

# October 11

## Announcements

- Section 2.1, 2.2 due tomorrow
- Section 2.3, 3.1 due next Thursday
- Worksheet 2 due Friday
- Midterm next week
- Worksheet 3 will be posted on Friday, it'll have some practice exam problems

## 3.1 Linear transformation

### One-to-one and Onto linear transformation

**Definition:** Let  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a linear transformation. Then

- $T$  is *one-to-one* if for every vector  $w \in \mathbb{R}^n$ , there exists at most one vector  $v \in \mathbb{R}^m$  such that  $T(v) = w$ .
- $T$  is *onto* if for every vector  $w \in \mathbb{R}^n$ , there is exists at least one vector  $v \in \mathbb{R}^m$  such that  $T(v) = w$ .

A linear transformation  $T$  is one-to-one if  $T(u) = T(v)$  implies  $u = v$ . In other words, if  $u \neq v$ , then  $T(u) \neq T(v)$ . (Two-to-two!)

Talk about the general idea of one-to-one and onto.

**Theorem:** Let  $T$  be a linear transformation  $T$  is one-to-one if  $T(u) = 0$  implies  $u = 0$ .

**Example:** Let  $T$  be the linear transformation defined by  $T(x) = Ax$ , where

$$\begin{bmatrix} 4 & -1 \\ -2 & 2 \\ 0 & 3 \end{bmatrix}$$

Is  $T$  one-to-one? Onto?

Let  $T$  be the linear transformation defined by  $T(x) = Ax$ , where

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

Is  $T$  one-to-one? Onto?

**Theorem:** Let  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a linear transformation. Let  $A$  be the matrix so that  $T(x) = Ax$ . Then

- $T$  is one-to-one if the columns of  $A$  are linearly independent.
- $T$  is onto if the columns of  $A$  span  $\mathbb{R}^n$

In particular, the dimension of  $A$  can sometimes implies that  $T$  cannot be one-to-one and onto.

**Theorem:** Let  $S = \{a_1, \dots, a_n\}$  with  $a_i \in \mathbb{R}^n$ ,  $A = [a_i]$ , and  $T(x) = Ax$ . (So  $A$  is square). Then the following are equivalent:

- $S$  spans  $\mathbb{R}^n$
- $S$  is linearly independent
- $Ax = b$  has a unique solution for all  $b \in \mathbb{R}^n$
- $T(xs) = b$  has a unique solution for all  $b \in \mathbb{R}^n$
- $T$  is onto
- $T$  is one-to-one.

## Geometry of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$

Lines go to lines (or points)! Why?  $T((1-s)u + sv) = (1-s)T(u) + sT(v)$ .

The columns of the matrix tells you where the standard basis goes.

Let's see what happens to the square  $\{(x, y) : 0 \leq x, y \leq 1\}$  under the following transforms

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$