$$_nq_x = \frac{l_x - l_{x+n}}{l_x} = \frac{21500 - 20323}{21500} = 0.0547.$$

c) Determine the central death rate at age 72.

$$_nm_x = \frac{_nd_x}{_nL_x} = \frac{1177}{250938} = 0.00469$$

d) the force of mortality at age $72\frac{1}{2}$.

$$\mu_x = -\frac{d}{dx}\log l_x$$

JANUARY 24, 2017 EXAM.

1.

a) $$\int_{\alpha}^{\beta} e^{-rx} p(x) m(x) = 1$$ characteristic equation.

Show that $\frac{\sigma^2 r^2}{2} - mr + \ln R_0 = 0$

$$\left(r = \frac{\ln R_0}{\mu - \frac{\sigma^2 r}{2}} \right)$$

Note that the given equation is similar to this equation. where $\mu \equiv m$

r - Lotka intrinsic rate of increase

Proof.

$$R_n = \int_{\alpha}^{\beta} x^n p(x) m(x)\, dx$$

$$R_1 = \int_{\alpha}^{\beta} x\, p(x) m(x)\, dx$$

1^{st} moment abt origin = mean.
1^{st} moment abt mean = variance.

$$R_2 = \int_{\alpha}^{\beta} x^2 p(x)\, m(x)\, dx$$

$$R_0 = \int_{\alpha}^{\beta} x^0 p(x)\, m(x)\, dx = \int_{\alpha}^{\beta} p(x)\, m(x)\, dx$$

$$\frac{R_1}{R_0} = \int_{\alpha}^{\beta} \frac{x\, p(x)\, m(x)}{R_0} = \text{mean } (m)$$

$$\frac{R_2}{R_1} = \int_{\alpha}^{\beta} \frac{x^2 p(x)\, m(x)}{R_0}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \frac{R_2}{R_0} \qquad [E(x)]^2 = m^2$$

$$\sigma^2 = \frac{R_2}{R_0} - m^2$$

$$\sigma^2 + m^2 = \frac{R_2}{R_0}$$

Expanding e^{-rx} using Taylor series

$$e^{-x} = \frac{(-x)^0}{0!} + \frac{(-x)^1}{1!} + \frac{(-x)^2}{2!} + \cdots = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

$$e^{-rx} = 1 - rx + \frac{r^2x^2}{2} + \cdots$$

Substituting in the characteristic equation.

$$1 = \int_{\alpha}^{\beta} e^{-rx} p(x)\, m(x)\, dx$$

$$\frac{1}{R_0} = \int_\alpha^\beta \frac{e^{-rx} p(x) m(x)}{R_0} dx$$

$$\int_\alpha^\beta \left\{1 - rx + \frac{r^2 x^2}{2} + \cdots\right\} \frac{p(x)\, m(x)}{R_0} dx \simeq \frac{1}{R_0}$$

$$\underbrace{\int_\alpha^\beta \frac{p(x) m(x)}{R_0} dx}_{1} - r \underbrace{\int_\alpha^\beta \frac{x\, p(x) m(x)}{R_0} dx}_{\mu(m)} + \frac{r^2}{2} \underbrace{\int_\alpha^\beta \frac{x^2 p(x) m(x)}{R_0}}_{\frac{x^2}{R_0} (\sigma^2 + m^2)} \simeq \frac{1}{R_0}$$

$$1 - rm + \frac{r^2}{2}(\sigma^2 + m^2) \simeq \frac{1}{R_0}$$

Introducing natural log.

$$\ln\left[1 - rm + \frac{r^2}{2}(\sigma^2 + m^2)\right] \simeq \ln \frac{1}{R_0}$$

$$\ln\left[1 - rm + \frac{r^2\sigma^2}{2} + \frac{r^2 m^2}{2}\right] \simeq -\ln R_0$$

$$-\log\Big[1 - \underbrace{\left\{rm - \frac{r^2\sigma^2}{2} - \frac{r^2 m^2}{2}\right)}_{\theta}\Big] \simeq \log R_0$$

using geometric series.

