

Chapter 4 Review

Theorem: Let $S = \{a_1, \dots, a_m\}$ be a set of vectors of \mathbb{R}^n . Let $A = [a_1 \dots a_n]$ be a matrix and $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be the linear transform defined by $T(x) = Ax$. Let B be an echelon form of A . Then the following objects are equal:

- The set of vectors killed by T ,
- $\{x : Ax = 0\}$ (this is the set of homogeneous solutions to A),
- $\text{null}(A)$,
- $\{x : T(x) = 0\}$,
- $\ker(T)$,
- number of rows of all zeros in B ,

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- The set of vectors hit by T ,
 - $\{T(x) : x \in \mathbb{R}^n\}$,
 - $\text{range}(T)$,
 - $\text{col}(A)$,
 - $\text{span}(S)$,

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- $\dim(\text{col}(A))$,
 - $\dim(\text{range}(T))$,
 - $\dim(\text{span}(S))$,
 - $m - \text{nullity}(A)$ (rank-nullity theorem),
 - $m - \dim(\ker(T))$,
 - $\dim(\text{row}(A))$, (think of this as maximal number of linear independent equations in $Ax = 0$),
 - number of pivots in B ,

Example: Let $T(x) = Ax$, where A is

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -4 & 1 & -3 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

and has reduce echelon form B given by

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- What is the range of T ?
- What is the kernel of T ?
- What is the row space of A ?
- What is the rank of A ?
- What is the nullity of A ?

- Write the columns corresponding the free variables as a linear combination of the pivot columns.
- What is the general solution to $Ax = 0$?
- What is the general solution to $Ax = [2 - 3, 4]^t$?
- What is a vector not in the range of T ?

Example: Answer all the same questions as above but for an invertible transform.

Example: Give an example of a linear transform $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T(1, 1, 0) = (1, 0)$ and $T(0, 1, 2) = (1, 2)$.

- What is the smallest possible rank such an example could be?
- What is the largest possible rank such an example could be?
- What is the smallest possible nullity such an example could be?
- What is the largest possible nullity such an example could be?