

## October 9

### Announcements

- Section 2.1, 2.2 due this Thursday
- Section 2.3, 3.1 due next Thursday
- Midterm next Wednesday in class
  - 1.1 - 3.1 (maybe 3.2)
- Worksheet 1 solutions has been posted
- Worksheet 2 has been posted, due this Friday
- Wednesday office hours to be held in classroom?

### Homogenous Systems

Let  $A$  be a matrix. Then  $A(x + y) = Ax + Ay$  and  $A(x - y) = Ax - Ay$ .

**Example:** Find a general solution for the linear system

$$2x_1 - 6x_2 - x_3 + 8x_4 = 7 \quad (1)$$

$$x_1 - 3x_2 - x_3 + 6x_4 = 6 \quad (2)$$

$$-x_1 + 3x_2 - x_3 + 2x_4 = 4. \quad (3)$$

Using row reduction, we see that a general solution is of the form  $x = (1, 0, -5, 0) + s_1(3, 1, 0, 0) + s_2(-2, 0, 4, 1)$ .

The solution to the homogenous system is  $x = s_1(3, 1, 0, 0) + s_2(-2, 0, 4, 1)$ .

Let  $x_p$  be a particular solution  $Ax = b$ . Then solutions have the form  $x_g = x_p + x_h$ , where  $x_p$  is a particular solution and  $x_h$  is the general solution to the homogenous equations.

### Linear Independence and Span

**Theorem:** Let  $A = [a_i]$  and  $b$  be a vector in  $\mathbb{R}^n$ . Then the following are equivalent (if one is true then they are all true, if one is false then they are all false).

- The set  $\{a_1, \dots, a_m\}$  are linearly independent.
- The vector equation  $x_1 a_1 + x_2 a_2 + \dots + x_m a_m = b$  has at most one solution.
- The linear system  $[a_1 \ a_2 \ \dots \ a_m | b]$  has at most one solution.
- The equation  $Ax = b$  has at most 1 solution.

**Example:** Consider the vectors  $a_1 = (1, 7, -2)$ ,  $a_2 = (3, 0, 1)$ , and  $a_3 = (5, 2, 6)$ . Set  $A = [a_i]$ . Show that the columns of  $A$  are linearly independent and that  $Ax = b$  has a unique solution for every  $b$  in  $\mathbb{R}^3$ .

**Example:** Let  $u_1 = (1, -1, 2)$ ,  $u_2 = (2, -1, 2)$ ,  $u_3 = (-2, 5, -10)$ ,  $u_4 = (3, -4, 8)$ . The associated matrix has reduced echelon form:

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is  $\{u_1, \dots, u_4\}$  linearly independent? Can we write  $u_1$  as a linear combination of  $u_2, \dots, u_4$ ?

If a set of vectors is not linearly independent, can every vector be written as a linear combination of the other vectors? In other words, is every vector in the span of the other vectors?

## Section 3.1 Linear Transformations

We can write linear equations as  $Ax = b$ . We can think of it as  $A$  sending  $x$  to  $b$ .

**Definition:** A function  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a linear transformation if for all vectors  $u, v \in \mathbb{R}^m$  and all scalars  $r$ , we have

- $T(u+v) = T(u) + T(v)$
- $T(ru) = rT(u)$ .

**Examples:**

- What are some examples of functions that aren't linear transforms? quadratic,  $ax+b$
- Consider the function given by  $T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 5x_2)$ . What is  $T(1, 2)$ ? Show that this is a linear transformation. Is it associated to a matrix?
- Projections are linear transforms.
- Let  $A$  be some matrix. Then  $T(x) = Ax$  is a linear transform. Make up some example in class.

A matrix,  $A$ , is said to be an  $n \times m$  matrix if it has  $n$  rows and  $m$  columns. If  $m = n$ , then  $A$  is a square matrix.

**Theorem:** Let  $A$  be an  $n \times m$  matrix, and define  $T(x) = Ax$ . Then  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a linear transformation. Moreover, all linear transformations are of this form.

**Example:** Consider the linear transformation with matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \end{bmatrix}.$$

Is  $(3, 4)$  in the range of  $A$ ?

**Theorem:** Let  $A = [a_1 \ a_2 \ \dots \ a_m]$  be a  $n \times m$  matrix, and let  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  with  $T(x) = Ax$  be a linear transformation. Then

- A vector  $w$  is in the range of  $T$  if and only if  $Ax = w$  is a consistent linear system.
- The range of  $T$  is the span of the columns (this is also called the column space).

If time, talk about 1-1 and onto