Worksheet 5 - Due 11/3

1. Extend $\{(1,-1,0,0),(1,0,-1,0)\}$ to a basis for the subspace, W, defined by $w+x+y+z=0$. In other words, find a basis for W that includes $(1,-1,0,0)$ and $(1,0,-1,0)$.

 Solution: The subspace W is 3-dimensional because the associated linear system has 3 free variables. $(1,0,0,-1)$ is not in the span of $(1,-1,0,0)$ and $(1,0,-1,0)$ so including in the set will form a basis.

2. Let P be the plane given by $2x+y+z=0$ in $\mathbb{R}^3$.
 (a) What is a normal vector to P?

 Solution: $(2,1,1)$.

 (b) Give a basis for $\mathbb{R}^3$ that includes a normal vector to P and 2 vectors that lie on P.

 Solution: $\{(2,1,1),(1,-2,0),(1,0,-2)\}$.

 (c) Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be the linear transform that reflects all vectors across P. This means that $T(n)=-n$ whenver n is normal to P and $T(v)=v$ if v lies on P. Find A such that $T(x)=Ax$.

 Solution: We know that

$$T(2,1,1)=(-2,-1,-1),\quad T(1,-2,0)=(1,-2,0),\quad T(1,0,-2)=(1,0,-2).$$

 So we can use the technique described in problem 2 of worksheet 4. We know that the linear transform with associated matrix

$$\begin{bmatrix}2 & 1 & 1\\ 1 & -2 & 0\\ 1 & 0 & -2\end{bmatrix}^{-1}$$

 sends $(2,1,1)$ to e_1, $(1,-2,0)$ to e_2, and $(1,0,-2)$ to e_3. The linear transform with associated matrix

$$\begin{bmatrix}-2 & 1 & 1\\ -1 & -2 & 0\\ -1 & 0 & -2\end{bmatrix}$$

 sends e_1 to $(-2,-1,-1)$, e_2 to $(1,-2,0)$, and e_3 to $(1,0,-2)$. It follows that

$$\begin{bmatrix}-2 & 1 & 1\\ -1 & -2 & 0\\ -1 & 0 & -2\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\ 1 & -2 & 0\\ 1 & 0 & -2\end{bmatrix}^{-1}=\begin{bmatrix}-\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3}\\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}\end{bmatrix}$$

 will send $(2,1,1)$ to $(-2,-1,-1)$, $(1,-2,0)$ to $(1,-2,0)$, and $(1,0,-2)$ to $(1,0,-2)$.

 (d) What is the rank of T? What is the nullity of T?

 Solution: The associated matrix to T was defined as a product of invertible matrices and is therefore invertible as well. This implies that T has rank 3 and nullity 0.

3. Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be the linear transform defined by $T(1,1,1)=(1,0)$, $T(1,0,1)=(1,1)$, and $T(1,1,0)=(0,2)$.

(a) Before doing a single computation, what can you already say about the rank and nullity of T?

> **Solution:** It should be clear that T is onto since $(1, 0)$ and $(1, 1)$ spans the codomain. Therefore, T has rank 2 and nullity 1.

(b) Give a matrix A such that $T(x) = Ax$. You may express A as a product of matrices and their inverses.

> **Solution:** This is similiar to the previous problem with the reflection. We have that
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> $$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}.$$

(c) What is the rank and nullity of T?

> **Solution:** See first part.

4. Give an example of each of the following. If it is not possible, write NOT POSSIBLE.

(a) Find an invertible 3×3 matrix A and a 3×3 matrix B such that $rank(AB) \neq rank(BA)$.

> **Solution:** NOT POSSIBLE. This is because since A is invertible, we know that $rank(AB) = rank(BA) = rank(B)$.
>
> Since A is invertible, it is equivalent to the identity matrix. This means that $A = E_1E_2 \dots E_mI_3$, where each E_i is some row operation matrix and I_3 is the identity matrix. We then have $AB = E_1E_2 \dots E_mB$. This means that AB is equivalent to B so $rank(AB) = rank(B)$.
>
> To establish $rank(BA) = rank(B)$, we will take advantage of tranposes. We have
>
> $$rank(BA) = rank((BA)^t) = rank(A^tB^t) = rank(B^t) = rank(B).$$
>
> The first equality is because $rank(X) = rank(X^t)$ for any matrix X because the dimension of the column space of X is the same as the dimension of the row space of X^t and these are both the rank. The second equality is a basic property of transposes. The third equality is because since A is invertible, A^t is also invertible so we can use the argument in the 2nd paragraph. The fourth equality is the same as the first equality.

(b) Find two 3×3 matrices A and B, each with nullity 1 such that AB is the zero matrix.

> **Solution:** NOT POSSIBLE. Let T be the linear transform defined by $T(x) = Ax$ and S be the linear transform defined by $S(x) = Bx$. We know that the dimension of the range of T is 2 while the dimension of the kernel of S is 1. This means some element in the range of T is not in the kernel of S. This means that $S \circ T$ is not the zero transformation. This means that AB is not the zero matrix.

(c) Find two 3×3 matrices A and B, each with rank 1 such that AB is the zero matrix.

> **Solution:** Let A be the matrix such that Ax is the projection onto the 1st coordination and B be the matrix such that Bx is the projection onto the 2nd coordination. Since AB is the zero matrix.

(d) Find two 3×3 matrices A and B, each with nullity 2 such that AB is the zero matrix.

> **Solution:** By the rank-nullity theorem, this problems in the same as part (c).

(e) Find two 3×3 matrices A and B, each with rank 2 such that AB is the zero matrix.

> **Solution:** By the rank-nullity theorem, this problems in the same as part (b).