TKE Turbulent Closure Scheme in NEMO

The objective of this document is to introduce the TKE closure as implemented in NEMO in its simplest form.

1 Continuous aspects

The first step is to add the following one-dimensional prognostic equation for the TKE which is a positive-definite quantity noted e:

$$\partial_t e = K_m \left[(\partial_z u)^2 + (\partial_z v)^2 \right] - K_t N^2 + \partial_z \left(K_e \partial_z e \right) - c_\epsilon \frac{e^{3/2}}{l_\varepsilon^2} \tag{1.1}$$

where K_m and K_t are respectively the eddy-viscosity and the eddy-diffusivity, and K_e a diffusion coefficient for TKE which is expressed as $K_e = c_e K_m$ with c_e a constant parameter. c_ϵ is also a constant parameter and l_ε a mixing length defined below. The eddy-viscosity and eddy-diffusivity are given by

$$K_m = \max\left(c_m l_m \sqrt{e}, K_{m0} \right), \qquad K_t = \max\left(K_m / \mathrm{Pr}_t, K_{t0} \right) \tag{1.2}$$

with Pr_t the turbulent Prandtl number defined as

$$\mathrm{Pr}_t = \max\left(0.1, \frac{\mathrm{Ri}_c}{\max(\mathrm{Ri}_c, \mathrm{Ri})} \right), \qquad \mathrm{Ri} = \frac{N^2}{[(\partial_z u)^2 + (\partial_z v)^2] + \varepsilon_{\mathrm{s}}}$$

and l_m a mixing length. The length scales l_m and l_ε are computed via two intermediate length scales l_{up} and l_{dwn} estimating respectively the maximum upward and downward displacement of a water parcel with a given initial kinetic energy. l_{up} and l_{dwn} are first initialized to

$$l_{\mathrm{up}}(z) = l_{\mathrm{dwn}}(z) = \sqrt{\frac{2e(z)}{\max(N^2, N_\varepsilon^2)}}.$$

The resulting length scales are then limited not only by the distance to the surface and to the bottom but also by the distance to a strongly stratified portion of the water column such as the thermocline. This limitation amounts to control the vertical gradients of $l_{\mathrm{up}}(z)$ and $l_{\mathrm{dwn}}(z)$ such that they are not larger that the variations of depth.

Once l_{up} and l_{dwn} are known, the length scales l_m and l_ε are computed via

$$l_\varepsilon = \sqrt{l_{\mathrm{up}} l_{\mathrm{dwn}}}, \qquad l_m = \min\left(l_{\mathrm{up}}, l_{\mathrm{dwn}} \right)$$

The boundary conditions for the different terms involved are:

$$e(z = \eta, t) = \max(C_{\mathrm{sfc}} \|\boldsymbol{\tau}\| / \rho_0, e_0^{\mathrm{sfc}}), \qquad e(z = -H, t) = e_0$$

Parameter	Value	Parameter	Value
c_m	0.1	c_e	1
c_ϵ	0.7	Ri_c	$\frac{2}{2+(c_\epsilon/c_m)} \approx 0.22$
$\varepsilon_{\rm s}$	1×10^{-20} s^{-2}	N^2_ε	1×10^{-20} s^{-2}
K_{m0}	1.2×10^{-4} m^2 s^{-1}	K_{t0}	1.2×10^{-5} m^2 s^{-1}
l_0	0.04 m	e_0	$\frac{\sqrt{2}}{2} \times 10^{-6}$ m s^{-2}
$C_{\rm sfc}$	67.83	C_l	2×10^5
$e_0^{\rm sfc}$	10^{-4} m s^{-2}		

Table 1
Caption

with η the free-surface and H the total depth of the water column at rest.

$$l_{\rm up}(z=\eta,t) = \max\left(l_0, \frac{\kappa C_l}{\rho_0 g}\|\boldsymbol{\tau}\|\right), \qquad l_{\rm dwn}(z=-H,t) = l_0$$

where κ is the Von Karman constant, ρ_0 the Boussinesq reference density, and $\boldsymbol{\tau}$ the wind-stress vector.

2 Discrete aspects

2.1 Space/time discretization

We consider a Lorenz grid in the vertical (with u and ρ located at cell centers, a.k.a. ρ-points) meaning that the vertical shear and Brunt-Vaisala frequency in the rhs of the TKE equation are naturally located at cell interfaces (a.k.a. w-points). Moreover since K_m and K_t are expected to be given at w-points, it is customary to locate e, l_m and l_ε at w-points. As far as the time discretization is concerned, a backward Euler scheme is generally used with the forcing terms $K_m\left[(\partial_z u)^2 + (\partial_z v)^2\right]$ and $K_t N^2$ taken at time n and the dissipation term $e^{3/2}$ discretized as $\sqrt{e^n}e^{n+1}$.

