Projection and the Gram-Schmidt Process

The goal this class is to find orthonormal basis for a subspace.

Example:

- What is an orthonormal basis for the span of $\{(2,0),(1,1)\}$?
- How would we do it with 3 vectors?

Definition: Let $u, v \in \mathbb{R}^n$ with v nonzero. Then the *projection of u onto v* is given by $\mathrm{proj}_v u = \frac{u \cdot v}{\|v\|^2} v$.

Theorem: Let $u, v \in \mathbb{R}^n$ and c be a nonzero scalar. Then

- $\mathrm{proj}_v u$ is in span$v$.
- $u - \mathrm{proj}_v u$ is orthogonal to v
- if u is in span$v$, then $u = \mathrm{proj}_v u$.
- $\mathrm{proj}_v u = \mathrm{proj}_c vu$.

Let S be a nontrivial subspace with orthogonal basis $\{v_1, \dots, v_k\}$. Then the projection of u onto S is given by $\mathrm{proj}_S u = \sum_{i=1}^{k} \mathrm{proj}_{v_i} u$.

Theorem: Let S be a nonzero subspace of $\mathbb{R}^n$ with orthogonal basis $\{v_1, \dots, v_k\}$, and let u be a vector in $\mathbb{R}^n$. Then

- $\mathrm{proj}_S u$ is in S.
- $u - \mathrm{proj}_S u$ is orthogonal to S.
- if u is in S, then $u = \mathrm{proj}_S u$.
- $\mathrm{proj}_S u$ is independent of the choice of orthogonal basis for S.

Theorem: (The Gram-Schmidt Process) Let S be a subspace with basis $\{s_1, \dots, s_n\}$. Define $v_1, \dots, v_k$ by

- $v_1 = s_1$,
- $v_2 = s_2 - \mathrm{proj}_{v_1} s_2$,
- $v_3 = s_3 - \mathrm{proj}_{v_1} s_3 - \mathrm{proj}_{v_2} s_3$,
- $\dots$
- $v_k = s_k - \mathrm{proj}_{v_1} s_k - \dots - \mathrm{proj}_{v_{k-1}} s_k$

Then $\{v_1, \dots, v_k\}$ is an orthogonal basis. To make it orthonormal, just normalize each element.

Example: Find an orthonormal basis for the subspace $(1,0,1,1), (0,2,0,3), (-3,-1,1,5)$.