5.1/5.2 - Determinants

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transform and A be the square matrix defined so that $T(x) = Ax$. Then the absolute value of the determinant of A, denoted $\det(A)$, measures the change in volume under T. This means that if S is a shape of volume V then $T(S) = \{T(s) : s \in \mathbb{R}^n\}$ has volume $|\det(A)|$V.

Definition: For any $n \in \mathbb{N}$, The **determinant** is the unique function from the set of square matrices of size n so the real numbers with the following 3 properties:

- $\det(I_n) = 1$,
- When viewing a square matrices as a list of n column vectors, the determinant is n-linear. This means $\det([a_1, \ldots, ca_i + db_i, \ldots, a_n]) = c\det([a_1, \ldots, a_i, \ldots, a_n]) + d\det([a_1, \ldots, b_i, \ldots, a_n])$.
- If this there is a column of zero, then the determinant is zero.

The determinant of a linear transform $T : \mathbb{R}^n \to \mathbb{R}^n$ is the determinant of its associated matrix.

Properties:

- $\det(AB) = \det(A)\det(B)$,
- $\det(A^t) = \det(A)$,
- If A is upper (or lower) triangular, then $\det(A)$ is the product of the diagonal,
- $\det(cA) = c^n \det(A)$,
- If A has positive nullity, then $\det(A) = 0$.

Computation: There is a thing called cofactor expansion. It is terrible but we will learn it. In practice, another method is used like LU-decomposition. There LU decompositon of a matrix A is $A = LU$ where L is lower trianguluar and U is upper triangular. Then $\det(A) = \det(L)\det(U)$, where $\det(L)$ and $\det(U)$ is the product of the diagonal. Give $n = 2$ and $n = 3$ shortcuts and the general cofactor formula. Note that you can expand along any row or column.

Notation: Often you denote the determinant of a matrix by replacing the square brackets by straight lines.

Examples: Compute the determinant in the following cases:

- $T(x, y) = (x + y, y)$
- $T(x, y) = (3x + y, -y)$. Also what is the determinant of T^{-1} here?
- Here is a 3×3 example
- Here is a 4×4 example that I won't actually work out fully

Theorem: Let $S = \{a_1, \ldots, a_n\}$ be a set of vectors in $\mathbb{R}^n$, let $A = [a_1 \ \ldots \ a_n]$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(x) = Ax$. Then the following are equivalent:

- S spans $\mathbb{R}^n$,
- S is linearly independent,

- S is a basis for $\mathbb{R}^n$,
- $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$,
- T is onto,
- T is one-to-one,
- $\ker T = \{0\}$,
- $range(T) = \mathbb{R}^n$,
- $col(A) = \mathbb{R}^n$,
- $row(A) = \mathbb{R}^n$,
- $rank(A) = n$,
- $nullity(A) = 0$,
- Any echelon form of A has no zero entries on the diagonal,
- The reduced echelon form of A is the identity matrix,
- $\det(A) \neq 0$,