$$\frac{1}{1-\theta} = 1 + \theta + \theta^2 + \theta^3 + \cdots$$

Integrating both sides

$$\int \frac{d\theta}{1-\theta} = \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3} + \cdots$$

$$-\log(1-\theta) = \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3} + \cdots$$

$$\left(rm - \frac{r^2\sigma^2}{2} - \frac{r^2 m^2}{2}\right) + \frac{1}{2}\left(rm - \frac{r^2\sigma^2}{2} + \frac{r^2 m^2}{2}\right)^2 \simeq \log R_0$$

$$rm - \frac{r^2\sigma^2}{2} - \frac{r^2m^2}{2} + \frac{1}{2}\left[r^2m^2 + \text{---}\right] \simeq \log R_0$$

$$rm - \frac{r^2\sigma^2}{2} \simeq \log R_0$$

$$rm - \frac{r^2\sigma^2}{2} - \log R_0 \simeq 0$$

$$-rm + \frac{r^2\sigma^2}{2} + \log R_0 \simeq 0$$

$$\frac{\sigma^2 r^2}{2} - rm + \log R_0 = 0 \quad \text{Hence the prof.}$$

(b) The table below show the births per woman ${}_5F_x$ for Kenya for the period 2005 – 2010 accompanied with some life table functions.

mean no. of births

Age	${}_5F_x$	${}_5L_x$	Mid age (x)	$x \cdot {}_5F_x$
15-19	0.085	448061	17.5	1.4875
20-24	0.23	440761	22.5	5.175
25-29	0.243	431961	27.5	6.6825
30-34	0.2	422542	32.5	6.5
35-39	0.135	411106	37.5	5.0625
40-44	0.057	396183	42.5	2.4225
45-49	0.012	377555	47.5	0.57
	$\sum {}_5f_x = 0.962$			$\sum x f_x = 27.875$

$l_0 = 100,000$ SRB = 1.05

Assuming the population of Kenya has been stable, obtain the following

(i) Mean age at child bearing.

Mean age at child bearing for discrete $= \frac{\sum x f_x}{\sum f_x} = \frac{27.875}{0.962}$

$= 28.976 \simeq 29$ years

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(ii) Mean length of generation

Mean length of generation = $\frac{\sum x f_x L_x}{\sum f_x L_x}$

Age	${}_5f_x \cdot L_x$	$x\, {}_5f_x \cdot L_x$
15-19	38085.185	666490.7375
20-24	101375.03	2280938.175
25-29	104966.523	2886579.383
30-34	84508.4	2746523
35-39	55507.41	2081527.875
40-44	22582.431	959753.3175
45-49	4530.66	213847.152
	$\sum$ 411584.6151	$\sum$ 11836499.25

$$= \frac{11836499.25}{411584.6151} = 28.758$$

(iv) When will the results in (i) and (iii) be equal?
They will be equal if L_x (person years lived) is the same across all age groups.

(iii) Intrinsic growth rate

$$NRR = \sum ASFR * \frac{{}_5L_x}{l_0} = \frac{1}{l_0} \sum ASFR\, {}_5L_x$$

$$= \frac{1}{100,000} * 411584.6151 = 4.115846151$$

$$r = \frac{\ln NRR}{T} = \frac{\ln(4.115846151)}{28.758} = 0.0492$$

Growth reproductive rate (GRR)

$$TFR = \sum_{x=15-19}^{45-49} 5 * ASFR = \sum_{x=15}^{49} ASFR$$

$$GRR = TFR\left(\frac{1}{1+SRB}\right)$$

$$TFR = 5\sum_{x=15-19}^{45-49} ASFR = 5 * 0.962 = 4.81$$

$$GRR = 4.81\left(\frac{1}{1+1.05}\right) = 2.346$$

QUESTION 2

$$\frac{dN}{dt} = rN - \lambda N^2 \quad \text{show that} \quad N(t) = \frac{r/\lambda}{1+\left(\frac{r}{\lambda N(0)}-1\right)e^{-rt}}$$

$$\frac{dN}{dt} = N(r-\lambda N)$$

$$\frac{1}{N}\frac{dN}{dt} = r - \lambda N \qquad \text{let } N = \frac{1}{u} \qquad \frac{dN}{du} = -\frac{1}{u^2}$$

$$\frac{1}{N}\frac{dN}{du}\cdot\frac{du}{dt} = r - \lambda N \qquad \text{but } N = N(t), \quad u = u(t)$$

substituting for N

$$u \cdot -\frac{1}{u^2}\frac{du}{dt} = r - \frac{\lambda}{u}$$

$$-\frac{1}{u}\frac{du}{dt} = r - \frac{\lambda}{u}$$

multiply through by $-u$.

