Midterm Review

2018-07-18

Throughout, let V, W be vector spaces over a field F and $T : V \to W$ a linear map.

1. Prove that the intersection of 2 subspaces is a subspace.
2. Let $V_1, V_2 \subseteq V$ be subspaces. Find necessary and sufficient conditions for $V_1 \cup V_2$ to be a subspace.
3. Let $V_1, V_2 \subseteq V$ be subspaces. Find necessary and sufficient conditions for $V_1 \setminus V_2$ to be a subspace.
4. Let $V_1, V_2 \subseteq V$ be subspaces. Find necessary and sufficient conditions for $V_1 + V_2 = \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$ to be a subspace.
5. Prove that $V_1 + V_2 := \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$ is the smallest subspace of V containing both V_1 and V_2.
6. Prove that $V \times W$ is a vector space with the addition law and scalar multiplication law derived from the addition law and scalar multiplication law from V and W so
$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \quad c(v_1, w_1) = (cv_1, cw_1).$$
7. Suppose V_1, V_2 are subspaces of V, prove
$$\dim V_1 + \dim V_2 - \dim V \geq \dim V_1 \cap \dim V_2.$$
8. Prove that $S = \{p \in P_5(F) : p'' + 2p' = 0\}$ is a subspace of $P_5(F)$. What is $\dim S$?
9. Prove that $S = \{A \in M_n(F) : \mathrm{tr}(A) = 0\}$ is a subspace of $M_n(F)$. What is $\dim S$?
10. Is the set of invertible $n \times n$ matrices a subspace?
11. Is the set of symmetric $n \times n$ matrices a subspace?
12. Is the set of 3×3 rank 2 matrices a subspace?
13. Suppose $T : V \to W$ and $S : W \to V$ are linear maps so that $S \circ T$ is an isomorphism. Prove that S is onto and T is one-to-one. Give an example where S is not one-to-one and T is not onto.
14. Prove that T is onto if and only if $T(S)$ is spanning whenever S is spanning.
15. Prove that T is one-to-one if and only if $T(S)$ is linearly independent whenever S is linearly independent.
16. Prove that T is an isomorphism if and only if $T(B)$ is a basis for any basis B.
17. Suppose $\{u, v\}$ is a basis for V. Is $\{u - v, u + v\}$ a basis for V?
18. Suppose $\{u, v, w\}$ is a basis for V. Is $\{u - v, v - w, w - u\}$ a basis for V?
19. (Definitely not on exam) Let B be a basis for $\mathbf{R}$ as a $\mathbf{Q}$ vector space. Prove that B is uncountable.