$$= \frac{-1}{{}_n d_x} \left\{ n * l_{x+n} - {}_n L_x \right\}$$

$$= \frac{{}_n L_x - n * l_{x+n}}{{}_n d_x} \equiv {}_n a_x$$

FERTILITY MEASURES

- Children Ever Born (CEB)
- Crude Birth Rate (CBR)
- General Fertility Rate (GFR)
- Age Specific Fertility Rate (ASFR)
- Total Fertility Rate (TFR)
- Gross Reproductive Rate (GRR)
- Net Reproductive Rate (NRR)
- Child - Woman Ratio (CWR)
- Parity Progression Ratio (PPR)
- Birth Interval (BI)

Crude Birth Rate (CBR)

$$CBR = \frac{\text{Total Number of Birth in a given period}}{\text{Total Population in that period}}$$

Remark 1: By definition of a rate CBR is NOT A RATE because not everybody in the population is exposed to birth. A better fertility measure follows :-

General Fertility Rate (GFR)

$$GFR = \frac{\text{Total Number of Births in a given period}}{\text{Total Number of women in the reproductive ages in that period}}$$

$$= \frac{B}{W}$$

Age Specific Fertility Rate (ASFR)

$$ASFR = \frac{\text{Number of births for woman in a certain age-group in a given period}}{\text{Number of women in that age-group for the given period}}$$

= Number of births per woman per year

Notations

Let $i = 1, 2, 3, 4, 5, 6, 7$

= the number given to an age-group

B_i = the number of births for the i^{th} age-group

W_i = the number of women in the i^{th} age-group

f_i = the ASFR for the i^{th} age group

B = the total number of births

W = the total number of women in the child-bearing ages

P = The total population size

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Relations

1). $CBR = \frac{B}{P}$

$= \frac{\sum_i B_i}{P}$

$= \sum_i \frac{B_i}{P}$

$= \sum_i \left(\frac{B_i}{W_i}\right)\left(\frac{W_i}{P}\right)$

2) $f_i = \frac{B_i}{W_i}$

$\therefore CBR = \sum_i \left(\frac{B_i}{W_i}\right)\left(\frac{W_i}{P}\right)$

$= \sum_i f_i \left(\frac{W_i}{P}\right)$.

where f_i = ASFR for the i^{th} age-group

$\frac{W_i}{P}$ = the proportion of women in the i^{th} age-group with respect to the total population

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3) $$CBR = \frac{B}{P}$$

$$= \frac{B}{W}\frac{W}{P}$$

$$= GFR * \frac{W}{P}$$

where $\frac{W}{P}$ = the proportion of women in the child-bearing ages with respect to the total population

4) $$GFR = \frac{B}{W} = \frac{\sum_i B_i}{W}$$

$$= \sum_i \frac{B_i}{W}$$

$$= \sum_i \frac{B_i}{W_i}\frac{W_i}{W}$$

$$= \sum_i f_i \frac{W_i}{W}$$

where

$\frac{W_i}{W}$ = the proportion of women in the i^{th} age-group with respect to all women in the child-bearing ages.

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In a table form we have the following :-

i	Age-group	W_i	B_i	f_i
1	15-19	W_1	B_1	f_1
2	20-24	W_2	B_2	f_2
3	25-29	W_3	B_3	f_3
4	30-34	W_4	B_4	f_4
5	35-39	W_5	B_5	f_5
6	40-44	W_6	B_6	f_6
7	45-49	W_7	B_7	f_7
Total		W	B	$\sum_i f_i$

Total Fertility Rate (TFR):

TFR = Average number of births a woman would have at the prevailing age-specific fertility rates throughout her lifetime

= the sum of ASFR over all age-groups

$$= \sum_i n_i f_i$$

Where n_i = the length of the i^{th} interval

For the 5year-groups

$$TFR = \sum_{i=1}^{7} 5 f_i = 5 \sum_{i=1}^{7} f_i$$

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Gross Reproductive Rate (GRR) → mortality not taken into account.

GRR = the average number of daughters a woman bears, neglecting mortality, assuming the current age-specific fertility rates through her life-time.

$$= TFR * \frac{F}{F+M}$$

$$= TFR\left(\frac{1}{1+\frac{M}{F}}\right)$$

$$= TFR\left(\frac{1}{1+SRB}\right).$$

where SRB = Sex Ratio at Birth.

Net Reproductive Rate (NRR).

NRR = the number of daughters a woman would have if she experiences a given set of age-specific rates throughout her reproductive ages with allowance for mortality of women over that period

Notations

f_i = ASFR for i^{th} age-group when all children (both female and male births) are considered.

$m(a)$ = the proportion of women at age 'a' who bear a female child

$p(a)$ = the probability of surviving from birth to age a

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Therefore

$$GRR = \int_{\alpha}^{\beta} n(a)\, da$$

where α and β are the lower and upper ages of child-bea

$$NRR = \int_{\alpha}^{\beta} p(a)\, n(a)\, da$$

If $p(a) = 1$ the $NRR = GRR$

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POPULATION GROWTH MODELS

1. Geometric Growth Model

Let P_t = the population size at time t

$\Rightarrow$ P_{t+1} = the population size at time t+1

Absolute change = $P_{t+1} - P_t$

Relative change = $\dfrac{P_{t+1} - P_t}{P_t}$

Suppose $r = \dfrac{P_{t+1} - P_t}{P_t}$

$\therefore rP_t = P_{t+1} - P_t$

$\therefore P_{t+1} = P_t(1+r)$

$\therefore P_{t+2} = P_{t+1}(1+r)$

$= P_t(1+r)^2$

$\therefore P_{t+n} = P_t(1+r)^n$

In financial Terms

P = Principal $\qquad$ r = Interest Rate

A = Amount

I = Interest

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let r the interest rate per year, after 1 year

$$A_1 = P+I$$
$$= P+rP$$
$$= P(1+r)$$

At the end of the second yr

$$A_2 = A_1 + I = A_1 + rA_1 = A_1(1+r)$$

$$\therefore A_2 = P(1+r)^2$$

$$\vdots$$

After n years

$$A_n = P(1+r)^n$$

In demography, we have

$$P_{t_2} = P_{t_1}(1+r)^{t_2-t_1}$$
$$= P_{t_1}(1+r)^t$$

where $t_2 - t_1 = t$ or simply

$$\boxed{P_t = P_0(1+r)^t}$$

which is called Geometric growth model make r the subject

$$r = \left(\frac{P_t}{P_0}\right)^{1/t} - 1$$

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* Doubling Time

Let $P_t = 2P_0$

$\therefore 2P_0 = P_0 (1+r)^t$

$\therefore t = \dfrac{\log 2}{\log(1+r)}$

* Tripling Time

Let $P_t = 3P_0$

$3P_0 = P_0 (1+r)^t$

$t = \dfrac{\log 3}{\log(1+r)}$

In general if $P_t = kP_0$

then

$t = \dfrac{\log k}{\log(1+r)}$

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2. Exponential Growth Model

Method 1 : Compounding

Let the interest be $\frac{r}{4}$ per quarter. After 1st quarter, the amount:

$$A_1 = P + \frac{r}{4}P$$

$$= P\left(1+\frac{r}{4}\right)$$

~~At the end of the year~~

At the end of the 2nd quarter,

$$A_2 = P\left(1+\frac{r}{4}\right)^2$$

At the end of the year we have 4 quarters

$$A_4 = P\left(1+\frac{r}{4}\right)^4$$

At the end of 2years (8quarters).

