Worksheet 8 - Never due

1. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE".

   (a) Give an example of a basis of $\mathbb{R}^4$ such that each element lies in the hyperplane $2w + 3x + y + z = 0$.

   Solution: NOT POSSIBLE. The hyperplane $2w + 3x + y + z = 0$ is a 3-dimensional subspace. The span of any set of vectors in a 3-dimensional subspace is at most 3-dimensional.

   (b) Give an example of a basis of $\mathbb{R}^4$ such that each element lies in the hyperplane $2w + 3x + y + z = 1$.

   Solution: Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for $\mathbb{R}^4$. A basis for the hyperplane is given by $\{e_1/2, e_2/3, e_3\}$.

   (c) Give an example of a matrix that is orthogonally diagonalizable but not diagonalizable.

   Solution: NOT POSSIBLE. Any orthogonally diagonalizable matrix is diagonalizable.

   (d) Give an example of a matrix that is diagonalizable but not orthogonally diagonalizable.

   Solution: A matrix is orthogonally diagonalizable if and only if it is symmetric. This problem then amounts to finding a diagonalizable matrix that is not symmetric. For example,

   $$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

   This matrix is obviously not symmetric. We can see that it has nullity 1 so 0 is an eigenvalue. It fixes e_1 so 1 is an eigenvalue. It has 2 distinct eigenvalues so we know it is diagonalizable.

   (e) Give an example of a nonzero matrix A such that $A^2 = 0$.

   Solution:

   $$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   (f) Give an example of a nonzero matrix A such that $A^2 = I$.

   Solution: The easier example is

   $$\begin{bmatrix} -1 \end{bmatrix}.$$

   Any reflection matrix would work.

   (g) Give an example of a nonzero matrix A such that $A^2 = I$ and the nullity of A is 1.

   Solution: NOT POSSIBLE. If $A^2 = I$ then $A^{-1} = A$ so A is invertible and must have nullity 0.

   (h) Give an example of an orthogonal set that is not linearly independent.

Solution: $\{e_1, 0\}$.

(i) Give an example of an orthogonal set that is not spanning.

Solution: $\{0\}$.

(j) Give an example of a 2×3 matrix whose rank is equal to its nullity.

Solution: NOT POSSIBLE. Let A be a matrix. By the rank-nullity theorem, we know that $\text{rank}(A) + \text{nullity}(A) = 3$. Since 3 is odd, we know that $\text{rank}(A)$ cannot be $\text{nullity}(A)$.

(k) Give an example of 2 matrices A and B such that $A^3 = B^3$.

Solution: Let A be the 2d-rotation matrix by $2\pi/3$ and B be the 2d-rotation matrix by $-2\pi/3$. Then $A^3 = B^3 = I$.

(l) Give an example of 2 matrices A and B such that A and B each have nullity 1 but AB has nullity 0.

Solution: NOT POSSIBLE. The nullity of AB is always at least the nullity of B.

(m) Give an example of 2 matrices A and B such that A and B each have nullity 0 but AB has nullity 1.

Solution: NOT POSSIBLE. The main idea is that the composition of two one-to-one functions is one-to-one.

Suppose A and B are matrices with nullity 0. We will show that AB has nullity 0 as well. Let $x \in \text{null}(AB)$. Then $ABx = 0$. This means $A(Bx) = 0$ so $Bx \in \text{null}(A)$. But $\text{null}(A) = \{0\}$ so $Bx = 0$. This means $x \in \text{null}(B)$ but $\text{null}(B) = \{0\}$ so $x = 0$. This proves that the only vectors in $\text{null}(AB)$ is the zero vector so the nullity of AB is 0.

(n) Give an example of a diagonalizable matrix that is not invertible.

Solution: Any matrix with eigenvalue 0 is an example. For instance, the zero matrix.

(o) Give an example of an invertible matrix that is not diagonalizable.

Solution: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

This matrix is upper triangular so determining the eigenvalues amounts to reading off the diagonal. We have that A has eigenvalue 1 with multiplicity 2. But the eigenspace corresponding to 1 has dimension 1. This means A is not diagonalizable.

The matrix A is invertible because the determinant is 1.

(p) Give an example of a symmetric matrix that is not diagonalizable.

Solution: NOT POSSIBLE. All symmetric matrices are orthogonally diagonalizable and hence diagonalizable.

(q) Give an example of a symmetric matrix that is not invertible.

Solution: Zero matrix.

(r) Give an example of an orthogonal matrix that is not invertible.

Solution: NOT POSSIBLE. All orthogonal matrices, Q, are invertible with inverse Q^t.

(s) Give an example of an invertible matrix that is not orthogonal.

Solution:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

(t) Give an example of a matrix with distinct eigenvalues that is not invertible.

Solution:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(u) Give an example of a 3×3 orthogonal matrix with only one eigenvalue.

Solution: The identity matrix.

(v) Give an example of a 3×3 matrix whose only eigenvalue is 2.

Solution: The diagonal matrix with only 2's along the diagonal.

(w) Give an example of a 3×3 invertible matrix whose only eigenvalue is 2.

Solution: The diagonal matrix with only 2's along the diagonal.

(x) Give an example of a matrix A and an eigenvalue λ such that the algebraic multiplicity of λ is less than the geometric multiplicity.

Solution: NOT POSSIBLE. THe algebraic multiplicity is always at least the geometric multiplicity.

(y) Give an example of a matrix A and an eigenvalue λ such that the geometric multiplicity of λ is less than the algebraic multiplicity.

Solution: Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalue 0 has algebraic multiplicity 2 but geometric multiplicity 1.

(z) Give an example of a matrix A and an eigenvalue λ such that the eigenspace is 0-dimensional.

Solution: NOT POSSIBLE. A number is an eigenvalue if and only if the corresponding eigenspace is positive-dimensional.