Assignment 6 (Extra Problems)

Problems:

1. (IP: 6.2.3) A spider and a fly move along a straight line in unit increments. The spider always moves towards the fly by one unit. The fly moves towards the spider by one unit with probability 0.3, moves away from the spider with one unit with probability 0.3, and stays in place with probability 0.4. The initial distance between the spider and the fly is integer. When the spider and the fly land in the same position, the spider captures the fly.

   (a) Construct a Markov chain that describes the relative location of the spider and the fly.

   (b) Identify the transient and recurrent states.

2. (IP: 6.3.10) Alvin likes to sail each Saturday to his cottage on a nearby island off the coast. Alvin is an avid fisherman, and enjoys fishing off his boat on the way to and from the island, as long as the weather is good. Unfortunately, the weather is good on the way to or from the island with probability p, independently of what the weather was on any past trip (so the weather could be nice on the way to the island, but poor on the way back). Now if the weather is nice, Alvin will take one of his n fishing rods for the trip, but if the weather is bad, he will not bring a fishing rod with him. We want to find the probability that on a given leg of the trip to or from the island the weather will be nice, but Alvin will not fish because all his fishing rods are at his other home.

   (a) Formulate an appropriate Markov chain model with $n+1$ states and find the steady-state probabilities.

   (b) What is the steady-state probability that on a given trip, Alvin sails with nice weather but without a fishing rod?

3. (IP: 6.3.13) **Ehrenfest model of diffusion.** We have a total of n balls, some of them black, some white. At each time step, we either do nothing, which happens with probability ϵ, where $0 < \epsilon < 1$, or we select a ball at random, so that each ball has probability $(1-\epsilon)/n > 0$ of being selected. In the latter case, we change the color of the selected ball (if white it becomes black, and vice versa), and the process is repeated indefinitely. What is the steady-state distribution of the number of white balls?

4. (IP: 6.3.14) **Bernoulli-Laplace model of diffusion.** Each of two urns contains m balls. Out of the total of $2m$ balls, m are white and m are black. A ball is simultaneously selected from each urn and moved to the other urn, and the process is indefinitely repeated. What is the steady-state distribution of the number of white balls in each urn?

5. (IP: 6.3.16) The parking garage at TAMU has installed a card-operated gate, which, unfortunately, is vulnerable to absent-minded faculty and staff. In particular, in each day, a car crashes the gate with probability p, in which case a new gate must be installed. Also a gate that has survived for m days must be replaced as a matter of periodic maintenance. What is the long-term expected frequency of gate replacements?

6. (IP: 6.3.18) **Uniqueness of solutions to the balance equations.** Consider a Markov chain with a single recurrent class, plus possibly some transient states.

   (a) Assuming that the recurrent class is aperiodic, show that the balance equations together with the normalization equation have a unique nonnegative solution. *Hint:* Given a solution different from the steady-state probabilities, let it be the PMF of X_0 and consider what happens as time goes to infinity.

   (b) Show that the uniqueness result of part (a) is also true when the recurrent class is periodic. *Hint:* Introduce a self-transitions in the Markov chain, in a manner that results in an equivalent set of balance equations, and use the result of part (a).

Hint The answer to the last problem is on page 363 of "Introduction to Probability" by D.P. Bertsekas and J.N. Tsitsiklis. Try to solve the problem on your own before you look it up.