#1 $\frac{dP(t)}{dt} = rP(t) - \lambda [P(t)]^2, \quad t \geq 0$

i).

$$u(t) = \frac{1}{P(t)}$$

$$P(t) * \frac{u(t)}{u(t)} = \frac{1}{u(t)}$$

$$P(t) = \frac{1}{u(t)}$$

$$\frac{dP(t)}{du(t)} * \frac{du(t)}{dt} = rP(t) - \lambda [P(t)]^2$$

$$\frac{dP(t)}{du(t)} = \frac{-1}{[u(t)]^2}$$

$$-\frac{1}{[u(t)]^2} \frac{du(t)}{dt} = rp(t) - \lambda [P(t)]^2$$

But $P(t) = \frac{1}{u(t)}$

$$\Rightarrow -\frac{1}{[u(t)]^2} \frac{du(t)}{dt} = \frac{r}{u(t)} - \lambda \left[\frac{1}{u(t)}\right]^2$$

$$\cancel{-}\frac{du(t)}{dt} = \frac{r u(t) - \lambda}{-}$$

$$= \frac{du(t)}{dt} = \lambda - r u(t). \text{ Ans.}$$

1 ii): $\frac{du(t)}{dt} + ru(t) = \lambda$

Using Integrating factor

$$e^{\int r\,dt} = e^{rt}$$

$$e^{rt}\frac{du(t)}{dt} + re^{rt}u(t) = \lambda e^{rt}$$

$$\frac{d}{dt}e^{rt}u(t) = \lambda e^{rt}$$

$$de^{rt}u(t) = \int \lambda e^{rt}\,dt$$

$$e^{rt}u(t) = \frac{\lambda e^{rt}}{r} + c$$

Replacing $t = 0$

$$e^{r(0)}u(0) = \frac{\lambda e^{r(0)}}{r} + c$$

$$u(0) = \frac{\lambda}{r} + c$$

$$\therefore \quad c = u(0) - \frac{\lambda}{r}$$

$$e^{rt}u(t) = \frac{\lambda e^{rt}}{r} + u(0) - \frac{\lambda}{r}$$

$$\Rightarrow \frac{\lambda e^{rt}}{r} + \frac{u(0)}{1} - \frac{\lambda}{r} = \frac{\lambda e^{rt} + r u(0) - \lambda}{r}$$

$$e^{rt}u(t) = \frac{\lambda e^{rt} - \lambda + r u(0)}{r}.$$

$$e^{-rt}\left(e^{rt}u(t)\right) = \left\{\frac{\lambda e^{rt} - \lambda + r u(0)}{r}\right\} e^{-rt}$$

$$u(t) = \frac{\lambda - \lambda e^{-rt} + r u(0) e^{-rt}}{r}$$

$$u(t) = \frac{\lambda + r u(0) e^{-rt} - \lambda e^{-rt}}{r}$$

$$u(t) = \frac{\lambda + (r u(0) - \lambda) e^{-rt}}{r} \Rightarrow \text{hence}$$

solved the equation for $u(t)$

Since $p(t) = \dfrac{1}{u(t)}$

$$\therefore \quad p(t) = \frac{r}{\lambda + (r u(0) - \lambda) e^{-rt}}$$

But $u(0) = \dfrac{1}{p(0)}.$

$$\Rightarrow P(t) = \frac{r}{\lambda + \left(\frac{r}{P(0)} - \lambda\right)e^{-rt}}$$ ⇒ hence

Solved for the formular for $P(t)$ in terms of $P(0)$, r and λ (in addition to time t)

1 (iii)

$$\lim_{t \to \infty} P(t) = \frac{r}{\lambda}$$ ⇒ hence the solution as t tends to ∞ $(t \to \infty)$

1.