$$\frac{du}{dt} = \lambda - ru$$

$$\frac{du}{dt} + ru = \lambda$$ Integrating factor $= e^{\int r dt} = e^{rt}$

multiply through by integrating factor.

$$\underbrace{e^{rt}\frac{du}{dt} + rue^{rt}} = \lambda e^{rt}$$

$$\frac{d}{dt} ue^{rt} = \lambda e^{rt}$$

$$d\, ue^{rt} = \lambda e^{rt} dt$$

Integrate both sides

$$ue^{rt} = \frac{\lambda e^{rt}}{r} + c$$

when $t = 0$

$$u(0) = \frac{\lambda}{r} + c$$

$$c = u(0) - \frac{\lambda}{r}$$

substituting for c

$$u(t) e^{rt} = \frac{\lambda e^{rt}}{r} + u(0) - \frac{\lambda}{r}$$

Divide through by e^{rt}

$$u(t) = \frac{\lambda}{r} + \left(u(0) - \frac{\lambda}{r}\right) e^{-rt}$$

Taking the reciprocal

$$\frac{1}{u(t)} = \frac{1}{\frac{\lambda}{r} + \left(u(0) - \frac{\lambda}{r}\right) e^{-rt}}$$

But $N(t) = \frac{1}{u(t)}$. substitute.

$$N(t) = \frac{1}{\frac{\lambda}{r} + \left(\frac{1}{N(0)} - \frac{\lambda}{r}\right)e^{-rt}}$$

multiply R.H.S by r/λ

$$N(t) = \frac{r/\lambda}{1 + \left(\frac{r}{\lambda N(0)} - 1\right)e^{-rt}}$$

Repeat the above using partial fractions.

$$\frac{dN}{dt} = rN - \lambda N^2$$

$$\frac{dN}{dt} = N(r - \lambda N)$$

$$\frac{dN}{dt} = Nr\left(1 - \frac{\lambda N}{r}\right) \qquad \text{let } \frac{\lambda}{r} \text{ be } \frac{1}{k}$$

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right)$$

$$\frac{dN}{N\left(1-\frac{N}{k}\right)} = r\,dt$$

Integrate both sides.

$$\int \frac{dN}{N\left(1-\frac{N}{k}\right)} = rt + C \Rightarrow \int\left[\frac{A}{N} + \frac{B}{1-\frac{N}{k}}\right]dN = rt + C$$

let $\frac{1}{N\left(1-\frac{N}{k}\right)} = \frac{A}{N} + \frac{B}{1-\frac{N}{k}} = \frac{A\left(1-\frac{N}{k}\right) + BN}{N\left(1-\frac{N}{k}\right)}$

it follows

$$1 = A\left(1-\frac{N}{K}\right) + BN$$

$$1 + 0*N = A - \frac{AN}{K} + BN$$

$$1 + 0*N = A + \left(B - \frac{A}{K}\right)N$$

Implying that

$A = 1$

$B - \frac{A}{K} = 0$ $\quad B - \frac{1}{K} = 0$ $\quad B = \frac{1}{K}$

$$\int\left[\frac{A}{N} + \frac{B}{1-\frac{N}{K}}\right]dN = \int\left[\frac{1}{N} + \frac{\frac{1}{K}}{1-\frac{N}{K}}\right]dN = rt + C$$

$$\int \frac{1}{N}\,dN + \int \frac{\frac{1}{K}}{1-\frac{N}{K}}\,dN = rt + C$$

Taking natural logs

$$\ln N + \frac{\frac{1}{K}\ln\left(1-\frac{N}{K}\right)}{-\frac{1}{K}} = rt + C$$

