

Math 308L - Autumn 2017  
Final Exam  
December 14, 2017

# KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
Total:	84	

- There are 7 problems on this exam. Be sure you have all 7 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

**Conventions:**

- I will often denote the zero vector by  $0$ .
- When I define a variable, it is defined for that whole question. The  $A$  defined in Question 2 is the same for each part.
- I often use  $x$  to denote the vector  $(x_1, x_2, \dots, x_n)$ . It should be clear from context.
- Sometimes I write vectors as a row and sometimes as a column. The following are the same to me.

$$(1, 2, 3) \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

- I write the evaluation of linear transformations in a few ways. The following are the same to me.

$$T(1, 2, 3) \quad T((1, 2, 3)) \quad T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

1. Give an example of each of the following. If it is not possible, write “NOT POSSIBLE”.

- (a) (2 points) Give an example of a  $2 \times 3$  matrix  $A$  and a vector  $b \in \mathbb{R}^2$  such that  $Ax = b$  has no solutions but  $Ax = 0$  has infinitely many solutions.

**Solution:** Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and  $b = (0, 1)$ .

- (b) (2 points) Give an example of a linear system in 3 variables whose solution space is the intersection of the  $x + y + z = 0$  plane and the  $xy$ -plane.

**Solution:** The linear system given by

$$\begin{aligned} x + y + z &= 0 \\ z &= 0 \end{aligned}$$

- (c) (2 points) Give an example of a  $2 \times 2$  matrix  $A$  such that  $A^4 = I_2$  but  $A^2 \neq I_2$ . If possible, give the matrix  $A$  explicitly.

**Solution:** Let  $A$  be the rotation by  $\pi/2$  matrix. This is given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

- (d) (2 points) Give an example of 2 linear transformations  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\text{range}(T) = \ker(S)$ .

**Solution:** Let  $T(x, y) = (x, y)$  and  $S(x, y) = (0, 0)$ .

- (e) (2 points) Give an example of an orthogonal matrix that is not invertible.

**Solution:** NOT POSSIBLE. The inverse of an orthogonal matrix is its transpose.

- (f) (2 points) Give an example of an diagonalizable matrix that is not orthogonally diagonalizable.

**Solution:**

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

2. Let  $A$  be defined by

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) (4 points) Find a basis for the solution space  $Ax = 0$ .

**Solution:**  $\{(2, -1, 0)\}$

(b) (4 points) What is the general solution to  $Ax = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$ ?

**Solution:**  $(6, 0, -3/2) + s_1(2, -1, 0)$ .

(c) (4 points) Is there a vector  $y \in \mathbb{R}^3$  such that  $Ax = y$  has no solutions? If so, give an example. If not, why not?

**Solution:** Yes. Many possibilities.

3. Let  $A$  and  $B$  be equivalent matrices defined by

$$A = \begin{bmatrix} -3 & 3 & -1 & -9 & 3 \\ 2 & -2 & 1 & 7 & -1 \\ 4 & -4 & 5 & 23 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

- (a) (4 points) Find a basis for the solution space of  $Ax = 0$ .

**Solution:**  $\{(1, 1, 0, 0, 0), (-2, 0, -3, 1, 0), (2, 0, -3, 0, 1)\}$

- (b) (4 points) Let  $a_1, a_2, a_3, a_4, a_5$  be the columns of  $A$ . Define  $C = [a_1 \ a_2 \ a_3 \ a_4]$ . What is a particular solution to  $Cx = a_5$ ?

**Solution:**  $(-2, 0, 3, 0)$ .

- (c) (4 points) Using the same variables as (b), what is the general solution to  $Cx = 2a_4 - a_5$ ?

**Solution:**  $(8, 0, 6, 0) + s_1(1, 1, 0, 0) + s_2(-2, 0, -3, 1)$ .

4. Let  $S$  be a subspace of  $\mathbb{R}^4$  defined by

$$S = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- (a) (3 points) What is a basis for  $(S^\perp)^\perp$ ?

**Solution:**  $\{(1, 1, 0, 0), (0, 0, 1, 1)\}$ .

- (b) (3 points) What is a basis for  $S^\perp$ ?

**Solution:**  $\{(1, -1, 0, 0), (0, 0, 1, -1)\}$ .

- (c) (3 points) Does there exist a rank 2 matrix  $A$  such that  $\text{null}(A) = S$ ? If so, give an example. If not, why not?

**Solution:** If  $\text{null}(A) = S$  then  $\text{row}(A) = S^\perp$  so we can take

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

- (d) (3 points) Does there exist a rank 3 matrix  $A$  such that  $\text{null}(A) = S$ ? If so, give an example. If not, why not?

**Solution:** No. By the rank-nullity theorem,  $\text{rank}(A) + \text{null}(A) = 4$ . Since  $\dim S = 2$ , the rank of  $A$  must be 2.

5. Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transform defined by the following properties:

- $T(0, 0, 1) = (0, 0, 0)$ ,
- If  $v$  is in the  $xy$ -plane, then  $v$  is reflected across the  $x + y = 0$  plane.

There is a matrix  $A$  such that  $T(x) = Ax$ . The goal of this problem is to understand  $A$ .

- (a) (3 points) Find a basis  $\{u, v, w\}$  where the action of  $T$  is well-understood. Give also  $T(u)$ ,  $T(v)$ , and  $T(w)$ .

**Solution:**

$$\begin{aligned} u &= (0, 0, 1), T(u) = (0, 0, 0) \\ v &= (1, 1, 0), T(v) = (-1, -1, 0) \\ w &= (1, -1, 0), T(w) = (1, -1, 0) \end{aligned}$$

- (b) (3 points) Find the eigenvalues of  $A$  and a basis for each eigenspace of  $A$ . (Think geometrically.)

**Solution:** Part (a) gives the answer.

$\lambda = 0$  is an eigenvalue with eigenspace spanned by  $u$ .

$\lambda = -1$  is an eigenvalue with eigenspace spanned by  $v$ .

$\lambda = 1$  is an eigenvalue with eigenspace spanned by  $w$ .

- (c) (3 points) What is  $A$ ? You may express it as product of matrices and their inverses.

**Solution:** Using the theory of diagonalization,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$$

- (d) (3 points) What is  $A^2$ ? Give it explicitly as a single matrix. (Think geometrically.)

**Solution:** We can see that  $A^2$  is projecting onto the  $xy$ -plane. So

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

6. Let  $A$  be the symmetric matrix defined as

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}.$$

(a) (3 points) Find the eigenvalues of  $A$  and a basis for each eigenspace of  $A$ .

**Solution:**  $\lambda = -1$  is an eigenvalue with  $\{(1, 1, 1)\}$  as a basis for its eigenspace.  
 $\lambda = 2$  is an eigenvalue with  $\{(-1, 0, 1), (-2, 1, 1)\}$  as a basis for its eigenspace.

(b) (3 points) Find a basis for each of the following subspaces.

- $\text{null}(A)$

**Solution:** Since 0 is not an eigenvalue,  $\text{null}(A) = \{0\}$  with basis  $\emptyset$ .

- $\text{null}(A - I)$

**Solution:** Since 1 is not an eigenvalue,  $\text{null}(A - I) = \{0\}$  with basis  $\emptyset$ .

- $\text{null}(A - 2I)$ .

**Solution:** We have that  $\text{null}(A - 2I) = E_2$  which has basis  $\{(-1, 0, 1), (-2, 1, 1)\}$ .

(c) (3 points) Find an orthogonal matrix  $Q$  and a diagonal matrix  $D$  such that  $A = QDQ^{-1}$ .

**Solution:** We use Gram-Schmidt to perform an orthogonal basis for each eigenspace.  
 An orthonormal basis for the eigenspace corresponding to  $\lambda = -1$  is  $\{(1/3, 1/3, 1/3)\}$ .  
 An orthonormal basis for  $\lambda = 2$  is  $\{\frac{1}{\sqrt{2}}(-1, 0, 1), \sqrt{\frac{2}{3}}(-1/2, 1, -1/2)\}$ .

(d) (3 points) Find all  $k \in \mathbb{R}$  such that  $A - kI_3$  is not invertible.

**Solution:**  $k = -1, 2$ .

7. Let  $v = (2, 1, 2)$  and  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be defined by  $T(x) = \text{proj}_v x$ .

- (a) (4 points) Find an orthogonal basis for  $\mathbb{R}^3$  that contains  $v$ . (Hint: first find a basis for  $\mathbb{R}^3$  that contains  $v$ .)

**Solution:**  $\{(2, 2, 1), (1, 0, -2), (0, 1, -2)\}$ .

- (b) (4 points) There exists a matrix  $A$  such that  $T(x) = Ax$ . Find the eigenvalues of  $A$  and a basis for each eigenspace of  $A$ . (Hint: see part (a).)

**Solution:** The eigenspace corresponding to 1 is spanned by  $(2, 2, 1)$ .  
The eigenspace corresponding to 0 is spanned by  $(1, 0, -2), (0, 1, -2)$ .

- (c) (4 points) Let  $e_1 = (1, 0, 0)$ . Evaluate the following:

- $Ae_1$

**Solution:** This is  $T(e_1) = \text{proj}_v e_1 = (4/9, 4/9, 2/9)$ .

- $A^2 e_1$

**Solution:** Doing two projections is the same as one.

- $A^{100} e_1$

**Solution:** Doing one hundred projections is the same as one.