Math 308H - Winter 2018
Final
2018-03-15

KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
Total:	72	

- There are 6 problems on this exam. Be sure you have all 6 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 2 is the same for each part.
- I treat row and column vectors as the same.
- For any linear transformation T, there exists a matrix A such that $T(x) = Ax$. I defined the determinant, rank, and nullity of T using A. This means,

$$\det(T) = \det(A), \quad \operatorname{rank}(T) = \operatorname{rank}(A), \quad \operatorname{nullity}(T) = \operatorname{nullity}(A).$$

1. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE". You do not need to justify your answers.

   (a) (2 points) If possible, give an example of a linear system of equations whose solution space is the $(1, 2, 3) + s_1(1, 0, 0)$ line.

   Solution:
   $$y = 2, \ z = 3$$

   (b) (2 points) If possible, give an example of a 2×2 matrix A such that $A \neq 0, I$ and $A(A - I) = 0$.

   Solution: This means that $A^2 = A$ so any projection matrix would work. For example,
   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

   (c) (2 points) If possible, give an example of a 2×2 invertible matrix, A, such that $e_1 - e_2 \notin \text{col}(A)$.

   Solution: NOT POSSIBLE. An invertible matrix must have spanning columns.

   (d) (2 points) If possible, give an example of two invertible 2×2 matrices A and B such that $A + B$ is not invertible.

   Solution: Let $A = -B = I$.

   (e) (2 points) If possible, give an example of two 2×2 matrices A and B that are neither zero nor the identity matrix such that $AB = BA$.

   Solution: Take any two diagonal matrices that are not zero or the identity.

   (f) (2 points) If possible, give an example of two linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that 2 is an eigenvalue of T and 3 is an eigenvalue of S but 6 is not an eigenvalue of $T \circ S$.

   Solution: $T(x, y) = (2x, 0)$, $S(x, y) = (0, 3y)$.

2. (a) (6 points) Let

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}.$$

1. What is the characteristic polynomial of A^{-1}?

Solution: $(1 - \lambda)(1/2 - \lambda)$.

2. The matrix A is diagonalizable so it can be written as $A = UDU^{-1}$. What is U and D?

Solution:

$$U = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

(b) (6 points) Let

$$B = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}.$$

1. What is the reduced echelon form of B?

Solution:

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

2. What is the general solution to $Bx = (6, 3, 6)$?

Solution:

$$(1, 1, 1) + s_1(-1/2, -2, 1).$$

3. Let A and B be equivalent matrices defined by

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Let a_1, a_2, a_3, a_4 denote the columns of A.

(a) (3 points) Do not write express a basis as a matrix.

1. Give a basis for $\mathrm{col}(2A^t)$.

Solution: The first thing to note that is $\mathrm{col}(2A^t) = \mathrm{row}(A)$. A basis is then

$$\{(1, 0, 0, -7), (0, 1, 0, 3), (0, 0, 1, -3)\}$$

2. Give a basis for $\mathrm{null}(A)$.

Solution:

$$\{(7, -3, 3, 1)\}$$

3. Give a basis for $\mathrm{row}(A)$.

Solution:

$$\{(1, 0, 0, -7), (0, 1, 0, 3), (0, 0, 1, -3)\}$$

(b) (3 points) These should be quick questions.

1. What is $\mathrm{rank}(A)$?

Solution: 3

2. What is $\mathrm{nullity}(A^t D^{-1})$, where D is the 4×4 diagonal matrix consisting of $1, 2, 3, 4$ along the diagonal.

Solution: 1

3. What is $\det(2A)$?

Solution: 0

(c) (3 points) Give a nontrivial linear combination of the columns of A that sum to zero. You may use a_1, a_2, a_3, a_4 to denote the columns of A.

Solution: $7a_1 - 3a_2 + 3a_3 + a_4 = 0$.

(d) (3 points) Let C be the 4×3 matrix given by $C = [a_1\ a_2\ a_3]$. So C is the submatrix of A consisting of the first 3 columns. Give the general solution for $Cx = a_1 + a_4$.

Solution: From the previous part, we know that $a_4 = -7a_1 + 3a_2 - 3a_3$. This means that $a_1 + a_4 = -6a_1 + 3a_2 - 3a_3$. The general solution is then

$$x = (-6, 3, -3, 0).$$

There is no homogenous part because the columns of C are linearly independent.