$$A_8 = P\left(1+\frac{r}{4}\right)^8$$

$$= P\left[\left(1+\frac{r}{4}\right)^4\right]^2$$

At the end of t years (4t quarters)

$$A_{4t} = P\left[\left(1+\frac{r}{4}\right)^4\right]^t$$

Divide a year into j parts. At the end of a year

$$A_1 = P\left[\left(1+\frac{r}{j}\right)^j\right]$$

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After t years we have

$$A_t = P\left[\left(1+\frac{r}{j}\right)^j\right]^t$$

In demography,

$$P_t = P_0\left[\left(1+\frac{r}{j}\right)^j\right]^t$$

Lemma: In calculating we can prove that

$$\lim_{j\to\infty}\left(1+\frac{r}{j}\right)^j = e^r$$

$$P_t = \lim_{j\to\infty} P_0\left[\left(1+\frac{r}{j}\right)^j\right]^t$$

$$= P_0\left[\lim_{j\to\infty}\left(1+\frac{r}{j}\right)^j\right]^t$$

$$\boxed{P_t = P_0 e^{rt}}$$

which is exponential

Proof:

$$\log\left(1+\frac{r}{j}\right)^j = j\log\left(1+\frac{r}{j}\right)$$

Geometric Series

$$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots; \quad |x|<1$$

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Integrate both sides wrt x i.e

$$\int \frac{1}{1-x}\,dx = \int \left[1+x+x^2+x^3+\cdots\right] dx$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$\therefore \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

let $t = -x \Rightarrow x = -t$

$$\therefore \log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} \cdots$$

let $t = \frac{r}{j}$

$$\log\left(1+\frac{r}{j}\right) = \frac{r}{j} - \frac{1}{2}\left(\frac{r}{j}\right)^2 + \frac{1}{3}\left(\frac{r}{j}\right)^3 - \frac{1}{4}\left(\frac{r}{j}\right)^4 \pm$$

$$\therefore \log\left(1+\frac{r}{j}\right)^j = j\left\{\frac{r}{j} - \frac{1}{2}\frac{r^2}{j^2} + \frac{1}{3}\frac{r^3}{j^3} - \frac{1}{4}\frac{r^4}{j^4} \pm \cdots\right\}$$

$$= r - \frac{1}{2}\frac{r^2}{j} + \frac{1}{3}\frac{r^3}{j^2} - \frac{1}{4}\frac{r^4}{j^3} \pm \cdots$$

$$\lim_{j\to\infty} \log\left(1+\frac{r}{j}\right)^j = r$$

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$$\therefore \lim_{j\to\infty} \log\left(1+\frac{r}{j}\right)^j = \lim_{j\to\infty} j \log\left(1+\frac{r}{j}\right)$$

$$\therefore \lim_{j\to\infty} \log\left(1+\frac{r}{j}\right)^j = r$$

$$\log\left[\lim_{j\to\infty}\left(1+\frac{r}{j}\right)^j\right] = r$$

Use base e (natural log)

$$\ln\left[\lim_{j\to\infty}\left(1+\frac{r}{j}\right)^j\right] = r$$

$\ln y = a$

$y = e^a$

$$\therefore \lim_{j\to\infty}\left(1+\frac{r}{j}\right)^j = e^r \quad *$$

Method 2 : Differential Equation Approach

Let

$$r = \frac{\Delta P}{P\Delta t}$$

$$r = \frac{dP}{Pdt}, \quad P = P(t)$$

This is ordinary Differential Eqn

$$= \frac{d}{dt}\log P$$

Take log to be ln

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$$r = \frac{d}{dt} \ln P(t)$$

$$\int r\,dt = \int d \ln P(t)$$

$$rt + c = \ln P(t)$$

$$P(t) = e^{rt+c}$$

$$P(t) = e^{c} e^{rt}$$

The ~~Ial~~ Initial Condition at time $t = 0$ we have

$$P(0) = P_0$$

$$\therefore P(0) = e^{c}$$

$$\therefore \boxed{P_t = P_0 e^{rt}}$$

which is the exponential growth model

Remark: Demographers call the exponential growth model as the Malthusian Law. In full, the Malthusian Law postulates that the population is growing exponentially while the food production is growing arithmetically.

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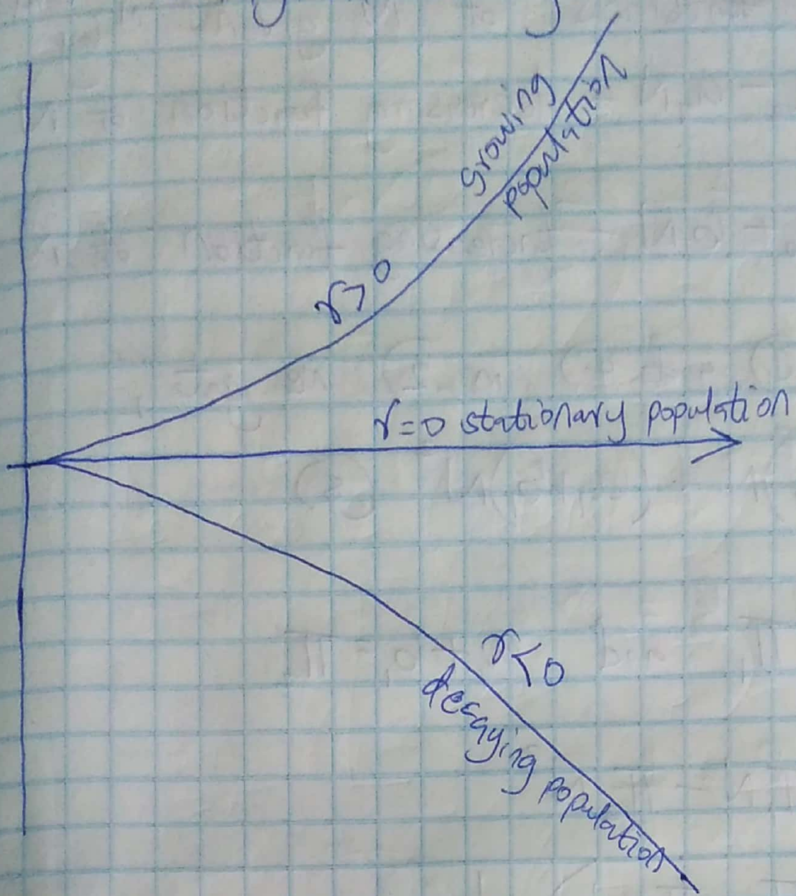
Diagramatically



