CS 206 Homework 3

Fall 2020

1. We will use generating functions to determine how many ways there are to use pennies, nickels, dimes, quarters, and half-dollars to give n cents change.

   (a) Write the generating function $P(x)$ for the number of ways to use only pennies to make n cents.

   (b) Write the generating function $N(x)$ for the number of ways to use only nickels to make n cents.

   (c) Write the generating function for the number of ways to use only nickels and pennies to change n cents.

   (d) Write the generating function for the number of ways to use pennies, nickels, dimes, quarters, and half-dollars to give n cents change.

   (e) Explain how to use this function to find out how many ways are there to change 50 cents; you do not have to provide the answer or actually carry out the process.

2. We are interested in generating functions for the number of different ways to compose a bag of n donuts subject to various restrictions. For each of the restrictions below, find a closed form for the corresponding generating function.

   (a) All the donuts are chocolate and there are at least 3.

   (b) All the donuts are glazed and there are at most 2.

   (c) All the donuts are coconut and there are exactly 2 or there are none.

   (d) All the donuts are plain and their number is a multiple of 4.

(e) The donuts must be chocolate, glazed, coconut, or plain with the numbers of each flavor subject to the constraints above.

(f) Now find a closed form for the number of ways to select n donuts subject to the above constraints.

3. Miss McGillicuddy never goes outside without a collection of pets. In particular:

   - She brings a positive number of songbirds, which always come in pairs.
   - She may or may not bring her alligator, Freddy.
   - She brings at least 2 cats.
   - She brings two or more chihuahuas and labradors leashed together in a line.

   Let P_n denote the number of different collections of n pets that can accompany her, where we regard chihuahuas and labradors leashed in different orders as different collections.

   For example, $P_6 = 4$ since there are 4 possible collections of 6 pets:

   - 2 songbirds, 2 cats, 2 chihuahuas leashed in line
   - 2 songbirds, 2 cats, 2 labradors leashed in line
   - 2 songbirds, 2 cats, a labrador leashed behind a chihuahua
   - 2 songbirds, 2 cats, a chihuahua leashed behind a labrador

   (a) Let
   $$P(x) = P_0 + P_1 x + P_2 x^2 + P_3 x^3 + \cdots$$
   be the generating function for the number of Miss McGillicuddy's pet collections.

   Verify that
   $$P(x) = \frac{4x^6}{(1-x)^2(1-2x)}.$$

   (b) Find a closed form expression for P_n.

4. Generating functions provide an interesting way to count the number of strings of matched brackets. To do this, we'll use a description of these strings as the set GoodCount of strings of brackets with a good count.

Namely, one precise way to determine if a string is matched is to start with 0 and read the string from left to right, adding 1 to the count for each left bracket and subtracting 1 from the count for each right bracket. For example, here are the counts for two strings:

```
  [ ]  ] [ [ [ [ [ ] ] ] ]
0 1 0 -1 0 1 2 3 4 3 2 1 0

  [ [ [ ] ] [ ] ] [ ]
0 1 2 3 2 1 2 1 0 1 0
```

A string has a *good count* if its running count never goes negative and ends with 0. So the second string above has a good count, but the first one does not because its count went negative at the third step.

Let GoodCount equal the set of strings of square brackets that have good counts. The matched strings can now be characterized precisely as this set of strings with good counts.

Let c_n be the number of strings in GoodCount with exactly n left brackets, and let $C(x)$ be the generating function for these numbers:

$$C(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

(a) The *wrap* of a string s is the string, `[` s `]` , that starts with a left bracket followed by the characters of s, and then ends with a right bracket. Explain why the generating function for the wraps of strings with a good count is $xC(x)$.

Hint: The wrap of a string with good count also has a good count that starts and ends with 0 and remains *positive* everywhere else.

(b) Explain why, for every string s with a good count, there is a unique sequence of strings $s_1, \ldots, s_k$ that are wraps of strings with good counts and $s = s_1 \cdots s_k$. For example, the string $r =$ `[ [ ] ] [ ] [ [ ] [ ] ]` $\in$ GoodCount equals $s_1 s_2 s_3$ where

$s_1 =$ `[ [ ] ]`

$s_2 =$ `[ ]`

$s_3 =$ `[ [ ] [ ] ]`

and this is the only way to express r as a sequence of wraps of strings with good counts.

(c) Conclude that

$$C = 1 + xC + (xC)^2 + \cdots + (xC)^n + \cdots , \tag{1}$$

so

$$C = \frac{1}{1 - xC}, \tag{2}$$

and hence

$$C = \frac{1 \pm \sqrt{1-4x}}{2x}. \tag{3}$$

Let $D(x) = 2xC(x)$. Expressing D as a power series

$$D(x) = d_0 + d_1 x + d_2 x^2 + \cdots ,$$

we have

$$c_n = \frac{d_{n+1}}{2}. \tag{4}$$

(d) Use equations (3), (4), and the value of c_0 to conclude that

$$D(x) = 1 - \sqrt{1-4x}.$$

(e) Prove that

$$d_n = \frac{(2n-3)\cdot(2n-5)\cdots 5\cdot 3\cdot 1\cdot 2^n}{n!}.$$

Hint: $d_n = D^{(n)}(0)/n!$

(f) Conclude that

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$