iv. $\frac{dP(t)}{dt} = rP(t) - \lambda e^{-rt}[P(t)]^2, \quad t \geq 0$

$$u(t) = \frac{1}{P(t)}$$

$$P(t) = \frac{1}{u(t)}$$

$$\frac{dP(t)}{du(t)} * \frac{du(t)}{dt}$$

$$\frac{dP(t)}{du(t)} = -\frac{1}{[u(t)]^2}$$

$$\frac{-1}{[u(t)]^2} * \frac{du(t)}{dt} = \frac{r}{u(t)} - \lambda e^{-rt}\left[\frac{1}{u(t)}\right]^2$$

$$\frac{-du(t)}{dt} = ru(t) - \lambda e^{-rt}$$

$$\frac{du(t)}{dt} = \lambda e^{-rt} - ru(t)$$

$$\frac{du(t)}{dt} + ru(t) = \lambda e^{-rt} \quad \ldots i)$$

$$e^{\int r\,dt} = e^{rt} \quad \text{---- Integrating factor}$$

Multiplying through i) with e^{rt} we get

$$e^{rt}\frac{du(t)}{dt} + re^{rt}u(t) = \lambda$$

$$\frac{d}{dt}e^{rt}u(t) = \lambda$$

$$d\,e^{rt}u(t) = \int \lambda\,dt$$

$$e^{rt}u(t) = \lambda t + c$$

Replacing $t=0$

$$e^{r(0)}u(0) = \lambda * 0 + c$$

$$u(0) = c \quad \text{(making } c \text{ the subject).}$$

$$\Rightarrow e^{rt}u(t) = \lambda t + u(0)$$

$$e^{-rt}\left[e^{rt}u(t)\right] = e^{-rt}\left[\lambda t + u(0)\right]$$

$$u(t) = \left[\lambda t + u(0)\right]e^{-rt}$$

But $p(t) = \dfrac{1}{u(t)}$

$$P(t) = \frac{1}{[\lambda t + u(0)]e^{-rt}}$$

Note : $u(0) = \frac{1}{P(0)}$

$$P(t) = \frac{1}{\left[\lambda t + \frac{1}{P(0)}\right]e^{-rt}}$$ $\Rightarrow$ hence Solved for expression of $P(t)$

$$\lim_{t \to \infty} P(t) = \text{Undefined}$$

#2 a) $\frac{dP}{dt} = rP - \lambda P^3$; $t \geq 0$

$u = \frac{1}{P^2} \Rightarrow P^2 \frac{\cancel{u}}{\cancel{u}} = \frac{1}{u} \Rightarrow P^2 = \frac{1}{u}$

$P = \frac{1}{u^{1/2}}$

$\frac{dP}{dt} = \frac{dP}{du} \cdot \frac{du}{dt}$ $\therefore$

$\frac{dP}{du} = -\frac{1}{2} u^{-3/2}$

$-\frac{1}{2} u^{-3/2} \frac{du}{dt} = r\left(\frac{1}{u^{1/2}}\right) - \lambda \left(\frac{1}{u^{1/2}}\right)^3$.

$-\frac{1}{2} u^{-3/2} \frac{du}{dt} = r u^{-1/2} - \lambda u^{-3/2}$

$-\frac{1}{2} \frac{du}{dt} = ru - \lambda$

$\frac{du}{dt} = -2ru + 2\lambda$

$\frac{du}{dt} = 2\lambda - 2ru$.

$\frac{du}{dt} + 2ru = 2\lambda$.

Note ; $u = u(t)$.

a) $\therefore \frac{du(t)}{dt} + 2\gamma u(t) = 2\lambda$. Ans. .... i).

#2b) Introducing Integrating factor

$$e^{\int 2\gamma dt} = e^{2\gamma t}$$

Multiplying through i) with $e^{2\gamma t}$ we get;

$$e^{2\gamma t}\frac{du}{dt} + 2\gamma u(t)e^{2\gamma t} = 2\lambda e^{2\gamma t}$$

$$\frac{d}{dt} u(t)e^{2\gamma t} = 2\lambda e^{2\gamma t}.$$

$$d\,u(t)e^{2\gamma t} = \int 2\lambda e^{2\gamma t}\,dt.$$

$$u(t)e^{2\gamma t} = \frac{\lambda e^{2\gamma t}}{\gamma} + c$$

When $t = 0$, we have

$$u(0)e^{2\gamma \times 0} = \frac{\lambda e^{2\gamma \times 0}}{\gamma} + c.$$

$$= u(0) = \frac{\lambda}{\gamma} + c$$

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Making c the subject;

$$c = u(0) - \frac{\lambda}{r}$$

$$\Rightarrow u(t)e^{2rt} = \frac{\lambda e^{2rt}}{r} + u(0) - \frac{\lambda}{r}$$