$r = b - d$

where b = birth rate
d = death rate

3, Logistic Curve Model:

The exponential model implies that a population can grow forever (indefinitely). ~~The~~ This is not true in reality. A ~~bio~~ Mathematical biologist by the name Verlhust (1934) showed ~~exponentially~~ experimentally that a population levels off at some point in time due to what is called carrying capacity. Alfred Lotka (1926) and Vito Volterra (1927) independently came up with the following population growth model.

$$\frac{dN}{dt} = P_N \cdot N \qquad 1)$$

where $N = N(t) = P(t)$ and $P_N = \lambda_N - \mu_N$

$$\therefore \frac{dN}{dt} = (\lambda_N - \mu_N) N \qquad 2)$$

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They further assumed that the birth rate λ_N and death rate μ_N are linear functions of N given by

$$\lambda_N = a_0 - a_1 N \text{ - decreasing function of } N \quad 3)$$

and

$$\mu_N = b_0 + b_1 N \text{ - increasing function of } N \quad 4)$$

Substituting 3) and 4) in 2) we get,

$$\frac{dN}{dt} = (a_0 - b_0)N - (a_1 + b_1)N^2 \quad (5)$$

Let $a_0 - b_0 = \Pi_1$ and $a_1 + b_1 = \Pi_2$

$$\therefore \frac{dN}{dt} = \Pi_1 N - \Pi_2 N^2$$

$$= (\Pi_1 - \Pi_2 N) N$$

$$= \Pi_1 \left(1 - \frac{\Pi_2}{\Pi_1} N\right) N$$

$$= \Pi_1 \left(1 - \frac{N}{\Pi_1 / \Pi_2}\right) N$$

$$= \Pi_1 \left(1 - \frac{N}{K}\right) N$$

where $K = \dfrac{\Pi_1}{\Pi_2}$

Put $\Pi_1 = r$

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$$\therefore \frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) \quad 6)$$

Where r = Intrinsic growth rate

and

K = the carrying capacity

The problem is to solve the above to obtain a logistic curve model.

Method 1 : Partial fraction Technique

$$\int \frac{dN}{N\left(1-\frac{N}{K}\right)} = \int r\,dt$$

$$= rt + c$$

$$\int \left[\frac{A}{N} + \frac{B}{1-\frac{N}{K}}\right] dN = rt + c$$

Where A and B are constants to be determined

$$\frac{1}{N\left(1-\frac{N}{K}\right)} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}}$$

$$\frac{1}{N\left(1-\frac{N}{K}\right)} = \frac{A\left(1-\frac{N}{K}\right) + BN}{N\left(1-\frac{N}{K}\right)}$$

$$1 = A\left(1-\frac{N}{K}\right) + BN$$

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$$1 = A - \frac{AN}{K} + BN$$

i.e

$$1 + 0.N = A - \left(\frac{A}{K} - B\right)N$$

Comparing Coefficient

$$A = 1$$

$$\frac{A}{K} - B = 0$$

$$\Rightarrow B = \frac{1}{K}$$

$$\therefore \int \left[\frac{1}{N} + \frac{1/K}{1 - N/K}\right] dN = rt + c$$

$$\int \frac{dN}{N} + \int \frac{1/K}{1 - \frac{N}{K}} dN = rt + c$$

$$\ln N + \frac{1/K \ln\left(1 - \frac{N}{K}\right)}{-1/K} = rt + c$$

$$\ln N - \ln\left(1 - \frac{N}{K}\right) = rt + c$$

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$$\therefore \ln \frac{N}{1-\frac{N}{K}} = rt + c$$

$$\therefore \frac{N}{1-\frac{N}{K}} = e^{rt+c} = e^{c} e^{rt}$$

i.e $$\frac{N(t)}{1-\frac{N(t)}{K}} = e^{c} e^{rt}$$

when $t=0$, we have

$$\frac{N(0)}{1-\frac{N(0)}{K}} = e^{c}$$

$$\therefore \frac{N(t)}{1-\frac{N(t)}{K}} = \frac{N(0)}{1-\frac{N(0)}{K}} e^{rt}$$

$$\therefore N(t) = \frac{N(0)}{1-\frac{N(0)}{K}} e^{rt} \left[1-\frac{N(t)}{K}\right]$$

$$= \frac{N(0)\, e^{rt}}{1-\frac{N(0)}{K}} - \frac{N(0)}{1-\frac{N(0)}{K}} \frac{N(t)}{K} e^{rt}$$

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$$\therefore \left[1 + \frac{N(0)}{K-N(0)} e^{rt}\right] N(t) = \frac{N(0)e^{rt}}{1-\frac{N(0)}{K}}$$

$$\therefore N(t) = \frac{N(0)e^{rt}}{\left[1+\frac{N(0)}{K-N(0)}e^{rt}\right]\left[1-\frac{N(0)}{K}\right]}$$

$$= \frac{e^{-rt}}{e^{-rt}} \frac{N(0)e^{rt}}{\left[1+\frac{N(0)}{K-N(0)}e^{rt}\right]\left[1-\frac{N(0)}{K}\right]}$$

$$= \frac{N(0)}{\left[e^{-rt}+\frac{N(0)}{K-N(0)}\right]\left[\frac{K-N(0)}{K}\right]}$$

$$N(t) = \frac{K\,N(0)}{[K-N(0)]e^{-rt}+N(0)}$$

$$N(t) = \frac{K}{1+\left(\frac{K}{N(0)}-1\right)e^{-rt}}$$

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$\therefore \lim_{t\to\infty} N(t) = K$

