PROBLEMS ON POPULATION GROWTH

#1. Let $P(t)$ denote the estimated population of a certain country at time t years from the present. A demographer believes that $P(t)$ will satisfy the logistic differential equation:

$$\frac{dP(t)}{dt} = rP(t) - \lambda[P(t)]^2, \quad t \geq 0$$

where r and λ are known positive constants. The present population estimate, $P(0)$, is known and $P(0) < r/\lambda$.

(i) By making the substitution $u(t) = [P(t)]^{-1}$, find a first-order linear differential equation for $u(t)$.

(ii) Solve this equation for $u(t)$ and hence find a formula for $P(t)$ in terms of $P(0)$, r and λ (in addition to time, t).

(iii) What is the limit as $t \to \infty$ of the estimated population?

(iv) The demographer has now decided to revise her model. The logistic equation is to be replaced by the equation

$$\frac{dP(t)}{dt} = rP(t) - \lambda e^{-rt}[P(t)]^2, \quad t \geq 0.$$

Find an expression for $P(t)$ in terms of $P(0)$, and r and λ (in addition to time, t) and hence or otherwise determine the behaviour of $P(t)$ as $t \to \infty$.

#2 Let $P(t)$ denote the population of a certain country at time t $(t \geq 0)$.
$P = P(t)$ is known to satisfy the equation

$$\frac{dP}{dt} = rP - \lambda P^3; \quad t \geq 0$$

where the present population, $P(0)$, is known and r, λ are positive constants.

a) By substituting ~~$u=P^2$~~, $u=P^{-2}$, find a first-order linear differential equation for $u=u(t)$.

b) Solve the differential equation.

c) Hence solve the original equation for $P(t)$.

d) What is the limit of $P(t)$ as $t \to \infty$?

#3. The mid-year population of Kenya was 18 million in 1982. Between 1970 and 1982 the average annual rate of growth was 4%. The World Bank estimate that, in mid-1990, Kenya's population was ~~26~~ 26 million, and that by the middle of the year 2000 it would be 40 million.

a) Assuming that the growth in the population of Kenya 1982 and 1990, and between 1990 and 2000, is exponential, calculate the annual growth rates using the World Bank's estimates of population.

b) Assume that the World Bank's estimate of 40 million in 2000 is correct. If Kenya's population continues to increase after 2000 at the same rate as the World Bank assumed it would increase between 1990 and 2000, when will it reach 80 million?

#4 Table 0 below gives the populations of Algeria and Jordan in various years between 1950 and 1985.

a) Plot the population against time for both countries

b) Plot the natural logarithms of the population against time. Is the exponential model a good description of the past growth of these two countries' populations?

#4 c) Project the populations of these two countries in the years 1995 and 2005

Table O.

Year	Population (thousands) Algeria	Jordan
1950	8753	1237
1955	9715	1447
1960	10 800	1695
1965	11923	1962
1970	13746	2299
1975	16018	2600
1980	18740	2923
1985	21788	3407

APPLICATIONS OF STABLE POPULATION THEORY

#5 A University buys new P micro-computers for its staff. Studies by the Department of Computer Services have shown that the chance, λ, that any one of these computers will break down irreparably at any time, given that it is still working at that time, does not depend on the age of the computer. It is the university's policy to sell off to students at a cheap rate all computers that are still in service on the 5th anniversary of their date of purchase.

(a) Find an expression, in terms of P and λ, for the number of new computers the University must buy each year in order to maintain a constant number in service.

b) The Department of Computer Services decides to upgrade the university's standard spreadsheet package, which is installed on all its micro-computers. Unfortunately, it turns out that the up-graded version of the package will only run on the operating system fitted to computers bought during the last three years. What proportion of the computers in service will be unable to run the new package?

#6 A certain university runs a degree in mathematics which lasts four years. Examinations are held at the end of each year, and students who fail at the end of each of the first three years are not permitted to continue their studies (i.e., they must leave the university). For many years now, an average of 10% of the students taking each examination have failed.

a) By what percentage must the number of students admitted to the degree course increase each year in order that the total number of students on the degree course doubles every 20 years?

b) Suppose that, after many years during which admissions have increased at the rate calculated in the answer to part (a), the university decides to freeze the total number of students on the course for one year. Because of government pressure to maintain the increase in admissions, it is suggested that this is done by increasing the failure rate by the same amount in all examinations. By how much must the failure rate be increased in order to achieve a freeze in the total numbers, while maintaining the annual increase in admissions?

#7. A population of small furry animals lives on a remote island. These animals have a maximum life-span of five years. Studies by naturalists ~~of the mortality of these animals~~ have shown that mortality of these animals ~~have shown that~~ has been constant for many years, and it is the same for males and females of the species, and is represented by the life-table shown below.

a) ~~e~~

Age ~~x~~ x	Number of animals surviving to exact age x
0	1000
1	800
2	700
3	500
4	200

a) Each year, the naturalist takes a census of these animals. The evidence from a long run of censuses is that their number has been increasing by 2% per year for many years. Estimate the age structure of the animal population (in percentage terms).

b) Female small furry animals bear young between exact ages one and three years. Their age-specific fertility rate at age 1 last birthday is twice that at age 2 last birthday. Assuming that equal numbers of males and females are born, estimate their age specific fertility rates at ages 1 and 2 last birthday.