Multiplying through by e^{-2rt} we get;

$$e^{-2rt}u(t)e^{2rt} = \left\{\frac{\lambda e^{2rt} + ru(0) - \lambda}{r}\right\}e^{-2rt}$$

$$u(t) = \left\{\frac{\lambda e^{2rt} - \lambda + ru(0)}{r}\right\}e^{-2rt}$$

$$\Rightarrow \frac{\lambda - \lambda e^{-2rt} + ru(0)e^{-2rt}}{r}$$

$$u(t) = \frac{\lambda - [\lambda - ru(0)]e^{-2rt}}{r} \quad \text{Ans.}$$

#2c

Note:

$$u(t) = \frac{1}{[P(t)]^2}$$

$$[P(t)]^2 = \frac{r}{\lambda - [\lambda - r u(0)] e^{-2rt}}$$

$$P(t) = \frac{r^{1/2}}{\left(\lambda - [\lambda - r u(0)] e^{-2rt}\right)^{1/2}}$$

Ans

#2d

$$\underset{t \to \infty}{P(t)} = \frac{r^{1/2}}{\lambda^{1/2}}$$

Ans

3a)

Using $P_t = P_o e^{rt}$

we have:

$P_t = 26$ million.

$P_o = 18$ million.

time $(t) = 8$ yrs.

$\therefore \Rightarrow \quad 26 = 18e^{8r}$

$\Rightarrow \quad 26/18 = e^{8r}$

$\ln 1.4444 = \ln e^{8r}$

$\ln 1.4444 = 8r \ln e$.

But $\ln e = 1$.

$$r = \frac{\ln 1.4444}{8}$$

$r = 0.046$

$\therefore$ Annual growth rates using the world Bank's estimates will be 4.6% Ans.

#3b

40 million = 2000
26 million = 1990
Time = 10 yrs

Using $P_t = P_o e^{rt}$

P_t = 40 million
P_o = 26 million
t = 10 yrs

$$\frac{40}{26} = \frac{\cancel{26} e^{10r}}{\cancel{26}}$$

$$1.538 = e^{10r}$$

$$\ln 1.538 = 10r \ln e$$

But $\ln e = 1$

$$\frac{\ln 1.538}{10} = \frac{\cancel{10} r}{\cancel{10}}$$

$$r = 0.043$$

$= \underline{4.3}\ \%$ ⇒ Population growth rate between 1990 and 2000

⇒ P_t = 80 million
P_o = 40 million
t = ?
r = 4.3 %

$$P_t = P_o e^{rt}$$

$$\frac{80}{40} = \frac{\cancel{40} e^{0.043t}}{\cancel{40}}$$

$$2 = e^{0.043t}$$

$$\ln 2 = 0.043t \ln e$$

But $\ln e = 1$.