Method 2: Using a simpler differential eqn Consider

$$\frac{1}{N}\frac{dN}{dt} = r\left(1-\frac{N}{K}\right)$$

$$\therefore \frac{1}{N}\cdot\frac{dN}{du}\frac{du}{dt} = r\left(1-\frac{N}{K}\right)$$

let $u = \frac{1}{N} \Rightarrow N = \frac{1}{u} \Rightarrow \frac{dN}{du} = -\frac{1}{u^2}$

$$\therefore u\left(-\frac{1}{u^2}\right)\frac{du}{dt} = r\left(1-\frac{1}{uK}\right)$$

$$\therefore -\frac{1}{u}\frac{du}{dt} = r - \frac{r}{uK}$$

Multiply by $-u$

$$\therefore \frac{du}{dt} = -ru + \frac{r}{K}$$

$$\therefore \boxed{\frac{du}{dt} + ru = \frac{r}{K}}$$

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which is ordinary differential equation which can be solved using Integrating factor Technique

$$\text{Integrating factor} = e^{\int r\,dt} = e^{rt}$$

Multiplying the ODE by e^{rt}

$$\therefore \boxed{e^{rt}\frac{du}{dt} + re^{rt}u} = \frac{re^{rt}}{K}$$

$$\frac{d}{dt}\left[e^{rt}u\right] = \frac{re^{rt}}{K}$$

$$de^{rt}u = \frac{r}{K}\int e^{rt}\,dt$$

$$e^{rt}u = \frac{r}{K}\left(\frac{e^{rt}}{r}\right) + C$$

$$ue^{rt} = \frac{e^{rt}}{K} + C$$

i.e $$u(t)e^{rt} = \frac{e^{rt}}{K} + C$$

when $t = 0$

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$$u(0) = \frac{1}{k} + c$$

$$\therefore \boxed{c = u(0) - \frac{1}{k}}$$

$$\therefore u(t)e^{rt} = \frac{e^{rt}}{k} + u(0) - \frac{1}{k}$$

$$e^{-rt}\left[u(t)e^{rt}\right] = e^{-rt}\left\{\frac{e^{rt} - 1 + k\,u(0)}{k}\right\} \text{ ---- i)}$$

Multiplying both sides by e^{-rt} on i).

$$u(t) = \frac{1 + [k\,u(0) - 1]e^{-rt}}{k}$$

$$\therefore \frac{1}{u(t)} = \frac{k}{1 + [k\,u(0) - 1]e^{-rt}}$$

$$N(t) = \frac{k}{1 + \left[\frac{k}{N(0)} - 1\right]e^{-rt}}$$

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THE MATHEMATICS OF STABLE POPULATION

1. Age distribution, Death Rate and Birth Rate

Suppose;

$n(a,t)$ = the number of persons aged a at time t

$N(t)$ = the total number of persons at time t

$c(a,t)$ = the proportion of those who are aged a at time t

$$\therefore \quad c(a,t) = \frac{n(a,t)}{N(t)} \qquad 1)$$

which we shall refer to as age distribution

Next, suppose

$\mu(a,t)$ = the annual death rate for persons aged a at time t

and

$D(t)$ = the total number of deaths for all ages at time t.

$\therefore D(t) = ?$

Number of deaths for those aged at time t
$= n(a,t)\,\mu(a,t)$

$$\therefore D(t) = \int_0^{\omega} n(a,t)\,\mu(a,t)\,da$$

$\therefore$ Crude Death Rate at time t is

$$d(t) = \frac{D(t)}{N(t)} = \int_0^{\omega} \frac{n(a,t)\,\mu(a,t)}{N(t)}\,da$$

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$$\therefore\ d(t) = \int_0^{\omega} c(a,t)\mu(a,t)\, da \qquad 2)$$

Now, let

$m(a,t)$ = the annual rate of bearing a female birth for a woman aged a at time t.

The number of female births by those aged a at time t $= n(a,t)\, m(a,t)$

$\therefore$ Total number of female births at time t is

$$B(t) = \int_0^{\omega} n(a,t)\, m(a,t)\, da \qquad 3)$$

The crude birth rate

$$b(t) = \frac{B(t)}{N(t)} = \int_0^{\omega} \frac{n(a,t)}{N(t)}\, m(a,t)\, da$$

$$\therefore\ b(t) = \int_0^{\omega} c(a,t)\, m(a,t)\, da \qquad 4).$$

Formula 3) can also be expressed as follows:

Those who are aged a at time t were born at time $t-a$

$n(a,t)$ = the survivors of those who were born a years ago

$$= B(t-a)\, p(a,t)$$

$$\therefore\ B(t) = \int_0^{\omega} n(a,t)\, m(a,t)\, da$$

$$= \int_0^{\omega} B(t-a)\, p(a,t)\, m(a,t)\, da \quad 5)$$

2 The Concept of Stable Population

Lopez (1967) derived that two populations with the same sequence of fertility and mortality schedules over a long period of time, but with different age distributions a long time ago, have the same current age distribution.

Alternatively, the age distribution of a closed population is determined by the history of its fertility and mortality in the recent past and does not depend on age distribution or fertility and mortality in the remote past

Definition of a Stable Population

A population that is established by a prolonged regime of unchanging fertility and mortality is called a stable population

Remark: By this definition, it implies that age distribution, fertility and mortality schedules are independent of time

$$\therefore \quad c(a,t) = c(a)$$
$$b(t) = b$$
$$d(t) = d$$
$$m(a,t) = m(a)$$
$$\mu(a,t) = \mu(a)$$

Also assuming exponential law

$$N(t) = N(0)e^{rt}$$

$$B(t) = B(0)e^{rt}$$

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$$D(t) = D(0)e^{rt}$$

Age Distribution:

$$c(a,t) = \frac{n(a,t)}{N(t)}$$

$$= \frac{B(t-a)\,p(a,t)}{N(t)}$$

$$= \frac{B(t-a)\,p(a,t)}{N(t-a)\,e^{ra}}$$

$$= b(t-a)e^{-ra}\,p(a,t)$$

For a stable pop:

$$\boxed{c(a) = be^{-ra}\,p(a)} \quad 1).$$

$p(a)$ = Mortality
b = fertility
$c(a)$ = Age distribution

$$\int_0^{\omega} c(a)\,da = \int_0^{\omega} be^{-ra}\,p(a)\,da$$

$$1 = b\int_0^{\omega} e^{-ra}\,p(a)\,da$$

$$\boxed{\underbrace{b}_{\text{birth rate}} = \frac{1}{\int_0^{\omega} e^{-ra}\,p(a)\,da}} \quad 2).$$

