# PROBLEMS AND SOLUTIONS

# IN

# STABLE POPULATION THEORY



PREPARED

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PROBLEM 1.

A follow-up study of migrants to a work area finds that all of them leave before their fourth "anniversary" in the area; 40 percent leave in the first year; 30 percent in the second; 20 percent in the third year and 10 percent in the fourth year. The number of migrants entering the area each year is 80 percent of the total migrant population. This pattern has been fixed for many years.

(a) What is the average length of time that a migrant remains in the work area? Assume that migration is distributed evenly through the year.

(b) For migrants who have remained in the area exactly one year, what is the probability of remaining exactly two more years?

(c) If the migrants who have been in the area between two and three years represent 15.22% of the total migrant population, what is the rate of growth of the migrant population?

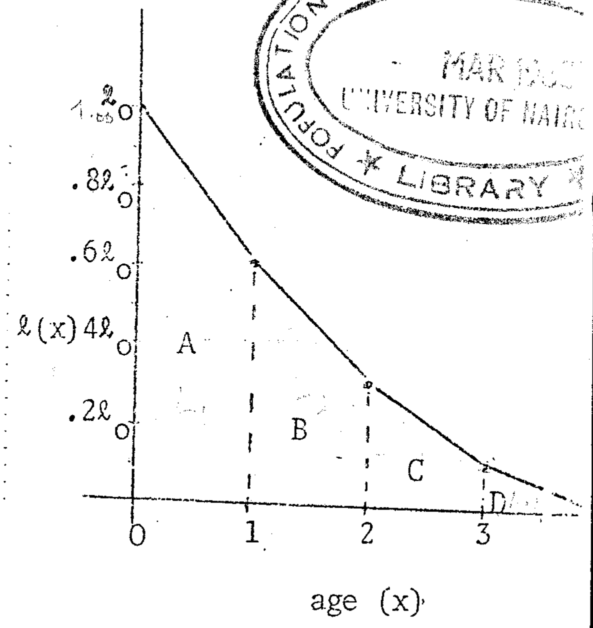
(d) What proportion of migrants leave the area each year?

In your calculations assume no deaths to migrants.

SOLUTION

Consider duration of stay analogous to age; migrants entering are analogous to births, and migrants leaving are analogous to deaths. So we have the following life table functions

| X | d(x) | $\ell(x)$ |
|---|---|---|
| 0 | 0 | $\ell_o$ |
| 1 | $.4\ell_o$ | $.6\ell_o$ |
| 2 | $.3\ell_o$ | $.3\ell_o$ |
| 3 | $.2\ell_o$ | $.1\ell_o$ |
| 4 | $.1\ell_o$ | 0 |






(a) $$e_o^o = \frac{T_o}{\ell_o}$$

$$= (\text{Area A} + \text{Area B} + \text{Area C} + \text{Area D}) \frac{1}{\ell_o}$$

$$= \frac{1}{\ell_o} \quad (\tfrac{1}{2} \times 1x1.6\ell_o) + (\tfrac{1}{2} x1x0.9\ell_o) + (\tfrac{1}{2}x1x0.4\ell_o) + (\tfrac{1}{2}x1$$

$$= 0.80 + 0.45 + 0.20 + 0.05$$

$$= \underline{\underline{1.5}}$$

(b) $${}_2p_1 = \frac{\ell_3}{\ell_1} = \frac{.1\ell_o}{.6\ell_o} = \frac{1}{6}$$

(c) Using $C(a) = b\, e^{-ra}\, p(a)$, we have

$$C(2.5) = b\, e^{-2.5r}\, p(2.5)$$

i.e., $$0.1522 = 0.80\, e^{-2.5r}\, p(2.5)$$

$$= 0.80\, e^{-2.5r}\, \frac{\ell(2.5)}{\ell_o}$$

i.e., $$0.1522 = 0.80\, e^{-2.5r}\, \frac{\ell_2 + \ell_3}{2\ell_o}$$

Therefore

$$0.1522 = 0.80\, e^{-2.5r}\, \frac{(.3\ell_o + .1\ell_o)}{2\ell_o}$$

$$= (0.80\, e^{-2.5r})\, 0.20$$

$$\therefore \quad e^{-2.5r} = \frac{0.1.522}{0.1600}$$

$$\therefore \quad r = -\frac{1}{2.5} \ln 0.95125$$

So

$$\therefore \quad r \simeq \underline{\underline{0.02}}$$

(d) Proportion leaving is equivalent to death rate d.

Thus

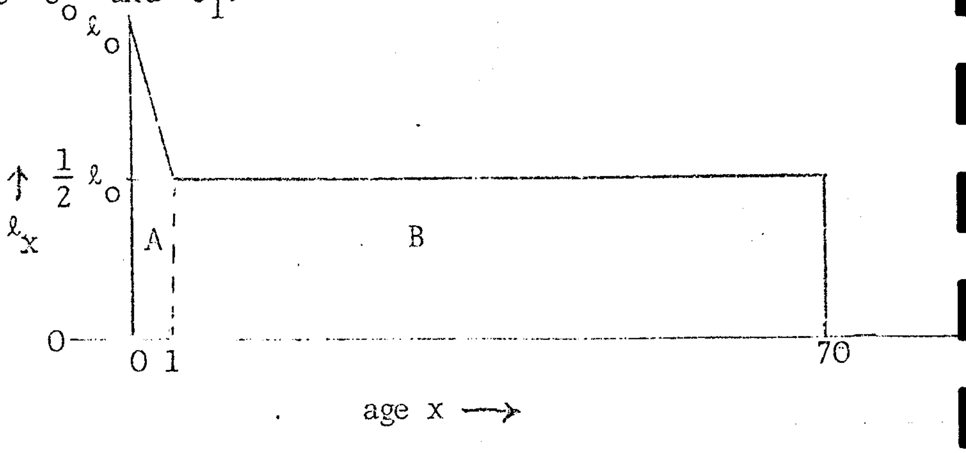
$$\begin{aligned} d &= b - r \\ &= .80 - .02 \\ &= \underline{\underline{0.78}} \end{aligned}$$

<u>PROBLEM 2</u>

Imagine a female life table according to which half of the initial cohort dies during the first year of life. Deaths are evenly distributed through this year. Then no deaths occur until age 70, when all survivors die. Sketch $\ell_x$. Determine $e_o^o$ and $e_1^o$.

<u>Solution</u>

(a)




$$\text{(b)} \quad e_o^o = \frac{T_o}{\ell_o} = (\text{Area A} + \text{Area B}) \frac{1}{\ell_o}$$

$$= (\text{Area of trapezium} + \text{Area of rectangle})$$

$$= \left[\frac{1}{2}\,(\ell_o + \frac{1}{2}\,\ell_o) \times 1 + \frac{1}{2}\,\ell_o \times 69\right] \frac{1}{\ell_o}$$

$$= \frac{3}{4} + \frac{69}{2} = \underline{\underline{35.5}}$$

(c) $$e_1^o = \frac{T_1}{\ell_1}$$

$$= \frac{\text{Area B}}{\ell_1}$$

$$= \frac{\frac{1}{2}\,\ell_o \times 69}{\frac{1}{2}\,\ell_o} = \underline{\underline{69}}$$



PROBLEM 3

In mortality schedule R, no one dies until age 100, with 100% mortality at that age. In schedule S, the annual death rate at every age from 0 to 100 is 6.9 per thousand; at exact age 100 all survivors die.

(a) Sketch $\ell(x)$ for R and S.

(b) Suppose R and S prevail in two stable female populations in both of which women bear an average of 0.1 female offspring per year from age 45 to 55. What is the rate of increase in each?

(c) Sketch the two stable age distributions, and determine the birth rate and death rate in each.

(d) What is the expectation of life at birth in R and S?

Solution

For R,

$$\ell(a) = \ell_o \quad , \quad 0 \le a \le 100$$

$$= 0 \quad , \quad \text{otherwise}$$

This implies

$$p(a) = \frac{\ell(a)}{\ell(a)} = 1, \quad 0 \le a \le 100$$

$$= 0, \text{ otherwise.}$$

For S,

$$p(a) = e^{-\int_0^a \mu(x)dx}$$

$$= e^{-\mu a} \quad \text{for } \mu(x) = \mu, \text{ constant}$$

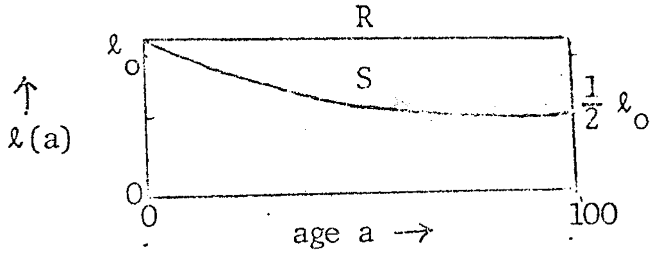
Therefore

$$p(a) = e^{-0.0069a}, \quad 0 \le a \le 100$$

$$= 0, \quad \text{otherwise}$$

So

$$\ell(a) = \ell(o)\, e^{-.0069a}, \quad 0 \le a \le 100$$




[ Note: $e^{-.69} \simeq \frac{1}{2}$ ]

(b) $\quad NRR = e^{rT}$

which implies that

$$r = \frac{\ln NRR}{T}$$

Also

$$NRR = \int_\alpha^\beta p(a)\, m(a)\, da$$

i.e.

$$NRR = \int_{45}^{55} p(a)\, m(a)\, da = \int_{45}^{55} p(a) \times (0.1) da$$

For R,

$$p(a) = 1$$

Therefore

$$NRR = \int_{45}^{55} 1 \times 0.1\, da$$

$$= 1$$

Thus

$$r = \frac{\ln 1}{T} = 0$$

which implies that R is stationary.

For S,

$$p(a) = \frac{\ell(a)}{\ell(o)} = e^{-.0069a} , \quad 0 < a < 100$$

$$= 0, \quad \text{otherwise}$$

Therefore

$$NRR = 0.1 \int_{45}^{55} e^{-.0069a} \, da$$

$$= \frac{0.1}{0.0069} \left[ \exp(-.0069 \times 45) - \exp(-.0069 \times 55) \right]$$

$$= \frac{0.1}{0.0069} (0.7331 - 0.6842)$$

$$= 0.7087$$

So

$$r = \frac{\ln NRR}{T} = \frac{\ln 0.7087}{50} \simeq -\underline{\underline{0.0069}}$$

(c) The formula for age distribution is given by

$$C(a) = b \, e^{-ra} \, p(a)$$

For R, $r = 0$ and $p(a) = 1$

Therefore $e^{-ra} \, p(a) = 1$

which implies that $C_R(a) = b_1$

For S, $r = -0.0069$ and $p(a) = e^{-.0069a}$

$$e^{-ra} p(a) = e^{.0069a} \, e^{-.0069a} = e^{o} = 1$$

$$\therefore \quad C_S(a) = b_2$$

In a stable population, birth rate is given by

$$b = \frac{1}{\int_0^{\omega} e^{-ra} p(a)\, da}$$

So

$$b_1 = \frac{1}{\int_0^{100} 1\, da} = \frac{1}{100} = \underline{\underline{0.01}}$$

and

$$b_2 = \frac{1}{\int_0^{100} 1\, da} = 0.01$$

$$\therefore \quad b_1 = b_2 = 0.01$$

Death rate is defined by

$$b - d = r \Rightarrow \boxed{d = b - r}$$

$$\therefore \quad d_1 = b_1 = 0.01 \text{ and } d_2 = b_2 - (-.0069) = 0.01 + .0069 = \underline{\underline{.016}}$$

Alternatively, recall the theorem which states that:-

Two stable populations with a constant difference in mortality schedules and same fertility schedule have the same age distribution.

In this problem,

$m(a) = 1$ (same fertility schedule)

For R,

$b(a) = 1 = e^0$

and for S

$b(a) = e^{-.0069a}$

This implies, force of mortality

$$\mu_1(a) = 0 \quad \text{for R}$$

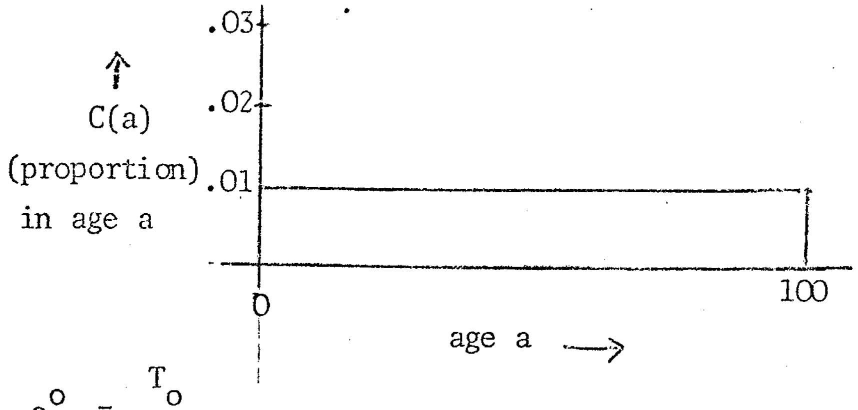
and

$$\mu_2(a) = .0069, \quad \text{for S;}$$

$$\therefore \quad \mu_2(a) - \mu_1(a) = .0069 \quad \text{(Constant)}$$



Then sketch for the age distribution is




(d) $$e_o^o = \frac{T_o}{\ell_o}$$

For R which is stationary

$$e_o^o = \frac{1}{b} = \frac{1}{.01} = 100$$

For S

$$e_o^o = \frac{\int_0^{100} \ell(a)\, da}{\ell_o}$$

$$= \frac{1}{\ell_o}\left[\int_0^{100} \ell_o \; e^{-.0069a}\, da\right]$$

$$= \int_0^{100} e^{-.0069a}\, da$$

$$= -\frac{1}{.0069}\; e^{-.0069a}\Big|_0^{100}$$

$$= \frac{1}{.0069}\;(1 - e^{-.0069 \times 100})$$

$$= \frac{1}{.0069} (1 - e^{-.69})$$

$$= \frac{1}{.0069} (1 - \frac{1}{2})$$

$$= \frac{1}{0.0138}$$

$$= \underline{\underline{72.5}}$$

PROBLEM 4

(a) Suppose a life table in which the number of survivors declines linearly from 100,000 at age zero to 50,000 at age 20, remains at 50,000 until age 60, and then declines linearly to zero at age 80.

(i) What is the expectation of life at age zero? Age 20? Age 60?

(ii) What is the average death rate of persons under 20 in this stationary population? Of persons over 60?

(b) Two stable populations embody the same fertility schedule, but different mortality schedules. In both populations, 20 percent of women between exact ages 45 to 55 bear a female child annually; no childbearing occurs outside this span. In population A, there is no mortality until age 100, where all who reach that age die. In population B, one percent of each cohort dies within each single year of age, with no survivors beyond age 100.

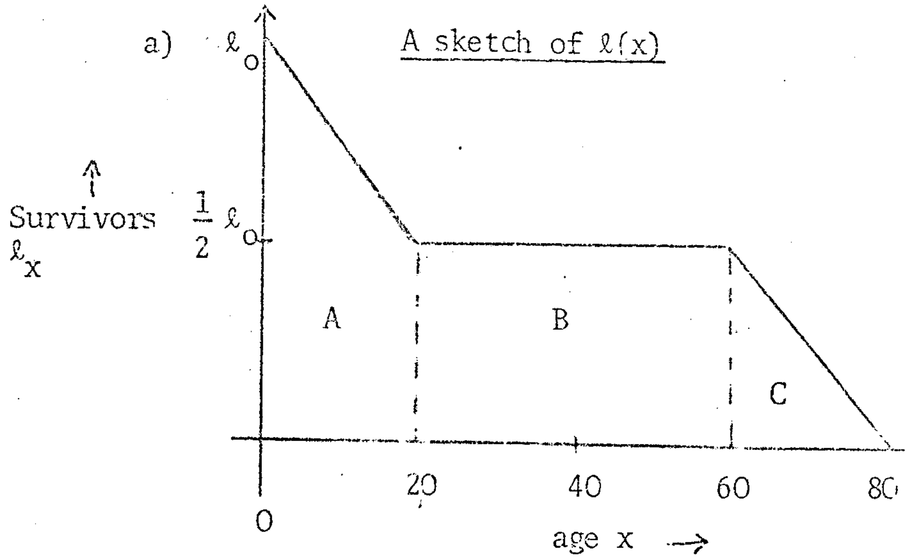
(i) Calculate GRR, NRR and r for each population.

(ii) Sketch the age distribution of each.

~~(iii) Determine the birth rate and the death rate of each.~~

Solution

a) A sketch of $\ell(x)$






Area A = Area of the trapezium

$$= \frac{1}{2}(\ell_o + \frac{1}{2}\ell_o) \times 20$$

$$= \underline{\underline{15\ \ell_o}}$$

Area B = Area of the rectangle

$$= \frac{1}{2}\ell_o \times 40$$

$$= \underline{\underline{20\ \ell_o}}$$

Area C = Area of the triangle

$$= \frac{1}{2} \times 20 \times \frac{1}{2}\ell_o$$

$$= \underline{\underline{5\ \ell_o}}$$

Also, expectation of life at age x in a life table is given by

$$e_x^o = \frac{T_x}{\ell_x}$$

Thus

(i) $$e_o^o = \frac{T_o}{\ell_o}$$

$$= \frac{1}{\ell_o}\ (\text{Area A + Area B + Area C})$$

$$= \frac{1}{\ell_o} (15\ell_o + 20\ \ell_o + 5\ell_o)$$

$$= \underline{\underline{40}}$$

$$\overset{o}{e}_{20} = \frac{T_{20}}{\ell_{20}} = \frac{1}{\ell_{20}} (\text{Area B} + \text{Area C})$$

$$= \frac{1}{\frac{1}{2}\ell_o} (20\ell_o + 5\ell_o)$$

$$= \underline{\underline{50}}$$

$$\overset{o}{e}_{60} = \frac{T_{60}}{\ell_{60}} = \frac{\text{Area C}}{\frac{1}{2}\ell_o}$$

$$= \frac{5\ \ell_o}{\frac{1}{2}\ell_o} = \underline{\underline{10}}$$

(ii) Death Rate under 20 $= \dfrac{\text{No. of deaths under 20}}{\text{No. of person-years lived in this period}}$

$$= \frac{\frac{1}{2}\ell_o}{{}_{20}L_o} = \frac{\frac{1}{2}\ell_o}{\text{Area A}}$$

$$= \frac{\frac{1}{2}\ell_o}{15\ell_o} = \frac{1}{30}$$

Death rate over 60 $= \dfrac{\text{No. of deaths over 60}}{{}_{20}L_{60}}$

$$= \frac{\frac{1}{2}\ell_o}{\text{Area C}}$$

$$= \frac{\frac{1}{2}\ell_o}{5\ell_o} = \frac{1}{10}$$

(b) $$m(a) = \frac{20}{100}$$

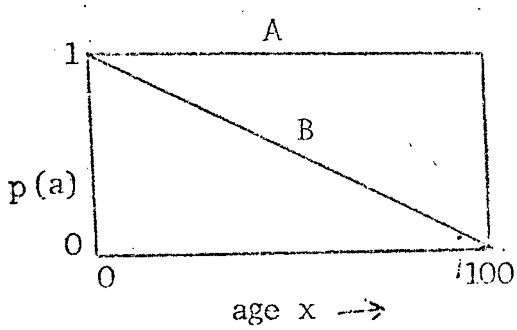
$$\therefore \quad GRR = \int_{\alpha}^{\beta} m(a)\, da$$

$$= \int_{45}^{55} \frac{20}{100}\, da$$

$$= (55 - 45)\, \frac{20}{100}$$

$$= \frac{200}{100} = \underline{\underline{2}}$$

<u>A Sketch for p(a), the probability of survival</u>




For A,

$$p(a) = 1 \;,\; 0 \le a \le 100$$
$$= 0 \text{ elsewhere}$$

For B,

$$p(a) = 1 - \frac{1}{100}a \;,\; 0 \le a \le$$
$$= 0, \text{ otherwise}$$

(i) $$NRR = \int_{\alpha}^{\beta} p(a)\, m(a)\, da$$

$$\therefore \quad NRR_A = \int_{45}^{55} 1 \times \frac{20}{100}\, da$$

$$= \underline{\underline{2}}$$

$$NRR_B = \int_{45}^{55} \left(1 - \frac{a}{100}\right) \times \frac{20}{100}\, da$$

$$= \frac{20}{100} \left[ a - \frac{a^2}{200} \right]_{45}^{55}$$

$$= \frac{20}{100} \left[ \left(55 - \frac{55^2}{200}\right) - \left(45 - \frac{45^2}{200}\right) \right]$$

$$= \frac{20}{100} \left[ 10 - \frac{(55^2 - 45^2)}{200} \right]$$

$$= \frac{20}{100}\left[10 - \frac{(55 + 45)(55 - 45)}{200}\right]$$

i.e. $$NRR_B = \frac{20}{100}\left(10 - \frac{100 \times 10}{200}\right) = \underline{\underline{1}}$$

Alternatively

$$NRR = \int_{\alpha}^{\beta} p(a)\, m(a)\, da$$

$$= p(\bar{m}) \int_{\alpha}^{\beta} m(a)\, da$$

$$= p(\bar{m})\ GRR$$

where $p(\bar{m})$ is the probability of survival from birth up to the mean age of child bearing which is $p(50)$ in this problem.

Therefore

$$NRR = p(50)\ GRR$$

So

$$NRR_A = 1 \times 2 = \underline{\underline{2}}$$

and

$$NRR_B = \left(1 - \frac{50}{100}\right) \times 2 = \frac{1}{2} \times 2 = \underline{\underline{1}}$$

To determine 'r', we use the formula

$$NRR = e^{rT}$$

which implies

$$r = \frac{\ln NRR}{T}$$

where T is the mean length of a generation.
In this case we shall take T = 50

Therefore

$$r_A = \frac{\ln 2}{50} = \underline{\underline{0.01386}}$$

and

$$r_B = \frac{\ln 1}{50} = 0$$

which shows that B is a stationary population.

