ESP Math 294

Week 7 Solutions
Induction I

February 23, 2021

Spring 2021

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1. Prove that for all $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

   Solution:

   Proof. We proceed by induction on n. $P(n)$ is the statement $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

   Base case: We want to prove $P(0)$, that is, that $\displaystyle\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$. The left hand side is $\sum_{i=0}^{0} 2^i = 2^0 = 1$, while the right hand side is $2^{0+1} - 1 = 2 - 1 = 1$. So both sides are equal and $P(0)$ is true.

   Induction step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true, that is, that $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ (this is the induction hypothesis).

   We want to prove that $P(n+1)$ is true, that is, $\displaystyle\sum_{i=0}^{n+1} 2^i = 2^{n+2} - 1$

   We see that

   $$\begin{aligned} \sum_{i=0}^{n+1} 2^i &= \sum_{i=0}^{n} 2^i + 2^{n+1} \\ &= 2^{n+1} - 1 + 2^{n+1} && \text{by induction hypothesis} \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1 \end{aligned}$$

   as desired. Thus by the principle of mathematical induction, for all $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$. $\square$

2. Let $x \in \mathbb{R}$ with $x > -1$. Prove that for all $n \in \mathbb{N}$, $1 + nx \leq (1+x)^n$. (Hint: at some point, you might want to consider two cases, one where $x \geq 0$, and one where $-1 < x < 0$.)

   Solution:

   Proof. We proceed by induction on n. $P(n)$ is the statement $1 + nx \leq (1+x)^n$.

   Base case: We want to prove $P(0)$, that is, $1 + 0x \leq (1+x)^0$. The left hand side is $1 + 0x = 1$, while the right hand side is $(1+x)^0 = 1$. So both sides are equal and $P(0)$ is true.

   Induction step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true, that is, that $1 + nx \leq (1+x)^n$ is true (this is the induction hypothesis).

   We want to prove that $P(n+1)$ is true, that is, $1 + (n+1)x \leq (1+x)^{n+1}$.

   Examining the left hand side, we have

   $$\begin{aligned} 1 + (n+1)x &= 1 + nx + x \\ &\leq (1+x)^n + x && \text{by induction hypothesis} \end{aligned}$$

On the other hand, the right hand side is equal to

$$(1+x)^{n+1} = (1+x)^n \cdot (1+x) = (1+x)^n + x \cdot (1+x)^n,$$

where we distribute over the parentheses for the second equality.

So it's enough for me to prove that $(1+x)^n + x \leq (1+x)^n + x \cdot (1+x)^n$. Cancelling the $(1+x)^n$ from both sides, we can see that it's actually enough for us to show that $x \leq x \cdot (1+x)^n$.

Now, we split into two cases:

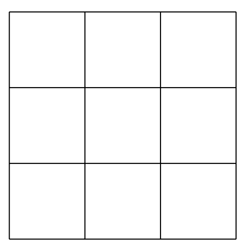
Case 1: $x \geq 0$. Then $(1+x)^n \geq (1+0)^n \geq 1$. Since x is positive, multiplying x by a number that's at least 1 will either keep it the same or increase it, so it is true that $x \leq x \cdot (1+x)^n$.

Case 2: $-1 < x < 0$. Then $0 < 1 + x < 1$, so $0 < (1+x)^n < 1$. Multiplying x by a number between 0 and 1 will decrease its absolute value. However, since x is negative, decreasing the absolute value means that you are increasing the number. So $x \leq x \cdot (1+x)^n$.

Since in both cases, we found that $x \leq x \cdot (1+x)^n$, we are done. This proves that $P(n+1)$ is true, and so by the principle of mathematical induction, for $x \in \mathbb{R}$ with $x > -1$, and for all $n \in \mathbb{N}$, $1 + nx \leq (1+x)^n$. $\square$

3. Recall that $\dfrac{d^n}{dx^n} f(x)$ means that you take the derivative of $f(x)$ n times. Prove that for all $n \in \mathbb{N}$, $\dfrac{d^n}{dx^n}(xe^x) = (n+x)e^x$. (Hint: use the product rule.)

4. Consider an $n \times n$ grid of squares. For example, here's a picture of a 3×3 grid:



   Within an $n \times n$ grid, how many squares can you find? For example, in the 3×3 case, there are nine 1×1 squares, four 2×2 squares, and one 3×3 square, so there are 14 squares total.

   Write your answer as a summation, and use induction to prove that this summation is $\dfrac{n(n+1)(2n+1)}{6}$.

5. **(Challenge)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
   (a) Prove that there is a real number a such that $f(n) = an$ for all $n \in \mathbb{N}$.
   (b) Deduce that $f(n) = an$ for all $n \in \mathbb{Z}$.
   (c) Deduce that $f(x) = ax$ for all $x \in \mathbb{Q}$.