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$$\frac{B(t)}{N(t)} = \int_0^{\omega} \frac{n(a,t)}{N(t)} m(a,t)\, da$$

$$= \cancel{\int_0^{\omega} B(t-a)}$$

$$b(t) = \int_0^{\omega} c(a,t)\, m(a,t)\, da$$

for stable population

$$b = \int_0^{\omega} c(a)\, m(a)\, da$$

from 1) : $c(a) = b e^{-ra} p(a)$

$$\therefore b = \int_0^{\omega} b e^{-ra} p(a)\, m(a)\, da$$

$$\boxed{1 = \int_0^{\omega} e^{-ra} p(a)\, m(a)\, da} \quad 3)$$

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Next:

$\boxed{b - d = r}$ 4)

3) * Constructing Probability Distribution Based on Demographic

- from Life-Table function

A function $f(x)$ is a pdf if

i) $f(x) > 0$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = \underline{1}$

from fertility Measures

Let

$m(a)$ = the proportion of women age "a" years bearing a female child

= annual rate of ~~child~~ bearing a female child by those aged "a" years

Total number of female births

$$GRR = \int_{0}^{\infty} m(a)\,da = \int_{\alpha}^{\beta} m(a)\,da$$

Assuming no mortality

$$1 = \int_{\alpha}^{\beta} \frac{m(a)}{GRR}\,da$$

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$$\therefore \frac{m(a)}{GRR} = \frac{m(a)}{\int_\alpha^\beta m(a)\,da} , \quad \alpha \le a \le \beta$$

is a pdf with mean

$$\bar{m} = \frac{\int_\alpha^\beta a\, m(a)\,da}{\int_\alpha^\beta m(a)\,da}$$

= Mean age of child bearing

when we take mortality into accant then we have:

$$NRR = \int_\alpha^\beta p(a)\, m(a)\, da$$

$$\therefore 1 = \int_\alpha^\beta \frac{p(a)\, m(a)}{NRR}\, da$$

$$\therefore \frac{p(a)\, m(a)}{NRR} = \frac{p(a)\, m(a)}{\int_\alpha^\beta p(a)\, m(a)\, da} , \quad \alpha \le a \le \beta$$

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is a pdf with mean

$$\mu = \frac{\int_{\alpha}^{\beta} a\,p(a)\,m(a)\,da}{\int_{\alpha}^{\beta} p(a)\,m(a)\,da}$$

*Constructing pdfs from Stable Population Models

a) $c(a) = b e^{-ra} p(a)$

$$\therefore \int_0^{\omega} c(a)\,da = \int_0^{\omega} b e^{-ra} p(a)\,da$$

$$1 = \int_0^{\omega} b e^{-ra} p(a)\,da$$

$$\therefore b e^{-ra} p(a) = \frac{e^{-ra} p(a)}{\int_0^{\omega} e^{-ra} p(a)\,da} \; ; \quad a > 0$$

is a pdf with mean

$$\bar{a}_s = \frac{\int_0^{\omega} a e^{-ra} p(a)\,da}{\int_0^{\omega} e^{-ra} p(a)\,da}$$

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b) from the characteristic function

$$1 = \int_0^{\omega} e^{-ra} p(a)\, m(a)\, da$$

$$= \int_{\alpha}^{\beta} e^{-ra} p(a)\, m(a)\, da$$

$\therefore e^{-ra} p(a)\, m(a)$ is a pdf for $\alpha \le a \le \beta$

where mean $= \dfrac{\int_{\alpha}^{\beta} a e^{-ra} p(a)\, m(a)\, da}{\int_{\alpha}^{\beta} e^{-ra} p(a)\, m(a)\, da}$

c) $$d = \int_0^{\omega} c(a) \mu(a)\, da$$

$$\therefore 1 = \int_0^{\omega} \frac{c(a)\mu(a)}{d}\, da$$

The pdf $= \dfrac{c(a)\mu(a)}{d} = \dfrac{c(a)\mu(a)}{\int c(a)\mu(a)\, da}$, $a>0$

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with mean age of dying being

$$\frac{\int_0^{\omega} a\, c(a)\mu(a)\, da}{\int_0^{\omega} c(a)\mu(a)\, da}$$

d) $$b = \int_{\alpha}^{\beta} \mu(a) m(a)\, da$$

$$\therefore 1 = \int_{\alpha}^{\beta} \frac{c(a) m(a)}{b} da$$

$$\text{pdf} = \frac{c(a) m(a)}{b} = \frac{c(a) m(a)}{\int_{\alpha}^{\beta} c(a) m(a)\, da}, \quad \alpha \le a \le \beta$$

with mean age $= \dfrac{\int_{\alpha}^{\beta} a\, c(a) m(a)\, da}{\int_{\alpha}^{\beta} c(a) m(a)\, da}$

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4. Stable Population in five-year-Age-Intervals

In practice, demographic data are given in age-groups/intervals. We shall consider five-year-age-intervals

$$_5C_a = \int_a^{a+5} c(x)\,dx$$

$$= \int_a^{a+5} b e^{-rx} p(x)\,dx$$

$$= b \int_a^{a+5} e^{-rx} p(x)\,dx$$

Remark: The interval is not easy to obtain explicitly we resolve to some approximation.

Take the mid-value between a and $a+5$ which is $a+2.5$

$$_5C_a \simeq b \int_a^{a+5} e^{-r(a+2.5)} p(x)\,dx$$

$$= b e^{-r(a+2.5)} \int_a^{a+5} p(x)\,dx$$

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$$= b e^{-r(a+2\cdot 5)} \int_{a}^{a+5} \frac{l(x)}{l(0)} \, da$$

$$= \frac{b e^{-r(a+2\cdot 5)}}{l(0)} \int_{a}^{a+5} l(x) \, dx$$

$$\boxed{{}_{5}C_{a} \simeq \frac{b e^{-r(a+2\cdot 5)}}{l(0)} \; {}_{5}L_{a}} \quad i)$$

$$\int_{0}^{\omega} {}^{5}C_{a} \, da = \int_{0}^{\omega} \frac{b e^{-r(a+2\cdot 5)}}{l(0)} \; {}_{5}L_{a} \, da$$

$$\underline{1} = b \int_{0}^{\omega} \frac{e^{-r(a+2\cdot 5)} \; {}_{5}L_{a}}{l(0)} \, da$$

$$b = \boxed{\frac{1}{\int_{0}^{\omega} \frac{e^{-r(a+2\cdot 5)} \; {}_{5}L_{a}}{l(0)}}} \quad ii).$$

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From i) and ii) we have

$$\boxed{{}_sC_a = \frac{e^{-r(a+2-5)}\,{}_sL_a}{\int_0^{\omega} e^{-r(a+2-5)}\,{}_sL_a\,da}}$$ iii)

In discrete form

$${}_sC_a = \frac{e^{-r(a+2-5)}\,{}_sL_a}{\sum_{a=0}^{\omega-5} e^{-r(a+2-5)}\,{}_sL_a}$$

Determining r from iv).

