Math 308G/H - Winter 2018
Midterm 2
2018-02-21

KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 2 is the same for each part.
- I treat row and column vectors as the same.
- For any linear transformation T, there exists a matrix A such that $T(x) = Ax$. I defined the determinant, rank, and nullity of T using A. This means,

$$\det(T) = \det(A), \quad \operatorname{rank}(T) = \operatorname{rank}(A), \quad \operatorname{nullity}(T) = \operatorname{nullity}(A).$$

1. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE". You do not need to justify your answers.

(a) (3 points) If possible, give an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ with $\text{rank}(T) = 3$.

Solution: NOT POSSIBLE.

(b) (3 points) If possible, give an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ with $\text{nullity}(T) = 3$.

Solution: Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by $T(w, x, y, z) = (w, 0)$. The nullspace is the $w = 0$ hyperplane which is 3-dimensional.

(c) (3 points) If possible, give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T is one-to-one but T^2 is not onto.

Solution: NOT POSSIBLE. Since T is a one-to-one linear transformation from the same dimension to the same dimension, it is invertible. The composition of invertible maps is invertible so T^2 is invertible and thus onto.

(d) (3 points) If possible, give an example of a basis for $\mathbb{R}^3$ where each basis element lies in the $2x + y + z = 1$ plane. (Hint: Note that $1 \neq 0$.)

Solution: The first thing to observe is that the $2x+y+z = 1$ plane is not a subspace. A basis for $\mathbb{R}^3$ that lies in the plane is $\{\frac{1}{2}e_1, e_2, e_3\}$.

2. (a) (6 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with the property that

$$T(2,0) = (3,0), \quad T(1,1) = (0,2).$$

What is $|\det(T)|$? (Hint: A picture could be helpful here.)

Solution: The linear transformation T sends a triangle with area 2 to a triangle with area 3. Therefore, $|\det(T)| = 3/2$.

(b) (6 points) Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 21 \\ 3 & 8 & 2018 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- What is $\det(B)$?

Solution: -6

- What is $\det(C)$?

Solution: 3

- What is $\det(2A^{-1}B^TAC^2)$? Here B^T denotes the transpose of B. You may leave your answer as a product of numbers and their powers.

Solution: Using properties of determinants,

$$\det(2A^{-1}B^TAC^2) = \det(2I_3)\det(B)\det(C)^2 = (2^3)(-6)(3^2).$$

3. Let A and C be equivalent matrices given by

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & 6 \\ 4 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = C.$$

Let T be the linear transformation defined by $T(x) = Ax$.

(a) (6 points) For each of the following subspaces, write down a basis for the subspace. If you write your answer as a matrix, I will draw frowny faces on your exam and will consider taking off points

- col(A)

Solution:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 8 \end{bmatrix} \right\}$$

- ker(T)

Solution:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

- row(A)

Solution:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(b) (6 points) Let $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. You do not need to justify the following answers.

- What is rank(T)?

Solution: 2

- What is nullity(AU)?

Solution: 1

- What is nullity$(2C^T)$?

Solution: 2

4. Let A and C be equivalent matrices defined by

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 3 \\ 3 & 1 & 4 & 0 \\ 4 & 1 & 5 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C.$$

(a) (4 points) What is a basis for row(A)?

Solution:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) (4 points) Denote the basis for row(A) found in the previous part by $\mathcal{B}$. If possible, write the following vectors as linear combinations of the element of $\mathcal{B}$. Otherwise, write "NOT POSSIBLE" and justify why is it not possible.

- $(2, 2, 4, 0)$

Solution: The membership problem is really easy with this basis. You can essentially read off the answer.

$$\begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

- $(2, 1, 0, 1)$

Solution: NOT POSSIBLE. The associated linear system to this membership problem is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

This linear system has no soluton because the first 3 lines are claiming:

$$x_1 = 2, \quad x_2 = 1, \quad x_1 + x_2 = 0.$$

(c) (4 points) Hopefully, you determined that $\mathcal{B}$ does not span $\mathbb{R}^4$. Give an example of a vector that could be added to $\mathcal{B}$ so that together they span $\mathbb{R}^4$. Be sure to justify your answer.

Solution: Since A has rank 3, we just need to add one vector not row(A). For example, $(2, 1, 0, 1)$.

5. Let

$$u_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, u_2 = \begin{bmatrix}3\\0\\8\end{bmatrix}, u_3 = \begin{bmatrix}2\\2\\1\end{bmatrix}.$$

Let $U = [u_1\ u_2\ u_3]$ and $\mathcal{B} = \{u_1, u_2, u_3\}$. We have that U is invertible. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with the property:

$$T(u_1) = u_1, \quad T(u_2) = 2u_2, \quad T(u_3) = 2u_1.$$

From lecture, we know that

$$T(x) = U \begin{bmatrix}1 & 0 & 2\\0 & 2 & 0\\0 & 0 & 0\end{bmatrix} U^{-1}x.$$

(a) (4 points) You do not need to show work for the following quick questions:

- What is rank(T)?

Solution: 2

- What is det(T)?

Solution: 0

- What is nullity(T)?

Solution: 1

- Is T invertible?

Solution: No.

(b) (4 points) What is $T(2u_1 + u_2)$? You may express your answer as a linear combination of u_1, u_2, u_3.

Solution: Let C be the matrix between U and U^{-1}.

$$\begin{aligned} T(2u_1 + u_2) &= UCU^{-1}(2u_1 + u_2) \\ &= UC(2e_1 + e_2) \\ &= U(2e_1 + 2e_2) \\ &= 2u_1 + 2u_2 \end{aligned}$$

(c) (4 points) Suppose $[x]_{\mathcal{B}} = (2, 1, 0)$. What is $[T(x)]_{\mathcal{B}}$? (Hint: a intermediate step could be to determine what x is. Then look at the previous part.)

Solution: If $[x]_{\mathcal{B}} = (2, 1, 0)$, then $x = 2u_1 + u_2$. So by the last part, $T(x) = 2u_1 + 2u_2$. This means $[T(x)]_{\mathcal{B}} = (2, 2, 0)$.