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(w, x, y, z) = (w + y + z, x + y + z, x + y + z).$$

(a) (3 points) There is a matrix A such that $T(x) = Ax$. What is A?

Solution:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

(b) (3 points) Let $v = (0, 3, 0, 8)$. Give the general solution to $Ax = 2Av + (2, 1, 1)$.

Solution: A particular solution to $Ax = 2Av$ is $x = 2v$. A particular solution to $Ax = (2, 1, 1)$ is $(2, 1, 0)$. The general solution to the homogenous system $Ax = 0$ is $s_1(-1, -1, 1, 0) + s_2(-1, -1, 0, 1)$. The general solutin to $Ax = 2Av = (2, 1, 1)$ is then

$$2v + (2, 1, 0) + s_1(-1, -1, 1, 0) + s_2(-1, -1, 0, 1).$$

(c) (3 points) Does there exists a rank 2 linear transformation S such that $T \circ S$ is the zero transformation? If so, give an example. If not, why not?

Solution: Yes. If $T \circ S = 0$ then range$(S) \subseteq \ker(T)$. We know a basis for $\ker(T)$ so define

$$S(x) = \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x.$$

(d) (3 points) Does there exists a rank 3 linear transformation S such that $T \circ S$ is the zero transformation? If so, give an example. If not, why not?

Solution: No. If range$(S) \subseteq \ker(T)$, then rank$(S) \leq$ nullity(T).

5. Let

$$A = \begin{bmatrix} 0 & -1 & \frac{37}{3} & -\frac{253}{15} \\ 0 & 2 & 0 & -\frac{1}{5} \\ 0 & 0 & 2 & \frac{7}{5} \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

be a matrix which decomposes as $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & -1 & 18 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Let u_1, u_2, u_3, u_4 be the columns of U and $\mathcal{B} = \{u_1, u_2, u_3, u_4\}$.

(a) (6 points) Fill out this table.

Eigenvalue λ	Alg. Multiplicity of λ	Geo. Multiplicity of λ	Basis for E_λ
0	1	1	$\{u_1\}$
2	2	2	$\{u_2, u_3\}$
3	1	1	$\{u_4\}$

(b) (3 points) Let $x = u_1 + u_2 + u_3 + u_4$. Express $A^{18}x$ as a linear combination of u_1, u_2, u_3, u_4. You are allowed to have exponents of numbers in your answer. (Hint: x has been expressed as the sum of eigenvectors.)

Solution: $2^{18}u_2 + 2^{18}u_3 + 3^{18}u_4$.

(c) (3 points) What are the eigenvalues for $A^2 - 2A$?

Solution: $0, 3$

6. Let $T(x) = Ax$, where A is as defined in Question 5. Let u_1, u_2, u_3, u_4 also be as defined in Question 5.
   (a) (4 points) Give two vectors v, w such that the triangle with vertices $\{T(0), T(v), T(w)\}$ has 6 times the area as the triangle with vertices $\{0, v, w\}$. Be sure to justify your answer. (Hint: It is unnecessary to compute the area of these triangles.)

   Solution: Let $v = u_2$ and $w = u_4$. Then $T(u_2) = 2u_2$ and $T(u_4) = 3u_4$. So the area of the triangle increased by a factor of 6.

   (b) (4 points) Find a basis for each of the following subspaces. If a subspace is trivial, then write $\emptyset$ for its basis.
   - null$(A - 2I)$

     Solution: This is a basis for the eigenspace corresponding to 2, $\{u_2, u_3\}$.

   - null$(A^2 - 3I)$.

     Solution: Since 3 is not an eigenvalue of A^2, this subspace is trivial so a basis is $\emptyset$.

   (c) (4 points) Let $B = \{u_1, u_2, u_3, u_4\}$ be a basis.
   - What is the general solution to $Ax = u_2 + 2u_3$?

     Solution:
     $$x = (1/2u_2 + u_3) + s_1(u_1)$$

   - Let y be a particular solution to the above linear system. What is $[y]_B$?

     Solution:
     $$(0, 1/2, 1, 0)$$