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Note for Grader: I apologize for the long answers.

1. The policy that is more secure is the one that generates a set of passwords such that the set has a higher cardinality than the other set. It is harder to guess a specific password sequence in a set which contains more sequences than less.

   a. Policy 1: Allows lowercase and uppercase characters (26 + 26), a number (10), or some special characters (5). This available set from which we can choose an element has cardinality 26+26+10+5 = 67.
      i. Since the password length will be exactly 16 characters, our sequence will contain an element chosen from the set P_1, which has 67 characters, 16 times.
      ii. $|P^{16}| = |P|^{16}$ {Product Rule}
      iii. $|P^{16}| = 67^{16}$
      iv. Hence the total number of password sequences is 67^{16}.

   b. Policy 2: Allows only lowercase characters (26). The available set from which we can choose an element thus has cardinality 26.
      i. Since the password length will be 24 characters, our sequence will contain an element chosen from the set P_1, which has 26 characters, 24 times.
      ii. $|P^{24}| = |P|^{24}$ {Product Rule}
      iii. $|P^{24}| = 26^{24}$
      iv. Hence the total number of password sequences is 26^{24}.

   c. $67^{16} - 26^{24} = -9x10^{33}$. Hence 26^{24} is greater than 67^{16}.
      i. The set of passwords formed from policy 2 has a greater cardinality than the set of passwords formed from policy 1. Thus policy 2 is more secure.

2. How many numbers between 1 and 1,000,000 contain either a 1 or a 5 (or both).
    a. Considering the other numbers in the range [1,1,000,000), we have numbers which contain either 1,2,3,4,5, or 6 digits.

    b. We can simplify this problem to use the Difference formula |S| - |(S-A)| = |A| where set S is the set of numbers which contain 1-6 digits and the set S-A is the set of numbers which contain 1-6 digits but do not contain 1s and also do not contain 5s. Thus the set A would be the set of numbers which have 1-6 digits which contains either a 1 or a 5, or both.

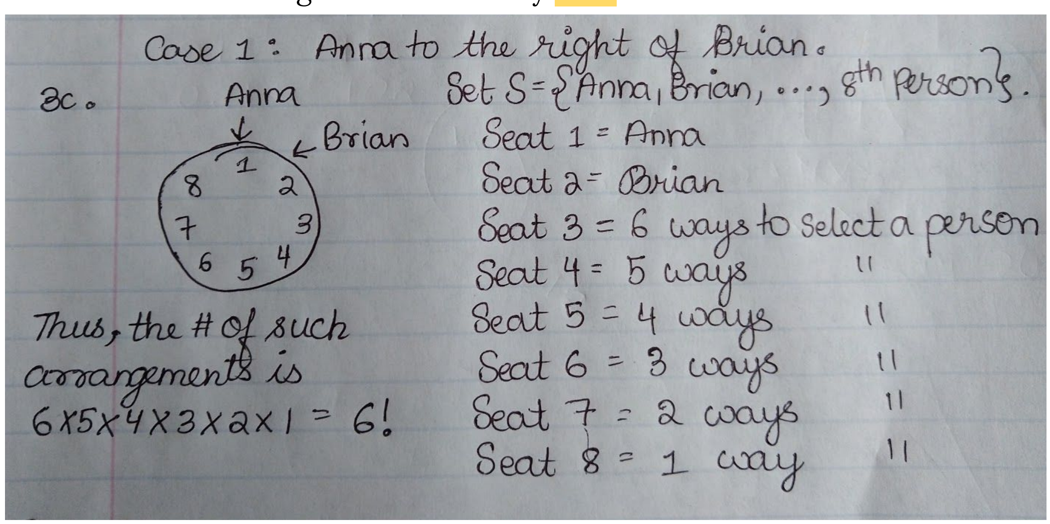
    c. There are approximately 1,000,000 numbers which contain either 1,2,3,4,5, or 6 digits [0, 999999]. So |S| = 1,000,000.

    d. _ _ _ _ _ _ Let the set P = {0,2,3,4,6,7,8,9}. |P| = 8.
    e. The cardinality of the set |P x P x P x P x P x P| is equal to $|P^6| = |P|^6$ by Product rule. The cardinality of such a set is 8^6.
    f. The set |P x P x P x P x P x P| is the same as the set (S-A) and hence the cardinality of S-A, |S-A|, is 8^6.

