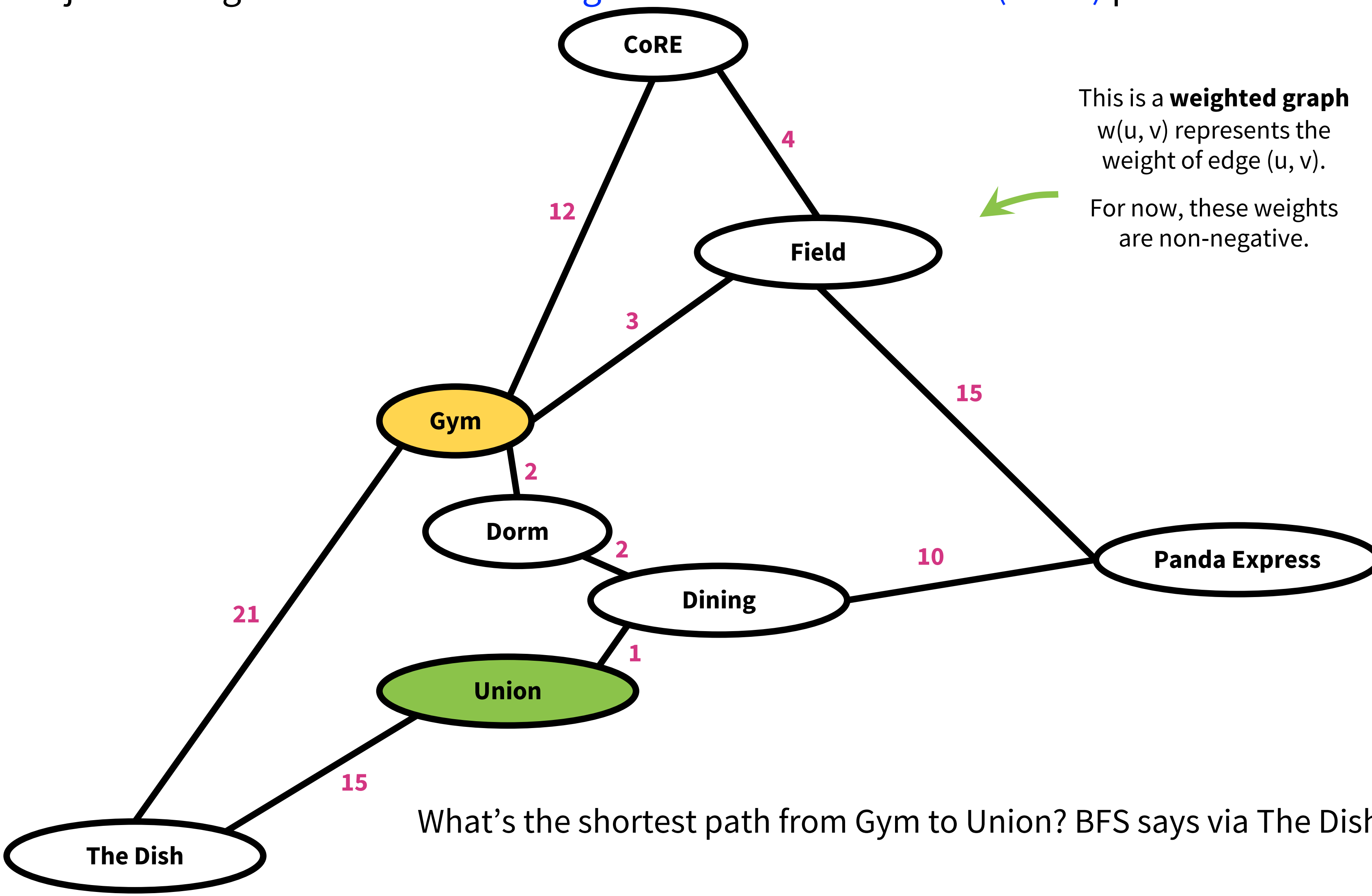
Dynamic Programming I

Outline for Today

- Dynamic Programming
  - DP graph algorithms
    - Bellman Ford
    - Floyd Warshall

Review: Dijkstra's Algorithm

Dijkstra's Algorithm solves the Single-Source Shortest Path (SSSP) problem.