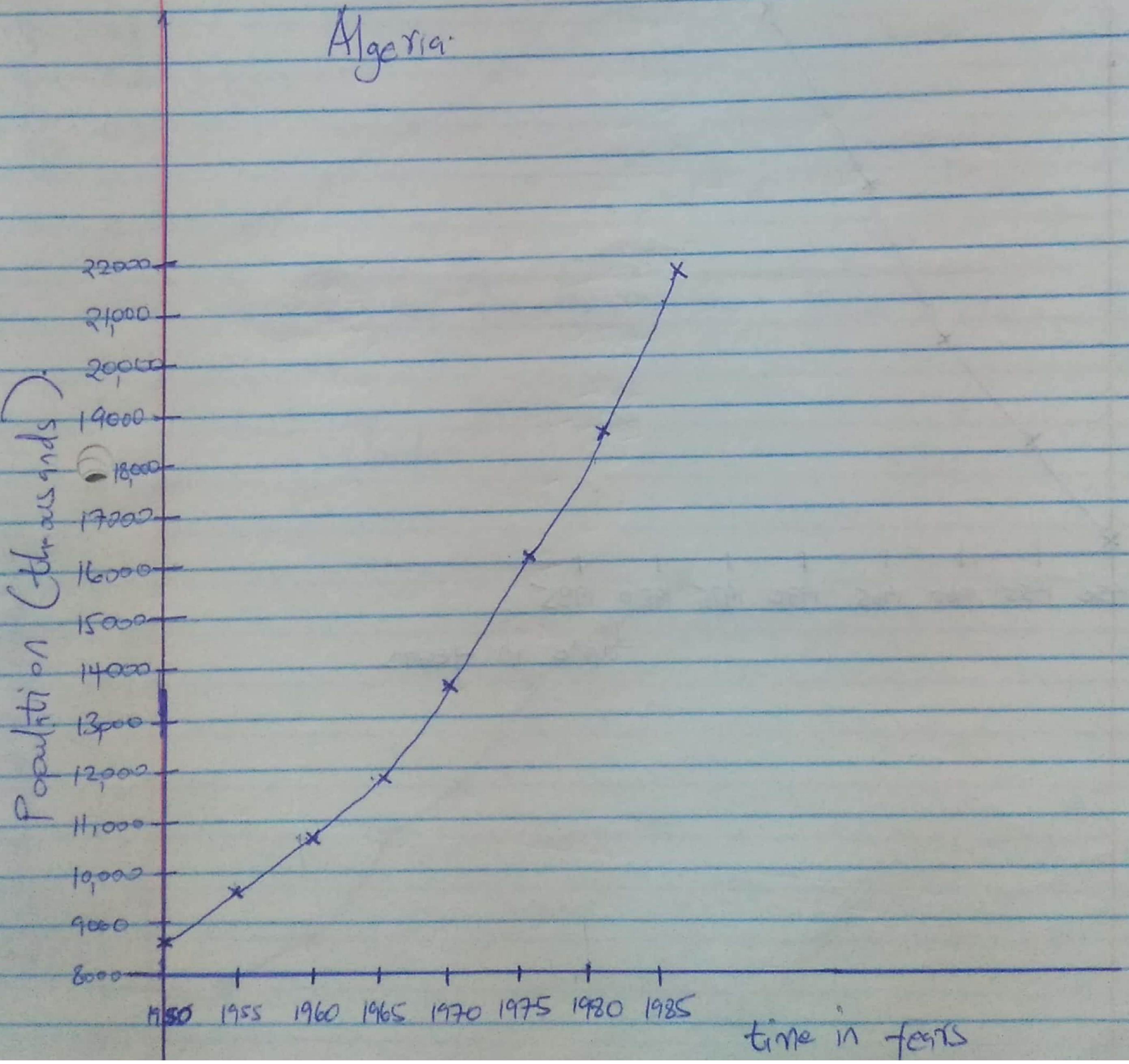
$$\frac{\ln 2}{0.043} = \frac{0.0\cancel{43}t}{0.0\cancel{43}}$$

