Probability: outcomes and sample spaces

CS 206: Discrete Structures II

Fall 2020

Monty Hall problem

There are three doors.

Behind one is a car. The other two have goats.

The game:

- you pick a door
- before it's opened, the host opens a different door
- it has a goat
- you can then stick with your original door or switch to the remaining unopened door

Is it better to switch, better to stay, or are they equivalent?

Four step method

1. Find the sample space
2. Define events of interest
3. Determine outcome probabilities
4. Compute event probabilities

Monty Hall problem

Step 1: find the sample space (what is random?)

Step 2: define events of interest

Step 3: determine outcome probabilities

Monty Hall problem

Step 4: compute event probabilities

Dice

If we roll a 6-sided die, what's the probability of getting a 4?

Dice

If we roll two 6-sided dice, what's the probability of getting a sum of 7?

Definitions

- a nonempty countable set S is our sample space
- $\omega \in S$ is an outcome
- probability function $\mathbb{P} : S \to \mathbb{R}$

$\mathbb{P}(\omega) \geq 0$

$\sum \mathbb{P}(\omega) = 1$

- probability space: sample space plus probability function
- event $E \subseteq S$

$\mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega)$

Rules

- for disjoint events, $\mathbb{P}\left(E \cup F\right) = \mathbb{P}\left(E\right) + \mathbb{P}\left(F\right)$
- $\mathbb{P}\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum \mathbb{P}\left(E_n\right)$ (for disjoint events)
- $\mathbb{P}\left(A\right) + \mathbb{P}\left(\bar{A}\right) = 1$
- $\mathbb{P}\left(\bar{A}\right) = 1 - \mathbb{P}\left(A\right)$
- inclusion-exclusion: $\mathbb{P}\left(A \cup B\right) = \mathbb{P}\left(A\right) + \mathbb{P}\left(B\right) - \mathbb{P}\left(A \cap B\right)$
- difference: $\mathbb{P}\left(B - A\right) = \mathbb{P}\left(B\right) - \mathbb{P}\left(A \cap B\right)$
- monotonicity: $A \subseteq B \Rightarrow \mathbb{P}\left(A\right) \leq \mathbb{P}\left(B\right)$
- union bound: $\mathbb{P}\left(E_1 \cup \cdots \cup E_n\right) \leq \mathbb{P}\left(E_1\right) + \cdots + \mathbb{P}\left(E_n\right)$

Probability of a full house

What's the probability of a full house in a 5-card hand?

Birthdays

Suppose there are 365 days in a year and we have 100 people in the room.

What's the probability that two people have the same birthday?

Uniform sample space

- Uniform sample space: all outcomes have the same probability

Then for an event E,

$$\mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega)$$

Marbles example

A bag holds 4 blue marbles and 5 red marbles.

If we pick one at random, what is the probability it's blue?

2d6 example

If we roll two 6-sided dice, what's the probability that their sum is 12?

Lottery example

A lottery chooses 4 numbers from 0 to 9.

- if you guess all correctly, you win the jackpot
- if you guess 3 correctly, you win a smaller prize

What are the probabilities of winning each prize?

Lottery example

There’s one way to win the jackpot:

Lottery example

Let S_i be the event where all but the ith number are correct.

Infinite sample spaces

Let's take turns flipping a coin until it comes up heads.

The player who flipped heads wins.

Infinite sample spaces

What is the probability that P_1 wins?

Infinite sample spaces

What is the probability that P_2 wins?