Physics 316—HW Set 7—Thursday, November 3, 2016

Due Thursday, November 10, 2016

I suggest starting with Problem 3, then doing Problem 1, and finally work to complete as much of Problem 2 as time and strength of will allows.

Problem 1 (20 points) Problem 7.13 in JS.

Problem 2 (30 points) This problem applies the Chirikov resonance overlap criterion to estimate the value of ϵ at which the last KAM torus is destroyed in the standard map. It will use the primary and secondary resonances centered about the period-one and period-two fixed points found in problem 1.

(a) To begin, change variables from time t to the map number $n = t/T$ in the kicked rotator Eqns. (7.86) in JS that we rewrite here:

$$\frac{d\phi}{dt} = \frac{1}{I}J \tag{1}$$

$$\frac{dJ}{dt} = \epsilon \sin\phi \sum_{m=-\infty}^{\infty} \delta(t - mT). \tag{2}$$

Show that when $I = T$ the Hamiltonian with respect to ϕ, J, n is

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}J^2 + \epsilon\cos\phi \sum_{m=-\infty}^{\infty} \delta(n - m) \\ &= \frac{1}{2}J^2 + \epsilon\cos\phi \sum_{q=-\infty}^{\infty} e^{2\pi i q n},\end{aligned} \tag{3}$$

where the second line replaces the "Dirac comb" $\sum_m \delta(n-m)$ with its Fourier expansion $\sum_q e^{2\pi i q n}$.

(b) Now we will make a first estimate for ϵ where the last KAM torus is destroyed by applying the Chirikov overlap criterion to the primary resonance. To isolate the primary resonance, we assume that ϕ is slowly varying, $d\phi/dn \ll 1$, and average the Hamiltonian (3) over n. You should find a simple pendulum Hamiltonian. The Chirikov resonance overlap criterion assumes that the last KAM torus is destroyed when the resonant region(s) fill the space. At this order of approximation, this occurs when the separatrices of the primary resonance meet each other, i.e., when $J_{\text{sep}}|_{\text{primary res.}} = \pi$. Find the height of the separatrices associated with the pendulum Hamiltonian of the primary resonance as a function of ϵ, and show that

$$J_{\text{sep}}|_{\text{primary res.}} = \pi \quad \Rightarrow \quad \epsilon = \left(\frac{\pi}{2}\right)^2 \approx 2.47. \tag{4}$$

(c) The condition found in part (b) is nearly 2.5 times too high (as the book states, the last KAM torus is destroyed when $\epsilon \approx 0.9716$). We can improve our estimate by adding the width of the second order resonance, which requires calculating the second order averaged Hamiltonian $\bar{H}_2$. We'll walk you through the calculation starting with the first order Lie perturbation equation for the generating function G_1:

$$\frac{\partial G_1}{\partial n} + \{G_1, H_0\} = \bar{H}_1 - H_1. \tag{5}$$

Note that this looks like Eq. (6.132b) from JS with $E_1 \to \bar{H}_1$ and a different sign for the Poisson bracket on the LHS. This may be due to some differing convention, but the formula I write above is what I have used before and have found in several references, so we're going to run with it here. The Hamiltonian (3) gives the $O(\epsilon^0)$ H_0 and $O(\epsilon)$ H_1; use this to show that $\bar{H}_1 = 0$ and

$$G_1 = \epsilon \sum_q \frac{\sin(\phi - 2\pi q n)}{2\pi q - J}. \tag{6}$$

This expression should remind you of the generating function that we used without derivation in class; while it has problems when $J = 2\pi q$ for integer q, it is okay near the secondary resonance $J \approx \pi$. Next, we turn to the second order equation

$$\frac{\partial G_2}{\partial n} + \{G_2, H_0\} = 2(\bar{H}_2 - H_2) - \{G_1, H_1\}, \tag{7}$$

which again resembles Eq. (6.132c) from JS with a few odd sign differences. Our problem has no second order perturbation, $H_2 = 0$, so that our task is to choose $\bar{H}_2$ to eliminate the slowly-varying part of $\{G_1, H_1\}$. Hence, show that

$$\bar{H}_2 = \frac{\epsilon^2}{4}\left\langle \sum_{q,q'} \frac{\cos[2\pi(q-q')n] - \cos[2\phi - 2\pi(q-q')n]}{(2\pi q - J)^2} \right\rangle, \tag{8}$$

where $\langle \cdot \rangle$ denotes an average over the unperturbed trajectories nearby the second-order resonance. As indicated in part (c) of Problem 1, the second-order resonance is centered at the period-two fixed point $J \approx \pi$. In this case, use the fact that the unperturbed motion in ϕ near $J \approx \pi$ is given by $\phi \approx Jn \approx \pi n$ to show that the averaged second order Hamiltonian is

$$\begin{aligned} \bar{H}_2 &= \frac{\epsilon^2}{4}\sum_q \frac{1}{(2\pi q - \pi)^2} - \frac{\epsilon^2}{4}\sum_q \frac{\cos[2\phi - 2\pi n]}{(2\pi q + \pi)^2} \\ &= \frac{\epsilon^2}{16} - \frac{\epsilon^2}{16}\cos[2\phi - 2\pi n]. \end{aligned} \tag{9}$$

