Deviation from the mean

IQ

Average IQ is 100.

How many people could have an IQ of 300 or more?

IQ

Let A be the people with IQ less than 300

Let B be the people with IQ at least 300

$$100 = \frac{1}{n}\left(\sum A + \sum B\right)$$

IQ

Let A be the people with IQ less than 300

Let B be the people with IQ at least 300

$$100 = \frac{1}{n} (np \cdot 300)$$

IQ

Let A be the people with IQ less than 300

Let B be the people with IQ at least 300

$$100 = \frac{1}{n}(np \cdot 300)$$

$$p = 1/3$$

So at most $1/3$ of the people can be in B.

IQ

How many could have at least 200?

$$100 = \frac{1}{n}\left(np \cdot 200\right)$$

$$p = 1/2$$

IQ

How many could have at least x?

$$100 = \frac{1}{n}\left(np \cdot x\right)$$

$$100 = p \cdot x$$

$$p = \frac{100}{x}$$

IQ

Given a random variable R with a mean of $\mathbb{E}(R)$, how often could it be at least x?

$$\mathbb{E}(R) = \frac{1}{n}(np \cdot x)$$

$$\mathbb{E}(R) = p \cdot x$$

$$p = \frac{\mathbb{E}(R)}{x}$$

Markov's theorem

For a nonnegative random variable R and $x > 0$:

$$\mathbb{P}(R \geq x) \leq \frac{\mathbb{E}(R)}{x}$$

Markov's theorem (proof)

For a nonnegative random variable R,

let I be an indicator random variable for the event that $R \geq x$.

Then

$$xI \leq R$$

$$x\mathbb{E}\left(I\right) \leq \mathbb{E}\left(R\right)$$

$$x\mathbb{P}\left(R \geq x\right) \leq \mathbb{E}\left(R\right)$$

$$\mathbb{P}\left(R \geq x\right) \leq \frac{\mathbb{E}\left(R\right)}{x}$$

Markov's theorem

Let R be the roll of a die.

What's the probability of getting at least a 6?

$$\mathbb{P}\left(R \geq x\right) \leq \frac{\mathbb{E}\left(R\right)}{x}$$

$$\mathbb{P}\left(R \geq 6\right) \leq \frac{7/2}{6} = \frac{7}{12}$$

Markov's theorem

Markov's theorem:

$$\mathbb{P}(R \geq x) \leq \frac{\mathbb{E}(R)}{x}$$

A corollary: let $x = c \cdot \mathbb{E}(R)$

$$\mathbb{P}(R \geq c \cdot \mathbb{E}(R)) \leq \frac{1}{c}$$

IQ

Suppose the average IQ of Rutgers CS students is 150.

How many could be over 200?

IQ

Suppose the average IQ of Rutgers CS students is 150.

How many could be at least 200?

$$\mathbb{P}\left(R \geq 200\right) \leq \frac{\mathbb{E}\left(R\right)}{200} = \frac{150}{200} = \frac{3}{4}$$

Chebyshev's theorem

Let's let $T = |R - \mathbb{E}(R)|$, another random variable.

Then Markov's theorem gives us:

$$\mathbb{P}(T \geq x) \leq \frac{\mathbb{E}(T)}{x}$$

Plugging in the definition of T:

$$\mathbb{P}(|R - \mathbb{E}(R)| \geq x) \leq \frac{\mathbb{E}(|R - \mathbb{E}(R)|)}{x}$$

Chebyshev's theorem

Using Markov:

$$\mathbb{P}\left(|R - \mathbb{E}\left(R\right)| \geq x\right) \leq \frac{\mathbb{E}\left(|R - \mathbb{E}\left(R\right)|\right)}{x}$$

We can square both sides of the first inequality:

$$\mathbb{P}\left(|R - \mathbb{E}\left(R\right)| \geq x\right) = P\left((R - \mathbb{E}\left(R\right))^2 \geq x^2\right)$$