(iii) Birth rate is given by

$$b = \frac{1}{\int_0^w e^{-ra}\, p(a)\, da}$$

For A,

$$b_1 = \frac{1}{\int_0^{100} e^{-.01386a}\, da}$$

since $p(a) = 1$

So

$$b_1 = \frac{.01386}{1 - e^{-1.386}}$$

$$= \frac{.01386}{1 - 0.2500} = \frac{.01386}{.75}$$

$$\therefore\ b_1 = \underline{\underline{0.01848}} \simeq \underline{\underline{0.02}}$$

Therefore the death rate which is defined by

$$d = b - r$$

becomes

$$d_1 = b_1 - r$$

$$= 0.01848 - 0.01386$$

$$= \underline{\underline{0.00462}}$$



For B,

$$d = b = \frac{1}{\int_0^{100} p(a)\, da}$$

$$= \frac{1}{\int_0^{100} (1 - \frac{a}{100})\, da}$$

$$= \frac{1}{\left[ a - \frac{a^2}{200} \right]_0^{100}}$$

$$= \frac{1}{100 - \frac{100^2}{200}} = \frac{1}{100-50}$$

$$\therefore \quad d = b = \frac{1}{50} = \underline{\underline{0.02}}$$

(ii) Age distribution is

$$C(a) = b\ e^{-ra}\ p(a).$$

For A,

$$C(a) = 0.02\ e^{-0.01386a}\left(1 - \frac{a}{100}\right) /$$

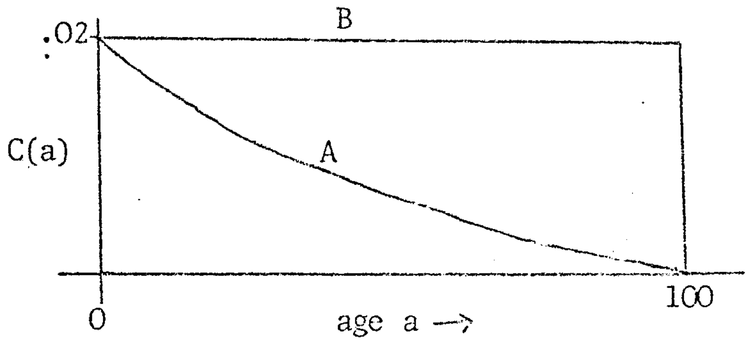
$$0 \leq a \leq 100$$

For B

$$C(a) = 0.02 \quad \text{for } 0 \leq a \leq 100$$



The sketch of age distribution




PROBLEM 5

With two quite different schedules of mortality (A and B), the average duration of life is the same. According to schedule A there are no deaths until age 50; within each cohort the number of deaths above 50 is uniform at each age until 100, beyond which there are no survivors. According to schedule B, within each cohort the number of deaths is the same at every age until age 50; from 50 to 100 there is no mortality; all survivors die on attaining age 100.

(i) Sketch $\ell_x$ for each life table.

(ii) What is the common $e_o^o$, and what is $\ell_{50}$ in B?

Female populations with life tables A and B are subject to the following fertility schedule: there are annually 0.120 female births for each woman between exact ages 20 and 30, and no births outside this span.

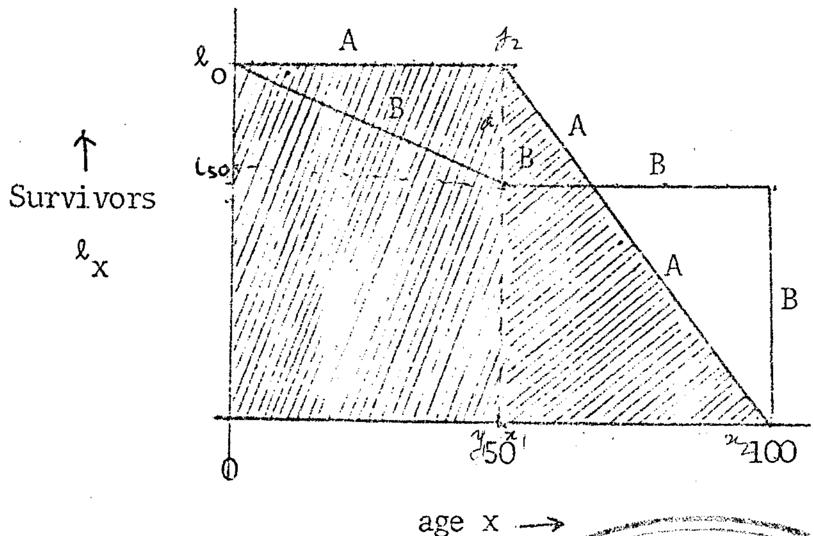
(iii) What is GRR, NRR, and the approximate value of r in the two populations?

(iv) What is the birth rate in stable population B?

(v) What is the approximate birth rate in population A?

Solution

(i)




(ii) By definition

$$e_o^o = \frac{T_o}{\ell_o}$$



Therefore from population A,

$$e_o^o = (\text{Area of shaded Rect.} + \text{Area of shaded triangle}) \frac{1}{\ell_o}$$

$$= \left(50\, \ell_o + \frac{1}{2} \cdot 50 \cdot \ell_o\right) \frac{1}{\ell_o}$$

$$= 75$$

which is the same in B.

Now using population B,

$$e_o^o = (\text{Area of trapezium} + \text{Area of rect}) \frac{1}{\ell_o}$$

i.e.,

$$75 = \left[\frac{1}{2}(\ell_o + \ell_{50}) \times 50 + 50\ell_{50}\right] \frac{1}{\ell_o}$$

$$= 25 + 75 \frac{\ell_{50}}{\ell_o}$$

Therefore

$$\frac{\ell_{50}}{\ell_o} = p(50) = \frac{50}{75} = \frac{2}{3}$$

$$\therefore \quad \ell_{50} = \underline{\underline{\frac{2}{3}\ell_o}}$$

(iii) Next, we are given

$$m(a) = 0.120\,, \quad 20 \leq a \leq 30$$
$$= 0, \text{ elsewhere}$$

$$\therefore \quad GRR = \int_\alpha^\beta m(a)\, da = \int_{20}^{30} 0.120\, da = \underline{\underline{1.2}}$$

$$NRR = \int_\alpha^\beta p(a)\, m(a)\, da$$

$$= \int_\alpha^\beta p(\bar{m})\, m(a)\, da$$

$$= p(\bar{m}) \int_\alpha^\beta m(a)\, da$$

$$= p(\bar{m})\ GRR$$

where $p(\bar{m})$ is the probability of survival from birth up to the mean age of child hearing which is p(25) in this case.

So

$$NRR = p(25)\ GRR$$

i.e.,

$$NRR_A = 1 \times 1.2 = 1.2$$

$$NRR_B = \frac{\frac{1}{2}(\ell_o + \ell_{50})}{\ell_o} \times 1.2$$

$$= \frac{\ell_o + \frac{2}{3}\ell_o}{2\ \ell_o} \times 1.2$$

$$= \frac{5}{6} \times 1.2 = 1$$

To obtain r, use

$$NRR = e^{rT}$$

which implies that

$$r = \frac{\ln NRR}{T}$$

$$r_A = \frac{\ln 1.2}{25} = 0.0073$$

$$r_B = \frac{\ln 1}{25} = 0.$$

(iv) Population B is stationary since $r = o$
But in a stationary population,

$$b = \frac{1}{e_o^o}$$

Therefore the birth rate for B is

$$b = \frac{1}{75} = 0.01333$$

(v) For population A, use

$$b = \frac{1}{\int_0^w e^{-ra} p(a)\, da}$$

From the sketch of $\ell_x$ shown above, for population A,

$$\ell(a) = \ell_o, \quad 0 \leq a < 50$$

$$= \ell_o - \frac{\ell_o}{50}(a - 50), \quad 50 \leq a < 100$$

Therefore $p(a) = \frac{\ell_a}{\ell_o}$ becomes

$$p(a) = 1, \quad 0 \leq a < 50$$

$$= 1 - \left(\frac{a - 50}{50}\right), \quad 50 \leq a < 100$$

i.e.,

$$p(a) = 1, \quad 0 \leq a < 50$$

$$= 2 - \frac{a}{50}, \quad 50 \leq a < 100$$

Therefore

$$b = \frac{1}{\int_0^{50} e^{-ra}\, da + \int_{50}^{100} e^{-ra} \left(2 - \frac{a}{50}\right) da}$$

Using the fact that

$$\int_0^x e^{-ra}\, da = \frac{1 - e^{-rx}}{r}$$

and

$$\int_0^x a\, e^{-ra}\, da = -\frac{x\, e^{-ra}}{r} + \frac{1 - e^{-rx}}{r^2}$$

then

$$\int_0^{50} e^{-ra}\, da = \frac{1 - e^{-50r}}{r}$$

and

$$\int_{50}^{100} e^{-ra}(2 - \frac{a}{50})da = \int_{0}^{100} e^{-ra}(2 - \frac{a}{50})da - \int_{0}^{50} e^{-ra}(2 - \frac{a}{50})da$$

$$= 2\int_{0}^{100} e^{-ra}\,da - \frac{1}{50}\int_{0}^{100} ae^{-ra}\,da - 2\int_{0}^{50} e^{-ra}\,da + \frac{1}{50}\int_{0}^{50} ae^{-}$$

$$= \frac{2(1 - e^{-100r})}{r} - \frac{1}{50}\left[- \frac{100\,e^{-100r}}{r} + \frac{1 - e^{-100r}}{r^2}\right]$$

$$- 2\,\frac{(1 - e^{-50r})}{r} + \frac{1}{50}\left[- \frac{50}{r}e^{-50r} + \frac{1 - e^{-50r}}{r^2}\right]$$

$$= \frac{2}{r}(e^{-50r} - e^{-100r})\,\frac{1}{50r}(100\,e^{-100r} - 50\,e^{-50r})$$

$$+ \frac{1}{50r^2}(1 - e^{-50r} - 1 + e^{-100r}$$

$$= \frac{1}{r}e^{-50r} - \frac{e^{-50r}}{50r^2} + \frac{e^{-100r}}{50r^2}$$

$$\therefore \int_{0}^{100} e^{-ra}\,p(a)\,da = \frac{1}{r} - \frac{e^{-50r}}{50r^2} + \frac{e^{-100r}}{50r^2}$$

Therefore putting $r = 0.0073$

$$b = 1/(136.9863 - 260.53543 + 180.86282)$$

$$= \frac{1}{57.31369} = 0.01744784$$

i.e. $b \simeq \underline{\underline{0.01745}}$

PROBLEM 6

Suppose there is a female stable population generated from a mortality schedule in which no one dies before age 60 and no one survives beyond age 60 and from a schedule of age specific maternity rates which are equal at each age from exact age 15 to axact age 45. In this population, the general fertility rate (with women 15 to 45 as the denominator) is 0.04483, the proportion of the population less than age 15 is 0.30861, and the proportion aged 15 to 30 is 0.26563:

(a) What is the death rate in this stable population? ✓

(b) What is the gross reproduction rate?
What is the net reproduction rate?

(c) Suppose there are two populations, A and B, with the above characteristics. In A women reduce their maternity rates by 50 percent at each age. In B women choose not to have children until age 30, at which age they continue to reproduce as in the past. After many years of the new reproduction patterns, does A or B have a higher crude death rate? Why?

(d) Suppose there is a population C which has the above initial characteristics at time t, and suppose that at time t age specific maternity rates begin to decline at an instantaneous rate of one percent per year for 15 years. What is the ratio of the population size at t + 15 to the population size at t?



Solution

Given: $p(a) = 1, \quad 0 \leq a < 60$

$= 0, \quad$ elsewhere

$m(a) = m$, constant, $15 \leq x \leq 45$

$$GFR = \frac{\text{total births}}{\text{women } 15 - 45} = 0.04483$$

$$C(<15) = {}_{15}C_{0} = 0.30861$$

$$C(15 - 30) = {}_{15}C_{15} = 0.26563$$

(a) Using the formula

$$C(a) = b\, e^{-ra}\, p(a) \ ,$$

$$C(<15) = b \int_0^{15} e^{-ra} p(a)\, da$$

i.e.,

$$0.30861 = b \int_0^{15} e^{-ra}\, da$$

$$= b \left[ - \frac{e^{-ra}}{r} \right]_0^{15}$$

$$= \frac{b}{r} \left[ 1 - \underline{e^{-15r}} \right] \qquad (1)$$

Next

$$C(15 - 30) = b \int_{15}^{30} e^{-ra}\, p(a)\, da$$

i.e. $$0.26563 = \frac{b}{r} \left( e^{-15r} - e^{-30r} \right) \qquad (2)$$

Therefore (1) $\div$ (2) becomes

$$\frac{0.30861}{0.26563} = e^{15r}$$

which implies

$$r = \frac{1}{15} \ln \frac{0.30861}{0.26563}$$

$$= 0.00999827$$

So

$$r \simeq \underline{\underline{0.01}}$$

From equation (1),

i.e.,

$$0.30861 = \frac{b}{r}\left[1 - e^{-15r}\right]$$

$$b = \frac{0.30861\ r}{1 - e^{-15r}}$$

$$\simeq \frac{0.30861 \times 0.01}{1 - 0.86070798}$$

$$= \frac{0.0030861}{0.13929202}$$

$$= 0.02215561$$

$$\therefore \quad b \simeq \underline{\underline{0.022}}$$

and

$$d = b - r$$

$$\simeq 0.022 - 0.010 = \underline{\underline{0.012}}$$

We can also obtain birth rate using

$$b = \frac{1}{\int_0^W e^{-ra}\, p(a)\, da}$$

$$= \frac{1}{\int_0^{60} e^{-ra}\, da}$$

$$= \frac{r}{1 - e^{-60r}}$$

$$= \frac{0.01}{1 - e^{-0.60}}$$

$$= \frac{0.01}{1 - 0.5488}$$

$$= \frac{0.01}{0.4512}$$

$$= 0.02216$$

$$\therefore \quad b \simeq 0.022$$

and

$$d \simeq b - r$$

$$= 0.022 - 0.010$$

$$= \underline{\underline{0.012}}$$

(b) $$GRR = \int_{\alpha}^{\beta} m(a)\, da$$

and

$$NRR = \int_{\alpha}^{\beta} m(a)\, p(a)\, da$$

In this problem,

$m(a) = m$ (constant) $15 \leq a \leq 45$

$p(a) = 1$ , $0 \leq a \leq 60$

zero otherwise.

Therefore

$$GRR = NRR = \int_{15}^{45} m\, da = 30m$$

To obtain 'm' we use the fact that

$$\underline{b = \int_{0}^{w} m(a)\, c(a)\, da} = \int_{\alpha}^{\beta} m(a)\, c(a)\, da,$$

In this case it becomes

$$b = m \int_{15}^{45} c(a)\, da$$

$$= m \int_{15}^{45} b\, e^{-ra}\, da$$

i.e. $$1 = m \int_{15}^{45} e^{-ra} da$$

Therefore, $$\frac{1}{m} = \left. \frac{e^{-ra}}{-r} \right|_{15}^{45}$$

i.e. $m = \dfrac{r}{e^{-15r} - e^{-45r}}$

Putting $r = 0.01$ we have

$$m = \frac{0.01}{0.86070798 - 0.63762815}$$

$$= \underline{\underline{0.044827}}$$

Thus

$$GRR = NRR = 30 \times 0.044827$$

$$= \underline{\underline{1.34481}}$$

Using the notion of

$$NRR = e^{rT}$$

then

$$GRR = NRR = e^{.01 \times 30} = e^{0.30} = \underline{\underline{1.3498588}}$$

Alternatively, since we are given GFR, we can use

$$GRF = \frac{TFR}{\beta-\alpha} + (\beta - \alpha)\, r_{f_i . c_i}\, \sigma_{f_i}\, \sigma_{c_i}$$

$$= \frac{TFR}{\beta - \alpha} \text{ assuming } r_{f_i\, c_i} = 0.$$

$$\therefore \; TFR = (\beta - \alpha)\, GFR$$

which implies

$$GRR = (\beta - \alpha)\, GFR$$

$$= 0.04483 \times 30$$

$$= \underline{\underline{1.3449}}$$

(c) $GRR_A = \int_{15}^{45} \frac{m}{2}\, da = 15m = \frac{30\ m}{2} = \frac{1}{2}\, GRR$

$GRR_B = \int_{30}^{45} m\, da = 15m = \frac{1}{2}\, GRR$

Therefore

$$NRR_A = NRR_B$$

i.e., $e^{r_A T_A} = e^{r_B T_B}$ ($T_A$, $T_B$: are mean length of generation)

This implies that

$$r_A T_A = r_B T_B$$

But

$$T_A < T_B \quad (T_A = 30 \text{ and } T_B = 37.5)$$

Therefore

$$r_A > r_B$$

which implies that $C_A(a) > C_B(a)$ at younger ages. i.e., Population A has larger proportion at younger ages. So the $C_D R_B$ is likely to be higher than $C_D R_A$.

Alternatively, since in both populations NRR is the same and the mean length of generation is longer in B, then r is smaller in B.

i.e.,

$$r_A > r_B$$

But

$$\frac{d\ b}{dr} = \bar{a}b > 0$$

which means that b increases with r.



Therefore

$$r_A > r_B$$

implies

$$b_A > b_B$$

In this problem the mean age $\bar{a}$ cannot be greater than a number, say, 45.

Therefore

$$\frac{db}{dr} = \bar{a}b < 45 \times 0.022 = 0.99 < 1$$

i.e.,

$$\frac{db}{dr} < 1$$

which implies that

$$\frac{b_A - b_B}{r_A - r_B} < 1$$

i.e.,

$$b_A - b_B < r_A - r_B$$

i.e.,

$$d_A < d_B$$

Thus

$$CDR_B > CDR_A$$

(d) Let $N_t$ or $N(t)$ be the population size at time t. Therefore

$$N_{t+x} = N_t e^{rx}$$

For

$$r = 0.01 \text{ and } x = 15$$

$$N_{t+15} = N_t e^{0.15}$$

The population over age 15 at time t + 15 is

$$N_{t+15} \times C(> 15) = N_t e^{0.15} \times (1 - C(< 15))$$

$$= N_t e^{0.15} \times (1 - 0.30861)$$

$$= N_t e^{0.15} \times 0.69139$$

$$= N_t \times 1.1618342 \times 0.69139$$

So Population over 15 at time t + 15 = $\underline{\underline{0.80328\ N_t}}$

Next, the population under 15 years are the survivors of births for the 15 years.

Suppose B(t) is the number of births at time t, then

$$\frac{B(t)}{N(t)} = b(t), \text{ birth rate at time } t.$$



Therefore

$$B(t) = b(t)\ N(t)$$

The population under 15 years

$$= B(t)\ p(15) + B(t+1)\ p(15-1) + \ldots + B_{t+x}\ p(15-x) + \ldots + B_{r+15}$$

In the continuous case, we have

$$\int_0^{15} B(t+x)\ p(15-x)\ dx$$

$$= \int_0^{15} B(t+x)dx, \text{ since } p(x) = 1, \quad 0 \leq x \leq 60$$

$$= \int_0^{15} b(t+x)\ N(t+x)dx$$

$$= \int_0^{15} b(t+x)\ N(t)\ e^{-rx}\ dx$$

But we are told that maternity rates decrease by 0.10 annually.
0.01

Thus

$$b(t + x) = b(t)\, e^{-0.10x} \qquad \left(b = \int_0^w m(a)\, c(a)da\right.$$

$$= m \int_0^w c(a)da$$

$$\left.= m\right).$$

Therefore, population under 15 is

$$\int_0^{15} b(t)\, N(t)\, dx = 15\, b(t)\, N(t)$$

since $r = 0.01$.

So the total population at time $t + 15$ is

$$N(t + 15) = 15\, b(t)\, N(t) + 0.80328\, N(t)$$

$$\therefore \quad \frac{N_{t+15}}{N_t} = \frac{N(t+15)}{N(t)}$$

$$= 15 \times 0.02215561 + 0.80328$$

$$= 0.33233415 + 0.80328$$

$$= \underline{\underline{1.1356142}}$$

PROBLEM 7

Suppose there is a female stable population with the following features:

The birth rate is 35.2 per thousand, the rate of increase is two percent per annum, and the proportion surviving to age 'a' is $1 - \frac{a}{100}$. As of January 1, 1976, fertility switches to a regime that maintains a constant stream of births. The mortality schedule is unaltered.

(a) What is the expectation of life at birth?

(b) How long will it take for the population to become exactly stationary?

(c) How large is the ultimate stationary population relative to the population at mid-night, December 31, 1975?

(d) If the proportion under age 34.5 years in the the initial stable population is 74.8 percent, how large is the population (with constant births) on June 30, 2010 relative to the population at midnight, December 31, 1975?

(e) What proportion is under 34.5 years in the stationary population?

(f) Show that the proportion under age 34.5 in the stable population is 74.8 percent, and that, on the basis of other data provided, the birth rate must be 35.2.

<u>Solution to problem 7</u>

We are given that

$$b = 0.0352$$
$$r = 0.02$$

Births = B is constant as of Jan. 1, 1976

$$p(a) = 1 - \frac{a}{100}, \quad 0 \le a \le 100$$

and zero otherwise

(a) $$e_o^o = \frac{T_o}{\ell_o} = \frac{\int_0^{100} \ell(a)\, da}{\ell_o}$$

$$= \int_0^{100} \frac{\ell(a)}{\ell(o)}\, da$$

$$= \int_0^{100} p(a)da$$

$$\doteq \int_0^{100} (1 - \frac{a}{100})da$$

$$= \left[ a - \frac{a^2}{200} \right]_0^{100}$$

$$= 50.$$

Thus the expectation of life at birth is 50 years.

(b) It will take 100 years for the stable population to become stationary.

(c) Let the number of births in Dec. 31, 1975 be the same as that of January 1st, 1976 = B, say.

