# October 27

## Announcements

- Webassign 3.3, 4.1, 4.2 due this Thursday
- Worksheet 4 solutions posted
- Worksheet 5 posted, due this Friday

## 4.2 Basis and Dimension

Recall that a subspace $S$ satisfies 2 properties.

- It contains the zero vector.
- It is closed under linear combinations. If $u, v \in S$ then $au + bv \in S$ as well.

It is always the case that $S$ is the span of a set of vectors because $S = span(S)$. What is the minimal number of vectors required to span $S$? What is the maximal number of vectors that are linearly indepedent in $S$? It turns out these numbers are equal and that number is called the dimension.

**Definition:** A set $B = \{u_1, \ldots, u_m\}$ is a *basis* for a subspace $S$ if

- $B$ spans $S$
- $B$ is linearly indepedent

In this case, the *dimension* of $S$ is $m$. Dimension is an invariant independent on the choice of basis.

The empty set is the basis for the zero subspace $\{0\}$. It has dimension 0.

**Theorem:**

- A set $B$ is a basis for $S$ if it is a maximal linearly independent set. This means you can't add any vectors to $B$ and have it still be linearly independent.

- A set $B$ is a basis for $S$ if it is a minimal set that spans $S$. This means you can't substract any vectors from $B$ and have it still be spanning.

**Example:**

Let $S$ be the subspace given by $w + x + y + z = 0$. What is the dimension? Give 2 3 different basis.

**Definition:**

The row space of a matrix $A$ is the span of the rows. It is denote $row(A)$.

**Theorem:**

Let $A$ and $B$ be equivalent matrices. Then the

Proof: The matrix $A$ is equivalent to $B$ if $A$ can be obtained from $B$ using elementary row opeartions. There are 3 such operations. None of them change the span.

**Example:**

Let $S$ be the subspace spanned by $u_1 = (-1, 2, 3, 1), u_2 = (-6, 7, 5, 2), u_3 = (4, -3, 1, 0)$. Find a basis for $S$. What is the dimension of $S$?

(just find rref, and use the rows)

Notice we just take the nonzero rows.

**Theorem:**

Suppose $U = [u_1\ u_2\ \ldots\ u_m]$ and $V = [v_1\ \ldots\ v_m]$ be equivalent matrices. Then any relation between the $u_i$ exists between the $v_i$. For example,

If $2u_1 - u_2 = u_3$ then $2v_1 - v_2 = v_3$.

Proof: Relations correspond to solutions to $Ux = 0$ and $Vx = 0$. Since they are equivalent, the relations must be the same.

**Example:**

Let $S$ be the subspace spanned by $u_1 = (-1, 2, 3, 1), u_2 = (-6, 7, 5, 2), u_3 = (4, -3, 1, 0)$. Find a basis for $S$. What is the dimension of $S$?

(use rref to determine relations between columns)

Notice we only take the columns with leading variables.

## 4.3 row and column spaces

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transform given by $T(x) = Ax$. Recall that the the range is the span of the columns of $A$ and the kernel is the space of solutions to $Ax = 0$. The dimension of the range is called the rank of $A$, the dimension of the kernel is called the nullity.

Do an example of this in class.