#1 a) Define the operator Δ on μ_x as

$$\Delta \mu_x = \mu_{x+1} - \mu_x$$

Then find $(1+\Delta)\mu_x$, $(1+\Delta)^2 \mu_x$, $(1+\Delta)^3 \mu_x$ and $(1+\Delta)^t \mu_x$.

b) Integrate μ_{x+t} for $0 \le t \le 1$ using
(i) the mean value approximation
(ii) the finite difference approximation of $(1+\Delta)^t \simeq (1+t\Delta)$.

Using these results integrate ~~the formula~~

$$\mu_{x+t} = -\frac{d}{dt} \log l_{x+t} \text{ for } 0 \le t \le 1.$$

What relationship do you get?

c) Integrate μ_{x+t} for $-1 \le t \le 1$ using
(i) the mean value approximation
(ii) $(1+\Delta)^t \simeq (1+t\Delta)$
Using these results integrate the formula for force of mortality given in (b) for $-1 \le t \le 1$.
What relationship do you get?

d) Using the above results, find
(i) q_{50} given $\mu_{50} = 0.01098$ and $\mu_{51} = 0.01173$

(ii) μ_{40} given $p_{39} = 0.99469$ and $p_{40} = 0.99438$.

#2 The Table below gives estimates of the life expectation e_x at various ages x, for females in the United States in 1988. Use them to estimate q_0, ${}_4q_1$, ${}_5q_5$ and ${}_5q_{10}$.

Age Interval $x - x+n$	e_x USA 1989
0-1	78.6
1-4	78.3
5-9	74.4
10-14	69.5
15-19	64.6

The mean age at death is given by $a_0 = 0.2$ and ${}_na_x \simeq \frac{n}{2}$ for $x = 1, 5, 10$ and 15. Take $l_0 = 100{,}000$.

#3 a) Using integration by parts, prove that
$${}_nL_x = \int_0^n l(x+t)\,dt = n\,l_{x+n} + {}_na_x \cdot {}_nd_x$$

b) Hence or otherwise, show that
$${}_na_x = n - \frac{n}{{}_nq_x} + \frac{1}{{}_nm_x}$$

c) Prove that the average age at death of those persons who die between age x and age y is
$$x + \frac{[T_x - T_y - (y-x)l_y]}{l_x - l_y}, \quad y > x.$$

#4 We have denoted the mean age at death between age x and age $x+n$ by ${}_n a_x$. Now life expectancy at age x, denoted by e_x is the mean age at death between age x and ∞. Show that $e_x = T_x / l_x$. Hence or otherwise prove that

$$T_x = T_0 \exp\left(-\int_0^x \frac{1}{e_y}\, dy\right)$$

#5 Prove that

a) $\mu_x = \frac{1}{e_x}\left(1 + \frac{d}{dx} e_x\right)$

b) $\mu_x \simeq \frac{1}{e_x}\left(1 + \frac{1}{2}(e_{x+1} - e_{x-1}\right)$.

#6 Express ${}_n d_x$ as an integral and show that this integral is equivalent to $l_x - l_{x+n}$.

#7. Selected values from a life-table:

x	l_x	e_x
0	100000	70
40	92000	35
70	65000	11

a) What is the death rate of
(i) the total population?
(ii) the population over 70?
(iii) the population under 40?
(iv) the population aged between 40 and 70?

b) What is the mean age at death of those who die between age 0 and age 40?

~~Rof~~ PROBLEM SET II:

ON STABLE POPULATION

#1 Table 1 ~~The table~~ below gives information on ASFR in England and Wales in 1991, together with data on female mortality.

a) Assuming that 105 boys are born for every 100 girls, estimate the GRR and the NRR.

b) What does your answer to part (a) tell you about population growth in England and Wales in the long run?

c) Calculate the mean age at child-bearing in England and Wales in 1991.

Table 1:

Age-Group	ASFR	Female Survivors to mid-point of age-group per 10 000 women born
15 – 19	0.033	9903
20 – 24	0.090	9890
25 – 29	0.120	9871
30 – 34	0.087	9850
35 – 39	0.032	9817
40 – 44	0.006	9766
45 – 49	0.000	9685

#2 Table 2 is the life table for ~~a~~ the population of a developing country.

a) Calculate the percentage of the population in each age-group, assuming that the population is stationary. (Note: you may assume in your calculations that 1.5% of the stationary population is aged 80 years and over, and the average age of people aged 80 years and over is 85 years).

b) Calculate the percentage of the population in each age-group, assuming that the population is growing exponentially at an annual rate of 2%.

c) Comment on the differences between the two age structures.

Table 2:

Age-Group	Number surviving to age x out of 1000 births
0	1000
1-4	950
5-9	900
10-19	880
20-29	850
30-39	790
40-49	700
50-59	580
60-69	450
70-79	340
80+	230

#3 In a stable population, you are told that the number of persons alive between exact ages 40 and $40\frac{1}{2}$ is the same as the number of persons alive between exact ages $40\frac{1}{2}$ and 41. Find an expression for the rate of growth, r, in terms of the constant set of l_x's representing the mortality of this population.

#2 d) Assuming that the population of this country is growing at an annual rate of 2%, estimate its TFR. State any assumptions you make.

e) If the country were to have a TFR of 6.0 (and the same mortality rates), estimate r it would experience.