

Eigenvalues and Eigenspaces

- Watch 13 and 14 of 3blue1brown

(insert picture of what a eigenvector is)

Definition: Let A be a $n \times n$ matrix. Then a nonzero vector u is an *eigenvector* if there exists a scalar λ such that $Au = \lambda u$. The scalar λ here is called the *eigenvalue*. Here u is an eigenvector associated to λ .

Examples:

- What are the eigenvalues and eigenvectors of a diagonal matrix?
- What are the eigenvalues and eigenvectors of problem 3 on the midterm?
- What are the eigenvalues and eigenvectors of reflection across a plane?
- Let $A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$. Determine if each of the following is an eigenvector for A . $u_1 = (5, 4)$, $u_2 = (4, -1)$, $u_3 = (-1, 1)$.

Theorem A square matrix is invertible if and only if 0 is not a eigenvalue.

Theorem/Definition: Let A be a $n \times n$ matrix with eigenvalue λ . Then the set of all eigenvectors associated to λ along with 0 forms a subspace, called the *eigenspace*, of \mathbb{R}^n . This is also the null space of $A - \lambda I$.

Theorem/Definition: Let A be an $n \times n$ matrix. Then λ is an eigenvalue if and only if $\det(A - \lambda I) = 0$. The polynomial $\det(A - \lambda I)$ is called the *characteristic polynomial* of A . The *multiplicity* of a eigenvalue is its multiplicity in the characteristic polynomial.

Example: Find the eigenvalues and a basis for each eigenspace for $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$.

It turns out that $\det(A - \lambda I)$ is $-\lambda^3 - \lambda^2 + \lambda + 1 = -(\lambda - 1)(\lambda + 1)^2$.

So we are just finding the basis for the nullspaces of $A - I$ and $A + I$ which we can do with row reductions.

Theorem: Let A be a square matrix with eigenvalue λ . Then the dimension of the associated eigenspace is less than or equal to the multiplicity of λ .