Again, this gives a pendulum Hamiltonian; show the height of the separatrices associated with the second-order resonance are

$$J_{\text{sep}}\big|_{\text{2nd res.}} = \frac{\epsilon}{2}. \tag{10}$$

Now, the primary and secondary resonances overlap when the sum of the heifghts of their separatrices equals π. Show that this results in the improved Chirikov resonance overlap criterion of

$$\epsilon = \left(\sqrt{4 + 2\pi} - 2\right)^{1/2} \approx 1.46. \tag{11}$$

Problem 3 (20 points) This problem looks at chaos in the Hamiltonian restriction of the Hènon map. The general Hènon map described in the book is non-Hamiltonian, and was first introduced as a simplification of the famous Lorenz equations. Its general form can exhibit many of the unusual properties of chaos including strange attractors. Our Hamiltonian restriction has no such attractors due to area conservation, but it does serve as a nice illustration of Hamiltonian chaos; the map is given by adding to the usual linear restoring force a quadratic nonlinearity via

$$x_{n+1} = x_n \cos\tau - (y_n - x_n^2)\sin\tau \tag{12}$$

$$y_{n+1} = x_n \sin\tau + (y_n - x_n^2)\cos\tau, \tag{13}$$

with τ an angle that gives the linear phase advance of small amplitude oscillations [these are essentially Eqns. (7.90) in JS]. The Hamiltonian Hènon map is a good toy model for nonlinear systems, including circular accelerators in which the linear restoring force is supplied by quadrupole magnets while the nonlinearity is introduced by sextupole magnets that are employed to eliminate chromatic aberations.

(a) Compute the Jacobian of the map (12)-(13) and show that it conserves area. Write a computer program that performs the Hènon map for $\tau = 2\pi\alpha$ with $\alpha = 0.2114$. Pick a few initial conditions and plot 600 iterations. Your results should look something like Figure 1. While the red and green islands are near where the map marches off to infinity and not terribly important, the fifth order resonance is. Using one initial condition on one of these islands, verify that a point visits each island in succession before returning to its initial island. Hence, the associated fixed point has a period equal to 5.

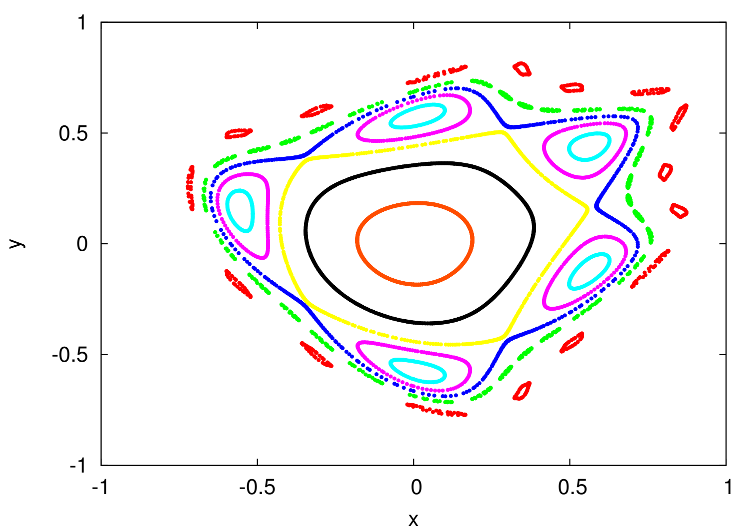

Figure 1: 50000 iterations of the Hènon map for $\tau = 2\pi \times 0.2114$.

(b) Now we look more closely at the complicated island structures that result as the rational tori break up, and at the chaotic sea that develops near the primary unstable fixed points. Plot the results of 50000 iterations of the map over the range $0.53 \leq x \leq 0.63$ and $0.1 \leq y \leq 0.2$ for the following initial conditions:

$$\begin{aligned} (x_0, y_0) &= (0.568, 0.124) & (x_0, y_0) &= (0.55, 0.1) \\ (x_0, y_0) &= (0.593, 0.17) & (x_0, y_0) &= (0.594, 0.15) \\ (x_0, y_0) &= (0.536, 0.1) & (x_0, y_0) &= (0.56, 0.1) \\ (x_0, y_0) &= (0.61, 0.175). \end{aligned}$$

Identify the three KAM trajectories and the four sets of resonant islands. These high-order resonant islands indicate elliptic orbits around fixed points with very large periods. I find that the periods of these fixed points are 140, 265, 3×81 and 5×71– verify (or disprove) that and identify which orbit has which period by finding the number of maps required to return to the same island. Finally, find another initial condition within the range that exhibits stochastic/chaotic behavior and plot 50000 iterates.

(c) This part investigates the development of the homoclinic tangle using two numerically derived facts. First, the hyperbolic fixed point used in part (b) is located at

$$(x_{\rm fp}, y_{\rm fp}) = (0.5689367002134077, 0.161844484385035) \tag{14}$$

to machine precision; second, one of the unstable manifolds associated with $(x_{\rm fp}, y_{\rm fp})$ can be locally approximated by the parametric line

$$x(t) = x_{\rm fp} + t \qquad\qquad y(t) = y_{\rm fp} - 0.785t \tag{15}$$

for $0 \leq t \ll 1$. Now, we turn to finding how the unstable manifold is mapped into the wildly oscillating homoclinic tangle by the Hènon map. To do this, plot the unstable manifold as a sequence of 5000 points for $0 \leq t \leq 0.02$, and then map this sequence of points forward until it approaches the fixed point $(x_{\rm fp}, y_{\rm fp})$ along the stable manifold. Show the resulting homoclinic tangle for 2 different number of mappings that you pick; for some guidance, you should find that the number of mappings to get an interesting picture is somewhere between 50 and 200.