Suppose you are given ${}_sL_x$ and ${}_sL_y$ we wish to obtain r as ~~footnotes~~ follows;

$$\frac{{}_sC_x}{{}_sC_y} = \frac{e^{-r(x+2-5)}\,{}_sL_x}{\sum_{x=0}^{\omega-5}\left[\cancel{e^{-r(x+2-5)}\,{}_sL_x}\right]} \cdot \frac{\sum_{y=0}^{\omega-5}\left[\cancel{e^{-r(y+2-5)}\,{}_sL_y}\right]}{e^{-r(y+2-5)}\,{}_sL_y}$$

$$\therefore \frac{{}_sC_x}{{}_sC_y} = \frac{e^{-r(a+2-5)}\,{}_sL_x}{e^{-r(y+2-5)}\,{}_sL_y} = \frac{e^{-rx}\,{}_sL_x}{e^{-ry}\,{}_sL_y}$$

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$$\frac{{}_sC_x}{{}_sC_y} = e^{r(y-x)} \frac{{}_sL_x}{{}_sL_y}$$

$$\therefore \ln \frac{{}_sC_x}{{}_sC_y} = r(y-x) + \ln \frac{{}_sL_x}{{}_sL_y}$$

$$\therefore \ln \frac{{}_sC_x}{{}_sC_y} - \ln \frac{{}_sL_x}{{}_sL_y} = r(y-x)$$

$$\ln\left(\frac{{}_sC_x}{{}_sC_y} \cdot \frac{{}_sL_y}{{}_sL_x}\right) = r(y-x).$$

$$\therefore \boxed{r = \frac{1}{y-x} \ln\left(\frac{{}_sC_x}{{}_sC_y} \frac{{}_sL_y}{{}_sL_x}\right)}$$

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Stationary Population

This is a special case of a stable population when $r = 0$

$\therefore$ $c(a) = b e^{-ra} p(a)$ becomes

$c(a) = b p(a)$

$$\Rightarrow \int_0^{\omega} c(a)\, da = b \int_0^{\omega} p(a)\, da$$

$$\text{i.e. } 1 = b \int_0^{\omega} p(a)\, da$$

$$= b \int_0^{\omega} \frac{l(a)}{l(0)}\, da$$

$$1 = \frac{b}{l(0)} \int_0^{\infty} l(a)\, da$$

$$= b \frac{T_0}{l_0}$$

$$= b \mathring{e}_0$$

$$\boxed{b = \frac{1}{\mathring{e}_0}}$$

$\therefore$ A life-table is a stable population with $r = 0$, $b = \frac{1}{\mathring{e}_0}$, l_0

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The characteristic function

$$\int_{\alpha}^{\beta} e^{-ra} p(a)\, m(a)\, da = 1$$

$$r = 0$$

$$\therefore \int_{\alpha}^{\beta} p(a)\, m(a)\, da = 1$$

$$NRR = 1$$

* Lotkas Method Of Determining Intrinsic Rate Of Increase

Lotka (1925) used the characteristic function of a stable population to determine r

$$\int_{\alpha}^{\beta} e^{-ra} p(a)\, m(a)\, da = 1$$

Define

$$R_n = \int_{\alpha}^{\beta} a^n p(a)\, m(a)\, da$$

$$\therefore R_0 = \int_{\alpha}^{\beta} p(a)\, m(a)\, da = NRR$$

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$$\int_{\alpha}^{\beta} \frac{p(a)\,m(a)}{R_0}\,da = 1$$

$$R_1 = \int_{\alpha}^{\beta} a\,p(a)\,m(a)\,da$$

$$\frac{R_1}{R_0} = \int_{\alpha}^{\beta} \frac{a\,p(a)\,m(a)}{R_0} = \mu, \text{ say}$$

$$R_2 = \int_{\alpha}^{\beta} a^2\,p(a)\,m(a)\,da$$

$$\therefore \frac{R_2}{R_0} = \int_{\alpha}^{\beta} \frac{a^2\,p(a)\,m(a)}{R_0}\,da$$

Note:

$$\boxed{\mu = E(A) = \frac{R_1}{R_0}}$$

$$E(A^2) = \frac{R_2}{R_0}$$

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$$\text{Var } A = E(A^2) - [E(A)]^2$$

$$= \frac{R_2}{R_0} - \left(\frac{R_1}{R_0}\right)^2$$

$$= \delta^2 \text{, say}$$

$$\frac{R_2}{R_0} - \left(\frac{R_1}{R_0}\right)^2 = \delta^2$$

$$\frac{R_2}{R_0} - \mu^2 = \delta^2$$

$$\therefore \boxed{\frac{R_2}{R_0} = \delta^2 + \mu^2}$$

The characteristic function

$$\int_\alpha^\beta e^{-ra} p(a)\, m(a)\, da = \underline{1}$$

Expanding e^{-ra} we get

$$e^{-ra} = 1 + \frac{(-ra)}{1!} + \frac{(-ra)^2}{2!} + \frac{(-ra)^3}{3!} + \ldots$$

$$\int_\alpha^\beta \left\{1 - ar + \frac{a^2 r^2}{2} + \ldots\right\} p(a)\, m(a)\, da = \underline{1}$$

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$$\therefore \int_{\alpha}^{\beta} \left\{ 1 - ar + \frac{a^2 r^2}{2} + \cdots \right\} \frac{p(a)\, m(a)}{R_0}\, da = \frac{1}{R_0}$$

$$\int_{\alpha}^{\beta} \frac{p(a)m(a)}{R_0}\, da - r \int_{\alpha}^{\beta} \frac{a p(a) m(a)\, da}{R_0} + \frac{r^2}{2} \int_{\alpha}^{\beta} a^2 p(a) m(a) = \frac{1}{R_0}$$

$$\boxed{1 - r\mu + \frac{r^2}{2}\left(\delta^2 + \mu^2\right) \simeq \frac{1}{R_0}}$$

$$\ln\left[1 - r\mu + \frac{r^2}{2}\left(\delta^2 + \mu^2\right)\right) \simeq \ln \frac{1}{R_0}$$

But logarithmic series:

Geometric series $\frac{1}{1-\theta} = 1 + \theta + \theta^2 + \theta^3 + \cdots$

~~Integration~~

Integrate wrt θ

$$\int \frac{d\theta}{1-\theta} = \int \left[1 + \theta + \theta^2 + \theta^3 + \cdots\right] d\theta$$

$$-\log(1-\theta) = \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3} + \frac{\theta^4}{4} + \cdots$$

