

KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
Total:	72	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0 .
- When I define a variable, it is defined for that whole question. The A defined in Question 2 is the same for each part.
- I treat row and column vectors as the same.
- For any linear transformation T , there exists a matrix A such that $T(x) = Ax$. I defined the determinant, rank, and nullity of T using A . This means,

$$\det(T) = \det(A), \quad \text{rank}(T) = \text{rank}(A), \quad \text{nullity}(T) = \text{nullity}(A).$$

1. Give an example of each of the following. If it is not possible, write “NOT POSSIBLE”. You do not need to justify your answers.

- (a) (2 points) If possible, give an example of a 2×2 matrix A that is not diagonalizable but A^2 is diagonalizable.

Solution: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = 0$ which is diagonalizable.

- (b) (2 points) If possible, give an example of a 2×2 matrix A such that $A^2 = I_2$ and $\text{nullity}(A) = 1$.

Solution: NOT POSSIBLE. If $A^2 = A$, then A is invertible so the nullity is 0.

- (c) (2 points) If possible, give an example of a 2×2 matrix with distinct eigenvalues that is not invertible.

Solution: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

- (d) (2 points) If possible, give an example of linear transformations $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\text{rank}(T) = \text{rank}(S) = \text{rank}(T \circ S) = 1$.

Solution: $T(x, y) = S(x, y) = (x, 0)$.

- (e) (2 points) If possible, give an example a 2×4 matrix A such that $\text{rank}(A) = \text{nullity}(A)$.

Solution: Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

- (f) (2 points) If possible, give an example of a 2×2 matrix A such that 1 is not an eigenvalue of A^2 but 1 is an eigenvalue of A^4 . (Think geometrically).

Solution: Let A be rotation by $\pi/2$.

2. Perform the following computations.

(a) (6 points) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}.$$

- What is the reduced echelon form of A ?

Solution:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Does $Ax = (1, 2, 1)$ have a solution? If so, give the general solution.

Solution: Yes.

$$(1, 1, 0) + s_1(-2, -1, 0)$$

- Does $Ax = (1, 1, 1)$ have a solution? If so, give the general solution.

Solution: No.

- What is a basis for $\text{row}(A)$?

Solution:

$$\{(1, 0, 2), (0, 1, 1)\}$$

(b) (6 points) Let

$$B = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- What is B^{-1} ?

Solution:

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

- What is the characteristic polynomial of B^2 ? (Hint: Be careful about the sign.)

Solution: The eigenvalues of B^2 are the square of the eigenvalues of B . The characteristic polynomial is

$$(4 - \lambda)(1 - \lambda)^2$$

3. Let A and B be equivalent matrices given by

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 1 & 1 & 3 \\ 2 & 5 & 6 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

Denote the columns of A by a_1, a_2, a_3, a_4 .

- (a) (3 points) Give a basis for $\text{null}(A)$.

Solution:

$$\{(-3, -8, 5, 1)\}$$

- (b) (3 points) Give the general solution to $Ax = a_2$.

Solution:

$$(0, 1, 0, 0) + s_1(-3, -8, 5, 1)$$

- (c) (3 points) Give the general solution to $2Ax - a_3 = a_1 + a_2$.

Solution:

$$(1/2, 1/2, 1/2, 0) + s_1(-3, -8, 5, 1)$$

- (d) (3 points) It turns out $e_1 = (1, 0, 0, 0) \notin \text{col}(A)$. Give a vector v such that $v \neq e_1$ and $Ax = e_1 - v$ has a solution.

Solution: Take e_1 with any vector in $\text{col}(A)$. For example,

$$v = e_1 + a_1$$

4. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ and $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformations given by

$$T(x, y, z) = (x + 2z, y + z, x + y + z, z)$$

and

$$S(x, y, z) = (2x + y + z, y + z, y + z)$$

- (a) (2 points) There exists a matrix A such that $T(v) = Av$. What is A ?

Solution:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- (b) (2 points) There exists a matrix B such that $S(v) = Bv$. What is B ?

Solution:

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- (c) (4 points) These should be quick questions.

- What is $\text{rank}(S)$?

Solution: 2

- What is $\det(S)$?

Solution: 0

- What is $\text{nullity}(2S)$?

Solution: 1

- What are 2 different eigenvalues of S ?

Solution: 0, 2

- (d) (4 points) Recall that $(T \circ S)(v) = T(S(v))$.

- What is the rank of $T \circ S$?

Solution: 2. See next part for explanation.

- What is a basis for $\text{range}(T \circ S)$? (Hint: Look at the 2nd two columns of B).

Solution: The range of $T \circ S$ is spanned by $\{(T \circ S)(e_1), (T \circ S)(e_2), (T \circ S)(e_3)\}$ but $S(e_2) = S(e_3)$. So the range of $T \circ S$ is spanned by $\{(T \circ S)(e_1), (T \circ S)(e_2)\}$. Then choose your favorite method to determine that these are linearly independent.

5. Let

$$A = \begin{bmatrix} 0 & -1 & 3 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

be a matrix which decomposes as $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Let u_1, u_2, u_3, u_4 be the columns of U and $\mathcal{B} = \{u_1, u_2, u_3, u_4\}$.

(a) (6 points) Fill out the following table.

Eigenvalue λ	Alg. Multiplicity of λ	Geo. Multiplicity of λ	A Basis for E_λ
0	1	1	$\{u_1\}$
-1	1	1	$\{u_2\}$
2	2	2	$\{u_3, u_4\}$

(b) (3 points) What is a basis for $\text{range}(A)$?

Solution: $\{(0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$.

(c) (3 points) Let $x = u_1 + u_2 + u_3$. What is $[A^{18}x]_{\mathcal{B}}$? You are allowed to have exponents in your answer.

Solution: $(0, 1, 2^{18}, 0)$.

6. Let A, u_1, u_2, u_3, u_4 be as defined in Question 5.

(a) (2 points) What is $\det(A)$?

Solution: 0.

(b) (2 points) What is $\det(A + 2I)$?

Solution: The eigenvalues of $A + 2I$ are 2, 1, 4, 4. The determinant is then $2 \cdot 1 \cdot 4 \cdot 4$.

(c) (2 points) What is $\text{rank}(A)$?

Solution: This is just the rank of D . Or the number of nonzero eigenvalues of A , counting multiplicities.

(d) (2 points) What is $\text{rank}(A - 2I)$?

Solution: This is just the rank of $[A - 2I]_B$. Or the number of nonzero eigenvalues of $A - 2I$, counting multiplicities.

(e) (2 points) Does $Ax = -u_2 + u_3 - 4u_4$ have a solution? If so, express it as a linear combination of u_1, u_2, u_3, u_4 .

Solution: $x = u_2 + \frac{1}{2}u_3 - 2u_4$.

(f) (2 points) Does $Ax = 2u_1 + u_2 + u_3$ have a solution? If so, express it as a linear combination of u_1, u_2, u_3, u_4 .

Solution: No. The column space of A does not contain any vector with a nonzero first component.