Then

$$b(\text{stable}) = \frac{B}{Pop(\text{stable})} = \frac{B}{P_s}$$

$$b(\text{stationary}) = \frac{B}{Pop(\text{stationary})} = \frac{B}{T_o}.$$

Therefore

$$\frac{T_o}{P_s} = \frac{b(\text{stable})}{b(\text{stationary})}$$

$$= \frac{0.0352}{1/e_o^o}$$

$$= 0.0352 \times 50$$

$$= \underline{\underline{1.76}}$$

(d) June 30, 2010 - Dec 31, 1975 = 34.5 years.
We can group the population in the year 2010 into those under 34.5 years (with constant birth) and population over 34.5 (with original stable Pop.)

$$\begin{aligned}
Pop_{2010}\ (> 34.5) &= Pop_{1975}\ (> 34.5) \times e^{rt} \\
&= Pop_{1975}\ (> 34.5) \times e^{0.02 \times 35.5} \\
&= Pop_{1975}\ (> 34.5) \times e^{0.69} \\
&= 2\ Pop_{1975}\ (> 34.5) \\
&= 2(1 - 0.748)\ Pop_{1975} \\
&= 2 \times 0.252\ Pop_{1975} \\
&= 0.504 \times \left(\frac{\text{Births}}{\text{birth rate}}\right)_{1975}
\end{aligned}$$

i.e.,

$$Pop_{2010}\ (> 34.5) = 0.504 \times \frac{B}{0.0362} \qquad *$$

Next

$$\begin{aligned}
Pop_{2010}\ (< 34.5) &= \int_0^{34.5} B \times p(a)da \\
&= B \int_0^{34.5} \left(1 - \frac{a}{100}\right) da \\
&= B \left[a - \frac{a^2}{200}\right]_0^{34.5} \\
&= 34.5\ B \left(1 - \frac{34.5}{200}\right) \\
&= 34.5B\ (1 - 0.1725) \\
&= 34.5 \times 0.8275B \qquad **
\end{aligned}$$



Therefore the total population in the year 2010 is given by summing up * and **.

i.e.,

$$Pop\ (2010) = \frac{0.5040}{0.0352} + 34.5 \times 0.8275)B$$

$$= 42.866932\ B$$

$$\simeq 42.87\ B$$

But

$$Pop\ (1975) = \frac{B}{0.0352}$$

Therefore

$$\frac{Pop\ (2010)}{Pop\ (1975)} = \frac{42.87B}{B/0.0352}$$

$$= 0.0352 \times 42.87$$

$$= \underline{\underline{1.509}}$$

(e) In a stationary population,

$$C(a) = b\ p(a) = \frac{1}{e_o^o}\ p(a)$$

Therefore the proportion under 34.5 years is

$$\int_o^{34.5} c(a)da = \frac{1}{e_o^o} \int_o^{34.5} p(a)da$$

$$= \frac{1}{50} \int_o^{34.5} (1 - \frac{a}{100})da$$

$$= \frac{34.5}{50} \times 0.8275$$

$$= \frac{28.54875}{50}$$

$$= 0.570975$$

$$= \underline{0.571}$$

Problem 8

(a) In a stationary population half of the deaths each year occur after exact age 69.5; the average age at death of those who die above 69.5 is 80 years, and of those who die at less than 69.5, 40 years.

(i) What is the birth rate of the stationary population?

(ii) What proportion is over 69.5?

(iii) What is the death rate of persons under 69.5?

(iv) In a stable population with the same mortality schedule and a growth rate of one per cent, the birth rate is 0.0233. What fraction of this stable population is between exact ages 69 and 70?

(b) A hypothetical stable population has an age distribution rising linearly from a value of 0.016 at age zero to 0.024 at age 50, at which exact age the death rate is infinite. Find an expression for p(a) as a function of age and the intrinsic rate of increase. What is the largest value of r compatible with the specified stable age distribution? (Hint : what is a necessary limit for the death rate at any age, and what is an expression for the slope of the stable age distribution?)

Solution : (a)

(i) $$b = \frac{1}{e_o^o}$$

But

$$e_o^o = \frac{T_o}{\ell_o} = \frac{69.5\ L_o + T_{69.5}}{\ell_o}.$$



If 'a' is the average age (length) at death for those who die between ages x and x+n, then ...

$$ {}_nL_x = a(\ell_x - \ell_{x+n}) + n\,\ell_{x+n} $$

$$ = a\,\ell_x + (n-a)\ell_{x+n} $$

Therefore

$$ {}_{69.5}L_o = 40\,\ell_o + (69.5 - 40)\ell_{69.5} $$

$$ = 40\,\ell_o + 29.5\,\frac{\ell_o}{2} $$

$$ = \frac{109.5}{2}\,\ell_o $$

$$ = 54.75\,\ell_o $$

Next

$$ {}_\omega L_{69.5} = T_{69.5} = (80 - 69.5)\,\ell_{69.5} + (\omega - a).0 $$

i.e.

$$ T_{69.5} = 10.5\,\frac{\ell_o}{2} = 5.25\,\ell_o $$

Alternatively

$$ T_{69.5} = e^o_{69.5}\,\ell_{69.5} $$



$$ = 10.5\,\frac{\ell_o}{2} = 5.25\,\ell_o $$

$$ \therefore \quad e^o_o = \frac{T_o}{\ell_o} = (54.75 + 5.25)\,\frac{\ell_o}{\ell_o} $$

$$ = 60. $$

$$ \therefore \quad b = \frac{1}{60} = \underline{\underline{0.01667}} $$

(ii) Proportion over 69.5 $= \dfrac{T_{69.5}}{T_o}$

$$ = \frac{5.25\,\ell_o}{60\,\ell_o} $$

$$ = 0.0875 $$

$$ = \underline{87.5\%} $$

(iii) Death rate under 69.5

$$= \frac{\text{Deaths } (< 69.5)}{\text{NO. of person - years lived} < 69.5}$$

$$= \frac{0.5\ \ell_o}{69.5 L_o}$$

$$= \frac{0.5\ \ell_o}{54.75\ \ell_o}$$

$$= \frac{1}{109.50}$$

$$= 0.009132$$

(iv) $r = 0.01$ and $b = 0.0233$

We shall assume the age between exact ages 69 and 70 to be 69.5.

Therefore using the formula

$$C(a) = b\ e^{-ra}\ p(a)$$

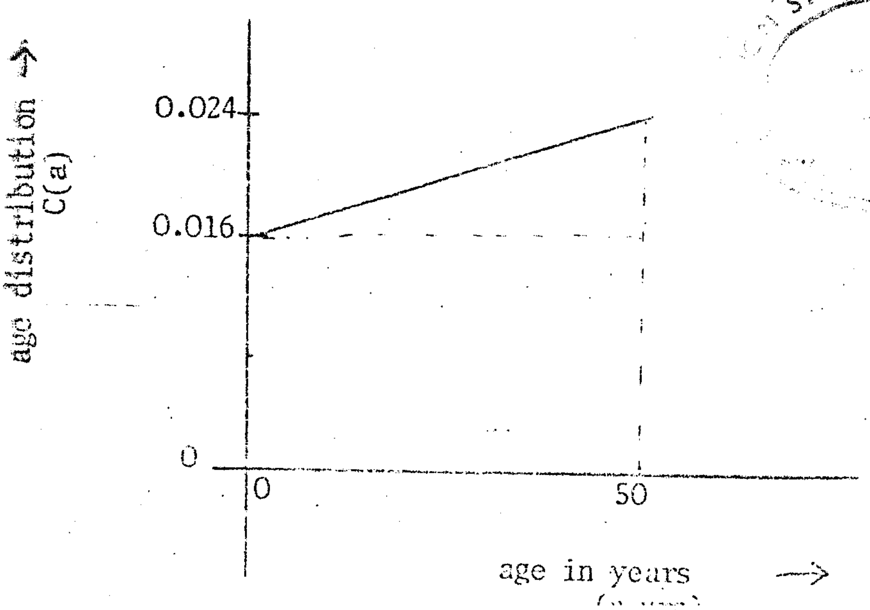
$$C(69.5) = 0.0233\ e^{-0.01 \times 69.5}\ p(69.5)$$

$$= 0.0233 \times \frac{1}{2} \times \frac{1}{2}$$

$$= \underline{\underline{0.005825}}$$



Solution : part (b)



From the diagram,

$$C(a) = 0.016 + \left(\frac{0.024 - 0.016}{50}\right)a$$

$$= 0.016 + 0.00016a \ , \ 0 \leq a \leq 50$$

In a stable population,

$$C(a) = b \, e^{-ra} \, p(a)$$

which implies that

$$p(a) = \frac{C(a)}{b} \, e^{ra}$$

But

$$b = C(o) = 0.016$$

Therefore

$$p(a) = \frac{C(a)}{0.016} \, e^{ra}$$

$$= (0.016 + 0.00016 \, a) \, \frac{e^{ra}}{0.016}$$

$$= (1 + 0.01 \, a) \, e^{ra}$$

We should recall that p(a) is the probability of survival. It is a non-increasing function of age.

Thus

$$\frac{d}{da} \, p(a) \leq 0$$

i.e.,

$$\frac{d}{da}\left[(1 + 0.01 \, a) \, e^{ra}\right] \leq 0$$

i.e.,

$$r \, e^{ra} \, (1 + 0.01 \, a) + 0.01 \, e^{ra} \leq 0$$

i.e.,

$$r(1 + 0.01 \, a) \leq -0.01$$

This implies

$$r \leq - \frac{0.01}{1 + 0.01a}$$

r will be greatest when a = o.

Putting

$a = o$ i.e. no mortality

$r < - 0.01.$

Problem 9

In a stationary population in which $e_o^o$ = 65 years, the proportion at ages 30 - 34 is 0.06742 and at 35-39, is 0.06638. In a stable population incorporating the same mortality schedule, the corresponding proportions are 0.06903 and 0.06477.

(a) The mean age of the stationary population is 37.5 years. What is the mean age of the stable population?

(b) If you had a table of exponentials ($e^x$ for different values of x), how would you calculate the rate of increase in the stable population? The birth rate?

Solution

For stationary population.

$C(30 - 34) = {}_5C_{30} = 0.06742$

$C(35-39) = {}_5C_{35} = \underline{0.06638}$.

So

$C(30-39) = {}_{10}C_{30} = 0.13380$

For the stable population,

$C(30-34) = {}_5C_{30} = 0.06903$

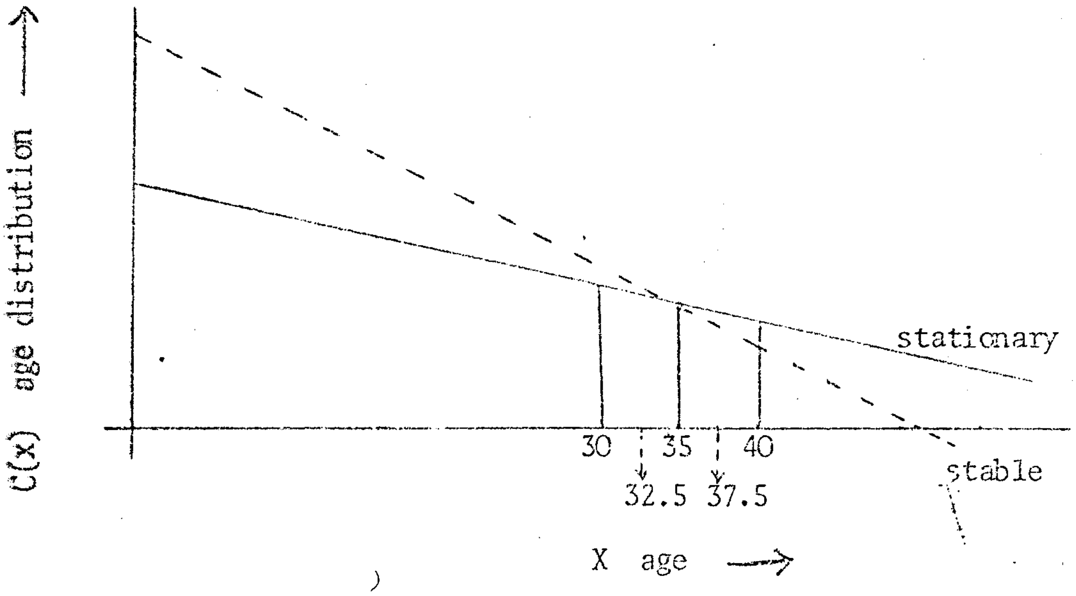
$C(35-39) = {}_5C_{35} = 0.06477$

So

$$C(30 - 39) = {}_{10}C_{30} = 0.13380$$

Assuming linearity in the age distributions between ages 30 - 39,

$$C(35)_{stat} = C(35)_{stable}.$$




Thus 35 years is the mean of the mean ages of a stable and a stationary population.

If $\bar{a}_o$ is the mean age of the stationary population and $\bar{a}_1$ is the mean age of the stable population, then

$$\frac{1}{2}(\bar{a}_o + \bar{a}_1) = 35$$

i.e.,

$$\bar{a}_o + \bar{a}_1 = 70$$

which implies that

$$\bar{a}_1 = 70 - \bar{a}_o$$
$$= 70 - 37.5$$
$$= \underline{\underline{32.5}}$$

(b) From stationary data,

$$\frac{{}_5C_{35}}{{}_5C_{30}} = \frac{{}_5L_{35}/\ell_o}{{}_5L_{30}/\ell_o}$$

This implies

$$\frac{{}_5L_{35}}{{}_5L_{30}} = \frac{0.06638}{0.06742} \qquad (1)$$

From stable population,

$$\frac{{}_5C_{35}}{{}_5C_{30}} = \frac{b\ e^{-37.5r}\ {}_5L_{35}/\ell_o}{b\ e^{-32.5r}\ {}_5L_{30}/\ell_o}$$

$$= e^{-5r}\ \frac{{}_5L_{35}}{{}_5L_{30}}$$

i.e.,

$$\frac{0.06477}{0.06903} = e^{-5r}\ \frac{{}_5L_{35}}{{}_5L_{30}} \qquad (2)$$

Substituting (1) in (2),

$$\frac{0.06477}{0.06903} = e^{-5r} \times \frac{0.06638}{0.06742}$$

i.e.,

$$r = \frac{1}{5}\ \ln\left(\frac{6903 \times 6638}{6477 \times 6742}\right)$$

$$= 0.00963055$$

So

$$r \simeq 0.0096$$

To determine birth rate, consider

$$\frac{{}_5C_{30}\ \text{(stationary)}}{{}_5C_{30}\ \text{(stable)}} = \frac{0.06742}{0.06903} = \frac{b_1\ ({}_5L_{30}/\ell_o)}{b_2\ ({}_5L_{30}/\ell_o) \times e^{-32.5r}}$$

i.e.,

$$\frac{6742}{6903} = \frac{b_1}{b_2\, e^{-32.5r}}$$

But

$$b_1 = \frac{1}{e_o^o} = \frac{1}{65}$$

Therefore

$$b_2 = \frac{6903}{6742} \times \frac{b_1}{e^{-32.5r}}$$

i.e.,

$$b_2 = \frac{6903}{6742} \times \frac{e^{32.5 \times .0096}}{65}$$

$$= \underline{\underline{0.0215}}$$



Problem 10

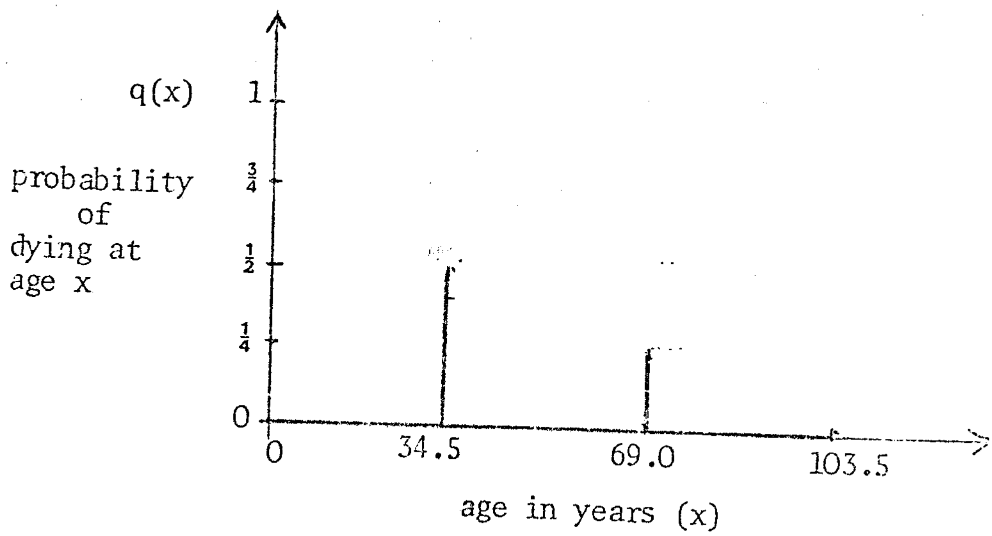
(a) Imagine a female population in which half of those reaching exact age 34.5 die at that instant; half reaching exact age 69 die then; and all survivors die on reaching age 103.5. There are no other deaths. Construct a graph of the $_1q_x$ schedule, and another of $\ell_x$ function of the life table. What is the expectation of life at birth? At age 40? What is the birth rate of the stationary population?

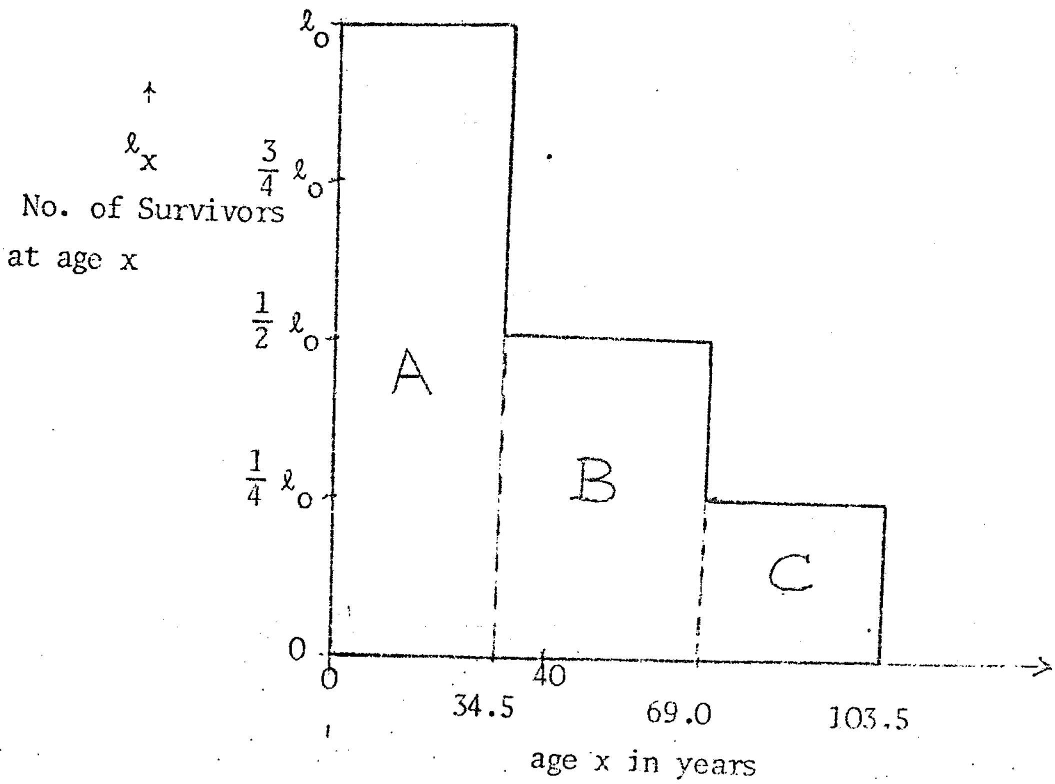
(b) Now suppose that one-half the women in this population bear a child between the ages of 33.5 and 34.5; that all women 34.5 to 35.5 bear a child during this year of their lives, and that exactly one half of the births are female. What is the growth rate of the stable population with these schedules of fertility and mortality?

(c) Construct a graph of the age distribution of the stable population described in (b). What is the birth rate?

What fraction of the population annually dies at age 103.5?
(Hint : Determine the ratio of annual births to annual deaths!)

Solution: Part (a)



$$e_o^o = \frac{T_o}{\ell_o} = \frac{\text{Area A} + \text{Area B} + \text{Area C}}{\ell_o}$$

$$= \frac{1}{\ell_o}\left(34.5\ \ell_o + 34.5\ \frac{\ell_o}{2} + 34.5\ \frac{\ell_o}{4}\right)$$

$$= 34.5\ \left(1 + \frac{1}{2} + \frac{1}{4}\right)$$

i.e.,

$$e_o^o = \frac{7}{4} \times 34.5$$

$$= \underline{\underline{60.375}}$$

$$e_{40}^o = \frac{(69-40)\ \frac{\ell_o}{2} + \text{Area C}}{\ell_{40}}$$

$$= \frac{29\ \frac{\ell_o}{2} + 34.5 \times \frac{\ell_o}{4}}{\frac{\ell_o}{2}}$$

$$= 29 + \frac{34.5}{2}$$

$$= 29 + 17.25$$

$$= \underline{\underline{46.25}}$$

$$b = \frac{1}{e_o^o} = \frac{1}{60.375} = 0.01656315$$

i.e.

$$b \simeq \underline{\underline{0.0166}}$$



Part (b)

Let f(a) be the age specific fertility rate and m(a) be the female births per woman per year.

Thus

$$f(34) = \frac{\frac{1}{2}\ell_o}{\ell_o} = \frac{1}{2}$$

$$f(35) = \frac{\frac{1}{2}\ell_o}{\frac{1}{2}\ell_o} = 1$$

$$m(34) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

and

$$m(35) = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$NRR = \int_{\alpha}^{\beta} p(a)\ m(a)da$$

$$= p(34)m(34) + p(35)m(35)$$

$$= 1. \ \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

Using the formula

$$NRR = e^{rT}$$

implies that

$$r = \frac{\ln NRR}{T}$$

where T is the mean length of generation. In this case

$$T \simeq 34.5$$

Therefore

$$r = \frac{\ln 0.5}{34.5} = -\underline{0.02}$$

<u>Part (c)</u>

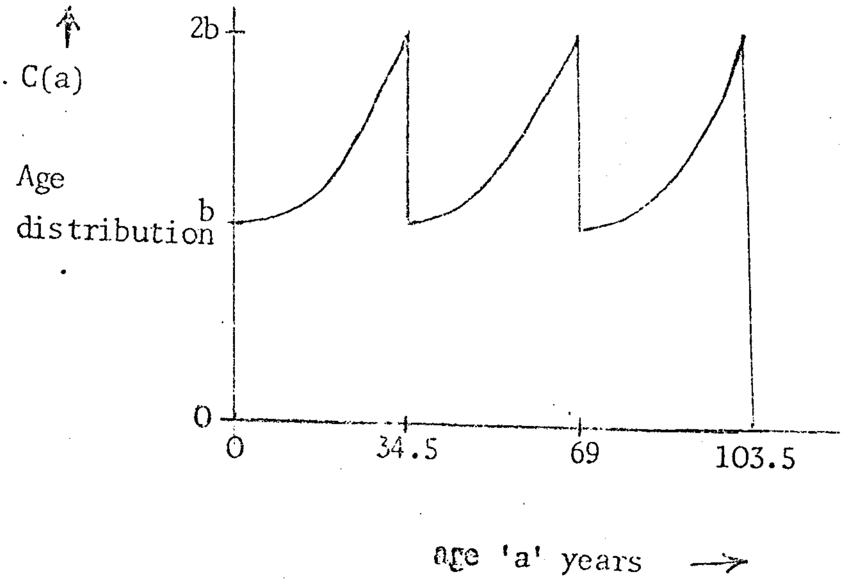
$$C(a) = b\ e^{-ra}\ p(a)$$

$$= b\ e^{0.02a}\ p(a)$$

So

$$C(a) = \begin{cases} b\, e^{0.02a} & , \quad 0 \leq a \leq 34.5 \\ \dfrac{b}{2}\, e^{0.02a} & , \quad 34.5 < a \leq 69 \\ \dfrac{b}{4}\, e^{0.02a} & , \quad 69 < a \leq 103.5 \end{cases}$$

zero elsewhere.




