Rejection Sampling

September 23, 2021

Rejection sampling is a sampling algorithm to sample data from a complex multivariate distribution using a proxy distribution.

Let's set up the scenario. Assume our target distribution $p(x)$ is easy to sample from but only up to a normalizing constant Z. So,

$$p(x) = \frac{\tilde{p}(x)}{Z}$$

where $\tilde{p}(z)$ is easy to sample from. Let $q(z)$ be any distribution that is easy to sample from. $q(z)$ is called the *proposal distribution*. Define $kq(z)$ such that it 'blankets' $\tilde{p}(z)$ (in other words, $kq(z) \geq \tilde{p}(z)$). See Figure 1 below.

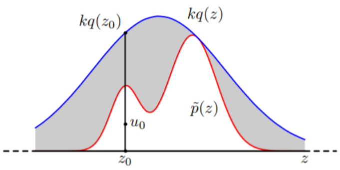

Figure 1: 'Blanket' distribution

We make use of the fact that $q(z)$ is (1) easy to sample from and (2) is never less than $\tilde{p}(z)$. (1) Sampling from $q(z)$ allows us to propose values (hence, *proposal distribution*) that may or may not be supported by the target distribution. (2) Because $kq(z) \geq \tilde{p}(z)$, every possible value of $\tilde{p}(z)$ has a positive probability of being sampled. Therefore, we do not need to worry about missing any values.

In short, the algorithm samples some $z_0 \sim q(z)$ and decide whether we want to accept or reject that sample based on some criteria. The resulting set of samples will be distributed according to the target distribution $\tilde{p}(z)$. See Algorithm 1 for more details.

Algorithm 1 Rejection Sampling

$z_0 \sim q(z)$
$u \sim U(0, kq(z_0))$
if $u \leq \tilde{p}(z_0)$ **then**
    Accept
else
    Reject
end if

1 Notes

- Complex target function f(x)
- Proposal function g(x)
- Scale g(x) with some constant c to 'blanket' the target function – cg(x)
- Sample some $x_0 \sim g(x)$
- x_0 will be a valid input to our distribution if f(x) and g(x) have the same range
- Compute $cg(x_0)$ and $f(x_0)$
- Sample $u \sim U(0, cg(x))$
- Our acceptance criterion is $u \leq f(x)$
- Alternate formulation (more common):
- Acceptance criterion: $u \leq f(x)/cg(x)$
- Sample $u \sim U(0, 1)$