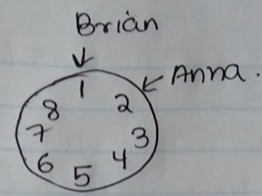
    g. Therefore the cardinality of A, |A|, is equal to |S| - |S-A| = 1,000,000 - 8^6 = 737,856. Factoring into account the only 7 digit number 1,000,000, we can say that there are 737,856 + 1 numbers between 1 and 1,000,000 which contain either a 1 or a 5 (or both).

    h. Answer: 737,857.

3. Eight students (Anna, Brian, Carol, ...) are to be seated around a cir-cular table with eight seats, and two seatings are considered the same arrangement if each student has the same student to their right in both settings.
    a. How many seatings of the eight students are there?
        i. There are 8! Seatings of the eight students {permutation without repetition}
    b. How many arrangements of the eight students are there?
        i. Seatings to Arrangements is a k to 1 function where k in this case is the number of students. This is a result of the fact that rotating the table at which the students are seated (student-1) times gives the same arrangement as the table initially. Hence |S| = k * |A|. |A| = |S|/k = 8!/8. Thus there are |A| = 7! arrangements.
    c. How many arrangements of the eight students are there with Anna sitting next to Brian?
        i. Seatings are considered the same arrangement if each student has the same student to their right in both settings (given) . Therefore (Anna, Brian, _, _, _, _, _, _) and (Brian, Anna, _, _, _, _, _, _) are different arrangements as in the first one, Anna is to the right of Brian and in the second one Brian is to the right of Anna. Therefore the # of arrangements with Anna and Brian sitting next to each other is the sum of two disjoint sets as illustrated (where Anna and Brian are pivots and there is no rotation of the table and thus no duplicates as the pivots implies a strict way of selecting the rest of the people). The number of such arrangements is exactly 2*6!.



3c.



Case 2: Brian to right of Anna

Total Arrangements: $\boxed{6!}$

Seat 3 : 6 ways to select a person
Seat 4 : 5 ways "
Seat 5 : 4 ways "
Seat 6 : 3 ways "
Seat 7 : 2 ways "
Seat 8 : 1 way "

(Anna, Brian, _ _ _ _ _ _) ∪ (Brian, Anna, _ _ _ _ _ _)

$= 6! + 6! = \boxed{2 \cdot 6!}$

3d. There are again two cases:
Case 1: (Anna, Brian, Carol, . . . , 8th Person)
Case 2: (Carol, Brian, Anna, . . . , 8th Person).
Choosing the remaining 5 people in both cases to make the above two arrangements is 5! in each case.

(Case 1) ∪ (Case 2) {disjoint}
= (Case 1) + (Case 2) {sum rule}
$= 5! + 5! = \boxed{2 \cdot 5!}$

d. How many arrangements of the eight students are there with Brian sitting next to both Anna and Carol?

i. Brian sits next to both Anna and Carol in the two cases in figure 3d. In order to sit next to both he must be in the middle of Anna and Carol. In case 1 we have a different arrangement than in case 2 because in case 1, Anna is to the right of Brian and Brian is to the right of Carol, while in case 2, Carol is to the right of Brian, and Brian is to the right of Anna. Using these three as the pivot (therefore no double counting) we find that there are 2*5! ways for Brian to sit next to both Anna and Carol.

e. How many arrangements of the eight students are there with Brian sitting next to either Anna or Carol.
    i. The number of arrangements of Brian sitting next to Anna is 2*6!. The number of arrangements of Brian sitting next to Carol is also 2*6! (same problem). Therefore, Brian can sit next to Anna or Carol in 2*6! + 2*6! ways = 2880 ways.
    ii. Suppose we only had 4 people, then there would be these seatings:
        1. ABCD
        2. ABDC
        3. BACD
        4. BADC
        5. CBAD
        6. CBDA
        7. BCAD
        8. BCDA
        9. Out of these the highlighted pairs are duplicates. Note if we had only 4 people, Brian could sit next to Anna 2*2! times and Brian could sit next to Carol 2*2! ways. The sum of those is 8. Here the number of duplicates is thus when Brian is sitting next to both Anna and Carol which is 2*2!/2 (duplicate). This is 2*2! + 2*2! - 2*2!/2 = 6 ways in which Brian can sit next to either Anna or Carol. Thus we subtracted 2*2!/2 = 2*1 = 2 from our initial answer of 8 to get 6. I.e We summed the union of Brian sitting next to either and subtracted from Brian sitting next to both.
    iii. Therefore from our initial answer of 2*6! + 2*6! = 2880, we must subtract our answer from part d, thus we get 2880-2*5! = 2880 - 240 = 2640.
    iv. There are 2640 ways for Brian to sit next to either Anna or Carol.