2.2 TKE positivity

As mentioned earlier, the TKE equation is discretized using a backward Euler scheme in time with a linearization of the dissipation term $\frac{c_\varepsilon}{l_\varepsilon}e^{3/2}$ which is discretized as $\frac{c_\varepsilon}{l_\varepsilon}\sqrt{e^n}e^{n+1}$. However, such discretization is not unconditionally positivity-preserving for TKE which could give rise to unphysical solutions. The overwhelming majority of atmospheric/oceanic models will simply put a minimum threshold on e^{n+1} once it is computed ($e^{n+1} = \max(e^{n+1}, e_0)$). Ignoring the diffusion term, the TKE prognostic equation (1.1) can be written as an ordinary differential equation (ODE) of the form

$$\partial_t e = S(\mathbf{u}_h, N^2) - D(e,t)\, e, \qquad \text{with} \qquad S(\mathbf{u}_h, N^2) = K_m\|\partial_z \mathbf{u}_h\|^2 - K_t N^2,\; D(e,t) = \frac{c_\varepsilon}{l_\varepsilon}\sqrt{e^n} \quad (2.1)$$

where the last term can be seen as a damping term. For ODEs like (2.1) it can be shown that for an initial condition $e(0) \geq 0$ and $S(\mathbf{u}_h, N^2) \geq 0$, the solution $e(t)$ keeps the same sign as $e(0)$ whatever the sign of the damping coefficient $D(e,t)$. Assuming that $S(\mathbf{u}_h, N^2)$ and $D(e,t)$ are positive, a backward Euler discretization of the damping term in (2.1) would lead to $e^{n+1} = \dfrac{e^n + \Delta t S(\mathbf{u}_h, N^2)}{1 + \Delta t D(e,t)}$ which preserves positivity since for $e^n \geq 0$ we obtain $e^{n+1} \geq 0$. However, there is no guarantee that the forcing term $S(\mathbf{u}_h, N^2)$ is positive in particular when the shear is weak and the stratification is large. When $S(\mathbf{u}_h, N^2)$ is negative a specific treatment (known as "Patankar trick") is required. In the event of a negative $S(\mathbf{u}_h, N^2)$, the idea is to move the buoyancy term from S to D after dividing it by e^n, such that $S(\mathbf{u}_h, N^2) = K_m \|\partial_z \mathbf{u}_h\|^2$ is now strictly positive and $D(e,t) = \dfrac{c_\varepsilon}{l_\varepsilon}\sqrt{e^n} + K_s \dfrac{N^2}{e^n}$. Such procedure is a sufficient condition to preserve the positivity of the TKE without ad-hoc clipping of negative values.

2.3 Length scales computation

Maybe the most delicate point is the discretization of the length scales $l_{\rm up}$ and $l_{\rm dwn}$. Let us introduce the index k to characterize the vertical layer k with thickness Δz_k and consider that the top-most layer is Δz_N and the bottom-most layer is Δz_1. The first step amounts to initialize $l_{\rm up}$ and $l_{\rm dwn}$ as

$$
\begin{aligned}
(l_{\rm up})_{k+1/2} &= \sqrt{2e_{k+1/2}/\max((N^2)_{k+1/2}\,, N_\varepsilon^2)} \\
(l_{\rm dwn})_{k+1/2} &= \sqrt{2e_{k+1/2}/\max((N^2)_{k+1/2}\,, N_\varepsilon^2)}
\end{aligned}
$$

Then to guarantee that the vertical gradients of $l_{\rm up}(z)$ and $l_{\rm dwn}(z)$ are not larger that the variations of depth the following logic is applied:

- For $l_{\rm dwn}$: initialize it with $(l_{\rm dwn})_{1/2} = l_0$ and going up from the bottom ($k = 1$) to the top ($k = N$)

$$
(l_{\rm dwn})_{k+1/2} = \min\left\{(l_{\rm dwn})_{k-1/2} + \Delta z_k, (l_{\rm dwn})_{k+1/2}\right\}
$$

- For $l_{\rm up}$: initialize it with the surface value $(l_{\rm up})_{N+1/2} = \frac{\kappa C_l}{\rho_0 g}\|\boldsymbol{\tau}\|$ and going down from the top ($k = N-1$) to the bottom ($k = 0$)

$$
(l_{\rm up})_{k-1/2} = \min\left\{(l_{\rm up})_{k+1/2} + \Delta z_k, (l_{\rm up})_{k-1/2}\right\}
$$