Applying Markov to the right-hand side:

$$\mathbb{P}\left((R - \mathbb{E}\left(R\right))^2 \geq x^2\right) \leq \frac{\mathbb{E}\left((R - E(R))^2\right)}{x^2}$$

Variance

The numerator here is called the *variance* of R:

$$\mathsf{Var}(R) = \mathbb{E}\left((R - \mathbb{E}(R))^2\right)$$

Chebyshev's theorem

Chebyshev's theorem:

$$\mathbb{P}\left(|R - \mathbb{E}\left(R\right)| \geq x\right) \leq \frac{\mathsf{Var}(R)}{x^2}$$

Variance

If R is the roll of a single die, what is $\mathsf{Var}(R)$?

$$\mathsf{Var}(R) = \mathbb{E}\left((R - \mathbb{E}(R))^2\right)$$

We know $\mathbb{E}(R) = 7/2$...

Variance

If R is the roll of a single die, what is $\mathsf{Var}(R)$?

$$\mathsf{Var}(R) = \mathbb{E}\left((R - 7/2)^2\right)$$

Consider $(R - 7/2)^2$ as a random variable.

Variance

Consider $(R - 7/2)^2$ as a random variable.

R	$R - 7/2$	$(R - 7/2)^2$
1	-2.5	6.25
2	-1.5	2.25
3	-0.5	0.25
4	0.5	0.25
5	1.5	2.25
6	2.5	6.25

So $\mathbb{E}\left((R - 7/2)^2\right) = \frac{1}{6} \cdot 6.25 + \frac{1}{6} \cdot 2.25 + \cdots + \frac{1}{6} \cdot 6.25 \approx 2.92$

Gambling games

Say there are two gambling games we can play:

Game A:

- Win \$2 with probability 2/3
- Lose \$1 with probability 1/3

Game B:

- Win \$1002 with probability 2/3
- Lose \$2001 with probability 1/3

Gambling games

Let A and B denote the payoffs for the two games.

$$\mathbb{E}(A) = 2 \cdot \frac{2}{3} + (-1) \cdot \frac{1}{3} = 1$$

$$\mathbb{E}(B) = 1002 \cdot \frac{2}{3} + (-2001) \cdot \frac{1}{3} = 1$$

Gambling games

But let's calculate Var(A):

A	$A-1$	$(A-1)^2$
2	1	1
-1	-2	4

$$\begin{aligned}\mathbb{E}\left((A - E(A))^2\right) &= 1 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} \\ &= 2\end{aligned}$$

And now Var(B):

B	$B-1$	$(B-1)^2$
1002	1001	1002001
-2001	-2002	4008004

$$\begin{aligned}\mathbb{E}\left((B-E(B))^2\right) &= 1002001 \cdot \frac{2}{3} + 4008004 \cdot \frac{1}{3} \\ &= 2004002\end{aligned}$$

Gambling games

Game	$\mathbb{E}(\cdot)$	$\mathrm{Var}(\cdot)$
A	1	2
B	1	2004002

The payoff of A is usually close to \$1.

The payoff of B can be very far from this.

Standard deviation

The square root of variance is called standard deviation:

$$
\begin{aligned}
\sigma_R &= \sqrt{\mathsf{Var}(R)} \\
&= \sqrt{\mathbb{E}\left((R - \mathbb{E}\left(R\right))^2\right)}
\end{aligned}
$$

Gambling games

Game	$\mathbb{E}(\cdot)$	Var$(\cdot)$	$\sigma(\cdot)$
A	1	2	1.414
B	1	2004002	1415.6

Gambling games

Let's modify the games slightly:

Game A:

- Win \$1 with probability 1/2
- Lose \$1 with probability 1/2

Game B:

- Win \$1000 with probability 1/2
- Lose \$1000 with probability 1/2

Both now have an expectation of 0.

