Generating functions

CS 206: Discrete Structures II

Fall 2020

Binomial theorem again

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Integer sums again

How many integer solutions for x_i are there to

$$x_1 + x_2 = 6$$

where all $x_i \geq 0$?

Crazy integer sums

How many integer solutions for x_i are there to

$$x_1 + x_2 + x_3 + x_4 = 27$$

where

- x_1 is even
- x_2 is a multiple of 5
- $x_3 \leq 4$
- x_4 is 0 or 1

Infinite series

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$

- We want the coefficient of x^n, denoted $[x^n]F(x) = f_n$

Infinite series

Let the coefficients be

$$\langle 1, 1, 1, 1, 1, \ldots \rangle$$

$$G(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Infinite series

Let the coefficients be

$$\langle 1, 2, 3, 4, 5, \ldots \rangle$$

$$N(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots = \sum_{n=0}^{\infty} (n+1)x^n$$

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{G(x)}{1-x} = \frac{1}{(1-x)^2}$$

Counting donuts

Select n donuts from 2 varieties: chocolate and glazed.

Counting donuts with generating functions

Select n donuts from 2 varieties: chocolate and glazed.

Products

If we know

- how to pick n apples
- how to pick n bananas

Can we figure out how to pick n apples or bananas?

Products

- Apples come in sets of 6
- Bananas have two types

Products (convolution)

$$A(x)B(x) = (a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1x + b_2x^2 + \cdots)$$
$$[x^n]A(x)B(x) = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$$

	b_0x^0	b_1x^1	b_2x^2	b_3x^3	$\cdots$
a_0x^0	$a_0b_0x^0$	$a_0b_1x^1$	$a_0b_2x^2$	$a_0b_3x^3$	$\cdots$
a_1x^1	$a_1b_0x^1$	$a_1b_1x^2$	$a_1b_2x^3$	$\cdots$	
a_2x^2	$a_2b_0x^2$	$a_2b_1x^3$	$\cdots$		
a_3x^3	$a_3b_0x^3$	$\cdots$			
$\cdots$	$\cdots$				

Rules

- $[x^n](c \cdot F(x)) = c \cdot [x^n]F(x)$
- $[x^n](x^m \cdot F(x)) = [x^{n-m}]F(x)$

Counting donuts yet again

Select n donuts from 2 varieties: chocolate and glazed.

Binomial theorem

$$[x^k](1+x)^n = \binom{n}{k}$$

Crazy counting problem

How many ways can we buy n fruits where:

- # of apples is even
- # of bananas is a multiple of 5
- # of oranges ≤ 4
- # of pears is 0 or 1

Crazy counting problem

How many ways can we buy n fruits where:

- # of apples is even
- # of bananas is a multiple of 5
- # of oranges ≤ 4
- # of pears is 0 or 1

$$\begin{aligned} A(x)B(x)O(x)P(x) &= \frac{1}{1-x^2}\frac{1}{1-x^5}\frac{1-x^5}{1-x}(1+x) \\ &= \frac{1}{(1-x)^2} \\ &= 1+2x+3x^2+4x^3+\cdots \end{aligned}$$

Partial fractions

If $p(x)$ is a polynomial with degree at most $n-1$, and a_i are distinct:

$$\frac{p(x)}{(1-a_1x)(1-a_2x)\cdots(1-a_nx)} = \frac{c_1}{(1-a_1x)} + \frac{c_2}{(1-a_2x)} + \cdots + \frac{c_n}{(1-a_nx)}$$

Partial fractions

$$\frac{x}{1-x-x^2} = \frac{x}{(1-a_1x)(1-a_2x)}$$

where

- $a_1 = \frac{1+\sqrt{5}}{2}$
- $a_2 = \frac{1-\sqrt{5}}{2}$

Partial fractions

$$\frac{x}{(1-a_1x)(1-a_2x)} = \frac{c_1}{1-a_1x} + \frac{c_2}{1-a_2x}$$

$$x = c_1(1-a_2x) + c_2(1-a_1x)$$

Letting $x = 1/a_1$ gives $c_1 = \frac{1}{\sqrt{5}}$.

Letting $x = 1/a_2$ gives $c_2 = -\frac{1}{\sqrt{5}}$.

Partial fractions

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-a_1x} + \frac{1}{1-a_2x}\right)$$

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left((1+a_1x+a_1^2x^2+\cdots) - (1+a_2x+a_2^2x^2+\cdots)\right)$$

$$\begin{aligned}[x^n]\frac{x}{1-x-x^2} &= \frac{a_1^n - a_2^n}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)\end{aligned}$$

Summary

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \cdots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$