Atividade 2 - Grupo 44

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1. Massa M de S

Suponha que o corpo S seja sólido cuja base é o cone $z^2 = x^2+y^2$ e a parte superior é a esfera $x^2+y^2+(z-1)^2 = 1$ e que tenha densidade de massa $\rho(x,y,z) = x^2+y^2+z^2$.

$$M = \int\int\int_S \rho(x,y,z)dV$$

Como $\rho(x,y,z) = x^2+y^2+z^2$

Vamos transformar estas coordenadas em coordernadas esféricas. Temos que

$x = \rho\, cos\theta\, sen\phi$

$y = \rho\, sen\theta\, sen\phi$

$z = \rho\, cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2\, sen\phi$$

Então

$x^2 + y^2 + z^2 = \rho^2\, cos^2\theta\, sen^2\phi + \rho^2\, sen^2\theta\, sen^2\phi + \rho^2\, cos^2\phi$

$\rho^2 sen^2\phi(cos^2\theta + sen^2\theta) + \rho^2 cos^2\phi$

$\rho^2 sen^2\phi + \rho^2 cos^2\phi = \rho^2(\cos^2\phi + sen^2\phi) = \rho^2$

Logo

$$M = \int\int\int_S \rho^2\, \rho^2\, sen\phi\; d\theta\, d\phi\, d\rho$$

Os limites de integração são tais que

$0 \leq \theta \leq 2\pi$, pois a figura gira 2π.

$0 \leq \phi \leq \frac{\pi}{4}$, é o resultado da solução $cos\phi = \frac{1}{\sqrt{2}}$.

$0 \leq \rho \leq 1$, pois o o raio da esfera é 1.

Portanto

$$M = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho^2\rho^2\, sen\phi\, d\theta\, d\phi\, d\rho$$

$$M = \int_0^1 \int_0^{\pi/4} \rho^4 sen\phi\, \theta|_{\theta=0}^{\theta=2\pi} d\phi d\rho$$

$$M = 2\pi \int_0^1 \int_0^{\pi/4} \rho^4\, sen\phi\, d\phi\, d\rho$$

$$M = 2\pi \int_0^1 \rho^4\; -cos(\phi)|_{\phi=0}^{\phi=\pi/4}\; d\rho$$

$$M = 2\pi(1 - \frac{1}{\sqrt{2}}) \int_0^1 \rho^4\, d\rho$$

$$M = 2\pi(1 - \frac{1}{\sqrt{2}})\; \frac{p^5}{5}\Big|_{\rho=0}^{\rho=1}$$

$$M = \frac{2\pi}{5}(1 - \frac{1}{\sqrt{2}})$$

$$M = \frac{2\pi}{5} - \frac{2\pi}{5\sqrt{2}}$$

$$\boxed{M = \frac{\pi}{5}(2 - \sqrt{2})}$$

2. Momentos em relação aos eixos coordernados

Seja S um corpo tridimensional de densidade $\rho(x,y,z)$. Os momentos em relação aos planos coordenados xy, yz e xz são dados, respectivamente, por

$$M_{xy} = \int\int\int_S z\rho(x,y,z)\, dV,$$

$$M_{xz} = \int\int\int_S y\rho(x,y,z)\, dV,$$

$$M_{yz} = \int\int\int_S x\rho(x,y,z)\, dV,$$

e que tenha densidade de massa $\rho(x,y,z) = x^2+y^2+z^2$.

Então em coordenadas esféricas temos

$$M_{xy} = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho cos\phi\, \rho^2\, \rho^2 sen\phi\; d\theta\, d\phi\, d\rho$$

$$M_{xy} = \int_0^1 \int_0^{\pi/4} \rho^5\, sen\phi\, cons\phi \quad \theta|_{\theta=0}^{\theta=2\pi}\, d\phi d\rho$$

$$M_{xy} = 2\pi \int_0^1 \int_0^{\pi/4} \rho^5\, sen\phi\, cons\phi\, d\phi\, d\rho$$

Para resolver $\int_0^{\pi/4} sen\phi\, cons\phi\, d\phi$, faremos a seguinte substituição:

$u = sen\phi,\ du = cos\phi$

$$\int_0^{\pi/4} sen\phi\, cons\phi\, d\phi = \int_0^{1/\sqrt{2}} u\, du = \left.\frac{u^2}{2}\right|_0^{1/2} = \frac{1}{4}$$

$$M_{xy} = \frac{1}{4} 2\pi \int_0^1 \rho^5\, d\rho$$

$$M_{xy} = \frac{1}{4} 2\pi \left.\frac{\rho^6}{6}\right|_0^1$$

$$\boxed{M_{xy} = \frac{\pi}{12}}$$

$$M_{xz} = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho\, sen\theta\, sen\phi\ \rho^2\ \rho^2 sen\phi\ d\theta\, d\phi\, d\rho$$

$$M_{xz} = \int_0^1 \int_0^{\pi/4} \rho^5\, sen^2\phi\ -cos\theta|_{\theta=0}^{\theta=2\pi}\ d\phi\, d\rho$$

$$M_{xz} = (cos0 - cos2\pi) \int_0^1 \int_0^{\pi/4} \rho^5\, sen^2\phi\ d\phi\, d\rho$$

$$\boxed{M_{xz} = 0}$$

$$M_{yz} = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho\, \cos\theta\, sen\phi\ \rho^2\ \rho^2 sen\phi\ d\theta\, d\phi\, d\rho$$

$$M_{yz} = \int_0^1 \int_0^{\pi/4} \rho^5\, sen^2\phi\ sen\theta|_{\theta=0}^{\theta=2\pi}\ d\phi\, d\rho$$

$$M_{yz} = (sen2\pi - sen0) \int_0^1 \int_0^{\pi/4} \rho^5\, sen^2\phi\ d\phi\, d\rho$$

$$\boxed{M_{yz} = 0}$$

3. Determine os momentos de inércia em relação aos eixos de S.

Os momentos de inércia com respeito aos eixos são dados pelas expressões

$$I_x = \int\int\int_S (y^2 + z^2)\rho(x,y,z)dV,$$

$$I_y = \int\int\int_S (x^2 + z^2)\rho(x,y,z)dV,$$

$$I_x = \int\int\int_S (x^2 + y^2)\rho(x,y,z)dV,$$

Transformando suas coordernadas para coordenadas esféricas:

$$I_x = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} [(\rho sen\theta sen\phi)^2 + (\rho\cos\phi)^2]\, \rho^2 \rho^2 sen\phi\, dV$$

$$I_x = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} (\rho\, sen\theta\, sen\phi)^2 (\rho^4 sen\phi)\, dV +$$

$$\int_0^1 \int_0^{\pi/4} \int_0^{2\pi} (\rho\ \cos\phi)^2 (\rho^4 sen\phi)\, dV$$

$$I_y = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} [(\rho\cos\theta sen\phi)^2 + (\rho\cos\phi)^2]\, \rho^2 \rho^2\, sen\phi\, dV$$

$$I_y = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} (\rho\ \cos\theta\, sen\phi)^2 (\rho^4 sen\phi)\, dV +$$

$$\int_0^1 \int_0^{\pi/4} \int_0^{2\pi} (\rho\ \cos\phi)^2 (\rho^4 sen\phi)\, dV$$

$$I_z = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} [(\rho\cos\theta\, sen\phi)^2 + (\rho\, sen\theta\, sen\phi)^2]\, \rho^2 \rho^2 sen\phi\, dV$$

$$I_z = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} (\rho\ \cos\theta\, sen\phi)^2 (\rho^4 sen\phi)\, dV +$$

$$\int_0^1 \int_0^{\pi/4} \int_0^{2\pi} (\rho\, sen\theta\, sen\phi)(\rho^4 sen\phi)\, dV$$

Como as integrais aparecem mais de uma vez vamos calculá-las separadamente e substituí-las na equação:

$$I) = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} (\rho\, sen\theta\, sen\phi)^2 (\rho^4 sen\phi)\, d\theta d\phi d\rho$$

$$= \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho^6\, sen^3\phi\, sen^2\theta\ d\theta d\phi d\rho$$

$$\int sen^2\theta\ d\theta = \int 1 - \frac{cos(2\theta)}{2}$$

$$= \theta - \frac{1}{2}(sen2\theta)$$

$$= \int_0^1 \int_0^{\pi/4} \rho^6\, sen^3\phi\ \frac{1}{2}(\theta - sen2\theta)\Big|_{\theta=0}^{\theta=2\pi}\ d\phi d\rho$$

$$= \pi \int_0^1 \int_0^{\pi/4} \rho^6\, sen^3\phi\ d\phi d\rho$$

$$= \pi \int_0^1 \rho^6\ -\cos\phi + \frac{\cos^3\phi}{3}\Big|_{\phi=0}^{\phi=\pi/4}\ d\rho$$

$$= \pi[-\frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}} + \frac{2}{3}] \int_0^1 \rho^6\ d\rho$$

$$\boxed{I) = \frac{\pi}{7}[-\frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}} + \frac{2}{3}]}$$

$$II) \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} (\rho\,\cos\phi)^2(\rho^4 sen\phi)\ d\theta d\phi d\rho$$

$$= \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho^6 \cos^2\phi\ sen\phi\ d\theta d\phi d\rho$$