4. The faculty of the department contains exactly 15 members. Of these six are women and nine are men. In order to find how many ways we can select a committee of 5 members (I will represent these with _ _ _ _ _) in which there is at least one woman, we can find the number of ways to form a committee and subtract from that number the number of ways to select a committee of all men.
    a. Let S be all possible formable committees. |S| is 15 choose 5. {no order, no rep} where 15 is derived from all men + women and 5 is the size of committee.
        i. |S| = (15!)/(5! * 10!) = 3003.
    b. Let S-A be all possible committees such that there are only men on the committee.
        i. Thus |S-A| is 9 choose 5
        ii. |S-A| = (9!)/(5!*4!) = 126
    c. Let set A be S - (S-A). Therefore set A is the set of all committees such that there is at least one woman (all committees - all male committees).
    d. |A| = 3003 - 126 = 2877.
    e. There are 2877 ways to select a committee of 5 members such that there is at least 1 woman on the committee.

5. Condensed Version to prepare you for the logic for the EXTENDED VERSION:

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}
each element %3 mapping exactly is below.
{1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0}

Set A Set where {1, 2, ... 15} %3 = 0 ⇒ {3, 6, 9, 12, 15}
Set B Set where {1, 2, ... 15} %3 = 1 ⇒ {1, 4, 7, 10, 13}
Set C Set where {1, 2, ..., 15} %3 = 2 ⇒ {2, 5, 8, 11, 14}

(a+b+c)%3 = ((a%3)+(b%3)+(c%3))%3
ex: (6+1+11)%3 = ((6%3)+(1%3)+(11%3))%3
⇒ (18)%3 = (0+1+2)%3
⇒ 0 = 0

{3, 6, 9, 12, 15} {1, 4, 7, 10, 13} {2, 5, 8, 11, 14}
0 1 2

_ _ _

Sum will only be divisible when we have a group (Situation) as follows: [0 0 0], [1 1 1], [2 2 2] or [0 1 2].

Since: (0+0+0) %3 = 0
(1+1+1) %3 = 1
(2+2+2) %3 = 2
(0+1+2) %3 = 0

Case 1 :

0 0 0 ; A group of 3 distinct elements such that each element is divisible by 3.

Since 0s represent an element from $\{3,6,9,12,15\}$ we have $\binom{|\{3,6,9,12,15\}|}{3} = \binom{5}{3} = \frac{5!}{3!(5-3)!} = \boxed{10}$ such groups

Case 2 : 111 : A group of 3 distinct elements such that each element is divisible by 3.

1s represent $\{1,4,7,10,13\}$

groups with this case = $\binom{5}{3} = \boxed{10 \text{ such groups}}$

Case 3 : 222. 2 represents $\{2,5,8,11,14\}$

groups with this case = $\binom{5}{3} = \boxed{10}$.

Case 4 : One Element from $\{3,6,9,12,15\}$, $\{1,4,7,10,13\}$, and $\{2,5,8,11,14\}$.

groups with this case = $\binom{5}{1}\binom{5}{1}\binom{5}{1}$

$\Rightarrow \frac{5!}{4!} \times \frac{5!}{4!} \times \frac{5!}{4!} = 5 \times 5 \times 5 = \boxed{125 \text{ such groups.}}$

Total such groups = $125 + 10 + 10 + 10 = \boxed{155}$

5. EXTENDED: How many ways can we pick three distinct integers from {1,2,...15} such that their sum is divisible by 3?

a. Note that 15 choose 3 is itself 455 so the number of ways that we pick three distinct integers such that their sum is divisible by 3 must be less than 455:
   i. 15!/(3!*12!) = 15*14*13/6 = 455.