$$t = 16.120 \text{ yrs.}$$

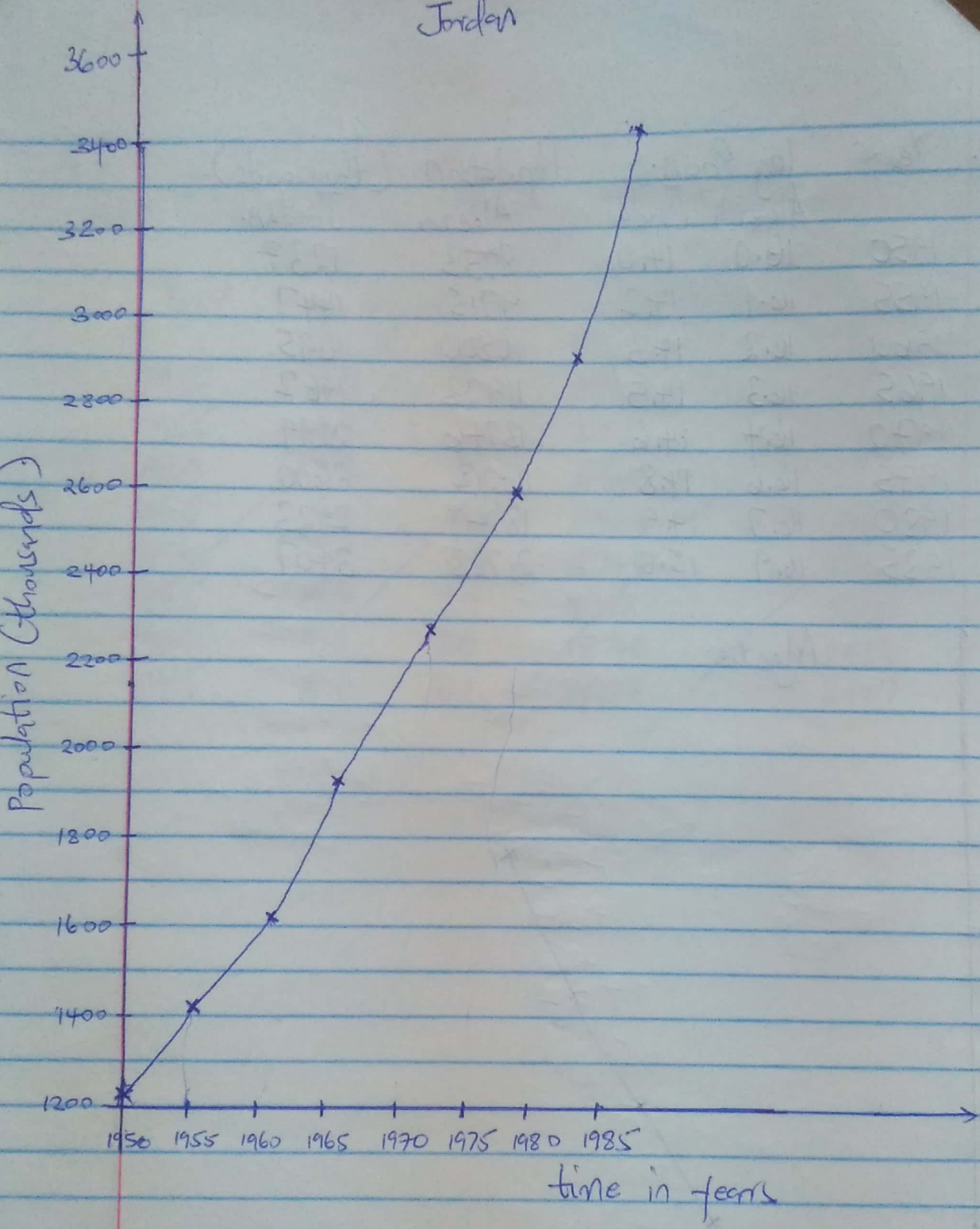
The population will reach 80 million after approximately 16 years. which will be the year 2016 Ans.

4a).

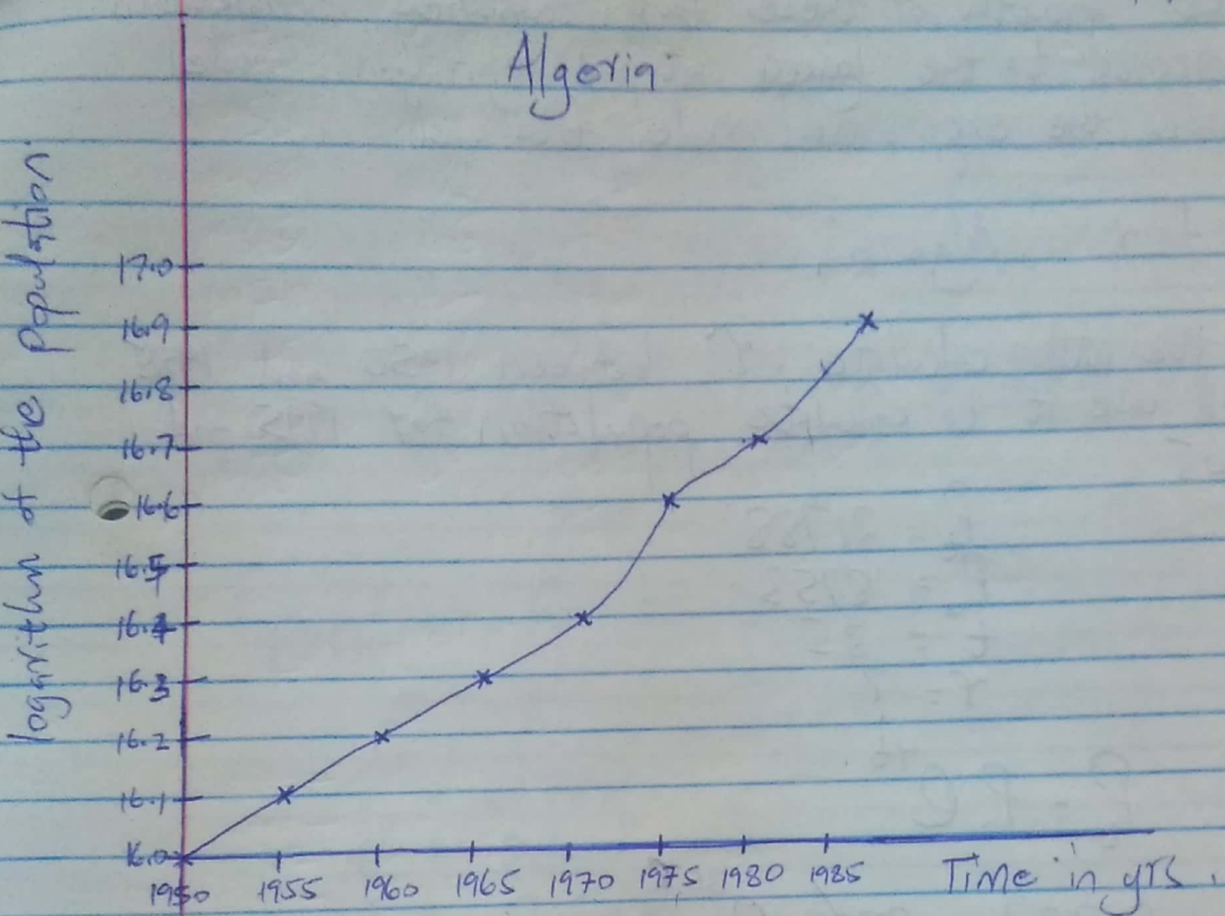
Year	log Popn.		Population (thousands).	
	Algeria	Jordan	Algeria	Jordan.
1950	16.0	14.0	8753	1237
1955	16.1	14.2	9715	1447
1960	16.2	14.3	10800	1695
1965	16.3	14.5	11923	1962
1970	16.4	14.6	13746	2299
1975	16.6	14.8	16018	2600
1980	16.7	14.9	18740	2923
1985	16.9	15.0	21788	3407