$$= \int_0^1 \int_0^{\pi/4} \rho^6 \cos^2\phi\ sen\phi\ \theta\big|_{\theta=0}^{\theta=\pi 2}\ d\phi d\rho$$

$$= 2\pi \int_0^1 \int_0^{\pi/4} \rho^6 \cos^2\phi\ sen\phi\ d\phi d\rho$$

$u = \cos\phi,\ du = -sen\phi$

$$\int cos^2\phi\, sen\phi\ d\phi = \int -u^2 du$$

$$- \int u^2 du = -\frac{u^3}{3}$$

$$= -\frac{\cos^3\phi}{3}$$

$$= 2\pi \int_0^1 \rho^6\ -\frac{\cos^3\phi}{3}\Big|_{\phi=0}^{\phi=\pi/4}\ d\rho$$

$$= 2\pi[-\frac{1}{6\sqrt{2}} + \frac{1}{3}] \int_0^1 \rho^6\ d\rho$$

$$= 2\pi[-\frac{1}{6\sqrt{2}} + \frac{1}{3}]\ \frac{\rho^7}{7}\Big|_{\rho=0}^{\rho=1}$$

$$\boxed{II) \frac{2\pi}{7}[-\frac{1}{6\sqrt{2}} + \frac{1}{3}]}$$

$$III) = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} (\rho\,\cos\theta\, sen\phi)^2(\rho^4 sen\phi)\, dV$$

$$= \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho^6 \cos^2\theta\, sen^3\phi dV$$

$$\int \cos^2\theta = \int \frac{1+\cos 2\theta}{2}$$

$$= \int 1 d\theta + \int \cos 2\theta d\theta$$

$$= \frac{1}{2}(x + \frac{1}{2}sen2\theta) + C$$

$$= \int_0^1 \int_0^{\pi/4} \rho^6\, sen^3\phi\ \frac{1}{2}(\theta + \frac{1}{2}sen2\theta)\Big|_{\phi=0}^{\phi=2\pi}\ d\phi d\rho$$

$$= \pi \int_0^1 \int_0^{\pi/4} \rho^6\, sen^3\phi\ d\phi d\rho$$

$$\int sen^3\phi\, d\phi = \int sen^2\phi\ sen\phi\ d\phi$$

$$= \int (1-\cos^2\phi)\ sen\phi\ d\phi$$

$u = cos\phi \rightarrow du = -sen\phi$

$$-u + \frac{u^3}{3} + C = -\cos\phi + \frac{\cos^3\phi}{3} + C$$

$$= \pi \int_0^1 \rho^6\ (-\cos\phi + \frac{\cos^3\phi}{3})\Big|_{\phi=0}^{\phi=\pi/4}\ d\rho$$

$$= \pi[(-\frac{1}{\sqrt{2}} + \frac{1^3}{3\sqrt{2}^3}) - (-1 + \frac{1}{3})] \int_0^1 \rho^6 d\rho$$

$$\boxed{III) = \frac{\pi}{7}[(-\frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}} + \frac{2}{3})]}$$

$I_x = I + II$

$$I_x = \frac{\pi}{7}[-\frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}} + \frac{2}{3}] + \frac{2\pi}{7}[-\frac{1}{6\sqrt{2}} + \frac{1}{3}]$$

$$\boxed{I_x = \frac{\pi}{84}(16 - 7\sqrt{2})}$$

$I_y = II + III$

$$I_y = \frac{2\pi}{7}[-\frac{1}{6\sqrt{2}} + \frac{1}{3}] + \frac{\pi}{7}[-\frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}} + \frac{2}{3}]$$

$$\boxed{I_y = \frac{\pi}{84}(16 - 7\sqrt{2})}$$

$I_z = III + I$

$$I_z = \frac{\pi}{7}[(-\frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}} + \frac{2}{3})] + \frac{\pi}{7}[-\frac{1}{\sqrt{2}} + \frac{1}{6\sqrt{2}} + \frac{2}{3}]$$

$$\boxed{I_z = \frac{\pi}{42}(8 - 5\sqrt{2})}$$

4. Encontre o centro de massa S

Os centros de massa são

$\bar{x} = M_{yz}/M$

$$\bar{x} = \frac{0}{\frac{\pi}{5}(2-\sqrt{2})}$$

$$\boxed{\bar{x} = 0}$$

$\bar{y} = M_{xz}/M$

$$\bar{y} = \frac{0}{\frac{\pi}{5}(2-\sqrt{2})}$$

$$\boxed{\bar{y} = 0}$$

$\bar{z} = M_{xy}/M$

$$\bar{z} = \frac{\frac{\pi}{12}}{\frac{\pi}{5}(2-\sqrt{2})}$$

$$\boxed{\bar{z} = \frac{5}{24-12\sqrt{2}}}$$