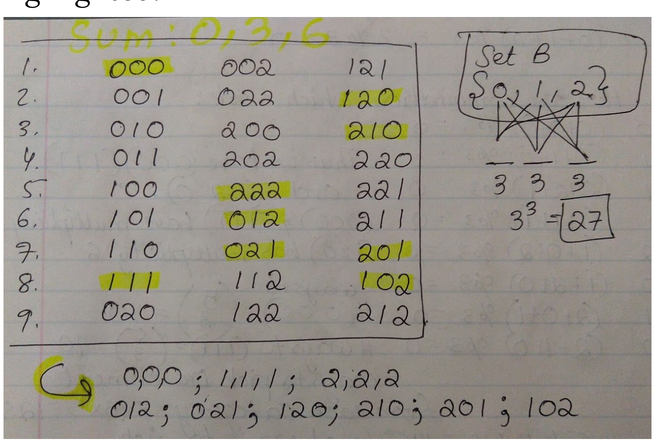
b. (a+b+c)%3 is equivalent to ((a mod 3) + (b mod 3) + (c mod 3)) mod 3.
   i. Take a=3, b=11, c=7 (chosen at random), then
   ii. (3+11+7)%3 = 21%3 = 0.
   iii. ((3 mod 3) + (11 mod 3) + (7 mod 3)) mod 3 = (0+2+1) mod 3 = 3%3 = 0.

c. Therefore we can simplify the problem by writing the set {1,2,...15} as:
   i. Set A = {1 mod 3, 2 mod 3, ... 15 mod 3}
   ii. Set A = {1,2,0, 1,2,0, 1,2,0, 1,2,0, 1,2,0}
   iii. We now have a set A which simplifies this problem.

Let us map each element:
{1, 2, 3,   4, 5, 6,   7, 8, 9,   10, 11, 12, 13, 14, 15}
{1, 2, 0,   1, 2, 0,   1, 2, 0,   1, 2, 0, 1, 2, 0}

d. Let us create three sets:
   i. A1 = {1,4,7,10,13} (the map from {1,2,...15}to 1 in A),
   ii. A2 = {2,5,8,11,14} (the map from {1,2,...15} to 2 in A),
   iii. A0 = {3,6,9,12,15} (the map from {1,2,...15) to 0 in A).

Since we must pick three distinct integers from {1,2,...15} to check whether their sum is divisible by 3, let us pick create a set of sequences of size 3 such that their elements come from either A0, A1, or A2, represented by (0,1,2) respectively. If set B = {0,1,2} the cardinality of the set of sequences of size 3, (_ , _ , _) is 3^3 = 27. From these set of sequences let us identify those which add up to multiples of 3, i.e the three members from the sequence should add up to a minimum of 0 (0,0,0), 3, or a maximum of 6 (2,2,2). Those whose sum does add up to either 0, 3, or 6 are highlighted.



e. Though there are 27 sequences, we want three distinct numbers from our set {1,2,..15}. Therefore the bottom line of the picture containing 012 021 120 210 201 102 are not distinct. And thus we have actually only four groups: 000 111 222 and 012 where the digits are in no particular order.

f. The mod of the sum of these digits is a multiple of three. Therefore their corresponding elements in the set {1,2,...15}summed together are also multiples of three.

   i. (0+0+0) mod 3 = 0. (1+1+1) mod 3 = 0. (2+2+2) mod 3 = 0. (One 0 + One 1 + One 2 (regardless of order)) mod 3 = 3 mod 3 = 0. Thus all these four groups' (000,111,222, 012(can be rearranged)) digits summed are divisible by 3.

g. Each digit in the group represents three distinct elements chosen from either the set A0, A1, A2 or the last group (012) which represents one element chosen from A0 and A1 and A2.

   i. 000 111 222 are three distinct elements (no ordering, no repetition) chosen from A0, A1, and A3 respectively.
   ii. 000 represents {3,6,9,12,15} choose 3. => 5!/(3! * 2!) = 10
   iii. 111 represents {1,4,7,10,13} choose 3. => 5!/(3! * 2!) = 10
   iv. 222 represents {2,5,8,11,14} choose 3. => 5!/(3! * 2!) = 10.
   v. 012 represents choosing one element {_ _ _} from each A0, A1, and A3. Thus we have 5 * 5 * 5 choices = 125.

h. In total we have 10+10+10+125 = 155 groups of 3 distinct integers from {1,2,...,15} such that their sum is divisible by 3.