Deaths occur only at exact ages 34.5, 69 and 103.5 years. Therefore the total death rate is given by

$$\begin{aligned} d &= c(34.5) + c(69) + c(103.5) \\ &= b + b + 2b \\ &= 4b \end{aligned}$$

But

$$b - d = r$$

i.e.,

$$b - 4b = r$$

i.e.,

$$b - 4b = -\,0.02$$

i.e.,

$$b = \frac{0.02}{3} = \underline{\underline{0.00667}}$$

Alternatively, use

$$b = \frac{1}{\int_o^W e^{-ra}\, p(a)da}$$

where

$$\int_o^W e^{-ra}\, p(a)da = \int_o^{34.5} e^{-ra}\, da + \int_{34.5}^{69} \frac{1}{2}\, e^{-ra}\, da + \int_{69}^{103.5} \frac{1}{4}\, e^{-ra} da$$

$$= -\frac{1}{r}\left[\frac{e^{-34.5r}}{2} - 1 + \frac{e^{-69r}}{4} + \frac{e^{-103.5r}}{4}\right]$$

$$= \frac{1}{0.02}\left[\frac{e^{0.69}}{2} - 1 + \frac{e^{(0.69)2}}{4} + \frac{e^{(0.69)3}}{4}\right]$$

$$= \frac{1}{0.02}\ (1 - 1 + \frac{2^2}{4} + \frac{2^3}{4})$$

$$= \frac{1}{0.02}\ (1 + 2) = \frac{3}{0.02}$$

Therefore

$$b = \frac{0.02}{3} = \underline{\underline{0.0067}}$$

The proportion dying at age 103.5 = C(103.5)

i.e.

$$C(103.5) = \frac{1}{4}\ b\ e^{(0.02)(103.5)}$$

$$= \frac{1}{4}\ b\ e^{0.69 \times 3}$$

$$= \frac{1}{4}\ b\ .\ 2^3$$

$$= 2b$$

$$= 2 \times 0.00667$$

$$= \underline{\underline{0.01334}}$$

* Problem 11.

Consider three hypothetical mortality schedules:

(1) There are no deaths before exact age 100 and all persons die exactly 100 years after birth.

(2) The deaths experienced by each cohort are perfectly evenly distributed over the range zero to exact age 100; and there are no survivors beyond 100.

(3) Age specific mortality rates are the same at every age from zero to infinity, and $e_o^o$ is 50 years.

(a) Determine $e_o^o$ and the mean age of the stationary population for each of these three mortality schedules.

(b) In a stable population with mortality schedule (1) and $r = 0.0069$, what is the birth rate? What is the mean age? (Hint: in this stable population what is the relation between c(100) and the death rate?)

(c) If $\ell_o$ is set at 100,000, what is the aoproximate (or exact if you will) value of $\ell_{70}$ in each life table? What is ${}_1q_{20}$ in each?

Solution: Part (a)

In a stationary population,

$$e_o^o = \frac{T_o}{\ell_o}$$

and

$$c(a) = \frac{\ell_a}{T_o}$$

where

$$T_o = \int_o^w \ell(a)da$$

The mean age is given by

$$E(a) = \int_0^{w} a\, c(a)\, da$$

<u>For the case of mortality schedule (1)</u>

$$\ell(a) = \ell_o \;, \quad 0 \leq a \leq 100, \text{ zero elsewhere}$$

Therefore

$$T_o = \int_0^{100} \ell_o \, da = 100\, \ell_o$$

Therefore

$$e_o^o = \underline{\underline{\frac{1}{100}}}$$

From $e_x^o = \frac{T_x}{l_x}$, $e_o^o = \frac{T_o}{l_o} = \frac{100 l_o}{l_o} = 1$

$$C(a) = \frac{\ell_o}{100\, \ell_o} = \frac{1}{100}$$

Therefore

$$E(a) = \int_0^{100} a \frac{1}{100}\, da$$

$$= \frac{1}{100} \left[ \frac{a^2}{2} \right]_0^{100}$$

$$= \underline{\underline{50}}$$

<u>For mortality schedule (2)</u> ,

$$\ell(a) = \ell_o \left(1 - \frac{a}{100}\right) , \quad 0 \leq a \leq 100$$

zero otherwise.

Therefore

$$T_o = \ell_o \int_0^{100} \left(1 - \frac{a}{100}\right) da$$

$$= \ell_o \left[ a - \frac{1}{2} \frac{a^2}{100} \right]_0^{100}$$

$$= 50\, \ell_o$$

Therefore

$$e_o^o = \frac{T_o}{\ell_o} = \underline{\underline{50}}$$

$$C(a) = \frac{\ell(a)}{T_o} = \frac{1}{50}\left(1 - \frac{a}{100}\right)$$

Therefore

$$E(a) = \int_o^{100} a\ c(a)da$$

$$= \int_o^{100} \frac{a}{50}\left(1 - \frac{a}{100}\right)da$$

$$= \int_o^{100} \left(\frac{a}{50} - \frac{a^2}{5000}\right)da$$

$$= \left[\frac{a^2}{100} - \frac{a^3}{15000}\right]_0^{100}$$

$$= \left[\frac{a^2}{100}\left(1 - \frac{a}{150}\right)\right]_0^{100}$$

$$= \frac{100^2}{100}\left(1 - \frac{100}{150}\right)$$

$$= 100\left(1 - \frac{2}{3}\right)$$

$$= \frac{100}{3}$$

<u>For mortality schedule (3)</u>

$$p(a) = e^{-\int_o^a \mu(x)dx}$$

i.e.,

$$\ell(a) = \ell_o\, e^{-\int_o^a \mu(x)dx}$$

If

$$\mu(x) = \mu \quad \text{(constant)},$$

then

$$\ell(a) = \ell_o e^{-\mu a}, \quad a \geq o$$

Next

$$C(a) = \frac{\ell(a)}{T_o}$$

$$= \frac{\ell_o e^{-\mu a}}{\int_o^\infty \ell_o e^{-\mu a} da}$$

$$= \mu e^{-\mu a}$$

$$E(a) = \int_o^\infty a\mu e^{-\mu a} da$$

$$= \frac{1}{\mu}$$

We are given that

$$e_o^o = 50$$

i.e.

$$\frac{T_o}{\ell_o} = 50$$

i.e.,

$$\frac{1}{\mu} = 50$$

which implies that

$$\mu = \frac{1}{50} = 0.02$$

$$E(a) = \frac{1}{\mu} = \underline{\underline{50}}$$

<u>Solution :</u> Part (b)

In general,

$$b = \frac{1}{\int_0^W e^{-ra} p(a)da}$$

$$c(a) = b\, e^{-ra}\, p(a)$$

and

$$E(a) = \int_0^W a\, c(a)da$$

$$= b \int_0^W a\, e^{-ra}\, p(a)da$$

$$= \frac{\int_0^W a\, e^{-ra}\, p(a)da}{\int_0^W e^{-ra}\, p(a)da}$$

In this particular problem, it is easier to use the fact that the death rate

$$d = c(100)$$

$$= b\, e^{-(0.0069)100} \times 1$$

$$= b\, e^{-0.69}$$

$$= \frac{1}{2} b$$

Also

$$d = b - r$$

This implies that

$$\frac{1}{2} b = b - r$$

i.e.,

$$\frac{1}{2} b = r$$

i.e.,

$$b = 2r$$

$$= 2 \times 0.0069$$

$$= \underline{\underline{0.0138}}$$

Next, the mean age is given by

$$E(a) = b \int_0^{100} a\, e^{-ra}\, p(a)da$$

$$= b \int_0^{100} a\, e^{-ra}\, da$$

Integrating by parts,

$$\int_0^w a\, e^{-ra}\, da = -\frac{w\, e^{-rw}}{r} - \frac{1}{r^2}\left(e^{-rw} - 1\right)$$

Therefore

$$E(a) = 0.0138 \left[ -\frac{100\, e^{-0.69}}{0.0069} - \frac{1}{(0.0069)^2}\left(e^{-0.69} - 1\right)\right]$$

$$= -100 + \frac{1}{0.0069}$$

$$= -100 + 144.93$$

$$= 44.93$$

**Solution :** Part (c)

**(i)** $\ell_a = \ell_o \;, \; 0 \le a \le 100 \;.$

Therefore

$$\ell_{70} = \ell_o = \underline{\underline{100,000}} \;,$$

**(ii)** $\ell_a = \ell_o \left(1 - \frac{a}{100}\right) \;, \; 0 \le a \le 100$

Therefore

$$\ell_{70} = \ell_o \left(1 - \frac{70}{100}\right)$$

$$= \frac{30}{100}\, \ell_o$$

$$= \frac{30}{100} \times 100,000$$

$$= \underline{\underline{30,000}}$$

(iii) $\ell_a = \ell_o e^{-\mu a}, \; a \geq 0$

For

$$\mu = 0.02$$

and

$$a = 70$$

$$\ell_{70} = \ell_o e^{-0.02 \times 70}$$

$$= 100,000 \times e^{-0.14}$$

$$\approx \underline{\underline{24,660}}$$

By definition

$$_1q_{20} = 1 - {_1p_{20}}$$

i.e.,

$$_1q_{20} = 1 - \frac{\ell_{21}}{\ell_{20}}$$

(i) $_1q_{20} = 1 - \frac{1}{1} = 0$

(ii) $_1q_{20} = 1 - \frac{\ell_o(1 - \frac{21}{100})}{\ell_o(1 - \frac{20}{100})}$

$$= 1 - \frac{79}{80}$$

$$= \frac{1}{80}$$

$$= \underline{0.0125}$$

(iii) $_1q_{20} = 1 - \frac{\ell_o e^{-21\mu}}{\ell_o e^{-20\mu}}$

$$= 1 - e^{-\mu}$$

$$= 1 - e^{-0.02}$$

$$= 1 - 0.9801$$

$$= \underline{\underline{0.0198}}$$

Problem 12

(A) Suppose there is a stationary population (no migration, constant fertility and mortality, rate of increase zero), in which the death rate of persons over 65 is 70 per thousand; over 75 the death rate is 120 per thousand. There are 150,000 persons reaching age 65 each year in this population, and 104,000 reaching age 75.

(1) What is the expectation of life at age 75 in this population?

(2) How many persons are there between 65 and 75?

(3) What fraction of the persons who reach 65 also attain age 75?

(4) What fraction of those over 65 at a given moment are still alive 10 years later?

(B) In a given life table, $\ell_0 = 100,000$, ${}_1L_{30} = 84,000$, and the population of persons surviving from exact age 10 to exact age 40.5 is 0.90. In a particular stable population experiencing this mortality, the ratio of persons between exact ages 40 and 41 to those between ages 9.5 and 10.5 is 0.60. The birth rate in this stable population is 24 per thousand. What is the proportion whose age (last birthday) was 30 years in the stable population? (Hint: In a stable population how does the ratio of persons 40-41 to persons 9.5 to 10.5 depend on r?)

(C) What is implied about the age composition of a female population if:

(1) The death rate in the stable population based on current fertility and mortality schedules) is lower than in the actual population.

(2) The ratio of births to women aged 15-44 in the stable population is lower than in the actual population.

(3) The death rate over 65 is less than $\frac{1}{e^o_{65}}$.

Solution:

A) $d(65^+) = 0.070$

$d(75^+) = 0.120$

$_1L_{65} = \ell_{65.5} = 150{,}000$

$_1L_{75} = \ell_{75.5} = 104{,}000$

1) $e^o_{75} = \frac{1}{d(75^+)} = \frac{1}{0.120} = \underline{\underline{8.33}}$

2) $_{10}L_{65} = T_{65} - T_{75}$

$= e^o_{65}\ \ell_{65} - e^o_{75}\ \ell_{75}$

$= \frac{\ell_{65}}{d(65)} - \frac{\ell_{75}}{d(75)}$

$= \frac{150{,}000}{0.07} - \frac{104{,}000}{0.12}$

$= 2143500 - 866320$

$= 1{,}277{,}180$

3) $$\frac{{}_1L_{75}}{{}_1L_{65}} = \frac{104,000}{150,000}$$

$$= \frac{104}{150}$$

$$= 0.6933$$

$$= 69.33\%$$

4) $$\frac{T_{75}}{T_{65}} = \frac{866320}{2,143,500}$$

$$= 0.4041$$

$$= 40.41\%$$

B) $$\frac{C(40 - 41)}{C(9.5 - 10.5)} = 0.60$$

**i.e.,**

$$\frac{c(40.5)}{c(10)} = 0.60$$

**i.e.,**

$$\frac{b\ e^{-40.5r}\ p(40.5)}{b\ e^{-10r}\ p(10)} = 0.60$$

**or**

$$\frac{b\ e^{-40.5r}\ {}_1L_{40}}{b\ e^{-10r}\ {}_1L_{9.5}} = 0.60$$

**i.e.,**

$$\frac{e^{-40.5r}}{e^{-10r}} \times 0.90 = 0.60$$

This implies

$$e^{-30.5r} = \frac{0.60}{0.90}$$

$$= \frac{2}{3}$$

Next,

$$c(30.5) = b\, e^{-30.5r}\, p(30.5)$$

$$= b\, e^{-30.5r} \times \frac{{}_1L_{30}}{\ell_0}$$

$$= 0.024 \times \frac{2}{3} \times 0.84$$

$$= 0.01344$$

$$= \underline{\underline{1.34\%}}$$



Problem 13

(A) Suppose mortality is constant, and the following values are from the life table:

| | | | |
|---|---|---|---|
| $\ell_0$ | = 100,000 | ${}_1L_{20}$ | = 71,600 |
| $\ell_{10}$ | = 75,000 | ${}_5L_0$ | = 410,000 |
| ${}_1L_{10}$ | = 74,850 | ${}_5L_5$ | = 380,000 |
| ${}_1L_{11}$ | = 74,530 | ${}_5L_{10}$ | = 370,000 |
| ${}_1L_{12}$ | = 74,210 | ${}_5L_{15}$ | = 361,000 |

A census taken on January 1, 1968, gives the following data:

| | |
|---|---|
| Population under exact age 5 | = 1,200,000 |
| Population under exact age 10 | = 1,900,000 |
| Persons aged 10 last birthday | = 120,000 |
| Persons aged 11 last birthday | = 110,000 |

Determine:

(a) The number of births in the calendar year 1957

(b) The population aged 20 (last birthday) on January, 1, 1978

(c) The population at ages 15 - 19 (last birthday) on January 1, 1978.

(B) Suppose there is a stable population, subject to a life table with these values:

$$e_0^o = 60.4 \text{ yrs.} \qquad e_5^o = 62 \text{ yrs.}$$

$$\ell_0 = 100{,}000 \qquad \ell_{65} = 60{,}000$$

$$\ell_5 = 90{,}000 \qquad T_{65} = 750{,}000$$

(a) The proportion of females under 5 years is 0.065. Is the population growing, stationary, or shrinking? How do you know?

(b) Give a valid numerical limit (upper or lower, as evidence permits) for the proportion over 65 in this stable population.

(c) Give a numerical limit (again upper or lower as warranted by the data) for the death rate of persons over 65.

(C) Assuming the population in B to be non-stationary and given that the mean of the mean ages of the stable and stationary population is 37.25 years, show how you could calculate the birth rate in the stable population.

Solution:

Part A.

(a) Persons born in 1957 are 10 years old by January 1, 1968 From the life table, $1^{L}10 = 74,850$. This means that out of 100,000 births, 74,850 are having their tenth birthday. The problem is therefore to find how many births there were (10 years ago) if there are 120,000 persons celebrating their 10th birthday.

Let x be the required number of births. Therefore

$$\frac{1^{L}10}{\ell_0} = \frac{120{,}000}{x}$$

This implies that

$$x = 120,000 \; \frac{\ell_0}{{}_1L_{10}}$$

$$= 120,000 \times \frac{100,000}{74,850}$$

$$= \underline{\underline{160,321}}$$

(b) Let y be the population aged 20 (last birthday) on January 1, 1978. Then

$$\frac{y}{120,000} = \frac{{}_1L_{20}}{{}_1L_{10}}$$

i.e.,

$$y = 120,000 \times \frac{{}_1L_{20}}{{}_1L_{10}}$$

$$= 120,000 \times \frac{71,600}{74,850}$$

$$= \underline{\underline{114790}}$$



(c) Let Z be the population at ages 15-19 (last birthday) on January 1, 1978.

Therefore

$$\frac{Z}{\left[\text{Pop} < 10 - \text{Pop} < 5\right]_{1968}} = \frac{{}_5L_{15}}{{}_5L_5}$$

i.e.,

$$\frac{Z}{1,900,000 - 1,200,000} = \frac{361,000}{380,000}$$

i.e.,

$$\frac{Z}{700,000} = \frac{361}{380}$$

Therefore

$$Z = \frac{361}{380} \times 700{,}000$$

$$\therefore = \underline{\underline{665{,}000}}$$

<u>Part B</u>

(a) In a stable population,

$$C(< 5) = 0.065$$

In a stationary population,

$$c(< 5) = \frac{{}_5L_0}{T_o}$$

$$= \frac{T_o - T_5}{T_o}$$

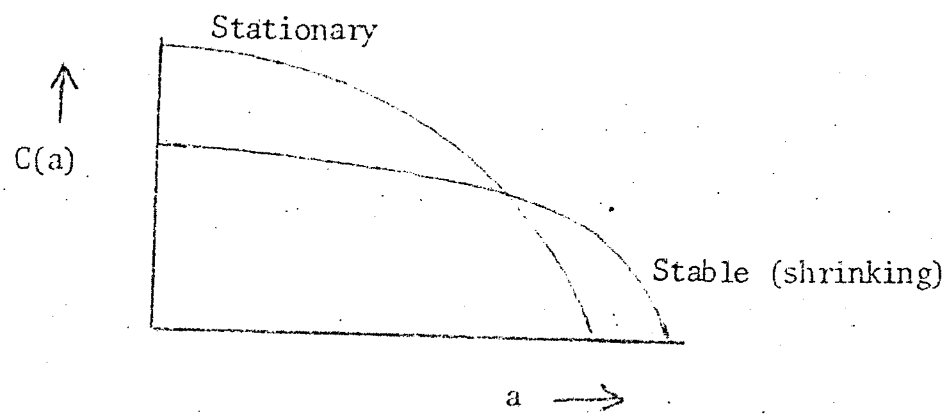
$$= 1 - \frac{T_5}{T_o}$$

$$= 1 - \frac{e^o_5\ \ell_5}{e^o_o\ \ell_o}$$

$$= 1 - \frac{62 \times 90{,}000}{60.4 \times 100{,}000}$$

$$= 1 - \frac{62 \times 9}{604}$$

$$= \underline{\underline{0.0762}}$$



Since

$$0.0762 > 0.065,$$

we have a shrinking stable population.

(b) In a shrinking stable population, proportion over 65 is greater than that in a stationary population.

That is

$$C(65^+) \text{ shrinking stable} > C(65^+) \text{ stationary}$$

$$= \frac{T_{65}}{T_o}$$

$$= \frac{750{,}000}{6{,}040{,}000}$$

$$= 0.1242$$



So 0.1242 is the lower limit for the proportion over 65 in this shrinking stable population.

(c) Death rate in a shrinking stable population is greater than the death rate in a stationary population.
This implies that

$$(\text{death rate } 65^+) \text{ shrinking stable} > \text{death rate } 65^+ \text{ stationary}$$

$$= \frac{1}{e^o_{65}}$$

$$= \frac{\ell_{65}}{T_{65}}$$

$$= \frac{60{,}000}{750{,}000}$$

$$= 0.08$$

Thus 0.08 is the lower limit for the death rate over 65.

<u>Part C:</u>

Since 37.25 years is the mean of mean ages,

then

$$\frac{C_1\ (37.25)}{C_2\ (37.25)} = 1$$

where $C_1$ is the proportion for stationary and $C_2$ for the non-stationary.

This implies

$$\frac{b_1}{b_2\, e^{-37.25r}} = 1$$

i.e.,

$$b_2 = b_1\, e^{37.25r} \qquad (1)$$

Next,

$$\frac{C_1\ (< 5)}{C_2\ (< 5)} = \frac{0.0762}{0.0650}$$

i.e.,

$$\frac{b_1}{b_2\, e^{-2.5r}} = \frac{0.0762}{0.0650}$$

Note: $C(< 5) = \int_0^5 b\, e^{-ra}\, p(a)da \simeq b\, e^{-2.5r}\, p(2.5)$

Therefore

$$b_2 = \frac{0.0650}{0.0762}\, b_1\, e^{2.5r} \qquad (2)$$

From (1) and (2)

$$e^{37.25r} = \frac{0.0650}{0.0762}\, e^{2.5r}$$

i.e.,

$$e^{34.75r} = \frac{0.0650}{0.0762}$$

i.e.,

$$r = \frac{1}{34.75} \ln \left( \frac{0.0650}{0.0762} \right)$$

$$= \frac{1}{34.75} \ln 0.8530$$

$$= - 0.0045748$$

Therefore

$$b_2 = b_1 e^{37.25r}$$

$$= \frac{e^{- 0.0045748 \times 37.25}}{60.4}$$

$$= 0.01396222$$

$$= \underline{\underline{0.01396}}$$

<u>Problem 14:</u>

There is a stable female population in which 50% survive to age 65, the expectation of life at birth is 55 years, and at age 65, 12.4 years. The proportion of women under 65 is 91%.

(a) Is the rate of increase positive, negative or zero? How do you know?

(b) It is possible to calculate a number (from the above information) that is described by one of the following statements:

(i) The number is the birth rate of the stable population.

(ii) The number is surely exceeded by the birth rate.

(iii) The number surely exceeds the birth rate.

Calculate the number in question, and explain why it fits a particular one of these statements.

(c) Is the death rate among women over 65 greater than, equal to, or less than 0.0806? Explain.

(d) Suppose that the mortality over 65 in the life table with $e_o^o = 55$ were accompanied by no mortality under 65, what would $e_o^o$ be? Suppose that the mortality under 65 with $e_o^o = 55$ were accompanied by no mortality 65 - 100 and there were no survivors beyond 100, what would be $e_o^o$?

(e) In another stable population, 50% survive to age 69. The ratio of the number annually reaching 69 in the population to the annual births is 0.250. What is the approximate annual rate of increase?



Solution

(a) In the stationary population,

$$T_o = e_o^o \ell_o = 55 \ell_o .$$

$$T_{65} = e_{65}^o \ell_{65} = e_{65}^o \frac{\ell_{65}}{\ell_o} . \ell_o$$

$$= e_{65}^o \, p(65) . \ell_o$$

$$= 12.4 \times 0.5 \, \ell_o$$

$$= 6.2 \, \ell_o$$

Therefore

$$C(65^+) = \frac{T_{65}}{T_o}$$

$$= \frac{6.2}{55} = 11.27\%$$

In the stable population, we are given that

$$C(65^+) = 9\%$$

So we have a case of more old people in a stationary population than is a stable population.