Gambling games

A	$A - 0$	$(A - 0)^2$
1	1	1
-1	-1	1

$$\begin{aligned}\mathbb{E}\left((A - \mathbb{E}(A))^2\right) &= 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \\ &= 1\end{aligned}$$

Gambling games

And now Var(B):

B	$B-0$	$(B-0)^2$
1000	1000	1000000
-1000	-1000	1000000

$$\begin{aligned}\mathbb{E}\left((B-\mathbb{E}(B))^2\right) &= 1000000 \cdot \frac{1}{2} + 1000000 \cdot \frac{1}{2} \\ &= 1000000\end{aligned}$$

Gambling games

Game	$\mathbb{E}(\cdot)$	$\text{Var}(\cdot)$	$\sigma(\cdot)$
A	0	1	1
B	0	1000000	1000

Corollary

Let σ be the SD of a random variable R, and let $x = c\sigma$.

Then Chebyshev's inequality gives:

$$\mathbb{P}\left(|R - \mathbb{E}\left(R\right)| \geq c \cdot \sigma\right) \leq \frac{1}{c^2}$$

IQ

Revisiting the IQ example, from Markov we know:

$$\mathbb{P}\left(R \geq 300\right) \leq \frac{1}{3}$$

Let's say we learn that the SD is 15.

IQ

We can rewrite this as:

$$\mathbb{P}\left(R \geq 300\right) = \mathbb{P}\left(|R - 100| \geq 200\right)$$

Then we can apply Chebyshev:

$$\begin{aligned}\mathbb{P}\left(|R - 100| \geq 200\right) &\leq \frac{\mathsf{Var}(R)}{200^2} \\ &= \frac{15^2}{200^2} \\ &\approx 0.56\%\end{aligned}$$

Properties

Let's set $\mu = \mathbb{E}\left(R\right)$. Then:

$$\begin{aligned}
\text{Var}(R) &= \mathbb{E}\left((R-\mu)^2\right) \\
&= \mathbb{E}\left(R^2 - 2\mu R + \mu^2\right) \\
&= \mathbb{E}\left(R^2\right) - 2\mu\mathbb{E}\left(R\right) + \mu^2 \\
&= \mathbb{E}\left(R^2\right) - 2\mu^2 + \mu^2 \\
&= \mathbb{E}\left(R^2\right) - \mu^2
\end{aligned}$$

Hence:

$$\begin{aligned}
\text{Var}(R) &= \mathbb{E}\left(R^2\right) - \left(\mathbb{E}\left(R\right)\right)^2 \\
&= \mathbb{E}\left(R^2\right) - \mathbb{E}^2(R)
\end{aligned}$$

Variance of a Bernoulli variable

What is the variance of a Bernoulli variable B?

We know $\mathbb{E}(B) = p$.

B	$B-p$	$(B-p)^2$	Prob.
0	$-p$	p^2	$1-p$
1	$1-p$	$(1-p)^2$	p

$$\begin{aligned}
\mathbb{E}\left((B-E(B))^2\right) &= (1-p)\cdot p^2 + p\cdot(1-p)^2 \\
&= p^2 - p^3 + p\cdot(1-2p+p^2) \\
&= p^2 - p^3 + p - 2p^2 + p^3 \\
&= p^2 + p - 2p^2 \\
&= p - p^2
\end{aligned}$$

Variance of a Bernoulli variable

What is the variance of a Bernoulli variable B?

We know $\mathbb{E}\left(B\right) = p$.

And $\mathbb{E}\left(B^2\right) = p$, since B is either 0 or 1.

Then by the previous property:

$$\begin{aligned}
\mathsf{Var}(B) &= \mathbb{E}\left(B^2\right) - \left(\mathbb{E}\left(B\right)\right)^2 \\
&= p - p^2 \\
&= p(1-p) \\
&= pq
\end{aligned}$$

Properties

$$\begin{aligned}
\mathsf{Var}(aR) &= \mathbb{E}\left((aR)^2\right) - (\mathbb{E}\,(aR))^2 \\
&= \mathbb{E}\left(a^2R^2\right) - (\mathbb{E}\,(aR))^2 \\
&= a^2\mathbb{E}\left(R^2\right) - (a\mathbb{E}\,(R))^2 \\
&= a^2\mathbb{E}\left(R^2\right) - a^2(\mathbb{E}\,(R))^2 \\
&= a^2(\mathbb{E}\left(R^2\right) - (\mathbb{E}\,(R))^2) \\
&= a^2\mathsf{Var}(R)
\end{aligned}$$

