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Linear combinations and span

Definition: If $u_1, u_2, \ldots, u_m$ are vectors and

$c_1, c_2, \ldots, c_m$ are scalars, then

$$c_1u_1 + c_2u_2 + \ldots + c_mu_m$$

is a *linear combination* of $u_1, \ldots, u_m$. Note that the constants can be negative or zero.

Definition: Let $S = \{u_1, u_2, \ldots, u_m\}$ be a set of vectors. Then the span of S, span$S$, is the set of all linear combinations of $u_1, u_2, \ldots, u_m\}$.

What vectors in $\mathbb{R}^2$ are a linear combination of $(1, 0)$ and $(0, 1)$? In other words, what vectors are in the span of $(1, 0)$ and $(0, 1)$?

What vectors in $\mathbb{R}^2$ are a linear combination of $(1, 2)$ and $(0, 1)$? Talk about lines and averages here.

Is $(3, 4)$ a linear combination of $(1, 2)$ and $(0, 1)$? In other words, is $(3, 4)$ in the span of $(1, 2)$ and $(0, 1)$? In other words, does there exists $x_1, x_2 \in \mathbb{R}$ such that $x_1(1, 2) + x_2(0, 1) = (3, 4)$? In other words, system of equations!

Every system of equation can be interpeted in this way.

Theorem: Let $u_1, \ldots, u_m$ and v be vectors in $\mathbb{R}^n$. Then $v \in \text{span}(\{u_1, \ldots, u_m\})$ if and only if the linear system with augmented matrix $[u_1\ u_2\ \ldots\ u_m|v]$ has a solution.

The solution space can be expressed as a linear combination.

Theorem: Let $u_1, u_2, \ldots, u_m$ be vectors in $\mathbb{R}^n$. If $u \in$span$(\{u_1, \ldots, u_m\})$, then span$(\{u_1, \ldots, u_m\})$ =span$(\{u_1, \ldots, u_m, u\})$.

When does a set of vectors span $\mathbb{R}^n$?

Theorem: Let $u_1, u_2, \ldots, u_m$ be vectors in $\mathbb{R}^n$. Let $A = [u_1\ u_2\ \ldots\ u_m]$ and $B \sim A$, where B is in echelon form. Then $\text{span}(\{u_1, \ldots, u_m\}) = \mathbb{R}^n$ if and only if B has a pivot position in every row.

We can write linear systems as $Ax = b$.