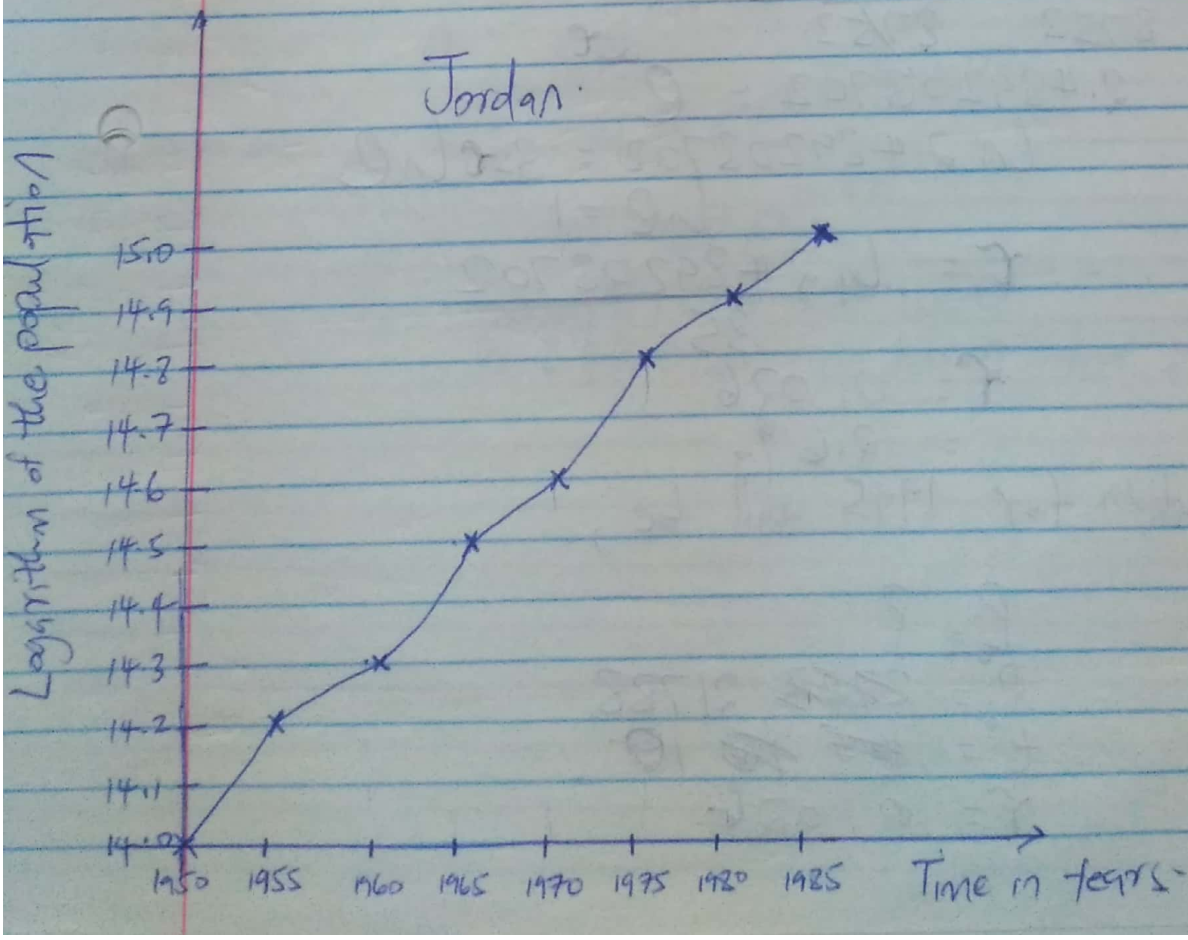


Jordan


3600
3400
3200
3000
2800
2600
2400
2200
2000
1800
1600
1400
1200
Population (thousands)
1950 1955 1960 1965 1970 1975 1980 1985
time in years

#46/. Plot for the natural logarithms of the population





Exponential model is a good description of the past growth of these two countries populations compared to the ~~logarit~~ natural logarithm model Since the exponential model fits well.

#4c) i) For Algeria;

We will calculate r between 1950 and 1985, and use it to calculate population for 1995 and 2005

$P_t = 21788$
$P_o = 8753$.
$t = 35$
$r = ?$

$$P_t = P_o e^{rt}$$

$$\frac{21788}{8753} = \frac{\cancel{8753}}{\cancel{8753}} e^{35r}$$

$$2.489203702 = e^{35r}$$

$$\ln 2.489203702 = 35r \ln e$$

$$\ln e = 1$$

$$\therefore r = \frac{\ln 2.489203702}{35}$$

$$r = 0.026$$

$$= 2.6\%$$

Population for 1995 will be;

$P_t = ?$
$P_o = \cancel{8753}$ 21788
$t = \cancel{45}$ $\cancel{10}$ 10
$r = 0.026$.

$P_t = 21788e^{10*0.026}$

$P_t = 21788*e^{0.26}$

$P_t = 28257.51273 \times 1000$

$P_t = 28,257,512.73 \simeq 28257512$ for 1995 in Algeria Ans.

Population for 2005

$P_t = ?$

$P_o = 21788$

$t = 20$

$r = 0.026$

$P_t = P_o e^{rt}$

$= 21788e^{20*0.026}$

$= 21788e^{0.52}$

$P_t = 36648.01843 \times 1000$

$= 36648018.$ population for 2005 Algeria

#4c) i) For Jordan:

$P_t = 3407$
$P_0 = 1237$
$r = ?$
$t = 35$

$P_t = P_0 e^{rt}$

$\Rightarrow 3407 = 1237 e^{35r}$

$\frac{3407}{1237} = \frac{\cancel{1237}}{\cancel{1237}} e^{35r}$

$\ln 2.754244139 = 35r \ln \cancel{e}$

$r = \frac{\ln 2.754244139}{35}$

$r = 0.029$

$r = 2.9\%$

Population 1995

$P_t = ?$
$P_0 = 3407$
$r = 0.029$
$t = 10$

$P_t = 3407 e^{10 * 0.029}$

$P_t = 4553.208452 \times 1000$

$P_t = 4{,}553{,}208.452 \simeq 4{,}553{,}208$ population for 1995

Population 2005.

$P_t = ?$
$P_0 = 3407$
$r = 0.029$.
$t = 20$.

$P_t = P_0 e^{rt}$.

$= 3407 e^{20 * 0.029} * 1000$

$P_t = 6{,}085{,}032.934 \simeq 6{,}085{,}032$ population for 2005 Ans.