Worksheet 2

Due 10/13

1. We know how to obtain the general solution from a linear system. Let's try to reverse it. Find a linear system who's general solution is

$$(x_1, x_2, x_3, x_4) = (1, 2, 3, 4) + s_1(5, 6, 7, 8) + s_2(9, 0, 1, 2).$$

ANSWER: This question turned out to be easier and harder than I thought.

The easier approach is to write out the 4 equations, solve for s_1 and to get rid of it, then do the same for s_2.

Here's the harder approach I had in mind. We have 2 free variables. I will choose x_2 and x_3 to be the free variables. The goal is to manipulate the form of the general solution until it looks like

$$(x_1, x_2, x_3, x_4) = (X, 0, 0, X) + s_1(X, 1, 0, X) + s_2(X, 0, 1, X),$$

where X is any number.

Let $u = (1, 2, 3, 4), v = (5, 6, 7, 8), w = (9, 0, 1, 2)$. The general solution is equivalent to $\{u + x : x \in \text{span}(v, w)\}$. We have some freedom. We can replace u with any particular solution and we can replace v, w with any other 2 vectors with the same span.

Let $v_2 = (v - 7w)/6$. Then v_2 and w has the same span as v, w and $v_2 = (-29/3, 1, 0, -1)$.

Let $u_2 = u - 2v_2 - 3w$. So $u_2 = (-20/3, 0, 0, 0)$.

We now have that

$$(x_1, x_2, x_3, x_4) = (-20/3, 0, 0, 0) + s_1(-29/3, 1, 0, -1) + s_2(9, 0, 1, 2)$$

is of the desired form. By setting $x_2 = s_1$ and $x_3 = s_2$, we see that $x_1 = -20/3 - 29/3x_2$ and $x_4 = -x_2 + 2x_3$.

2. Suppose A is a matrix. Let v, w be distinct (meaning $x \neq y$) vectors that solve $Ax = 0$ so $Av = 0$ and $Aw = 0$ (0 here of course means the zero vector!). Let L be the line that passes through v and w. If u is on L, then $Au = 0$. Why? This exercise suggests that solution spaces are convex.

ANSWER:

If u is on the line that passes through v and w, then u is of the form $sv + (1-s)w$. Then $A(u) = sA(v) + (1 - s)A(w) = s0 + (1 - s)0 = 0$.

3. Let $z_1, z_2 \in \mathbb{R}$ and let $S = \{(1, z_1, z_2), (2, 1, 0), (1, 0, -1)\}$.

- Find some values for z_1 and z_2 such that S spans $\mathbb{R}^3$.

- Find some values for z_1 and z_2 such that S does not span $\mathbb{R}^3$.
- Find all values for z_1 and z_2 such that S spans $\mathbb{R}^3$. (In the process of solving this problem, some of you will be tempted to divide by zero. Resist that temptation.)

ANSWER: The problem becomes much easier once you reorder the vectors. This does not affect linear indepedence! The matrix use to determined linear independence is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & z_1 \\ -1 & 0 & z_2 \end{bmatrix}$$

This matrix is much easier to use than the order the vectors were presented in. The nasty $(1, z_1, z_2)$ is in the last column and the topright entry is a 1. This matrix is equivalent to

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & z_1 \\ 0 & 2 & 1+z_2 \end{bmatrix}$$

by adding the first row to the last. We can then see that this matrix is has a pivot in each column whenever $(1, z_1)$ is not parallel to $(2, 1+z_2)$. So the vectors are linearly independent whenever $2z_1 \neq 1 + z_2$.

4. Consider the following linear system that came from the book and the lecture.

$$2x_1 - 6x_2 - x_3 + 8x_4 = 0 \tag{1}$$
$$x_1 - 3x_2 - x_3 + 6x_4 = 0 \tag{2}$$
$$-x_1 + 3x_2 - x_3 + 2x_4 = 0. \tag{3}$$

Using row reduction, we see that a general solution is of the form $x = s_1(3, 1, 0, 0) + s_2(-2, 0, 4, 1)$. Let $v_1 = (2, 1, -1), v_2 = (-6, -3, 3), v_3 = (-1, -1, 2), v_4 = (8, 6, 2)$.

- Is $\{v_1, v_2, v_3, v_4\}$ is linearly independent set? The answer should be no.
- Express v_1 as a linear combination of v_2, v_3, v_4.
- Express v_2 as a linear combination of v_1, v_3, v_4.
- Express v_3 as a linear combination of v_1, v_2, v_4.
- Express v_4 as a linear combination of v_1, v_2, v_3.

ANSWER:

By setting $s_1 = 1$ and $s_2 = 2$, we obtain a nontrivial solution to the system which implies $v_1 + v_2 + 4v_3 + v_4 = 0$. By solving for v_1 here, we can write v_1 as a linear combination of v_2, v_3, v_4. Same for the others.

5. Suppose $\{v_1, v_2, v_3\}$ is a linearly dependent set. Is it always the case that we can write v_1 as a linear combination of v_2 and v_3? If not, come up with a counterexample.

ANSWER:

No. Take $v_1 = (1, 0), v_2 = (0, 1), v_3 = (0, 2)$.

6. Come up with a inconsistent linear system whose associated homogenous linear system is consistent.

ANSWER:

A homogeneous linear system is always consistent so any inconsistent linear system is an example.