Assignment 3

Reading Assignment:

1. Section 1.5: Recurrence and Transience
2. Section 1.6: Recurrence and Transience of Random Walks

Problems:

1. Show that, for the Markov chain $\{X_n\}_{n\geq 0}$ in Exercise 1.3.4 we have

$$\Pr(X_n \to \infty \text{ as } n \to \infty) = 1.$$

   Suppose, instead, the transition probabilities satisfy

$$p_{i,i+1} = \left(\frac{i+1}{i}\right)^{\alpha} p_{i,i-1}.$$

   For each $\alpha \in (0, \infty)$ find the value of $\Pr(X_n \to \infty \text{ as } n \to \infty)$.

2. Denote by T_j the first passage time to state j and set

$$f_{ij}^{(n)} = \Pr_i(T_j = n).$$

   Justify the identity

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \quad \forall n \geq 1$$

   and deduce that

$$P_{ij}(s) = \delta_{ij} + F_{ij}(s) P_{jj}(s)$$

   where

$$P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n, \quad F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n.$$

   Hence show that $\Pr_i(T_i < \infty) = 1$ if and only if

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$$

   without using Theorem 1.5.3.

3. A random sequence of non-negative integers $\{F_n\}_{n\geq 0}$ is obtained by setting $F_0 = 0$ and $F_1 = 1$ and, once $F_0, \ldots, F_n$ are known taking F_{n+1} to be either the sum of the difference of F_{n-1} and F_n, each with probability 1/2. Is $\{F_n\}_{n\geq 0}$ a Markov chain?

   By considering the Markov chain $M_n = (F_{n-1}, F_n)$, find the probability that $\{F_n\}_{n\geq 0}$ reaches 3 before first returning to 0.

Draw enough of the flow diagram for $\{X_n\}_{n\geq 0}$ to establish a general pattern. Hence, using the strong Markov property, show that the hitting probability for (1,1), starting from (1,2), is $(3-\sqrt{5})/2$.

Deduce that $\{X_n\}_{n\geq 0}$ is transient. Show that, moreover, with probability 1, $F_n \to \infty$ as $n \to \infty$.