$$\ln N - \ln\left(1-\frac{N}{K}\right) = rt + C$$

$$\ln\left(\frac{N}{1-\frac{N}{K}}\right) = rt + C$$

Taking exponentials

$$\frac{N}{1-\frac{N}{K}} = e^{rt+C} \qquad \frac{N(t)}{1-\frac{N(t)}{K}} = e^{rt}\cdot e^{C}$$

when $t = 0$

$$\frac{N(0)}{1 - \frac{N(0)}{k}} = e^{0} e^{c} \qquad e^{c} = \frac{kN(0)}{k - N(0)}$$

$$\frac{N(t)}{1 - \frac{N(t)}{k}} = e^{rt}\left(\frac{kN(0)}{k - N(0)}\right)$$

$$N(t) = \frac{e^{rt} k N(0)}{k - N(0)}\left(1 - \frac{N(t)}{k}\right)$$

$$N(t) = \frac{e^{rt} \cancel{k} N(0)}{k - N(0)}\left(\frac{k - N(t)}{\cancel{k}}\right) = \frac{k e^{rt} N(0)}{k - N(0)} - \frac{e^{rt} N(0) N(t)}{k - N(0)}$$

$$N(t) + \frac{e^{rt} N(0)}{k - N(0)} N(t) = \frac{k e^{rt} N(0)}{k - N(0)}$$

$$N(t)\left[1 + \frac{e^{rt} N(0)}{k - N(0)}\right] = \frac{k e^{rt} N(0)}{k - N(0)} \qquad 1 + \frac{e^{rt} N(0)}{k - N(0)} = \frac{k - N(0) + e^{rt}N(0)}{k - N(0)}$$

$$N(t) = \frac{k e^{rt} N(0)}{\cancel{k - N(0)}} * \frac{\cancel{k - N(0)}}{k - N(0) + e^{rt} N(0)}$$

$$N(t) = \frac{k e^{rt} N(0)}{k - N(0) + e^{rt} N(0)}$$

Divide through by e^{rt} (the R.H.S)

$$N(t) = \frac{k N(0)}{[k - N(0)]e^{-rt} + N(0)}$$

Divide through by $N(0)$

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$$N(t) = \frac{K}{1 + \left(\frac{K}{N(0)} - 1\right)e^{-rt}}$$

but $K = r/\lambda$

$$N(t) = \frac{r/\lambda}{1 + \left(\frac{r}{\lambda N(0)} - 1\right)e^{-rt}}$$

1989 - 21.4M

1999 - 28.7M

2009 - 38.6

$$a = \frac{\frac{1}{N_1} + \frac{1}{N_3} - \frac{2}{N_2}}{\frac{1}{N_1 N_3} - \frac{1}{N_2^2}}$$

$$a = \frac{\frac{1}{21.4} + \frac{1}{38.6} - \frac{2}{28.7}}{\frac{1}{(21.4 \times 38.6)} - \frac{1}{(28.7)^2}} =$$

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JAN 2017

QUESTION 5

(a) In a population assumed to be stable, a census was taken and the population count at age x was found to be $c(x)$ while at age $y > x$ is $c(y)$. Express the rate of increase in terms of $c(x)$, $c(y)$ and the survivorship function $l(x)$

$${}_5C_a = \frac{b e^{-r(a+2.5)} \, {}_5L_a}{l_0}$$

$$b = \frac{1}{\int_0^{\omega} \frac{e^{-r(a+2.5)} \, {}_5L_a}{l_0}}$$

$${}_5C_a = \frac{e^{-r(a+2.5)} \, {}_5L_a}{\int_0^{\omega} e^{-r(a+2.5)} \, {}_5L_a}$$

Discrete case

$${}_5C_a = \frac{e^{-r(a+2.5)} \, {}_5L_a}{\sum_{a=0}^{\omega-5} e^{-r(a+2.5)} \, {}_5L_a}$$