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$$\log\left[1-\left(r\mu-\frac{r^2\delta^2}{2}-\frac{r^2\mu^2}{2}\right)\right]=\log\frac{1}{R_0}=-\log R_0$$

$$\therefore -\log\underbrace{\left[1-\left(r\mu-\frac{r^2\delta^2}{2}-\frac{r^2\mu^2}{2}\right)\right]}_{\Theta}=\log R_0$$

$$r\mu-\frac{r^2\delta^2}{2}-\frac{r^2\mu^2}{2}+\frac{1}{2}\left[r\mu-\frac{r^2\delta^2}{2}-\frac{r^2\mu^2}{2}\right]^2\simeq\log R_0$$

Upto r^2 we have

$$r\mu-\frac{r^2\delta^2}{2}-\frac{r^2\mu^2}{2}+\frac{1}{2}\left(r^2\mu^2+\cdots\right)\simeq\log R_0$$

$$\therefore r\mu-\frac{\delta^2r^2}{2}\simeq\log R_0$$

$$r\left(\mu-\frac{\delta^2}{2}r\right)\simeq\log R_0$$

$$\therefore \boxed{r=\frac{\ln R_0}{\mu-\frac{\delta^2}{2}r}}$$

By Newton Raphson Method

$$\boxed{r_{n+1}=\frac{\ln R_0}{\mu-\frac{\delta^2 r_n}{2}},\quad n=0,1,2,}$$

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Alternatively solve the quadratic eqn

$$r\mu - \frac{\delta^2 r^2}{2} = \ln R_0$$

$$\frac{\delta^2}{2} r^2 - \mu r + \ln R_0 = 0$$

$$r = \frac{\mu \pm \sqrt{\mu^2 - \frac{4r^2}{2} \ln R_0}}{\delta^2}$$

$$= \frac{\mu \pm \sqrt{\mu^2 - 2\delta^2 \ln R_0}}{\delta^2}$$

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Coales' Method of determining r

Concept of NRR:

NRR is the total number of daughters that each member of a birth cohort of women produces, after allowing for the mortality of the birth cohort.

If we consider the original birth cohort as one generation, then NRR measures the size of the next generation relative to the size of the present size.

If the NRR exceeds 1.0 then the next generation will be bigger than the present one and the population will grow.

If NRR is less than 1.0 then the next generation will be smaller than the present and the population will decline.

If the NRR is equal to 1.0 the next generation will be equally the same size as the present one and the population will remain constant.

The Link between NRR and r

Let ; T = the length of a generation in years.

By the exponential law

$$P(T) = P(0)e^{rT}$$

$$\frac{P(T)}{P(0)} = e^{rT}$$

$$\boxed{NRR = e^{rT}}$$

$$\boxed{r = \frac{\ln NRR}{T}}$$ ⇒ The problem is to solve for T

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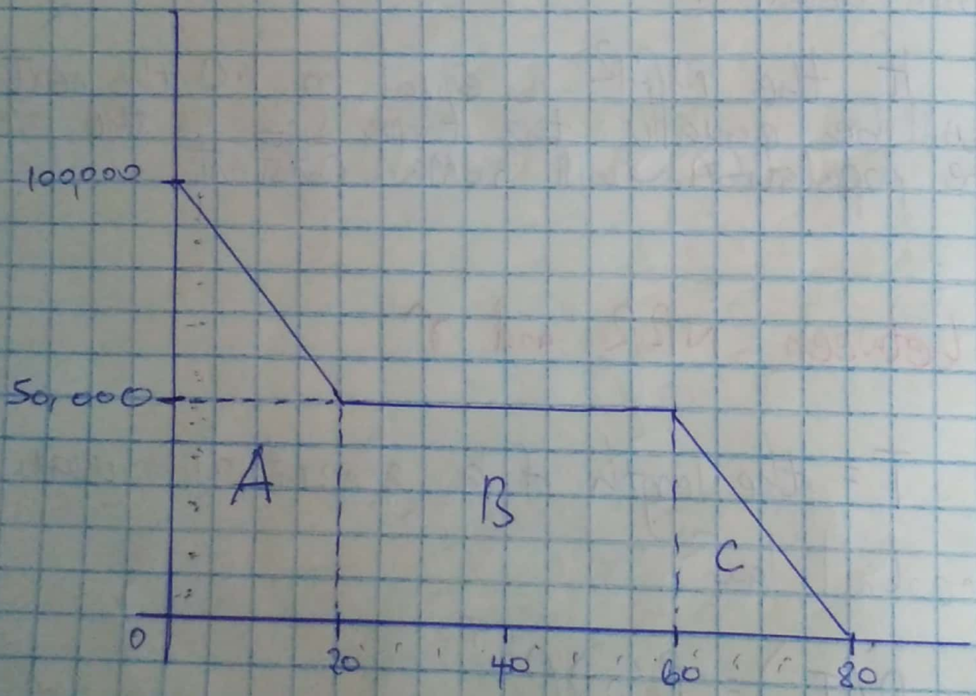
PROBLEM 4 Page 9

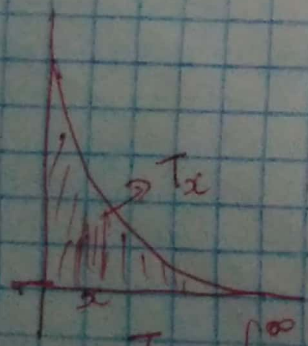
a) Suppose a life-table in which the number of survivors declines linearly from 100000 at age zero to 50[...] at age 20, remains at 50000 until age 60, and th[...] declines linearly to zero at age 80

i) What is the expectation of life at age zero? Age 2[...] Age 60?

ii) what is the average death rate of persons under 20 years in this stationary population of persons over 6[...]

i) Solution:





$T_x = \int_x^{\infty} l(y)\,dy$

$$\Rightarrow e_0 = \frac{T_0}{l_0} = \frac{\text{Area of } A+B+C}{l_0}$$

$$= \frac{\text{Area of A} + \text{Area of B} + \text{Area of C}}{l_0}$$

$$e_0 = \frac{\frac{20}{2}(100{,}000 + 50{,}000) + (50{,}000 \times 40) + \frac{1}{2} * 50{,}000}{100{,}000}$$

$e_0 = 40$ Ans.