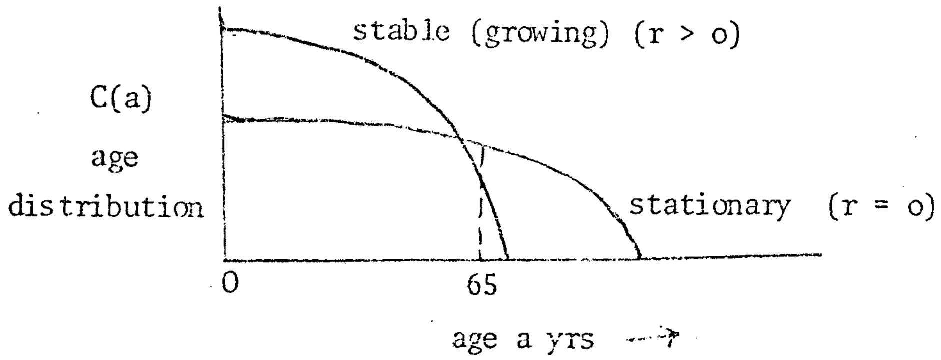
Thus we have a growing stable population

i.e.

$$r > 0.$$



Graphically, we have the following




(b) In a growing stable population,

$$C(0) \text{ stable} > c(0) \text{ stationary of diagram above}$$

i.e.,

$$b(\text{stable} > b(\text{stationary}) = \frac{1}{e_o^o}$$

i.e.

$$b(\text{stable}) > \frac{1}{55}$$

Therefore the number $\frac{1}{55}$ (birth rate of the stationary population) is surely exceeded by the birth rate of the growing stable population.

(c) In a growing stable population, death rate < death rate of stationary population

Thus

death rate ($65^+$) in stable < death rate ($65^+$) stationary

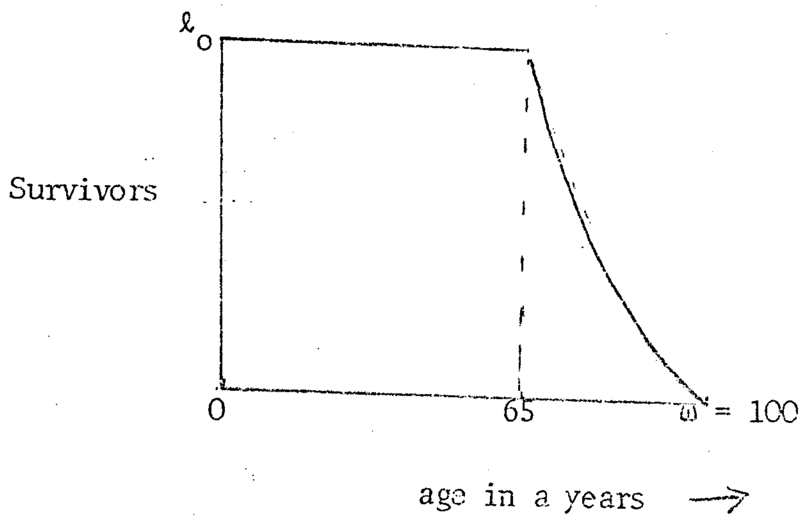
But

$$d(65^+) \text{ stationary } = \frac{1}{e^o_{65}} = \frac{1}{12.4}$$

Therefore

$$\text{death rate } (65^+) \text{ stable} < \frac{1}{12.4} = 0.0806$$

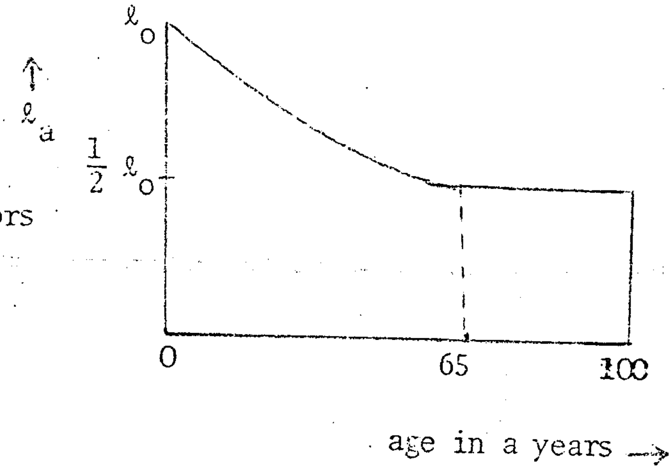
(d) (i)




$$e^o_o = \frac{\text{Area of Rectangle}}{\ell_o} + e^o_{65}$$

$$= 65 + 12.4 = \underline{\underline{77.4}}$$

(ii) Survivors




$$e_o^o = \frac{T_o - T_{65}}{\ell_o} + \frac{\text{Area of rect}}{\ell_o}$$

$$= \frac{T_o}{\ell_o} - \frac{T_{65}}{\ell_o} + \frac{\text{Area of rect}}{\ell_o}$$

$$= e_o - e_{65}^o \frac{\ell_{65}}{\ell_o} + \frac{1}{2}\ell_o \times \frac{35}{\ell_o}$$

$$= 55 - 12.4 \times \frac{1}{2} + 17.5$$

$$= 72.5 - 6.2$$

$$= \underline{\underline{66.3}}$$



(e) Using

$$C(a) = b\, e^{-ra}\, p(a)$$

we have

$$\frac{C(a)}{b} = e^{-ra}\, p(a)$$

But

$$\frac{C(69)}{b} = 0.250$$

Part B: Suppose that in a life table $\ell_{30}$ is 90,100 and $\ell_{31}$ is 89,900 and that in the stationary population the proportion of persons aged 30 is 1.5%. What is the expectation of life at birth?

Part C: Suppose that in a stable population with this life table and a rate of increase of 0.0228, the proportion at age 30 is 1.35%. What is the death rate?

**Solution:**

**Part A**

| $i$ | $f_i$ | $w_i$ (1000) | $w_i f_i$ |
|---|---|---|---|
| 1 | .100 | 400 | 40 |
| 2 | .300 | 350 | 105 |
| 3 | .350 | 300 | 105 |
| 4 | .300 | 250 | 75 |
| 5 | .200 | 200 | 40 |
| 6 | .050 | 200 | 10 |
| | 1.300 | 1700 | 375 |

(i) $$\text{CBR} = \frac{\text{Total Births}}{\text{Total Population}}$$

$$= \frac{10^3 \sum_i w_i f_i}{N}$$

$$= \frac{10^3 \times 375}{8,500,000}$$

$$= \underline{\underline{0.0441}}$$



(ii) $$\text{T.F.R.} = 5 \sum_i f_i$$

where $f_i$ is the ith age specific fertility rate i.e., births per annum per woman

Therefore

$$T.F.R = 5 \times 1.3 = \underline{\underline{6.5}}$$

(iii) $$GFR = \frac{\text{Total Births}}{\text{No. of Women (15.44)}}$$

$$= \frac{10^3 \times 375}{10^3 \times \sum_i w_i}$$

$$= \frac{375}{1700}$$

$$= 0.22058$$

$$\simeq \underline{\underline{0.221}}$$

(iv) GRR = Total female births per woman in the child-bearing age span without mortality under consideration

$$= TFR \times \frac{100}{206}$$

$$= \frac{TFR}{2.06}$$

$$= \frac{6.5}{2.06}$$

$$= \underline{\underline{3.1553}}$$



Part B

$$e_o^o = \frac{1}{b}.$$

From

$$C(a) = b\, e^{-ra} p(a)$$

we have

$$\frac{1}{b} = \frac{e^{-ra}\, p(a)}{C(a)}$$

In a stationary population,

$$\frac{1}{b} = \frac{p(a)}{c(a)}$$

since

$$r = o.$$

Therefore

$$e_o^o = \frac{1}{b} = \frac{p(a)}{c(a)}$$

We are given that

$$c_{30} = C(30.5) = 0.015$$

Therefore

$$e_o^o = \frac{p(30.5)}{c(30.5)}$$

$$= \frac{\ell(30.5)}{\ell_o\ c(30.5)}$$

$$= \frac{\frac{1}{2}\ (\ell_{30} + \ell_{31})}{\ell_o\ c(30.5)}$$

$$= \frac{90,100 + 89,900}{2\ \ell_o \times 0.015}$$

$$= \frac{90,000}{\ell_o \times 0.015} = \underline{\underline{60}}$$

assuming that $\ell_o = 100,000$.



<u>Part C</u>

From

$$C(a) = be^{-ra}\ p(a)$$

we have

$$b = \frac{C(a)\ e^{ra}}{p(a)}$$

If we put

$$a = 30.5$$

then

$$b = \frac{C(30.5)}{p(30.5)} \; e^{0.0228 \times 30.5}$$

$$= \frac{0.0135}{0.90} \; \exp(0.6954)$$

$$= 0.0301$$

Therefore

$$d = b - r$$

$$= 0.0301 - 0.0228$$

$$= \underline{\underline{0.0073}}$$



Problem 16 : A female stable population has the following characteristics:

| Life table values | | Intrinsic Rates | | |
|---|---|---|---|---|
| $\ell_o$ | 100,000 | | | |
| ${}_1L_{34}$ | 90,000 | r | = | 0.020 |
| $e^o_o$ | 66.67 yrs | b | = | 0.0285 |
| ${}_1L_o$ | 97,000 | | | |

(a) What proportion of the stationary population is between exact ages 34 and 35? Of the stable population?

(b) What proportion of the stationary population is under 1? Of the stable?

(c) Two per cent of another stable population is between exact age 5 and exact age 6. $e^o_o$ is 50 years. Is the growth rate positive, negative or zero? How do you know?

Solution.

(a) (i) $$\frac{{}_1L_{34}}{T_o} = \frac{90,000}{e_o^o \, \ell_o}$$

$$= \frac{90,000}{66.67 \times 100,000}$$

$$= \frac{9}{666.7}$$

$$= \underline{\underline{0.0135}}$$

(ii) $${}_1C_{34} = C(34.5)$$

$$= be^{-34.5r} \frac{{}_1L_{34}}{\ell_o}$$

$$= 0.0285\, e^{-34.5 \times 0.030} \times \frac{90,000}{100,000}$$

2

$$= \underline{\underline{0.0129}}$$

(b) (i) $$\frac{{}_1L_o}{T_o} = \frac{97000}{66.67 \times 100,000}$$

$$= \underline{\underline{0.0145}}$$

(ii) $${}_1C_o = C(0.5)$$

$$= be^{-0.5r} \frac{{}_1L_o}{\ell_o}$$

$$= 0.0285\, e^{-0.5 \times 0.02} \times \frac{97,000}{100,000}$$

$$= \underline{\underline{0.0274}}$$

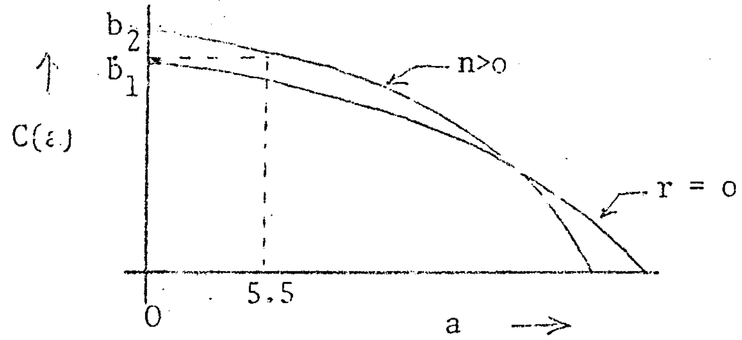
(c) In a stationary population (i.e., r = 0)

$$b_1 = \frac{1}{50} = 0.02$$

We are also given that in another stable population,

$$_1C_5 = c(5.5) = b_2 e^{-5.5r} \frac{_1L_5}{\ell_0} = 0.02$$

Diagramatically, we have




This implies that

$$b_1 \text{ (stationary)} < b_2 \text{ (stable)}$$

Thus we have a growing stable population. That is, r is positive.

Proble 17:

Prove that in a stationary population, if $e_0^o = e_1^o$, then the proportion of population under age 1 equals the infant mortality rate $_1q_o$.

Proof:
$$\frac{_1L_o}{T_o} = \frac{T_o - T_1}{T_o}$$

$$= 1 - \frac{T_1}{T_o}$$

$$= 1 - e_1^o \ell_1/T_o$$

$$= 1 - e_o^o \ell_1/T_o, \quad \text{since } e_1^o = e_o^o$$

$$= 1 - e_o^o \frac{\ell_o}{T_o} \cdot \frac{\ell_1}{\ell_o}$$

$$= 1 - \frac{\ell_1}{\ell_o}$$

$$= 1 - {}_1p_o$$

$$= {}_1q_o$$



Problem 18 :

In a closed population, where both the schedule of age specific fertility rates (that show a pattern of early child-bearing) and the schedule of mortality rates have been constant in the indefinite past, a survey finds that the number of children ever born per 100 women aged 50-54 is 480. The mortality of the female population is described by the following life table values:

| | | |
|---|---|---|
| $\ell_o$ = 10,000 | $\ell_{30}$ = 8,400 | $e^o_o$ = 60.0 |
| $\ell_{10}$ = 8,850 | $\ell_{40}$ = 7,950 | |
| $\ell_{20}$ = 8,600 | $\ell_{50}$ = 7,500 | $e^o_{50}$ = 25.0 |

Answer the following questions (seeking only as much precision as the available information and rough calculation permit):

(a) What is the female net reproduction rate?

(b) What is the female growth rate? What is the male growth rate?

(c) What, if anything, can be said about the level of the female birth rate and the female death rate?

(d) Assume that there is a sudden reduction in mortality above age 50 (say $e^o_{50}$ increases to 35.0 years) but all other elements of the demographic picture remain unchanged. How will this change affect the net reproduction rate, the growth rate, the birth rate and the death rate? Distinguish, if necessary, between short-run and long-run affects of the stipulated change on these indices.

Solution

(a) $$\text{TFR} = \frac{480}{100} = 4.80$$

$$\text{GRR} = \frac{\text{TFR}}{2.06} = \frac{4.80}{2.06}$$

$$\text{NRR} = p(\bar{m}) \times \text{GRR}$$

$$= p(\bar{m}) \times \frac{4.80}{2.06}$$

where $p(\bar{m})$ is the probability of surviving up to the mean age of child-bearing.

Assume

$$\bar{m} = 27.5 \quad \text{(though not given in the question)}$$

Then

$$p(\bar{m}) = p(27.5)$$

$$= \frac{\ell_{27.5}}{\ell_o}$$

$$= \frac{\ell_{25} + \ell_{30}}{2\,\ell_o}$$



But

$$\ell_{25} = \frac{\ell_{20} + \ell_{30}}{2}$$

Therefore

$$p(\bar{m}) = \frac{\frac{1}{2}.(\ell_{20} + \ell_{30}) + \ell_{30}}{2\ell_o}$$

$$= \frac{\ell_{20} + 3\,\ell_{30}}{4\,\ell_o}$$

$$= \frac{8600 + 3 \times 8400}{4\,\ell_o}$$

$$= \frac{86 + 3 \times 84}{400} \quad (\text{since } \ell_0 = 100{,}000)$$

$$= 0.845$$

Therefore

$$NRR = 0.845 \times \frac{4.80}{2.06}$$

$$= \underline{\underline{1.9689}}$$

(b) From

$$NRR = e^{rT}$$

we have

$$r = \frac{\ln NRR}{T}$$

where T is the mean length of generation and is obtained by using the formula

$$T \simeq \mu_1 - \frac{\mu_2}{2} \quad \frac{\ln NRR}{\mu_1}$$

$$\simeq \mu_1 - 0.8 \ln NRR$$

where

$$\mu_1 = \bar{m} = 27.5$$

which is the mean age of net fertility

Thus

$$T = 27.5 - 0.8 \ln NRR$$

Therefore

$$r = \frac{\ln NRR}{27.5 - 0.8 \ln NRR}$$

$$= \frac{\ln 1.9689}{27.5 - 0.8 \ln 1.9689}$$

$$= \frac{0.6780}{27.5 - 0.54242}$$

$$= 0.02515$$

$$\simeq \underline{\underline{0.025}}$$

Alternatively, we could simply use

$$r = \frac{\ln NRR}{T}$$

$$= \frac{\ln 1.9689}{27.5}$$

$$\simeq \underline{\underline{0.025}}$$

The growth rate is the same for both female and male population.

(c) From the life table (stationary population)

$$b = d = \frac{1}{e_o^o} = \frac{1}{60} = \underline{\underline{0.01667}}$$

Therefore

$$b \text{ (stable)} > \frac{1}{e_o^o} = 0.01667$$

and

$$d \text{ (stable)} < d \text{ (stat)} = b \text{ (stat)} = 0.01667$$

i.e.,

$$d \text{ (stable)} < 0.01667$$

which implies

$$d + r < 0.01667 + r$$

i.e.,

$$b \text{ (stable)} < 0.01667 + 0.025$$

which implies

$$b \text{ (stable)} < 0.042$$

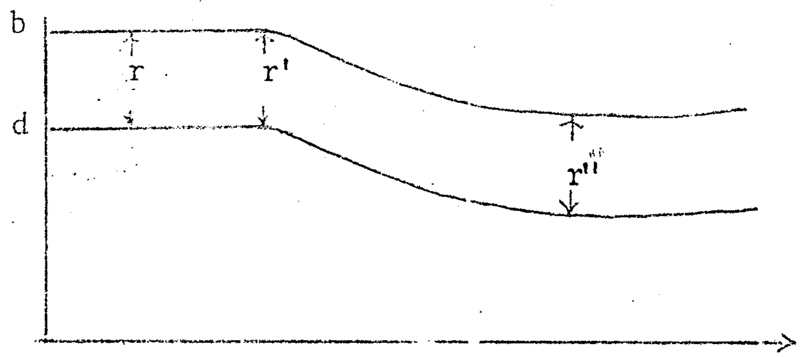
Therefore

$$0.01667 < b < 0.042$$

and

$$d < 0.01667$$

(d) NRR is the same because age above 50 is not child-bearing age.




In the short run, $r' > r$. We have an increasing growth rate. Since we have more people with same birth, birth rate decreases. The death rate decreases because we are told so. In the long run, $r''$ goes back to r for stability. Birth and death rates are still reduced.

Problem 19

A closed population has been subject to constant mortality and fertility schedules for many years, and its size has been constant. The number of female births this year is one million. The following values are from its female life table:

| | | |
|---|---|---|
| $\ell_0 = 10{,}000$ | $\ell_{80} = 200$ | $e^o_5 = 40$ years |
| $\ell_5 = 7{,}500$ | ${}_5L_0 = 39{,}500$ | $e^o_{50} = 18$ years |
| $\ell_{10} = 7{,}300$ | ${}_5L_{50} = 12{,}000$ | $e^o_{80} = 2$ years |

(a) What is the size of the female population?

(b) How many females are there under 5 years of age? Over 80? 50 to 55?

(c) Suppose there is now a reduction in mortality such that the proportion surviving for one year is increased at each age by 0.005. Fertility remains unchanged. What will be the new birth rate? The new death rate?

The proportion under 5 after many years of the new mortality regime?

Solution:

Since the number of female births this year is one million, the life table becomes

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\ell_0$ | · | $10{,}000 \times 10^2$ | $\ell_{80}$ | = | $200 \times 10^2$ | $e_5^o$ | = | 40 |
| $\ell_5$ | = | $7{,}500 \times 10^2$ | ${}_5L_0$ | = | $39{,}500 \times 10^2$ | $e_{50}^o$ | = | 18 |
| $\ell_{10}$ | = | $7{,}300 \times 10^2$ | ${}_5L_{50}$ | = | $12{,}000 \times 10^2$ | $e_{80}^o$ | = | 2 |

(a)

$$\begin{aligned} T_0 &= {}_5L_0 + T_5 \\ &= {}_5L_0 + e_5^o \ell_5 \\ &= (39{,}500 \times 10^2) + (40 \times 7{,}500 \times 10^2) \\ &= (39{,}500 + 40 \times 7{,}500) \times 10^2 \\ &= (39{,}500 + 300{,}000) \times 10^2 \\ &= 339{,}500 \times 10^2 \\ &= 33{,}950{,}000 \\ &= \underline{\underline{33.95 \text{ million}}} \end{aligned}$$

(b) (i) ${}_5L_0 = 39{,}500 \times 10^2$

$= \underline{\underline{3.95 \text{ million}}}$

(ii)

$$\begin{aligned} {}_\infty L_{80} &= T_{80} \\ &= e_{80}^o \ell_{80} \\ &= 2 \times 200 \times 10^2 \\ &= \underline{\underline{40{,}000}} \end{aligned}$$

(iii) $_5L_{50} = 12{,}000 \times 10^2$

$= \underline{1.2 \text{ million}}$

(c) The new mortality rate is

$$\mu^*(a) = \mu(a) - 0.005$$

i.e.,

$$\mu^*(a) - \mu(a) = -0.005 \quad \text{for all ages.}$$

<u>Thm:</u> Two stable populations with a constant difference in mortality schedules and some fertility schedules have the same age distribution.

So (i) the birth rate will be the same as the old one :

i.e.,

$$b = \frac{1}{e_o^o} = \frac{\ell_o}{T_o}$$

$$= \frac{10{,}000 \times 10^2}{33.95 \times 10^6} = \frac{1}{33.95}$$

(ii) The new death rate is

$$d^* = \int c(a)\mu^*(a)da$$

$$= \int c(a)\left[\mu(a) - 0.005\right]da$$

$$= \int c(a)\,\mu(a)da - 0.005 \int c(a)da$$

$$= \int c(a)\,\mu\,(a)da - 0.005$$

$$= d - 0.005$$

$$= \frac{1}{33.95} - 0.005$$

$$= 0.0295 - 0.005$$

$$= \underline{\underline{0.0245}}$$

$$r^* = b^* - d^*$$

$$= b^* - (d - 0.005)$$

$$= b - d + 0.005$$

$$= r + 0.005$$

(iii) Age composition is unaffected. Thus

$${}_5C_o = \frac{{}_5L_o}{T_o} = \frac{39{,}500 \times 10^2}{33.95 \times 10^6} = \frac{395}{3395} = \underline{\underline{0.1163}}$$

**Problem 20:**

Suppose the birth rate in a stable female population is 0.02. If the rate of increase is zero, what can be said about the expectation of life at birth? If the rate of increase is positive, what can be said about $e_o^o$?

