Binomials and multinomials

CS 206: Discrete Structures II

Fall 2020

Splitting into multiple subsets

How many ways can we split a 6-element set into subsets of size 1, 2, and 3?

Splitting into multiple subsets

In general,

$$\binom{n}{k_1, k_2, ..., k_n} = \frac{n!}{k_1!k_2!...k_n!}$$

where $\sum_i k_i = n$

This is called a multinomial coefficient.

Permutations with repetition

How many ways can we rearrange the letters in CAT?

How many ways can we rearrange the letters in APPLE?

Permutations with repetition

How many ways can we rearrange the letters in BOOKKEEPER?

We have:

- 2 Os
- 2 Ks
- 3 Es
- 1 B
- 1 P
- 1 R

Permutations with repetition

This is a $(1,2,2,3,1,1)$ split of $[10]$

Suppose we create a map from letter to position

By the subset split rule,

$$\frac{10!}{1!2!2!3!1!1!}$$

Permutations with repetition

Let $l_1, ..., l_m$ be distinct objects

If we have k_1 l_1's, k_2 l_2's, ..., k_m l_m's,

then the number of permutations is

$$\binom{k_1 + \cdots + k_m}{k_1, ..., k_m}$$

Example

If we take a 20 mile walk, but want 5 miles to be in each of the four cardinal directions (N, S, E, W).

Binomial theorem

$$(a+b)^2 =$$

$$(a+b)^3 =$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Binomial theorem

$$(a+b)^5$$

What is the coefficient of a^2b^3?

Multinomial theorem

$$(a+b+c+d)^5$$

What is the coefficient of ab^2c^2?

Multinomial theorem

$$(b+o+k+e+p+r)^{10}$$

What is the coefficient of $bo^2k^2e^3pr$?

Multinomial theorem

$$(x_1 + \cdots + x_m)^n = \sum_{k_1,\ldots,k_m} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

where $\sum_i k_i = n$

Symmetry:

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinatorial proofs

Say n people apply for a team, and k are selected.

Consider a particular applicant. Either they're selected or not.

- if selected, then there are $\binom{n-1}{k-1}$ ways to select other members
- if not, then there are $\binom{n-1}{k}$ ways to select other members

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Combinatorial proofs

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

can be visualized as Pascal's triangle:

Example

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Example

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

- n applicants
- hire r of them
- promote k to be managers

Example (proved algebraically)

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

$$\begin{aligned}\binom{n}{r}\binom{r}{k} &= \frac{n!}{r!(n-r)!}\frac{r!}{k!(r-k)!} \\ &= \frac{n!}{(n-r)!k!(r-k)!}\end{aligned}$$

$$\begin{aligned}\binom{n}{k}\binom{n-k}{r-k} &= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)!(n-k-(r-k))!} \\ &= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)!(n-r)!} \\ &= \frac{n!}{k!(r-k)!(n-r)!}\end{aligned}$$

Integer solutions to equations

How many non-negative integer solutions are there to $a + b + c = 12$?

Poker

- 52 card deck
- 4 suits (hearts, diamonds, spades, clubs)
- 13 ranks (ace, 2, 3, ..., 10, jack, queen, king)

Note: an ace is sometimes treated as low (before 2) and sometimes as high (after king)

Poker

How many 5 card hands are there?

General approach

- develop sequence that describes the hand
- enumerate number of sequences

Four of a kind

For example: 8h, 2h, 2d, 2s, 2c

Full house (pair + triple)

- 3 of one rank
- 2 of another rank

For example: 2h, 2c, 2s, Jd, Js

Two pairs

For example: 2h, 2d, Qs, Qc, 7c

Two pairs (another approach)

For example: 2h, 2d, Qs, Qc, 7c

Hand with every suit

For example: 7d, Kc, 3d, Ah, 2s