{1}{t}, \ldots, \frac{1}{t}\right) \prec \mathbf{p}$ for all $\mathbf{p}$. The symmetric function $f(\mathbf{p})$ defined on stochastic vectors is *Schur convex* (resp. *concave*) if $\mathbf{p} \prec \mathbf{q}$ implies $f(\mathbf{p}) \le f(\mathbf{q})$ (resp. $f(\mathbf{p}) \ge f(\mathbf{q})$). Under the assumption of symmetry and differentiability of f, a necessary and sufficient condition for $f(p_1, \ldots, p_n)$ to be Schur concave is

$$(p_1 - p_2)\left(\frac{\partial f}{\partial p_1} - \frac{\partial f}{\partial p_2}\right) \le 0 \tag{3}$$

(see [6, p.57]). Under these preliminaries we state the main theorem.

Theorem 1 *The probability $P(X_{\mathbf{p}} \le n)$ is a Schur concave function of $\mathbf{p}$.*

Now a random variable X is *stochastically smaller* than a random variable Y if $P(X > a) \le P(Y > a)$ for all real a (see [5, Chap. IV.1.1]).

Corollary 1 *If $\mathbf{p} \prec \mathbf{q}$ then $X_{\mathbf{p}}$ is stochastically smaller than $X_{\mathbf{q}}$. In particular $X_{\left(\frac{1}{t},\ldots,\frac{1}{t}\right)}$ is stochastically smaller than $X_{\mathbf{p}}$ for all $\mathbf{p}$.*

Proof. By Theorem 1 if $\mathbf{p} \prec \mathbf{q}$ we have $P(X_{\mathbf{p}} > n) \le P(X_{\mathbf{q}} > n)$ for all n. Since $\left(\frac{1}{t}, \ldots, \frac{1}{t}\right) \prec \mathbf{p}$ for all $\mathbf{p}$, we obtain the desired result.

Thus it is harder to collect all kinds of coupons if there is some bias for the probability of appearance of coupons.

Corollary 2 *The expectation $E(X_{\mathbf{p}})$ is a Schur convex function of $\mathbf{p}$. In particular $E\left(X_{\left(\frac{1}{t},\ldots,\frac{1}{t}\right)}\right) \le E(X_{\mathbf{p}})$ for all $\mathbf{p}$.*

Proof. By Corollary 1 if $\mathbf{p} \prec \mathbf{q}$ we have

$$E(X_{\mathbf{p}}) = \sum_{n=0}^{\infty} P(X_{\mathbf{p}} > n) \le \sum_{n=0}^{\infty} P(X_{\mathbf{q}} > n) = E(X_{\mathbf{q}}),$$

which implies Schur convexity.

Proof of Theorem 1.

For convenience, letting $f(\mathbf{p}) = P(X_{\mathbf{p}} \le n)$ we have

$$f(\mathbf{p}) = \sum_{\{i_1,\ldots,i_n\}=\{1,\ldots,t\}} p_{i_1} \cdots p_{i_n},$$

where the summation runs through all $(i_1, \ldots, i_n)$ such that there exists a surjection $g : \{1, \ldots, n\} \to \{1, \ldots, t\}$ satisfying $g(k) = i_k$ for $1 \le k \le n$. Accordingly we have

$$f(\mathbf{p}) \quad = \quad \sum_{(l_1,\ldots,l_t)} \binom{n}{l_1; \cdots; l_t} p_1^{l_1} p_2^{l_2} \cdots p_t^{l_t} = n! \sum_{(l_1,\ldots,l_t)} \frac{p_1^{l_1}}{l_1!} \cdot \frac{p_2^{l_2}}{l_2!} \cdots \frac{p_t^{l_t}}{l_t!},$$

the sum being taken over $l_k \ge 1$ and $\sum_{k=1}^{t} l_k = n$. Note that because of

$$\sum_{(l_1,\ldots,l_t)} \binom{n}{l_1; \cdots; l_t} = t! \left\{ {n \atop t} \right\}$$

we can confirm Equation (2) if $p_1 = \cdots = p_t = 1/t$ (see [3, II.3.4]). Hence we have

$$f(\mathbf{p}) = n![z^n] \prod_{i=1}^{t} (e^{zp_i} - 1),$$

where $[z^n]A(z)$ denotes the coefficient of z^n for $A(z)$. Because f is symmetric and differentiable with respect to $p_1, \ldots, p_t$, we check Equation (3) to show the Schur concavity of $f(\mathbf{p})$. Since

$$\frac{\partial f}{\partial p_1} = n![z^n] z e^{zp_1} \prod_{i=2}^{t} (e^{zp_i} - 1),$$

we have

$$\frac{\partial f}{\partial p_1} - \frac{\partial f}{\partial p_2} = n![z^{n-1}](e^{zp_2} - e^{zp_1}) \prod_{i=3}^{t} (e^{zp_i} - 1) = (p_2 - p_1) h(\mathbf{p}),$$

where $h(\mathbf{p})$ is some positive function. Hence

$$(p_1 - p_2) \left(\frac{\partial f}{\partial p_1} - \frac{\partial f}{\partial p_2} \right) = -(p_1 - p_2)^2 h(\mathbf{p}) \le 0$$

yields that $f(\mathbf{p})$ is Schur concave.

Note that by virtue of Corollary 1 we can prove Corollary 2 without applying directly the criterion Equation (3).

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