Properties

$$\begin{aligned}
\mathsf{Var}(R+b) &= \mathbb{E}\left((R+b)^2\right) - \left(\mathbb{E}\left(R+b\right)\right)^2 \\
&= \mathbb{E}\left(R^2 + 2bR + b^2\right) - \left(\mathbb{E}\left(R\right) + b\right)^2 \\
&= \mathbb{E}\left(R^2\right) + 2b\mathbb{E}\left(R\right) + b^2 - \left(\left(\mathbb{E}\left(R\right)\right)^2 + 2b\mathbb{E}\left(R\right) + b^2\right) \\
&= \mathbb{E}\left(R^2\right) - \left(\mathbb{E}\left(R\right)\right)^2 \\
&= \mathsf{Var}(R)
\end{aligned}$$

Properties

$$
\begin{aligned}
\sigma(aR+b) &= \sqrt{\mathsf{Var}(aR+b)} \\
&= \sqrt{\mathsf{Var}(aR)} \\
&= \sqrt{a^2\mathsf{Var}(R)} \\
&= |a|\sqrt{\mathsf{Var}(R)} \\
&= |a|\sigma(R)
\end{aligned}
$$

Properties

Given two *independent* random variables R and S,

$$\mathsf{Var}(R + S) = ?$$

Properties

Aside: we can assume $\mathbb{E}\left(R\right) = \mathbb{E}\left(S\right) = 0$.

If not, let $R' = R - \mathbb{E}\left(R\right)$ and $S' = S - \mathbb{E}\left(S\right)$.

Then $\mathrm{Var}(R') = \mathrm{Var}(R)$ and $\mathrm{Var}(S') = \mathrm{Var}(S)$.

And

$$\begin{aligned} \mathrm{Var}(R+S) &= \mathrm{Var}(R' + \mathbb{E}\left(R\right) + S' + \mathbb{E}\left(S\right)) \\ &= \mathrm{Var}(R' + S') \end{aligned}$$

Properties

Another aside: if $\mathbb{E}(R) = 0$, then

$$\begin{aligned} \text{Var}(R) &= \mathbb{E}\left(R^2\right) - (\mathbb{E}(R))^2 \\ &= \mathbb{E}\left(R^2\right) \end{aligned}$$

Properties

Given two *independent* random variables R and S,

Note that $\mathrm{Var}(R) = \mathbb{E}\left(R^2\right)$ and $\mathrm{Var}(S) = \mathbb{E}\left(S^2\right)$.

Then

$$\begin{aligned}
\mathrm{Var}(R+S) &= \mathbb{E}\left((R+S)^2\right) \\
&= \mathbb{E}\left(R^2 + 2RS + S^2\right) \\
&= \mathbb{E}\left(R^2\right) + 2\mathbb{E}\left(RS\right) + \mathbb{E}\left(S^2\right) \\
&= \mathbb{E}\left(R^2\right) + 2\mathbb{E}\left(R\right)\mathbb{E}\left(S\right) + \mathbb{E}\left(S^2\right) \\
&= \mathbb{E}\left(R^2\right) + \mathbb{E}\left(S^2\right) \\
&= \mathrm{Var}(R) + \mathrm{Var}(S)
\end{aligned}$$

Variance of a binomial distribution

If J has an (n, p)-binomial distribution,

$$J = \sum_{k=1}^{n} I_k$$

where the I_k are mutually independent and $\mathbb{P}\left(I_k = 1\right) = p$.

Variance of a binomial distribution

Then

$$
\begin{aligned}
\mathsf{Var}(J) &= \mathsf{Var}\left(\sum_{k=1}^{n} I_k\right) \\
&= \sum_{k=1}^{n} \mathsf{Var}(I_k) \\
&= \sum_{k=1}^{n} pq \\
&= npq
\end{aligned}
$$