**Solution:**

(a) In a stable population, the formular for birth rate is given by

$$b = \frac{1}{\int_o^w e^{-ra} p(a)da}$$

If

$$r = o \quad \text{(i.e. stationary population)}$$

then

$$b = \frac{1}{\int_o^w p(a)da}$$

$$= \frac{1}{\int_o^w \frac{\ell(a)}{\ell(o)} da}$$

$$= \frac{\ell(o)}{\int_o^w \ell(a)\,da}$$

$$= \frac{\ell_o}{T_o} = \frac{1}{T_o/\ell_o} = \frac{1}{e_o^o}$$

i.e.,

$$e_o^o = \frac{1}{b}$$

$$= \frac{1}{0.02}$$

$$= \underline{\underline{50}} \text{ years.}$$

(b) From

$$b = \frac{1}{\int_o^w e^{-ra}\, p(a)da}.$$

$$= \left[\int_o^w e^{-ra}\, p(a)da\right]^{-1}$$



$$\frac{d\,b}{dr} = -\left[\int_o^w e^{-ra}\, p(a)da\right]^{-2} \left(\int_o^w - a\, e^{-ra}\, p(a)da\right)$$

$$= \frac{\int_o^w a\, e^{-ra}\, p(a)da}{\left[\int_o^w e^{-ra}\, p(a)da\right]^2}$$

$$= \bar{a}\, b$$

where

$$\bar{a} = \frac{\int_o^w a\, e^{-ra}\, p(a)da}{\int_o^w e^{-ra}\, p(a)da}$$

is the mean age of a stable population

It follows that

$$\frac{d\,b}{d\,r} > 0$$

since

$$\bar{a} > 0$$

and

$$b > 0$$

Therefore birth rate is an increasing function of r. This implies that

$$\underset{\substack{\text{with} \\ r > o}}{\text{birth rate (stable)}} > \text{birth rate (stationary)}$$

Therefore

$$\frac{1}{b(\text{stable})} < \frac{1}{b(\text{stationary})}$$

i.e.,

$$\frac{1}{b(\text{stable})} < e_o^o$$

i.e.

$$e_o^o > \frac{1}{b(\text{stable})}$$

i.e.,

$$e_o^o > \frac{1}{0.02} = 50$$

$$\underline{\underline{e_o^o > 50}}$$



<u>Problem 21:</u>

Imagine a population with a stable age distribution that has arisen from its being closed and having had fixed age schedules of female fertility and male and female mortality throughout many decades. At present: The death rate is 15 per 1,000;
The birth rate is 35 per 1,000;
The average age is 24 years.

Now, without any other changes in the above conditions, suppose that the probability of survival p(x) increased by 0.5 per cent in each age instantaneously, thereafter remaining fixed.

Under the new conditions what will be:

(1) the birth rate;

(2) the rate of natural increase;

(3) the average age of the population, 5 and 10 years later respectively?

<u>Solution:</u>

If p(x) has increased by 0.5 per cent, then the new mortality rate is

$$\mu_2(x) = \mu_1(x) - 0.005$$

where $\mu_1(x)$ is the old mortality rate.

So $\mu_2(x)$ and $\mu_1(x)$ deffer by a constant at each age. Hence the age distribution remains unchanged. Therefore

(1) The birth rate remains the same

i.e.,

$$b = \frac{35}{1000} = \underline{\underline{0.035}}$$

(2) The new death rate is

$$\begin{aligned} d_2 &= \int c(x)\,\mu_2(x)dx \\ &= \int c(x)\left[\mu_1(x) - 0.005\right]dx \\ &= \int c(x)\,\mu_1(x)dx - 0.005 \int c(x)dx \\ &= d_1 - 0.005 \\ &= \frac{15}{1000} - 0.005 \\ &= 0.015 - 0.005 \\ &= 0.010 \end{aligned}$$

Therefore the rate of natural increase is

$$r_2 = b - d_2$$
$$= 0.035 - 0.010$$
$$= \underline{\underline{0.025}}$$

(3) Since the age distribution $C(x)$ remains the same, so does the average age at any time. Thus after 5 and 10 years, the average age is still 24 years.

Problem 22

Suppose the number of deaths at each age in a closed population is known for year t and year t + 5, and that age specific death rates during the five years are approximately constant. Explain how one might calculate:

(a) The distribution of deaths in the life table for the five year period.

(b) The $\ell_x$ function of this life table.

(c) The age distribution of the mean population (or the distribution of person - years lived).

Solution:

(a) With the usual notation,

$$N(x) = N(o)\ e^{-rx}\ p(x)$$
$$= N(o)\ e^{-rx}\ \frac{\ell(x)}{\ell(o)} \qquad (1)$$

Multiplying (1) by $\mu(x)$, we get

$$N(x)\ \mu(x) = \frac{N(o)}{\ell(o)}\ e^{-rx}\ \ell(x)\ \mu(x)$$

i.e.,

$$d(x) = \frac{N(o)}{\ell_o} e^{-rx} \ell(x) \mu(x) \qquad (2)$$

where

$$d(x) = N(x) \mu(x) \qquad (3)$$

is the number of deaths at age x

In a stationary population,

$$r = 0$$

and

$$N(o) = \ell_o \text{ (births)}$$

Therefore equation (2) becomes

$$d_o(x) = \ell(x) \mu(x) \qquad (4)$$

Divide (4) by (2) and we get

$$\frac{d_o(x)}{d(x)} = \frac{\ell_o}{N(o)} e^{rx}$$

i.e.,

$$d_o(x) = \frac{\ell_o}{N(o)} d(x) e^{rx}$$

$$= k\, d(x) e^{rx} \qquad (5)$$

where k is a constant to be determined.

Now

$$\int_o^w d_o(x)dx = k \int_o^w d(x) e^{rx} dx$$

But the sum of $d_o(x)$ over the whole life upon equals to $\ell_o$.
Therefore

$$\ell_o = k \int_o^w d(x) e^{rx} dx$$

Therefore

$$k = \frac{\ell_o}{\int_o^w d(x)\ e^{rx}\ dx}$$

Put

$$\ell_o = 1$$

Therefore

$$k = \frac{1}{\int_o^w d(x)\ e^{rx}\ dx} \qquad (6)$$

So formula (5) becomes

$$d_o(x) = \frac{d(x)\ e^{rx}}{\int_o^w d(x)\ e^{rx}\ dx} \qquad (7)$$

In the generalized stable population,

$$d_o(x) = \frac{d(x)\ e^{\int_o^x r(y)dy}}{\int_o^w d(x)\ e^{\int_o^x r(y)dy} dx} \qquad (8)$$

which is the required result.

(b) $$\ell(a) = \sum_{x=a}^{w} d_o(x)$$

or

$$\ell(a) = \int_{x=a}^{w} d_o(x)dx \qquad (9)$$



(c) Age distribution is given by

$$C(a) = b\ e^{-\int_o^a r(y)dy}\ p(a) \qquad (10)$$

To determine the age distribution of the mean population, we shall consider a 5-year age interval.

Thus

$$_5C_x = \int_x^{x+5} C(a)\ p(a)da$$

$$= \int_x^{x+5} b e^{-\int_0^a r(y)dy} p(a)da$$

$$\simeq \int_x^{x+5} b\, e^{-\int_0^{x+2.5} r(y)dy} p(a)da$$

$$= \left(b\, e^{-\int_0^{x+2.5} r(y)dy}\right)\int_x^{x+5} p(a)da$$

$$= b\, e^{-\int_0^{x+2.5} r(y)dy}\ \frac{{}_5L_x}{\ell_o} \qquad (11)$$

Equation (11) implies

$$\int_o^w {}_5C_x\, dx = b\int_o^w e^{-\int_o^{x+2.5} r(y)dy}\ \frac{{}_5L_x}{\ell_o}\, dx$$

But

$$1 = \int_o^w {}_5C_x\, dx$$

Therefore

$$b = \frac{1}{\int_o^w e^{-\int_o^{x+2.5} r(y)dy}\ \frac{{}_5L_x}{\ell_o}\, dx} \qquad (12)$$

Therefore equation (11) becomes

$${}_5C_x = \frac{e^{-\int_o^{x+2.5} r(y)dy}\ {}_5L_x}{\int_o^w e^{-\int_o^{x+2.5} r(y)dy}\ {}_5L_x\, dx} \qquad (13)$$

Further, assuming constant growth rates within the 5-year age interval so that ${}_5r_x$ becomes the growth rate between ages x and x+5, then

$$\int_o^{x+2.5} r(y)dy = \int_o^5 r(y)dy + \int_5^{10} r(y)dy + \ldots + \int_{x-5}^{x} r(y)dy + \int_x^{x+2.5} r(y)dy$$

$$= 5\left[{}_5r_o + {}_5r_5 + {}_5r_{10} + \ldots + {}_5r_{x-5}\right] + 2.5({}_5r_x) \qquad (14)$$

Formula (14) is then substituted in formula (13) and we get the age distribution of the mean population.

Problem 23

With a life table in which $e_o^o$ is 60 years, the birth rate of the stable population in which r = 0.01 is 0.0233.

(1) What is the death rate in the stationary and stable populations?

(2) What is the mean age of living persons and the mean age at death in the stable population when r = 0.005?

Solution

(1) (i) The death rate in the stationary population is

$$d = b = \frac{1}{e_o^o} = \frac{1}{60} = \underline{\underline{0.0167}}$$

(ii) The death rate in the stable population is

$$\begin{aligned} d &= b - r \\ &= 0.0233 - 0.01 \\ &= \underline{\underline{0.0133}} \end{aligned}$$



(2) If g is a characteristic of persons and G the prevalence of the characteristic in the populations, such that

$$G = \int_o^w g(a)\ c(a)da$$

in a stable population, then

$$\frac{dG}{dr} = G(\bar{a} - \bar{a}_g)$$

where $\bar{a}$ is the mean age of the population, and $\bar{a}_g$ is the mean age of persons with the characteristic g.

Special cases:

When

$$G = b$$

then

$$\bar{a}_g = \bar{a}_b = 0$$

Therefore, we have

$$\frac{db}{dr} = b\,\bar{a}$$

This implies

$$\Delta b \simeq \left(b + \frac{\Delta r}{2}\right) \bar{a}\,\Delta r$$

Now

$$\Delta b = b(\text{stable}) - b(\text{stationary})$$

$$= 0.0233 - 0.0167$$

$$= 0.0066$$

$$\Delta r = 0.005$$



Therefore

$$\bar{a} \simeq \frac{\Delta b}{\left(b + \frac{\Delta r}{2}\right) \Delta r}$$

$$= \frac{0.0066}{\left(0.0233 + \frac{0.005}{2}\right) \times 0.005}$$

$$= \frac{0.0066}{0.0258 \times 0.005}$$

$$= \frac{0.0066}{0.0001}$$

$$= \underline{\underline{66}}$$

Next, let

$$G = d$$

Thus

$$\frac{d\,d}{dr} = d(\bar{a} - \bar{a}_d)$$

This implies

$$\Delta d = (d + \frac{\Delta r}{2})\ (\bar{a} - \bar{a}_d)\Delta r$$

i.e.,

$$d(\text{stationary}) - d(\text{stable}) = \left[d(\text{stable}) + \frac{0.005}{2}\right](66 - \bar{a}_d) \times 0.005$$

i.e.,

$$0.0167 - 0.0133 \simeq (0.0133 + 0.0025)(66 - \bar{a}_d) \times 0.005$$

i.e.,

$$0.0034 \simeq 0.0001 \times (66 - \bar{a}_d)$$

Therefore

$$66 - \bar{a}_d = \frac{0.0034}{0.0001}$$

which implies that

$$\bar{a}_d = 66 - 34$$
$$= \underline{\underline{32}}$$



## Problem 24

In a female life table, the following values are found: $e^o_o = 70$ years; $e^o_{69.4} = 11$ years; $\ell_{69.4}|\ell_o = 0.65$.
The stable population incorporating this life table and a rate of increase of 0.010 has a birth rate of 0.020. In this stable population 58% of the deaths occur over age 69.4.

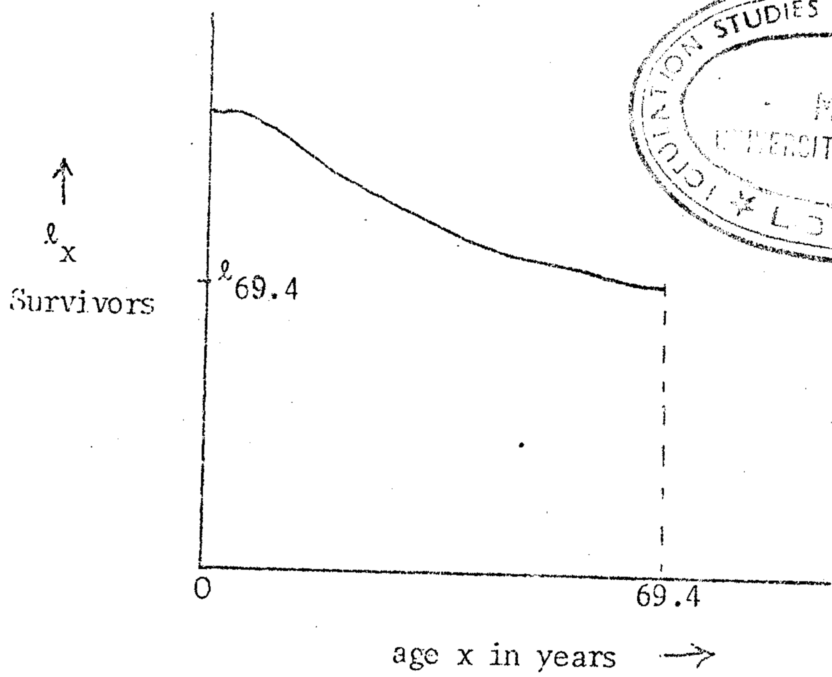
(a) If mortality up to age 69.4 were as in this life table, but everyone reaching exact age 69.4 died at that moment, what would the death rate in the new stationary population be?

(b) What would be death rate be in the stable population incorporating the modified life table and a rate of increase of 0.010?

(c) What is the proportion over 69.4 in the stable population with the original life table and rate of increase 0.010? (Hint: Is the birth rate different in the two stable populations; if so, by how much and why?)

Solution

(a)






In any stationary population,

$$d = b = \frac{1}{e_o^o}$$

So basically, the problem here, is to find the new expectation of life denoted by $e_o^{*o}$ say,

Thus

$$e_o^{*o} = \frac{T_o^*}{\ell_o} = \frac{T_o - T_{69.4}}{\ell_o}$$

where $T_o^*$ is the total population for the new stationary population.

So

$$e_o^{*o} = \frac{T_o}{\ell_o} - \frac{T_{69.4}}{\ell_o}$$

$$= e_o^o - \frac{e_{69.4}^o \ \ell_{69.4}}{\ell_o}$$

i.e.,

$$e_o^{*o} = 70 - \frac{11 \times 0.65 \ \ell_o}{\ell_o}$$

$$= 70 - (11 \times 0.65)$$

$$= 70 - 7.15$$

$$= 62.85$$

Therefore the death rate for the new stationary population is given by

$$d^* = \frac{1}{e_o^*}$$

$$= \frac{1}{62.85}$$

$$= \underline{\underline{0.0159}}$$

(b) In the stable population incorporated with the new stationary population,

$$C^* (69.4) = b^* e^{-69.4 \times 0.01} p(69.4)$$

$$\simeq b^* \times 0.50 \times 0.65$$

But

$$C^* (69.4) = d^* (69.4)$$

= death rate at age 69.4 in the new stable population.

i.e.,

$$d^*(69.4) = 0.5 \times 0.65b^*$$

In the original stable population, the ratio

$$\frac{d(< 69.4)}{b} = \frac{0.42d}{b}$$

$$= \frac{0.42\ (0.02 - 0.01)}{0.02}$$

$$= \underline{\underline{0.21}}$$

This ratio is the same for both the original and new stable population under 69.4

So, in the new stable population,

$$\frac{d^*(< 69.4)}{b^*} = 0.21$$

This implies that

$$d^*(< 69.4) = 0.21b^*$$

Therefore the total death rate of the new stable population is

$$\begin{aligned} d^* &= d^*(\leq 69.4) \\ &= d^*(< 69.4) + d^*(69.4) \\ &= 0.21b^* + 0.5 \times 0.65\ b^* \\ &= 0.535\ b^* \end{aligned}$$

But

$$b^* - d^* = 0.01$$

i.e.,

$$b^*(1 - 0.535) = 0.01$$

i.e.,

$$\begin{aligned} b^*\ \ b^* &= \frac{0.010}{0.465} \\ &= 0.0215 \end{aligned}$$

Therefore

$$\begin{aligned} d^* &= b^* - r^* \\ &= 0.0215 - 0.010 \\ &= \underline{\underline{0.0115}} \end{aligned}$$

(c) $$b^{*} = \frac{\text{Births}}{\text{Pop }(0 - 69.4)}$$

and

$$b = \frac{\text{Births}}{\text{Pop }(0 - \omega)}, \quad (\omega > 69.4)$$

Therefore

$$\frac{b}{b^{*}} = \frac{\text{Pop }(0 - 69.4)}{\text{Pop }(0 - \omega)}.$$

Thus

$$\frac{\text{Pop }(0 - 69.4)}{\text{Pop }(0 - \omega)} = \frac{b}{b^{*}}$$

$$= \frac{0.020}{0.0215}$$

This implies that

$$C(\leq 69.4) = \frac{0.020}{0.0215} = 0.9302$$

Therefore

$$C(> 69.4) = 1 - 0.9302$$

$$= \underline{\underline{0.0698}}$$

<u>Problem 25</u>

Selected values from a life table:

| $x$ | $\ell_x$ | $e^o_x$ |
|---|---|---|
| 0 | 100,000 | 70 |
| 40 | 92,000 | 35 |
| 70 | 65,000 | 11 |



(1) In the stationary population defined by this life table, what is the death rate:

(a) of the total population;

(b) of the population over 20

(c) of the population under 40

(d) of the population 40 - 70?

(e) what is the mean age at death of those who die between 0 and 40?

(2) In a stable population incorporating this life table, and a growth rate of 1%, which of the death rates (if any) in part 1 of this question would be higher, and which (if any) lower than in the stationary population? Explain.

Solution

Part 1

By definition,

$$\text{Death Rate} = \frac{\text{Deaths for particular age, period etc}}{\text{The Corresponding Population}}$$

(a) Death rate of the total population

$$d = \frac{\ell_o}{T_o} = \frac{1}{e_o^o} = \frac{1}{70} = \underline{\underline{0.0143}}$$

(b) Death rate of population over 70

$$d(70^+) = \frac{\ell_{70}}{T_{70}} = \frac{1}{e_{70}^o} = \frac{1}{11} = \underline{\underline{0.0909}}$$

(c) Death rate of population under 40

$$d(40^-) = \frac{\ell_o - \ell_{40}}{T_o - T_{40}}$$

$$= \frac{100{,}000 - 92{,}000}{e_o^o\,\ell_o - e_{40}^o\,\ell_{40}}$$

$$= \frac{100,000 - 92,000}{70x100,000 - 35x92,000}$$

$$= \frac{100 - 92}{70x100 - 35x92}$$

$$= \frac{8}{70(100 - 46)}$$

$$= \frac{8}{70 \times 54}$$

$$= \underline{\underline{0.0021}}$$

(d) Death rate of the population 40 - 70

$$d(40 - 70) = \frac{\ell_{40} - \ell_{70}}{T_{40} - T_{70}}$$

$$= \frac{\ell_{40} - \ell_{70}}{e^{o}_{40}\ \ell_{40} - e^{o}_{70}\ \ell_{70}}$$

$$= \frac{92,000 - 65,000}{35x92,000 - 11x65,000}$$

$$= \frac{27}{2505}$$

$$= \underline{\underline{0.0108}}$$



(e) Person-years lived between ages x and x+n is given by

$${}_{n}L_{x} = a(\ell_{x} - \ell_{x+n}) + n\ \ell_{x+n}$$

where 'a' is the average number of years lived for individuals who die between age x and x+n.

Therefore, with

$$x = o \quad \text{and} \quad n = 40$$

we have

$$_{40}L_o = a(\ell_o - \ell_{40}) + 40\,\ell_{40}$$

This implies that

$$T_o - T_{40} = a(\ell_o - \ell_{40}) + 40\ell_{40}$$

i.e.,

$$a = \frac{T_o - T_{40} - 40\,\ell_{40}}{\ell_o - \ell_{40}}$$

$$= \frac{70 \times 100{,}000 - 35 \times 92{,}000 - 40 \times 92{,}000}{100{,}000 - 92{,}000}$$

$$= \frac{7000 - 35 \times 92 - 40 \times 92}{8}$$

$$= (7000 - 3220 - 3680)/8$$

$$= \frac{100}{8} = \underline{\underline{12.50}}$$

So $12\frac{1}{2}$ years is the mean age at death for those who die between age 0 and 40.

Part 2

(a) We know that

$$C(0)_{r=o} < C(0)_{r>o}$$



i.e.,

$$b(r=o) < b(r>o)$$

But

$$b(r=o) = d(r=o) = 0.0143$$

as shown in part 1 (a)

Therefore

$$0.0143 < b(r>o)$$

It is empirically true that for $r > o$,

$$0 < d(r>o) < b(r>o)$$

and

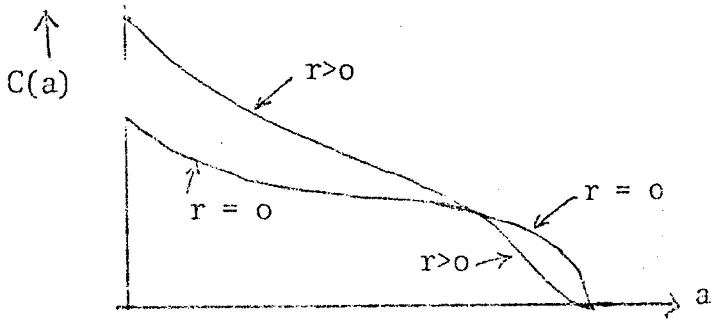
$$0 < d(r>o) < b(r=o)$$

i.e.,

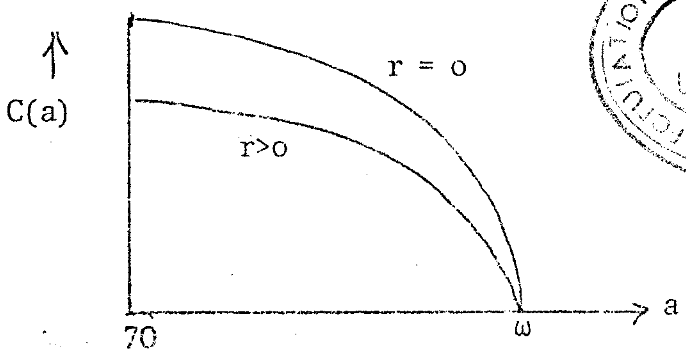
$$0 < d(r>o) < 0.0143$$

Therefore

$$d(r>o) < d(r=o)$$




(b)






From the diagram,

$$b^{70+}_{r=o} > b^{70+}_{r>o}$$

Since in a growing population

$$d < b$$

$$d^{70+}_{r>o} < b^{70+}_{r>o} < b^{70+}_{r=o} = d^{70+}_{r=o}$$

Therefore

$$d^{70+}_{r>o} < d^{70+}_{r=o}$$



(c) Left for an exercise

Hint: consider three cases

(1) when 40 is the intersection

(2) when 40 is less than the intersecting point

(3) when 40 is greater than the intersecting point.

