Homework 1

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Exercise 1

Let $X = (x_1, x_2, \dots, x_n)$
$x_i \sim Pois(\lambda)$ $\lambda \sim \Gamma(\alpha, \beta)$

We have that:

$$\begin{aligned} P(\lambda \mid X) &= \frac{P(X \mid \lambda)P(\lambda)}{P(X)} \\ &\propto P(X \mid \lambda)P(\lambda) \\ &\propto \lambda^{n\bar{x}} e^{-n\lambda} \lambda^{\alpha-1} e^{-\beta\lambda} \\ &= \lambda^{n\bar{x}+\alpha-1} e^{-(n+\beta)\lambda} \end{aligned}$$

Therefore, $P(\lambda \mid X) \sim \Gamma(n\bar{x} + \alpha, \beta + n)$ and the Gamma distribution is conjugate to the Poisson distribution.

Exercise 2

Let $s = (s_1, s_2, \ldots, s_n)$, $s' = (s'_1, s'_2, \ldots, s'_n)$ and $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$
Detailed balance equation: $\pi(s)P(s, s') = \pi(s')P(s', s)$

w.l.g consider an update for s_1

Case 1: $s_{-1} \neq s'_{-1}$

$$\pi(s)P(s, s') = \pi(s')P(s', s) = 0$$

Case 2: $s_{-1} = s'_{-1}$

$$\begin{aligned}
\pi(s)P(s, s') &= \pi(s)P(s'_1 \mid s'_{-1}) \\
&= \pi(s)\frac{\pi(s')}{\sum_z \pi(z, s'_{-1})} \\
&= \pi(s')\frac{\pi(s)}{\sum_z \pi(z, s_{-1})} \\
&= \pi(s')P(s_1 \mid s_{-1}) \\
&= \pi(s')P(s', s)
\end{aligned}$$

Consider the move from s to s'. The acceptance probability for MH will be

$$\frac{\pi(s')P(s', s)}{\pi(s)P(s, s')}$$

We have that

$$\begin{aligned}
\frac{P(s', s)}{P(s, s')} &= \frac{\frac{\pi(s)}{\sum_z \pi(z, s_{-1})}}{\frac{\pi(s')}{\sum_z \pi(z, s'_{-1})}} \\
&= \frac{\pi(s)}{\pi(s')}
\end{aligned}$$

So

$$\frac{\pi(s')P(s', s)}{\pi(s)P(s, s')} = \frac{\pi(s')\pi(s)}{\pi(s)\pi(s')} = 1$$