# October 25

## Announcements

- Worksheet 4 posted, due Friday
- Webassign 3.2 due Thursday

## 3.2 Matrix Algebra

**Definition:** The $i$th standard basis vector, denoted $e_i$, for $\mathbb{R}^n$ is the length $n$ vector consisting of all zeros except a one in the $i$th position. The set of all standard basis vectors for $\mathbb{R}^n$ is call the standard basis. When it matters, we will regard $e_i$ a column vector.

**Theorem** Let $A = [a_1 \ \ldots \ n]^t$ be an $n \times k$ matrix and $B$ be a $k \times m$ matrix. Then the rows of $AB$ are $a_1 B, \ldots, a_n B$.

**Example** Let's consider the square matrices $A = [2, -1; 1, 3]$ and $B = [4, -2; -1, 1]$. We can think of $AB$ in two ways. Either $A$ is acting on $B$ by combining the rows or $B$ is acting on $A$ by combining the columns.

We should think about what the standard basis does. So what is $e_i^t B$?

We should think of $[2, -1]$ as $2e_1^t - e_2^t$.

## 3.3 Inverses

The compositon of $f$ and $g$ is the function $(f \circ g)$ where $(f \circ g)(x) = f(g(x))$.

**Definition:** Let $A, B$ be sets and $f : A \to B$ be a function. Then the (two-sided) inverse of $f$ is a function $g : B \to A$ such that $g \circ f$ is the identity on $B$ (which means $(g \circ f)(x) = x$) and $f \circ g$ is the identity on $A$. We often denote $g$ here by $f^{-1}$.

The goal is to invert linear transforms. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be linear transform. When can $T$ be invert? It has to be one-to-one at least. Suppose $T(x_1) = T(x_2)$. Then by applying $T^{-1}$, we have $x_1 = x_2$. This implies $n \geq m$. By using the same argument, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ must be one-to-one as well so $m \geq n$. Therefore, $n = m$.

**Theorem:** Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transform such that $T(x) = Ax$. Then $T$ is invertible if and only if $m = n$ and the columns of $A$ are linearly independent (or spanning).

Let's $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transform given by $T(x) = [1, 2; 3, 4]x$. So we know $T(1, 0) = (1, 2)$ and $T(0, 1) = (3, 4)$. This implies $T^{-1}(1, 2) = (1, 0)$ and

$T^{-1}(3,4) = (0,1)$. To determine $T^{-1}$, we just need to know what $T^{-1}(1,0)$ and $T^{-1}(0,1)$ are. Row reduction!

The determinant of a 2,2 matrix is blah. Here's the formula for the inverse of a 2,2 matrix.

**Theorem:** Let $A$ and $B$ be invertible matrices and $C$ and $D$ be matrices. Then

- $A^{-1}$ is also invertible.
- $AB$ is invertible. The inverse is given by $(AB)^{-1} = B^{-1}A^{-1}$.
- If $AC = AD$ then $C = D$
- If $CA = DA$ then $C = D$

**Theorem:** Let $A$ be a $n \times n$ matrix. Let $S$ be the columns of $A$. Let $T(x) = Ax$. Then the following are equivalent:

- $S$ spans $\mathbb{R}^n$
- $S$ is linearly indepedent
- $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$ given by $x = A^{-1}b$.
- $T$ is onto
- $T$ is one-to-one
- $T$ is invertible
- $A$ is invertible

## Subspaces

Here's the defintion of subspaces. Here's an example of what is a subspace. Here's an example of what isn't a subspaces.

The solutions to homoegenous equations are subspaces. This is why we care.

Introduce kernel and range.