Problem 26

Many thousands of years ago in the country of Gynasia a terrible epidemic occurred. This epidenic, called Gynasia fever, was 100% fatal to all men. Before dying, however, the men were able to store many gallons of semen in a giant sperm bank. Unfortunately the refrigeration system failed and all Y sperm were destroyed. As a result of having only X sperm in the bank, the Gynasia population is totally female since all births are female. By rigid government control, age-specific fertility and mortality rates have remained constant for a long, long while.

The relevant data are:

Fertility - just before her 30th birthday each woman has one birth.

- just before her 40th birthday each woman has twins.

Mortality - on their 20th birthday exactly ½ the women in each cohort die.

- on their 30th birthday, ½ of those surviving to age 30 die.

- on their 50th birthday all the remainder die.

Questions

(a) What is total fertility in Gynasia?
(b) What is the gross reproduction rate (GRR)?
(c) What is the net reproduction rate (NRR)?
(d) What is the life expectancy at birth?
(e) What is the birth rate in Gynasia?
(f) In 1950 there were exactly 1000 births.

What can you say about the size of the total population in 1950?

Solutions

(a) Total Fertility = total number of children a woman has in the child bearing age span

i.e.,

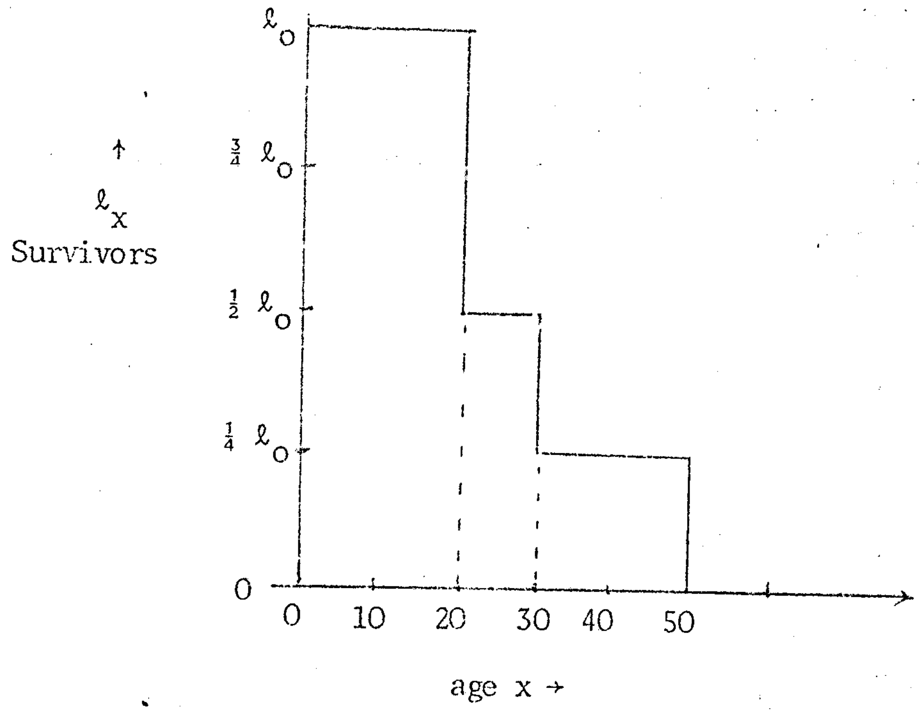
$$T.F. = 1 + 2$$
$$= \underline{\underline{3.0}}$$

(b) GRR is the total number of female births in the child-bearing age span.

In this case

$$T.F. = GRR$$
$$= \underline{\underline{3.0}}$$

(c) The mortality schedule is as follows:-




$$\text{NRR} = \int_{\alpha}^{\beta} p(a)m(a)da$$

$$= \int_{\alpha}^{30} p(a)m(a)da + \int_{30}^{40} p(a)m(a)da$$

Assuming

$$\alpha = 20$$

then we have



$$\text{NRR} = \int_{20}^{30} p(a)m(a)da + \int_{30}^{40} p(a)m(a)da$$

$$= p(25) \int_{20}^{30} m(a)da + p(35) \int_{30}^{80} m(a)da$$

$$= p(25) \times 1 + p(35) \times 2$$

$$= \frac{1}{2} \times 1 + \frac{1}{4} \times 2$$

i.e.,

$$\text{NRR} = \underline{\underline{1}}$$

This implies that

$$r = o$$

and thus we have a stationary population.

(d) $$e_o^o = \frac{T_o}{\ell_o}$$

$$= \frac{\text{Area A} + \text{Area B} + \text{Area C}}{\ell_o}$$

$$= \frac{20\ \ell_o + 10\ (\frac{1}{2}\ \ell_o) + 20\ (\frac{1}{4}\ \ell_o)}{\ell_o}$$

$$= \underline{\underline{30}}$$

(e) $$b = \frac{1}{e_o^o} = \frac{1}{30} = 0.0333$$

since the population is stationary

(f) $$T_o = e_o^o\ \ell_o$$

$$= 30 \times 1000$$

$$= \underline{\underline{30{,}000}}$$



Problem 27

Suppose a female population has the following characteristics.

(a) In each cohort one percent of the initial cohort dies within each year of age up to exact age 20, no deaths occur from 20 to 50, and above age 50 two percent of the original cohort die at each age until the cohort is extinguished.

(b) The rate of first marriage (first marriages at each age divided by person - years lived by the cohort at that age) is 10 percent per year from exact age 20 to exact age 30. There is no marital dissolution from 20 to 50.

(c) Marital fertility is 0.40 to age 30, and declines linearly to zero from 30 to 50.

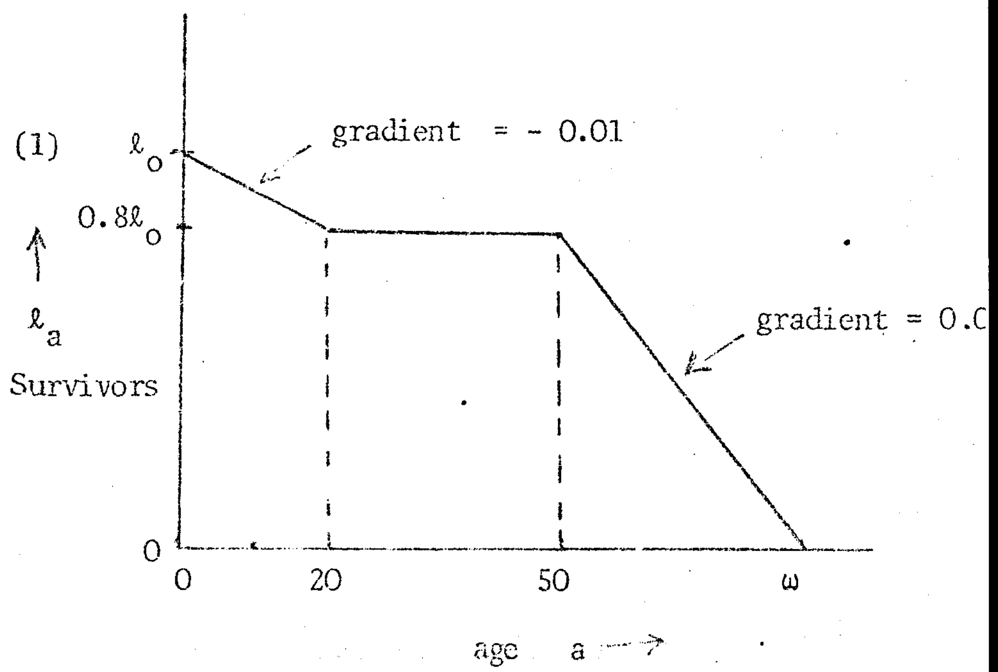
(d) The number of male and female births are equal.

(e) All births are legitimate.

1. Make a sketch of $\ell_x$. What is $e_o^o$?
2. Make a sketch of the age-specific fertility schedule. What is the total fertility rate?
3. What is the NRR? Is the rate of increase greater or less than two percent? How do you know?

Solution

(1)




The survivor function is defined by

$$\ell_x = \begin{cases} \ell_o - 0.01a\ \ell_o & , \quad 0 \leq a \leq 20 \\ \ell_o - 0.01 \times 20\ \ell_o = a8\ \ell_o, & 20 \leq a \leq 50 \\ 0.8\ell_o - 0.02\ (a - 50)\ \ell_o\ , & 50 \leq a \leq \omega \end{cases}$$

Note· Since $\ell_\omega = 0$, then the last expression given above becomes

$$0 = 0.8\ \ell_o - 0.02\ (\omega - 50)\ \ell_o$$

i.e.,

$$\omega = \frac{0.80}{0.02} + 50$$

$$= \underline{90}$$

Thus

$$\ell_a = \begin{cases} \ell_o\ (1 - \frac{a}{100})\ , & 0 \leq a < 20 \\ 0.8\ \ell_o & , \quad 20 \leq a < 50 \\ \ell_o(1.8 - 0.02a), & 50 \leq a \leq 90 \end{cases}$$

$$e_o^o = \frac{\text{Area of Trapezium + Area of Rectangle + Area of Triangle}}{\ell_o}$$

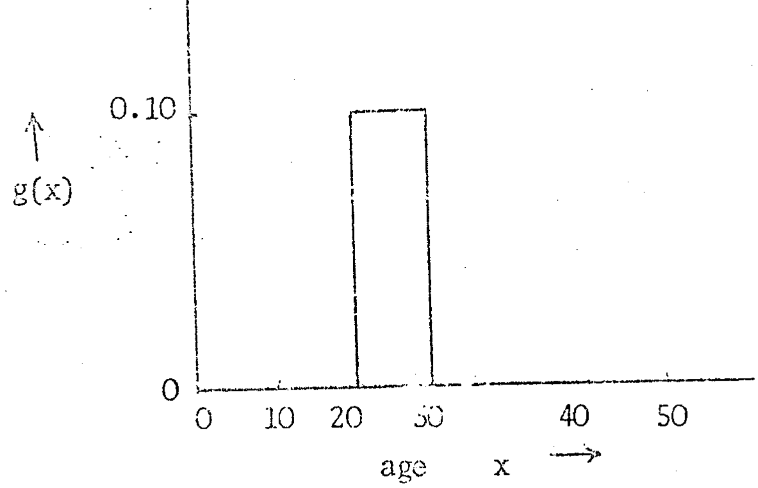
$$= \frac{1}{\ell_o}\left[\frac{1}{2}\ (\ell_o + 0.8\ell_o)\ 20 + 0.8\ell_o(50-20) + \frac{1}{2}\ (90-50)0.8\ell_o\right]$$

$$= \frac{1}{2} \times 1.8 \times 20 + 0.8 \times 30 + \frac{1}{2} \times 40 \times 0.8$$

$$= 18 + 24 + 16$$

$$= \underline{\underline{58}}$$

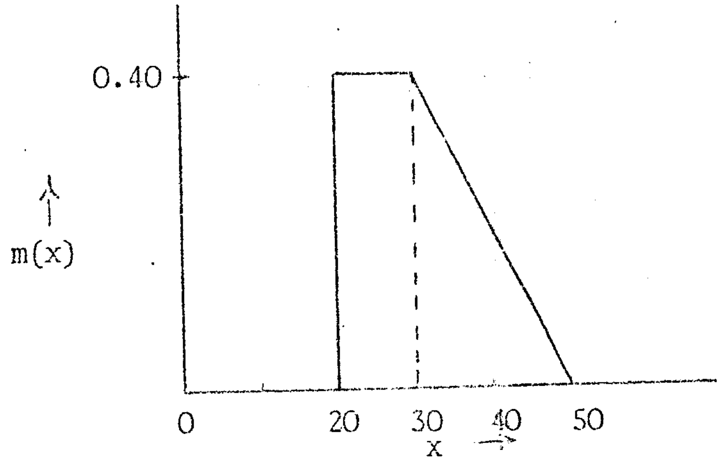
(2) (i) Rate of first marriage




$$g(x) = \text{rate of first marriage}$$

$$= \frac{\text{first marriages at age x}}{\text{Person-years lived by the cohort at that age}}$$

(ii) Marital fertility




$$m(x) = \text{marital fertility at age x}$$

$$= \frac{\text{Births to women aged x}}{\text{Women ever married to age x}}$$

(iii) Age specific fertility rate

$$f(a) = m(a)G(a)$$

$$= m(a) \int_{a}^{a} g(x)dx$$

But

$m(a) = 0.40$ for, $20 \leq a < 30$

and

$$m(a) = 0.40 - \frac{0.40}{50-30}(a - 30)$$

$$= 0.40 - \frac{0.40}{20}(a - 30)$$

$$= 0.40 - 0.02(a - 30), \quad 30 \leq a \leq 50.$$

Therefore

$$f(a) = 0.40 \int_{20}^{a} 0.10\, dx$$

$$= 0.40\,(a - 20)\,0.10$$

$$= 0.04\,(a - 20) \quad \text{for} \quad 20 \leq a < 30.$$

For $30 \leq a \leq 50$ we have

$$f(a) = \left[0.40 - 0.02\,(a - 30)\right] \int_{30}^{50} 0.10\, dx$$

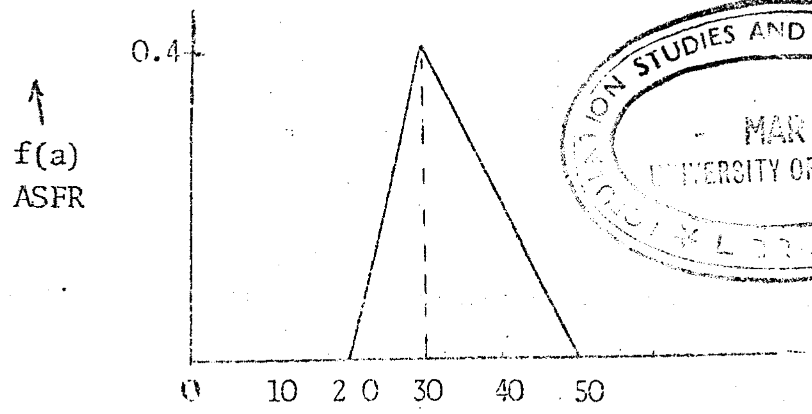
$$= \left[0.40 - 0.02\,(a - 30)\right](50 - 30) \times 0.10$$

$$= \left[0.40 - 0.02\,(a - 30)\right](20 \times 0.10)$$

$$= 2\left[0.40 - 0.02\,(a - 30)\right]$$

$$= 0.80 - 0.04\,(a - 30), \quad 30 \leq a \leq 50$$

Zero elsewhere.






$$\text{Total Fertility Rate} = \int_{20}^{50} f(a)\,da$$

$$= \text{Area of the big triangle}$$

$$= \frac{1}{2} \times 30 \times 0.4$$

$$= \underline{\underline{6}}$$

(3)(i) $\quad NRR = \int_{\alpha}^{\beta} m(a)\ p(a)da$

$$= 0.8 \int_{20}^{50} m(a)da$$

where m(a) is the maternity function and not the same as marital fertility described in (2).

$$p(a) = 0.8, \ 20 \leq a \leq 50$$

as shown in the first diagram.

Therefore

$$NRR = 0.8\ GRR$$

because by definition

$$GRR = \int_{\alpha}^{\beta} m(a)da$$



But

$$GRR = TFR \times \frac{\text{Female}}{\text{Female} + \text{Male}}$$

$$= 6 \times \frac{1}{2}$$

because female births and male births are equal and total fertility rate is 6.

Therefore

$$NRR = 0.8 \times 6 \times \frac{1}{2}$$

$$= \underline{\underline{2.4}}$$

(ii) Another definition of NRR is

$$NRR = e^{rT}$$

which implies that

$$r = \frac{\ln NRR}{T}$$

$$= \frac{\ln 2.4}{35}$$

$$= 0.0257 > 0.02$$

Thus the rate of increase is greater than two percent.

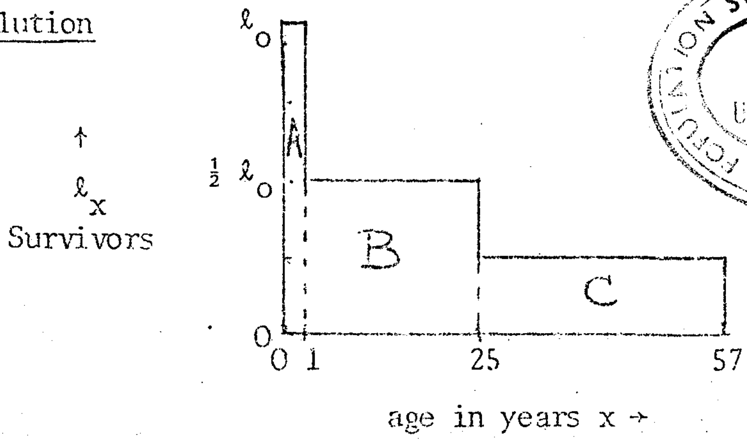
Problem 28

Suppose that in a closed female population deaths occur only at three exact ages : 1, 25 and 57. One half of those reaching age 1 die, one half of those reaching 25 die, and all who reach 57 die at that age. Compute $e_o^o$. If there is a constant rate at which female children are borne from age 15 to 25, what rate would yield a stationary population? What rate of female child-bearing would be required if the constant rate extended from 20 to 30? What rate of child-bearing (approximate) from 20 to 30 would yield an annual rate of increase of 2.76%?

Reduce all death rates by 50% and recompute $e_o^o$.



Solution




(i) $$e_o^o = \frac{T_o}{\ell_o}$$

$$= \frac{\text{Area A} + \text{Area B} + \text{Area C}}{\ell_o}$$

$$= \frac{(1 \times \ell_o) + 24(\frac{1}{2}\ell_o) + 32(\frac{1}{4}\ell_o)}{\ell_o}$$

$$= 1 + 12 + 8$$

$$= \underline{\underline{21}}$$

(ii) $NRR = \int_{\alpha}^{\beta} m(a)p(a)da$

If

$NRR = 1$ (i.e. stationary population)

and

$m(a) = m$ (constant), $15 \leq a \leq 25$

zero elsewhere

then

$$1 = m \int_{15}^{25} p(a)da$$

$$= m \int_{15}^{25} \frac{1}{2} da$$

$$= 5m$$

Therefore

$$m = \underline{\underline{0.2}}$$

(iii) If $\alpha = 20$ and $\beta = 30$

then

$$1 = \int_{20}^{30} m(a)p(a)da$$

$$= m\left[\int_{20}^{25} p(a)da + \int_{25}^{30} p(a)da\right]$$

$$= m\left[\int_{20}^{25} \frac{1}{2} da + \int_{25}^{30} \frac{1}{4} da\right]$$

$$= m\left[\frac{5}{2} + \frac{5}{4}\right]$$

$$= m \times \frac{15}{4}$$

Therefore

$$m = \frac{4}{15} \simeq \underline{\underline{0.27}}$$

(iv) $\quad NRR = \int_{\alpha}^{\beta} m(a)p(a)da$

Also

$$NRR = e^{rT}$$

Therefore

$$\int_{\alpha}^{\beta} m(a)p(a)da = e^{rT}$$

In this problem

$$m \int_{20}^{30} p(a)da = e^{0.0276 \times 25} \qquad (T = \frac{20 + 30}{2})$$

i.e.,

$$\frac{15}{4} m = e^{0.0276 \times 25}$$



This implies that

$$m = \frac{4}{15} e^{0.0276 \times 25}$$

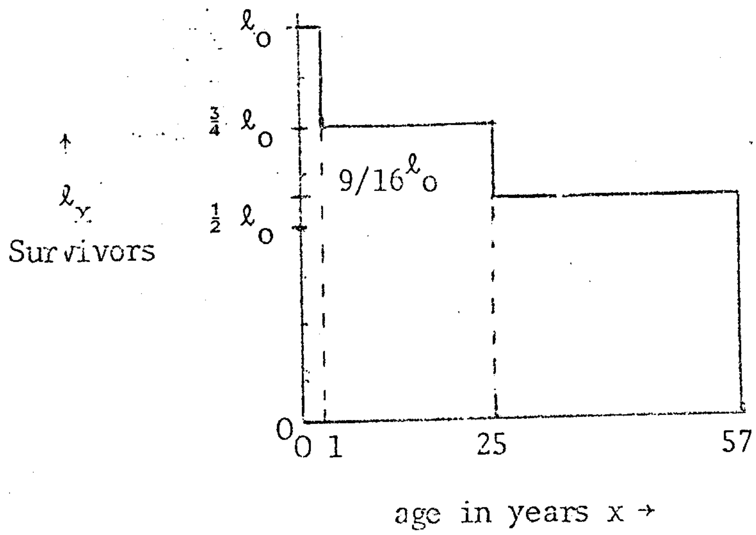
$$= \underline{\underline{0.5316}}$$

(v) Reducing all death rates by 50% implies that
$\frac{1}{4}$ of those reaching age 1 die

1/4 of those reaching age 25 die

All those who reach age 57 die.

The survivor function looks as shown below.




$$e_o^o = \frac{(1 \times \ell_o) + 24(\frac{3}{4}\ \ell_o) + 32(\frac{9}{16}\ \ell_o)}{\ell_o}$$

$$= 1 + 18 + 18$$

$$= \underline{\underline{37}}$$



## Problem 29

A. The age and age at death distributions for a stable population are shown below. Given the following two pieces of information: (a) $e_{60}^o = 13.4$ and (b) the death rate of persons less than age 20 equals 0.01744, is it possible to determine precisely the expectation of life at birth? State carefully your reasoning.

| Age | Proportion of the population in each age group. | Proportion of deaths in each age group. |
|---|---|---|
| 0 - 19 | 0.33 | 0.27 |
| 20 - 39 | 0.28 | 0.08 |
| 40 - 59 | 0.25 | 0.16 |
| 60 - 79 | 0.13 | 0.38 |
| $30^+$ | 0.01 | 0.11 |

B. A biologist keeps a colony of fruitflies under observation and supplies an environment such that, for a long time, the population increases at a constant daily rate. The death rate is independent of age and does not vary from day to day. The fertility rates vary with age, but not with time. The mean age at death in this population is 20 days

(a) From these data, calculate whatever ones of the following are possible:

(1) The average daily death rate for the colony,

(2) The average daily birth rate

(3) The fraction of the population over 20 days of age.

