

## October 11

### Announcements

- Section 2.3, 3.1 due next Thursday
- Write down name if you did worksheet 2
- Midterm next week
- Worksheet 3 will be posted tonight, it'll have some practice exam problems

### 3.1 Linear transformation

**Theorem:** Let  $S = \{a_1, \dots, a_n\}$  with  $a_i \in \mathbb{R}^n$ ,  $A = [a_i]$ , and  $T(x) = Ax$ . (So  $A$  is square). Then the following are equivalent:

- $S$  spans  $\mathbb{R}^n$
- $S$  is linearly independent
- $Ax = b$  has a unique solution for all  $b \in \mathbb{R}^n$
- $T(xs) = b$  has a unique solution for all  $b \in \mathbb{R}^n$
- $T$  is onto
- $T$  is one-to-one.

### Geometry of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$

Lines go to lines (or points)! Why?  $T((1-s)u + sv) = (1-s)T(u) + sT(v)$ .

The columns of the matrix tells you where the standard basis goes. Once you know this, you should know everything.

Let's see what happens to the square  $\{(x, y) : 0 \leq x, y \leq 1\}$  under the following transforms

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

## Piecing things together

**Theorem:** Let  $S = \{v_1, \dots, v_n\}$ . Let  $A$  be the matrix with the elements of  $S$  as columns. Let  $B$  be an echelon matrix equivalent to  $A$ . Let  $T$  be a linear transform with  $T(x) = Ax$ . Then the following are equivalent

- The set  $S$  is linearly independent.
- The linear equation  $x_1v_1 + \dots + x_nv_n = 0$  has only the trivial solution.
- Every columns of  $B$  has a pivot. (computationally useful)
- For any  $b \in \mathbb{R}^n$ , the equation  $x_1v_1 + \dots + x_nv_n = b$  has a unique solution.
- The homogenous equation  $Ax = 0$  has only the trivial solution.
- For any  $b \in \mathbb{R}^n$ , the equation  $Ax = b$  has at most one solution.
- For any  $b \in \mathbb{R}^n$ ,  $b$  can be expressed as a linear combination of elements in  $S$  in at most one way.
- The zero vector can be expressed as a linear combination of elements in  $S$  in only one way.
- $T$  is a one-to-one linear transformation.
- The only solution to  $T(x) = 0$  is  $x = 0$ . If  $T(x) = 0$ , then  $x = 0$ .
- There is at most one solution to  $T(x) = b$ .

**Theorem:** Let  $S = \{v_1, \dots, v_n\}$  be a set of vectors in  $\mathbb{R}^m$ . Let  $A$  be the matrix with the elements of  $S$  as columns. Let  $B$  be an echelon matrix equivalent to  $A$ . Let  $T$  be a linear transform with  $T(x) = Ax$ . Then the following are equivalent

- The set  $S$  spans  $\mathbb{R}^m$ .
- The linear equation  $x_1v_1 + \dots + x_nv_n = b$  always has a solution.
- Every row of  $B$  has a pivot. (computationally useful)
- For any  $b \in \mathbb{R}^n$ , the equation  $Ax = b$  has at least one solution.
- For any  $b \in \mathbb{R}^n$ ,  $b$  can be expressed as a linear combination of elements in  $S$  in at least one way.
- $T$  is an onto linear transformation.
- There is always a solution to  $T(x) = b$ .

### Examples:

Kristin DeVleming exam: Let  $u_1 = (4, 4, 2)$  and  $u_2 = (8, 5, -3)$ . Let  $v = (26, 17, -8)$ . Write  $v$  as a linear combination of  $u_1, u_2$ . Write a vector  $w$  that is not in the span of  $u_1, u_2$ .

Josh Swanson exam: Are the following sets spanning?

- $\{(1, 2, 3), (-1, -1, 2), (-1, 0, 7)\}$
- $\{(1, -1, 1), (0, 1, 2), (-2, 0, 2), (1, 3, 1)\}$ .