Given population x and y

$${}_5C_x = \frac{e^{-r(x+2.5)} \, {}_5L_x}{\sum_{x=0}^{\omega-5} e^{-r(x+2.5)} \, {}_5L_x}$$

$${}_5C_y = \frac{e^{-r(y+2.5)}\, {}_5L_y}{\sum_{y=0}^{w-5} e^{-r(y+2.5)}\, {}_5L_y}$$

~~Taking~~

Taking the ratio ${}_5C_x : {}_5C_y$

$$\frac{{}_5C_x}{{}_5C_y} = \frac{e^{-r(x+2.5)}\, {}_5L_x}{\cancel{\sum_{x=0}^{w-5}} e^{-r(x+2.5)}\, {}_5L_x} * \frac{\cancel{\sum_{y=0}^{w-5}} e^{-r(y+2.5)}\, {}_5L_y}{e^{-r(y+2.5)}\, {}_5L_y}$$

$$\frac{{}_5C_x}{{}_5C_y} = \frac{e^{-r(x+2.5)}\, {}_5L_x}{e^{-r(y+2.5)}\, {}_5L_y}$$

$$\frac{{}_5C_x}{{}_5C_y} * \frac{{}_5L_y}{{}_5L_x} = e^{-rx-2.5r} \cdot e^{ry+2.5r}$$

$$e^{r(y-x)} = \frac{{}_5C_x}{{}_5C_y} * \frac{{}_5L_y}{{}_5L_x}$$

Taking natural log

$$r(y-x) = \ln\left(\frac{{}_5C_x}{{}_5C_y} * \frac{{}_5L_y}{{}_5L_x}\right)$$

$$r = \frac{1}{y-x} \ln\left[\frac{{}_5C_x}{{}_5C_y} * \frac{{}_5L_y}{{}_5L_x}\right]$$

(b) The table below gives the age distribution of a certain Island without migration and life table function ${}_5L_x$

Age	${}_5C_x$	${}_5L_x$
10	19,118	467,241
20	9,393	461,603
30	8,025	451,857
40	7,194	416,635
All ages	154,931	

Use the results in a above to calculate the rate of increase, with all possible combination of age.

Age distributions for 10 and 20

(i) $x=10 \quad y=20$

$$r = \frac{1}{y-x} \ln\left[\frac{{}_5C_x}{{}_5C_y} * \frac{{}_5L_y}{{}_5L_x}\right]$$

$$r_1 = \frac{1}{20-10} \ln\left(\frac{19118}{9393} * \frac{461\,603}{467\,241}\right) = 0.0699$$

(ii) Age distributions for 10 and 30

$x=10 \quad y=30$

$$r_2 = \frac{1}{30-10} \ln\left\{\frac{19118}{8025} * \frac{451\,857}{467,241}\right\} = 0.0417$$

Question 3

(a) Using integration by parts, prove that

$${}_nL_x = \int_0^n l_{x+t}\,dt = n\,l_{x+n} + {}_na_x \cdot {}_nd_x$$

$${}_nd_x = \int_0^n l_{x+t}\,\mu_{x+t}\,dt$$

$$1 = \int_0^n \frac{l_{x+t}\,\mu_{x+t}}{{}_nd_x}\,dt \qquad \text{let } {}_nP_x(t) = \frac{l_{x+t}\,\mu_{x+t}}{{}_nd_x}$$

$$1 = \int_0^n {}_nP_x(t)\,dt \qquad \text{is a pdf} \qquad 0 < t \leq n$$

$$E(T) = \int_0^n t\,{}_nP_x(t)\,dt$$

$$= \int_0^n t\,\frac{\cancel{l_{x+t}}}{{}_nd_x}\left(-\frac{1}{\cancel{l_{x+t}}}\,\frac{d\,l_{x+t}}{\cancel{dt}}\right)\cancel{dt}$$

$$= -\frac{1}{{}_nd_x}\int_0^n t\,d\,l_{x+t}$$

Using integration by parts to integrate

$$\int_0^n t\,dl_{x+t} \qquad \text{let } u = t \quad \therefore du = dt$$

$$dv = dl_{x+t} \quad \therefore v = l_{x+t}$$

$$\int u\,dv = uv - \int v\,du$$

$$E(T) = -\frac{1}{{}_nd_x}\left[t\,l_{x+t}\Big|_0^n - \int_0^n l_{x+t}\,dt\right]$$