(4) $e_o^o$.

(b) Consider an additional datum: the colony doubles every 20 days. With this additional information, determine the expectation of life at age 20 days.



Solution

$$\text{Death rate under 20} = \frac{\text{deaths under 20}}{\text{Pop } (0 - 19)}$$

$$= \frac{\text{deaths under 20}}{\text{Total pop}} \times \frac{\text{Total pop}}{\text{pop } (0 - 19)}$$

$$= \frac{\text{deaths} < 20}{\text{total pop}} \Big/ \frac{\text{Pop } (0 - 19)}{\text{total pop}}$$

$$= \frac{0.27 \text{ total deaths}}{\text{total pop}} \Big/ 0.33$$

$$= \frac{0.27}{0.33} \; \frac{\text{total deaths}}{\text{total pop}}$$

i.e.,

$$0.01744 = \frac{0.27}{0.33} \; \frac{\text{total deaths}}{\text{total pop}}$$

which implies that

$$\frac{\text{total deaths}}{\text{total population}} = \frac{0.33}{0.27} \times 0.01744$$

$$= 0.0213$$

i.e.,

total death rate is given by

$$d = \underline{\underline{0.0213}}$$

Next,

$$\text{death rate} > 60 = \frac{\text{deaths} > 60}{\text{pop } (60^{+})}$$

$$= \frac{(0.38 + 0.11) \text{ total deaths}}{(0.13 + 0.01) \text{ total pop}}$$

$$= \frac{0.49}{0.14} \; \frac{\text{total deaths}}{\text{total pop}}$$

$$= \frac{0.49}{0.14} \times 0.0213$$

$$= \underline{\underline{0.07455}}$$

In a stationary population, if

$$d_{60} = 0.07455$$

then

$$e^{o}_{60} = \frac{1}{d_{60}} = \frac{1}{0.07455} = \underline{\underline{13.4}}$$

Conclusion: In a stable population, the receiprocal of death rate over 60 turns out to be equal to the expectation of life at age 60 in a stationary population. Therefore, the stable population must be stationary. Therefore the total death rate is

$$d = 0.0213$$

Therefore

$$e_0^0 = \frac{1}{d} = \frac{1}{0.0213} = \underline{\underline{46.9}}$$

B.

$$\mu(a, t) = \mu \quad \text{constant}$$

$$m(a, t) = m(a)$$

This implies that

$$d = \int_0^\omega c(a)\mu\, da$$

$$= \mu \int_0^\omega c(a)da$$

Therefore

(1) $$d = \underline{\underline{\mu}}$$

and

$$b = \int_0^\omega c(a)m(a)da.$$

(2) Mean age at death $$= \frac{\int_0^\omega a\, c(a)\mu(a)da}{\int_0^\omega c(a)\mu(a)da}$$

$$= \frac{\mu \int_0^\omega a\, c(a)da}{\mu \int_0^\omega c(a)da}$$

$$= \int_0^\omega a\, c(a)da$$

$$= \int_0^\omega abe^{-ra}\, p(a)da$$

But we are given that the mean age at death is 20.
Therefore

$$b \int_0^{\omega} ae^{-ra} p(a)da = 20$$

i.e.,

$$b \int_0^{\infty} a e^{-ra} e^{-\mu a} da = 20$$

Therefore

$$b \int_0^{\infty} a e^{-(r+\mu)a} da = 20$$

Using the fact that

$$\int_0^{\infty} t e^{-nt} dt = \frac{1}{n^2} .$$

then

$$b . \frac{1}{(r+\mu)^2} = 20$$

i.e.,

$$b \frac{1}{(r+d)^2} = 20$$

which implies

$$b \frac{1}{b^2} = 20$$

Thus

$$b = \frac{1}{20} = \underline{\underline{0.05}}$$

(3) $c(> 20) = \int_{20}^{\infty} c(a)da$

$$= \int_{20}^{\infty} b e^{-ra} p(a)da$$

$$= b \int_{20}^{\infty} e^{-(r+\mu)a} da$$

$$= b \int_{20}^{\infty} e^{-ba} \, da$$

$$= e^{-1}$$

$$= \underline{\underline{0.3679}}$$

(4) In a stationary population,

$$c(> 20) = \int_{20}^{\infty} b \; p(a) da$$

$$= b \int_{20}^{\infty} e^{-\mu a} \, da$$

$$= \frac{b}{\mu} \; e^{-20\mu}$$

Since

$$\mu = d = b \text{ in a stationary population}$$

then

$$C(> 20) = e^{-20b}$$

But if we put

$$b = \frac{1}{20}$$

then

$$C(> 20) = e^{-1}$$

$$= \underline{\underline{0.3679}}$$

This implies that the given population is actually a stationary population.

Therefore

$$e_o^o = \frac{1}{b} = \underline{\underline{20}}$$

(b)

$$P_t = p_o \, e^{rt}$$

This implies

$$P_{20} = P_o e^{20r}$$

But

$$P_{20} = 2 P_o \quad \text{given}$$

Therefore

$$P_o e^{20r} = 2P_o$$

i.e.,

$$e^{20r} = 2$$

Therefore

$$r = \frac{\ln 2}{20}$$

$$= 0.0347$$

Therefore

$$d = b - r$$

$$= 0.05 - 0.0347$$

$$= 0.0153$$

Therefore

$$e_o^o = \frac{1}{d} = \frac{1}{0.0153}$$

$$= \underline{\underline{65.359}}$$



Problem 30

The proportion surviving to age x, $\ell(x)$, in three life tables, is as follows:

A. $\ell(x)$ is 1.0 from zero to $\omega$, when all survivors die.
B. $\ell(x)$ is $1.0 - x/\omega$
C. $\ell(x)$ is $(1.0 - x/\omega)^2$

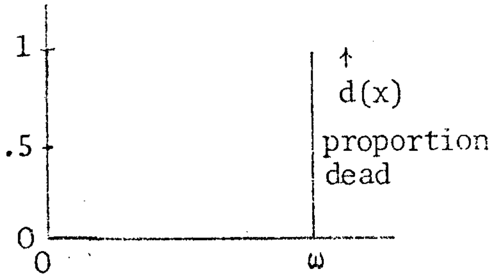
(1) The deaths in a short interval dx are d(x) dx; sketch the distribution of deaths d(x) for each life table.

(2) Determine an expression for the expectation of life at age x, e(x), for each life table. (Hint: it may be easier to consider the variable $y = \omega - x$).

(3) If $\omega = 100$, what is the death rate of the whole stationary population and of the population over 90 in each instance?

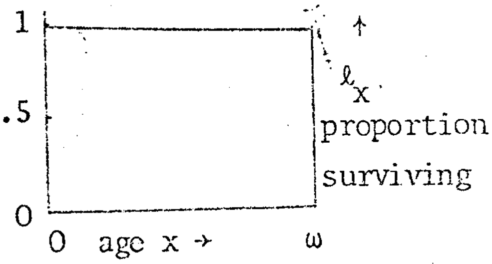
Solution

For A




(1) $d(x) = 0 \ , \quad 0 \leq x < \omega$

$\quad\quad = 1 \ , \quad x = \omega$




(2) $$e_x^o = \frac{T_x}{\ell_x} = \frac{\int_x^\omega \ell(x)\,da}{\ell_{(x)}}$$

$$= \int_x^\omega da$$

$$= \omega - x, \ 0 \leq x \leq \omega .$$

(3) Total death rate $$= \frac{1}{e_o^o}$$

$$= \frac{1}{\omega}$$

$$= \frac{1}{100}$$

$$= \underline{\underline{0.01}}$$

Death rate over 90 $$= \frac{\ell_{90} - \ell_\omega}{T}$$

$$= \frac{\ell_{90}}{T_o}$$

$$= \frac{1}{\int_0^{\omega} \ell(a)da}$$

$$= \frac{1}{\omega}$$

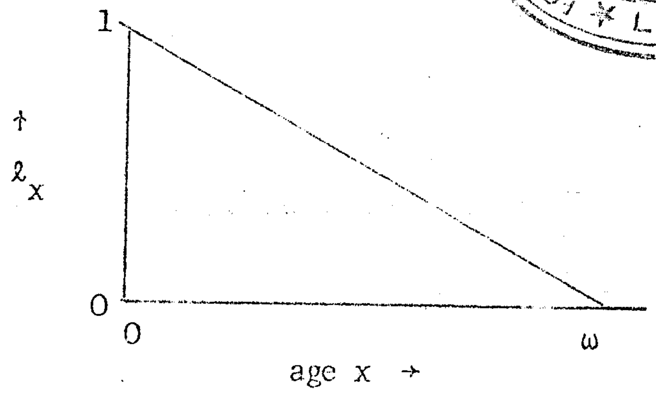
$$= \frac{1}{100}$$

$$= \underline{0.01}$$



For B

$$\ell(x) = 1 - \frac{x}{\omega} \quad , \quad 0 < x < \omega$$




$$\ell(x) - \ell(x + \Delta x) = d(x)\ \Delta x$$

i.e.,

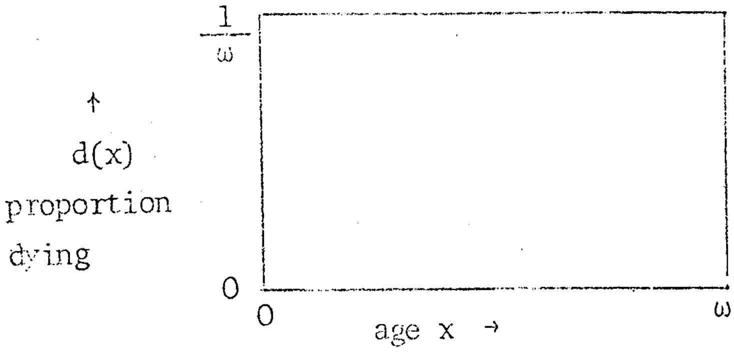
$$(1 - \frac{x}{\omega}) - (1 - \frac{x + \Delta x}{\omega}) = d(x)\ \Delta x$$

Therefore

$$\frac{\Delta x}{\omega} = d(x)\Delta x$$

i.e.,

(1) $$d(x) = \frac{1}{\omega} \ , \ 0 \leq x \leq \omega$$




$$(2) \qquad e_x^o = \frac{T_x}{\ell_x} = \frac{\int_x^{\omega} \ell(a)\,da}{\ell_x}$$

i.e.,

$$e_x^o = \frac{\int_x^{\omega} \left(1 - \frac{a}{\omega}\right) da}{1 - \frac{x}{\omega}}$$



$$= \frac{\left[a - \frac{a^2}{2\omega}\right]_x^{\omega}}{1 - \frac{x}{\omega}}$$

$$= \frac{\left(\omega - \frac{\omega^2}{2\omega}\right) - \left(x - \frac{x^2}{2\omega}\right)}{1 - \frac{x}{\omega}}$$

$$= \frac{\frac{\omega}{2} - x + \frac{x^2}{2\omega}}{1 - \frac{x}{\omega}}$$

$$= \frac{\omega^2 - 2\omega . x + x^2}{2\omega\left(1 - \frac{x}{\omega}\right)}$$

$$= \frac{\omega^2 - 2\omega x + x^2}{2(\omega - x)}$$

$$= \frac{(\omega - x)^2}{2(\omega - x)}$$

Therefore

$$e_x^o = \frac{\omega - x}{2}$$



(3) In particular, if

$$x = 0$$

$$e_o^o = \frac{\omega}{2}$$

Therefore

$$\text{the whole death rate} = \frac{1}{e_o^o} = \frac{2}{\omega} = \frac{2}{\omega} = \underline{\underline{0.02}}$$

$$\text{Death rate over } 90 = \frac{\ell_{90} - \ell_{\omega}}{T_o}$$

$$= \frac{\ell_{90}}{T_o}$$

$$\equiv \frac{1 - \frac{90}{\omega}}{e_o^o \ \ell_o}$$

$$= \frac{1 - 0.90}{50} = \frac{0.10}{50} = \underline{\underline{0.002}}$$

<u>For C</u>

$$\ell(x) = (1 - \frac{x}{\omega})^2$$

$$d(x) = \frac{\ell_{(x)} - \ell(x + \Delta x)}{\Delta x}$$

$$= \frac{(1 - \frac{x}{\omega})^2 - (1 - \frac{x + \Delta x}{\omega})^2}{\Delta x}$$

$$= \frac{\left[(1 - \frac{x}{\omega}) + (1 - \frac{x+\Delta x}{\omega})\right]\left[(1 - \frac{x}{\omega}) - (1 - \frac{x+\Delta x}{\omega})\right]}{\Delta x}$$

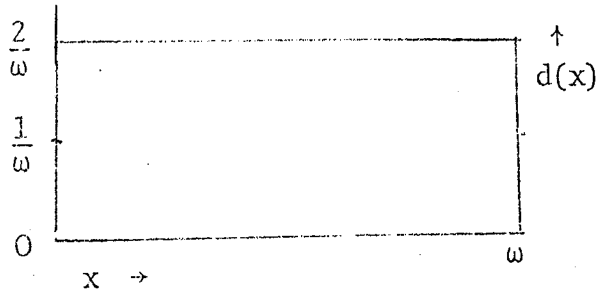
$$= \frac{\left(2 - \frac{\Delta x}{\omega}\right) \frac{\Delta x}{\omega}}{\Delta x}$$

$$= \frac{2 - \frac{\Delta x}{\omega}}{\omega}$$

Therefore

(1) $$d(x) = \frac{2}{\omega} - \frac{\Delta x}{\omega^2}$$

$$\simeq \frac{2}{\omega}, \quad 0 \leq x \leq \omega$$




(2)
$$e_x^o = \frac{T_x}{\ell_x} = \frac{\int_x^\omega \ell(a)\,da}{\ell(x)}$$

Therefore
$$e_x^o = \frac{\int_x^\omega \left(1 - \frac{a}{\omega}\right)^2 da}{\left(1 - \frac{x}{\omega}\right)^2}$$

$$= \frac{\int_x^\omega \left(1 - \frac{2a}{\omega} + \frac{a^2}{\omega^2}\right) da}{1 - \frac{2x}{\omega} + \frac{x^2}{\omega^2}}$$

$$= \frac{\left[a - \frac{2a^2}{2\omega} + \frac{a3}{3\omega^2}\right]_x^\omega}{1 - \frac{2x}{\omega} + \frac{x^2}{\omega^2}}$$

$$= \frac{\left[3\omega^2 a - 3a^2\omega + a^3\right]_x^{\omega}}{3\omega^2\left(1 - \frac{2x}{\omega} + \frac{x^2}{\omega^2}\right)}$$

$$= \frac{(3\,\omega^3 - 3\,\omega^2 + \omega^3) - (3\,\omega^2\,x - 3\omega\,x^2 + x^3)}{3\,\omega^2 - 6\,x\,\omega + 3\,x^2}$$

$$= \frac{(\omega - x)^3}{3(\omega - x)^2}$$

i.e.,

$$e_x^o = \frac{\omega - x}{3}\ , \quad 0 \leq x \leq \omega$$

(3) When $x = 0$, then

$$e_o^o = \frac{\omega}{3}$$



Therefore

$$\text{death rate} = \frac{1}{e_o^o} = \frac{3}{\omega} = \frac{3}{100} = \underline{0.03}$$

$$\text{Death rate over 90} = \frac{\ell_{90}}{T_o}$$

$$= \frac{\ell_{90}}{e_o^o\ \ell_o}$$

$$= \frac{\left(1 - \frac{90}{\omega}\right)^2}{\frac{100}{3} \times 1}$$

$$= \frac{3}{100}\ (0.10)^2$$

$$= 0.03 \times 0.01$$

$$= \underline{\underline{0.0003}}$$

Problem 31

In a life table with a radix of 10,000, $_5L_0$ is 43,684 and $_5L_5$ is 41,686. In the stationary population, the proportion under age 10 is 0.1626; in the stable population when r = 0.01, the proportion under 10 is 0.2134.

(a) What is $e_0^o$? Determine the birth rate in the stable population to a close approximation. Show your work.

(b) Determine the mean age of the population and the mean age at death in the stable population when r = 0.005.

Solution: Part (b) has already been solved in problem 2

a) Given.

$$\ell_0 = 10{,}000 \;, \quad {}_5L_0 = 43{,}684, \quad {}_5L_5 = 41{,}686,$$

$$\frac{{}_{10}L_0}{T_0} = C(< 10) \text{ in stationary pop } = 0.1626 \text{ (i.e. for } r = 0)$$

$$C(< 10) = 0.2134 \quad \text{for } r = 0.01$$

(i) $$e_0^o = \frac{T_0}{\ell_0}$$

From

$$\frac{{}_{10}L_0}{T_0} = 0.1626$$

we have

$$T_0 = \frac{{}_{10}L_0}{0.1626}$$

$$= \frac{{}_5L_0 + {}_5L_5}{0.1626}$$

$$= \frac{43{,}684 + 41{,}686}{0.1626}$$

$$= \frac{85370}{0.1626}$$

Therefore

$$e_o^o = \frac{T_o}{\ell_o} = \frac{8.5370}{0.1626} = \underline{\underline{52.50}}$$

(ii) $C(a) = b\ e^{-ra}\ p(a)$

We are given the ratio

$$\frac{C(<10) \text{ stationary } (r=0)}{C(<10) \text{ stable } (r = 0.01)} \qquad \frac{0.1626}{0.2134}$$

i.e.,

$$\frac{b_1}{b_2\ e^{-5r}} = \frac{0.1626}{0.2134}$$

Therefore

$$b_2 = b_1\ e^{5r} \times \frac{0.2134}{0.1626} = \frac{1}{e_o^o} \times e^{5 \times 0.01} \times \frac{0.2134}{0.1626}$$

$$= \frac{0.1626}{8.5370} \times e^{0.05} \times \frac{0.2134}{0.1626} = \underline{\underline{0.0263}}$$

Problem 32

(1) Suppose that successive birth cohorts experienced schedules of first marriage frequencies with the same mean age at first marriage and the same standard deviation (or horizontal scale), but that the proportion ever marrying steadily fell from cohort to cohort. Would the period (cross-sectional) mean age at first marriage rise, fall or remain constant as the reduction in the proportion ever marrying began? Explain.

Suppose a negligible proportion of marriages occur after age 35 and that the proportion ever marrying has been declining at the rate of b per year starting with the cohort now aged 35.

Let g(a) be the original first marriage frequency so that the proportion ever marrying is given by $C = \int_0^{35} g(a)da$. In this year's cross-sectional population, the first marriage frequency is $g_o(a) = g(a)\left[1 - b(35 - a)\right]$.

(2) What is the proportion every marrying in the cohort currently aged x?

(3) Show that the mean age at marriage in the cross-sectional population is

$$\bar{a}_o = \frac{\bar{a}(1 - 35b) + b(\sigma^2 + \bar{a}^2)}{1 - b(35 - \bar{a})}$$

where $\bar{a}$ and $\sigma_2$ are the mean and variance of the age at marriage in the pre-decline population. Calculation $a_o$ if b is 0.02, $\bar{a} = 24.0$ and $\sigma = 5.0$.

Solution

(1) The period mean age at first marriage will rise as the reduction on the proportion ever marrying began. This is because in this critical period, the period mean age at first marriage is more heavily weighted at the older ages, since the cohorts at the older ages have a higher schedule of first marriage frequencies.

(2) The cohort schedule of first marriage frequencies is given by

$$g_x(a) = 1 - b(35 - x)g(a)$$

Therefore the proportion ever marrying in the cohort currently aged x is given by

$$C_x = \int_0^{35} g_x(a)da$$

$$= \int_0^{35} \left[1 - b(35 - x)\right] g(a)da$$

$$= \left[1 - b(35 - x)\right] \int_0^{35} g(a)da$$

But

$$C = \int_0^{35} g(a)da \qquad \text{given}$$



Therefore

$$C_x = \left[1-b(35 - x)\right] C$$

(3) The mean age at marriage in the cross-sectional population is

$$\bar{a}_o = \int_0^{35} a\, g_o(a)da \Big| \int_0^{35} g_o(a)da$$

$$= \int_0^{35} ag(a)\left[1 - b(35-a)\right]da \Big| \int_0^{35} g(a)\left[1 - b(35-a)\right] da$$

$$= \frac{\int_0^{35}\left[ag(a) - 35bag(a) + ba^2g(a)\right]da}{\int_0^{35} g(a)da} \Bigg| \frac{\int_0^{35}\left[g(a)-35bg(a)+bag(a\right.}{\int_0^{35} g(a)da}$$

$$= \frac{(1 - 35b)\int_0^{35} ag(a)da + b\int_0^{35} a^2g(a)da \;\Big|\; \int_0^{35} g(a)da}{\left[(1 - 35b)\int_0^{35} g(a)da + b\int_0^{35} ag(a)da\right] \Big| \int_0^{35} g(a)da}$$

$$= \frac{(1 - 35b)\,\bar{a} + b(\sigma^2 + \bar{a}^2)}{(1 - 35b) + b\bar{a}}$$

$$= \frac{\bar{a}\,(1 - 35b) + b(\sigma^2 + \bar{a}^2)}{1 - b(35 - \bar{a})}$$

If

$$b = 0.02, \quad \bar{a} = 24.0 \text{ and } \sigma = 5.0$$

then

$$\bar{a}_o = \frac{24(1 - 35 \times 0.02) + 0.02(25 + 576)}{1 - 0.02(35 - 24)}$$

$$= \frac{24\,(1 - 0.7) + 0.02 \times 601}{1 - 0.22}$$

$$= \frac{7.20 + 12.02}{0.78}$$

$$= \frac{19.22}{0.78} = \underline{\underline{24.64}}$$



Problem 33

Suppose an archeologist discovers the records maintained by the administrator of the necropolis at Ephesus during the second century. The records appear to cover all of the deaths occurring in the area, and age at death seems to be accurately reported. The moving average of the annual number of deaths increases smoothly at an approximately constant rate for 100 years; the average annual number at the end of a century is 1.6 times the average number at the beginning. The proportionate distribution of deaths by age remains very nearly constant suppose the population may be assumed to be closed.

(a) Why is the average age at death not an acceptable estimate of $e_o^o$?

(b) How could an accurate estimate of $e_o^o$ be calculated?

(c) Show how the age distribution of the population could be constructed from these data on the number of deaths changing at a constant rate, and the distribution of deaths by age.

Problem 34

Two stable populations have the same proportion annually arriving at age 20, c(20).

(1) Suppose both populations include the same life table.