Problem 5

1.

   (a) False. Consider $n = \frac{3}{2}\pi + 2k\pi$, $k \in \mathbb{N} \Longrightarrow \sin n = -1$ and $n^{\sin n} = \frac{1}{n}$. We cannot find a constant c such that $\sqrt{n} \leq c \cdot \frac{1}{n}$ for all such n.

   (b) True. $f(n) = \Theta(g(n)) \Longrightarrow \exists c_0, c_1, n_0$ such that $c_0 \cdot g(n) \leq f(n) \leq c_1 \cdot g(n), \forall n \geq n_0$
   Take logarithm on both sides, we have

$$\frac{1}{2}\log g(n) \leq \log c_0 + \log g(n) \leq \log f(n) \leq \log c_1 + \log g(n) \leq 2\log g(n)$$

   which holds for all $n \geq n_0$, with $n_0 \in \mathbb{R}$ and $g(n_0) \geq c_1$ and $g(n_0) \geq \frac{1}{c_0^2}$

   (c) False. Let $f_1(n) = g_1(n) = e^n, f_2(n) = 2n, g_2(n) = n$, then $f_2(n) = O(g_2(n))$ and $f_1(n) = O(g_1(n))$, but $e^{2n} \neq O(e^n)$

   (d) True. We have $n + a \leq n + |a| \leq 2n$ for $n \geq |a|$, and $n + a \geq n - |a| \geq \frac{1}{2}n$ for $\frac{1}{2}n \geq |a|$
   So for $n \geq 2|a| \geq |a|$, $\frac{1}{2}n \leq n + a \leq 2n \Longrightarrow \frac{1}{2}^b n^b \leq (n+a)^b \leq 2^b n^b$.
   Take $c_0 = \frac{1}{2}^b, c_1 = 2^b, n_0 = 2|a|$

2.

   (a) We expand $T(n)$ to get $T(n) = \sum_{i=1}^{\frac{n}{127}} \frac{127}{\log(127 \cdot i)}$. It is clear that $\frac{n}{127} \cdot \frac{127}{\log n} \leq \sum_{i=1}^{\frac{n}{127}} \frac{127}{\log(127\cdot i)}$, so we have $T(n) = \Omega(\frac{n}{\log n})$
   Then we consider the big-O bound. Since 127 is only a constant and it does not change the asymptotic behavior of a function, we ignore it and get

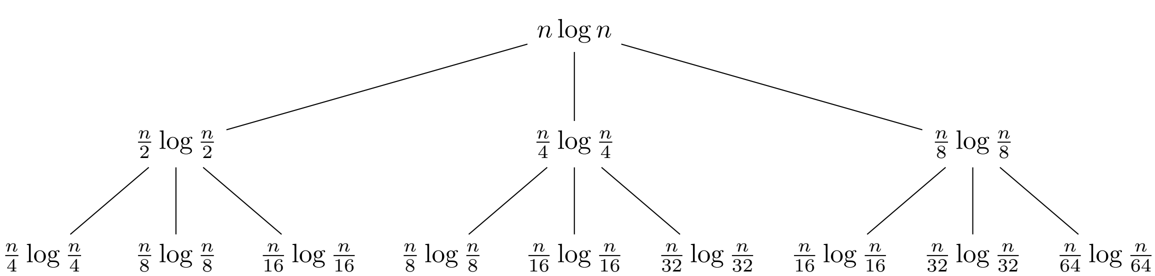
$$\sum_{i=1}^{\frac{n}{127}} \frac{1}{\log(127 \cdot i)} \leq 1 + \int_1^{\frac{n}{127}} \frac{1}{\log(127 \cdot x)} dx \overset{\text{change of variable}}{=} 1 + \frac{1}{127}\int_2^n \frac{1}{\log x} dx.$$

   Let k be a constant such that $\log x \leq 127 \cdot (\log x - 1)$ for $x \geq k$, then we have

$$\begin{aligned} 1 + \frac{1}{127}\int_2^k \frac{1}{\log x}dx + \frac{1}{127}\int_k^n \frac{1}{\log x}dx &\leq 1 + \frac{1}{127}\int_2^k \frac{1}{\log x}dx + \int_k^n \frac{\log x - 1}{\log^2 x}dx \\ &= 1 + \frac{1}{127}\int_2^k \frac{1}{\log x}dx + \frac{n}{\log n} - \frac{k}{\log k} \\ &= O(\frac{n}{\log n}) \end{aligned}$$

   Thus $T(n) = \Theta(\frac{n}{\log n})$.

   (b)



   The figure above shows a small part of the recursion tree.
   Let the base of logarithm be 2. Then the sum is at most $((\frac{7}{8})^i n \log n)$ at the i-th layer, for $i \in \{0\} \cup \mathbb{N}$.
   For the upper bound, we assume the tree has infinite layers with the same pattern, then we have

$$\sum_{i=0}^{\infty} (\frac{7}{8})^i \cdot n \log n = 8n \log n$$

   As for the lower bound, we already have $n \log n$ at the root, so we conclude that $T(n) = \Theta(n \log n)$

(c) By master theorem case 1, we have $a = 4, b = 2, f(n) = n \log n$. Let $\epsilon = 0.5$, then we have $f(n) = O(n^{log_b a - \epsilon})$, so $T(n) = \Theta(n^2)$

(d) Consider $S(n) = \frac{T(n)}{n}$, we have

$$\begin{aligned} &nS(n) = nS(\sqrt{n}) + n \\ \Longrightarrow &S(n) = S(\sqrt{n}) + 1 \\ \Longrightarrow &S(2^k) = S(2^{\frac{k}{2}}) + 1 \quad (k = \log_2 n) \end{aligned}$$

Further let $P(k) = S(2^k)$, we have $P(k) = P(\frac{k}{2}) + 1$.
By master theorem

$$\begin{aligned} &P(k) = \Theta(\log k) \\ \Longrightarrow &S(n) = \Theta(\log \log n) \\ \Longrightarrow &T(n) = \Theta(n \log \log n) \end{aligned}$$