601 - Statistical demography
602 -- modelling and analysis
604 - Advanced quantitative economics
608 - Professional issues in social statistics

$$E(T) = -\frac{1}{{}_nd_x}\left[n\, l_{x+n} - {}_nL_x\right]$$

$$E(T) = \frac{{}_nL_x - n * l_{x+n}}{{}_nd_x} = {}_na_x$$

$${}_nL_x - n * l_{x+n} = {}_na_x * {}_nd_x$$

$${}_nL_x = n * l_{x+n} + {}_na_x * {}_nd_x$$ Hence the proof.

(b) Hence or otherwise, show that

$${}_na_x = n - \frac{n}{{}_nq_x} + \frac{1}{{}_nm_x}$$

$${}_nm_x = \frac{{}_nd_x}{{}_nL_x} \qquad {}_nL_x = \frac{{}_nd_x}{{}_nm_x}$$

$${}_nL_x = n * l_{x+n} + {}_na_x * {}_nd_x = \frac{{}_nd_x}{{}_nm_x}$$

$$n * l_{x+n} + {}_na_x * {}_nd_x = \frac{{}_nd_x}{{}_nm_x}$$

Divide through by ${}_nd_x$

$$\frac{n\, l_{x+n}}{{}_nd_x} + {}_na_x = \frac{1}{{}_nm_x}$$

$${}_na_x = \frac{1}{{}_nm_x} - \frac{n\, l_{x+n}}{{}_nd_x}$$

$${}_na_x = \frac{1}{{}_nm_x} - n\left[\frac{l_{x+n}}{l_x - l_{x+n}}\right]$$

removing -1 from inside the brackets

$${}_na_x = \frac{1}{{}_nm_x} + n\left[\frac{l_{x+n}}{l_{x+n} - l_x}\right]$$

$${}_na_x = \frac{1}{{}_nm_x} + n\left[1 - \frac{l_x}{l_x - l_{x+n}}\right]$$

$${}_na_x = \frac{1}{{}_nm_x} + n\left[1 - \frac{1}{{}_nq_x}\right]$$

$${}_na_x = \frac{1}{{}_nm_x} + n - \frac{n}{{}_nq_x}$$ Hence the proof.

Que 7

	Whites		Non-Whites	
x	l_x	${}_5L_x$	l_x	${}_5L_x$
20	96714	481191	94885	470123
25	95758	476704	93075	460668
30	94932	442590	90889	448775
35	94068	467604	88526	435035
40	92889	460272	85452	418035
45	91074	448559	81579	395230
50	88111	430257	79222	365069
55	83718	402493	69489	326382
60	75906	361948	60790	280014
65	67459	308271	50979	229168
70	55524	243082	40637	172633
75	41510	169700	28667	117474
80	26499	100605	18521	74378
85+	13996	65805*	11465	71696*

(c) How would you measure the relationships between time spent in labour free and in retirement in the two populations assuming that everybody is employed between ages 20 and 65.

labour free → < 20
Work force → 20 - 65
Retired → over 65

		For whites (l_x)	For Non-whites (l_x)
labour free	< 20	0	0
work force	20-65	881629	791886
Retired	over 65	137529	99290

(i) Dependancy ratio for whites $= \frac{137529}{881629} = 0.1559 \times 100000$

15590 per 100,000

For every 100,000 white persons, there are 15,590 dependants.

(ii) Dependancy ratio for non-whites $= \frac{99290}{791886} = 0.12538 \times 100,000$

12,538

For every 100,000 non-white persons, there are 12,538 dependants

(b) Given values for the two population

$$\mathring{e}_{20} = \frac{T_{20}}{l_{20}}$$

$$\mathring{e}_x = \frac{T_x}{l_x}$$

$$\mathring{e}_{20} \text{ (whites)} = \frac{4858481}{96714} = 50.24$$

$$e^{\circ}_{20} \text{ (non-whites)} = \frac{426075}{94885} = 44.94$$