$$e_{20} = \frac{T_{20}}{l_{20}} = \frac{\text{Area of } B + \text{Area of } C}{l_{20}}$$

$$= \frac{(50{,}000 * 40) + \left(\frac{1}{2} * 20 * 50{,}000\right)}{50{,}000}.$$

$$= \frac{2500000}{50{,}000}$$

$$= 50 \text{ Ans.}$$

$$e_{60} = \frac{T_{60}}{l_{60}} = \frac{\text{Area of } C}{l_{60}}.$$

$$= \frac{\frac{1}{2} * 20 * 50000}{50{,}000}$$

$$= 10 \text{ Ans.}$$

a) $M_x = \frac{{}_{20}d_{20}}{L_x}$ $\quad x = 20 \rightarrow M_{20} = \frac{50{,}000}{1{,}500{,}000}$

for under 20 years.

average death rate = 0.0333 Ans

$\Rightarrow {}_{20}d_{20} = l_x - l_{x+n}$

$= 100000 - 50000$

${}_{20}d_{20} = 50{,}000$

${}_{20}L_{20} = \frac{1}{2}(l_x + l_{x+1}).$

$= \frac{20}{2}(100{,}000 + 50000)$

${}_{20}L_{200} = 1{,}500{,}000$

$*50{,}000$

$M_x = \frac{d_x}{L_x} = \frac{{}_{20}d_{60}}{{}_{20}L_{60}}$

$\cancel{L_x = \frac{1}{2}(l_x + l_{x+n})}$

$L_x = \frac{1}{2} * b * h = (\text{Area } C)$

$= \frac{1}{2} \times 20 \times 50000$

$= 500{,}000 = L_{60}$

${}_{20}d_{60} = 50000 - 0$

$= 50{,}000$

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$m_{60} = 50000 / 500{,}000 = 0.1$ Ans which is average death rate for 60 persons

(b) Two stable populations embody the same fertility schedule but different mortality schedules, In both populations 20% percent of women between exact age 45 to 55 bear a female child annually; no child bearing occurs outside this span. In population A there is no mortality until age 100 where all who reach that age die.
In population B, one percent of each cohort dies within each single year of age, with no survivors beyond age 100

i) Calculate GRR, NRR and r for each population.
ii) Sketch the age distribution of each
iii) Determine the birth rate and death rate of each pop

Soln.

$$GRR = \int_{\alpha}^{\beta} m(a)\,da$$

$\alpha = 45$

$\beta = 55$

$m(a) = 0.2 = 20\%$

$$= \int_{45}^{55} 0.2\,da$$

$$= 0.2a\Big]_{45}^{55}$$

$$= \underline{2} \text{ Ans}$$

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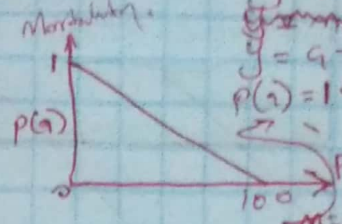
$$NRR_B = \int_{\alpha}^{\beta} p(a)\, m(a)\, da.$$

$\Rightarrow$



Using Eq of a st line

$y = c + mx$

$p(a) = 1 - \frac{1}{100}a.$

$p(a) = 1 - \frac{a}{100}$

$-m = \frac{\Delta y}{\Delta x} = \frac{1}{-100}$

$P(a) = 1 - 0.01 = 0.99$

$m(a) = 0.2$

$$= \int_{45}^{55} (0.99 * 0.2)\, da$$

$\Rightarrow$ CORRECTION

$$NRR_B = \int_{45}^{55} p(a)\, m(a)\, da.$$

$$\int_{45}^{50} \left(1 - \frac{a}{100}\right) \frac{20}{100}\, da.$$

$$\frac{20}{100} \int_{45}^{50} \left(1 - a/100\right) da.$$

$$\frac{20}{100} \left[a - \frac{a^2}{200} \Big|_{45}^{55} \right]$$

$$= \frac{20}{100} \left[10 - \left(\frac{55^2 - 45^2}{200} \right) \right]$$

$$\frac{20}{100} (10 - 5)$$

$= 1$ Ans.

$$= \int_{45}^{55} 0.198\, da$$

$$= 0.198a \Big]_{45}^{55}$$

$= 1.98$ ✗

$$NRR_A = \int_{\alpha}^{\beta} p(a)\, n(a)\, da.$$

No-mortality $p(a) = 1$

$$= \int_{45}^{55} (1 * 0.2)\, da$$

$$= \int_{45}^{55} 0.2\, da \Rightarrow 0.2a \Big]_{45}^{55}$$

$\Rightarrow 2$ Ans.

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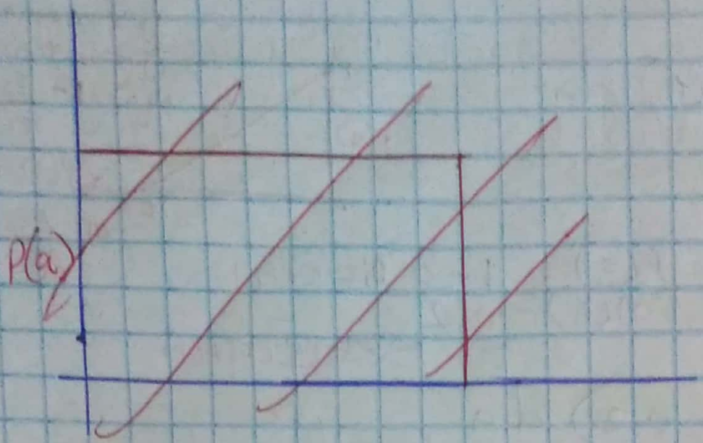

Take $T = \frac{1}{2}(45+55) = 50$

$$r = \frac{\ln NRR}{T}$$

for $r_A = \frac{\ln 2}{50} = 0.01386$ Ans

$$r_B = \frac{\ln NRR}{T}$$

$r_B = \frac{\ln 1}{50} = 0$ Ans

(iii) birth rate $= \dfrac{1}{\int_\alpha^\beta e^{-ra} p(a)\, da}$

b_A = birth rate for A $= \dfrac{1}{\int_{45}^{55} e^{-0.01386a}\, da}$ 0.00442 hrs

$b_A - d_A = r_A$

$\therefore \cancel{d_A = 0.01542}$

$$= \left. \frac{e^{-0.01386a}}{-0.01386} \right|_{45}^{55}$$

b_B = birth rate for B = $\dfrac{1}{\displaystyle\int_{45}^{55} e^{-0.a}\left(1-\frac{a}{100}\right) da}$ = $\underline{\underline{0.2}}$ Ans

$b_B - d_B = r_B$.

$0.2 - d = 0$

$d = \underline{\underline{0.2}}$

~~dea~~ $d_B = \underline{\underline{0.2}}$ Ans.

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