This is a **weighted graph**
w(u, v) represents the weight of edge (u, v).

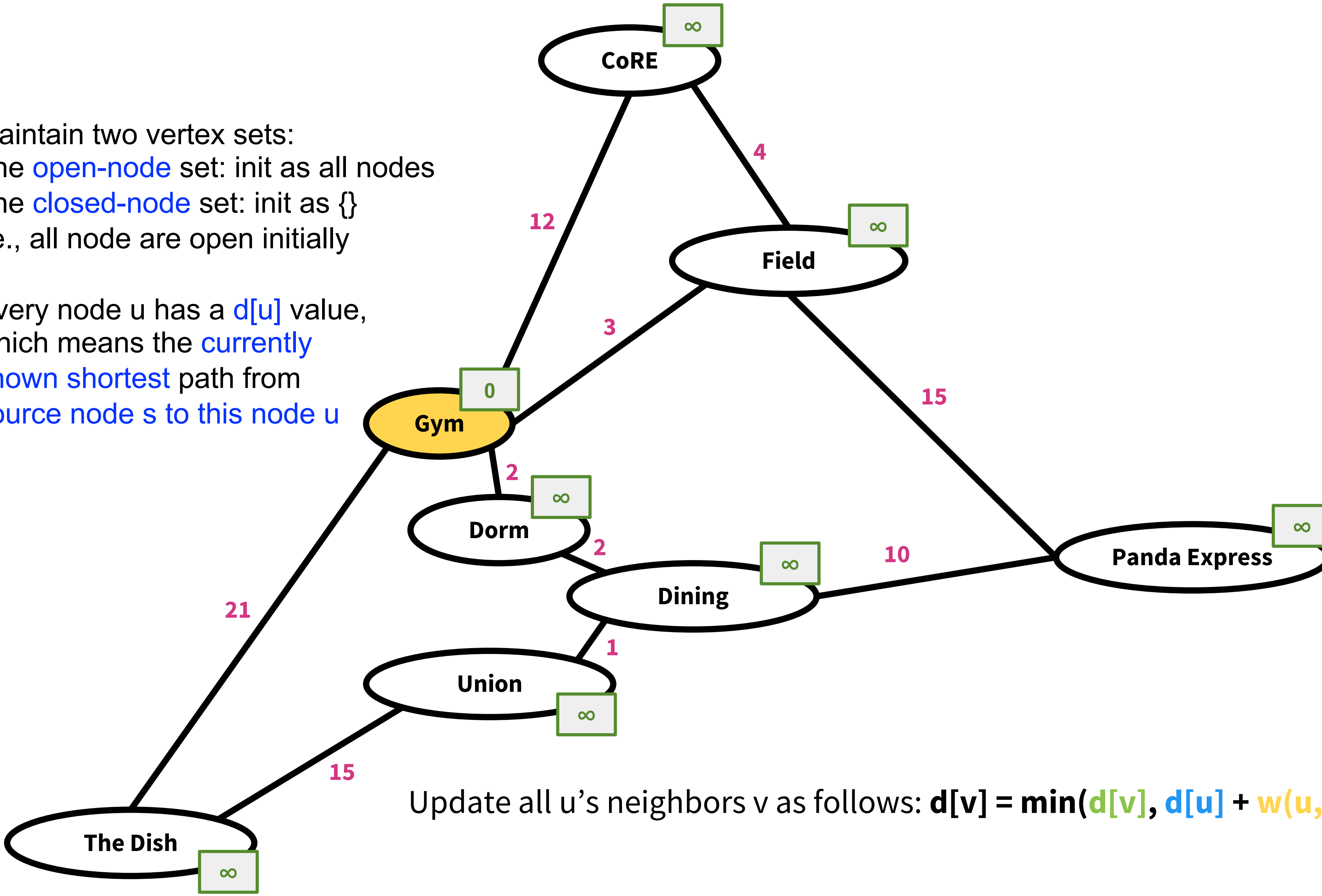
For now, these weights are non-negative.

What's the shortest path from Gym to Union? BFS says via The Dish.

Dijkstra's Algorithm

Maintain two vertex sets:
The open-node set: init as all nodes
The closed-node set: init as {}
i.e., all node are open initially

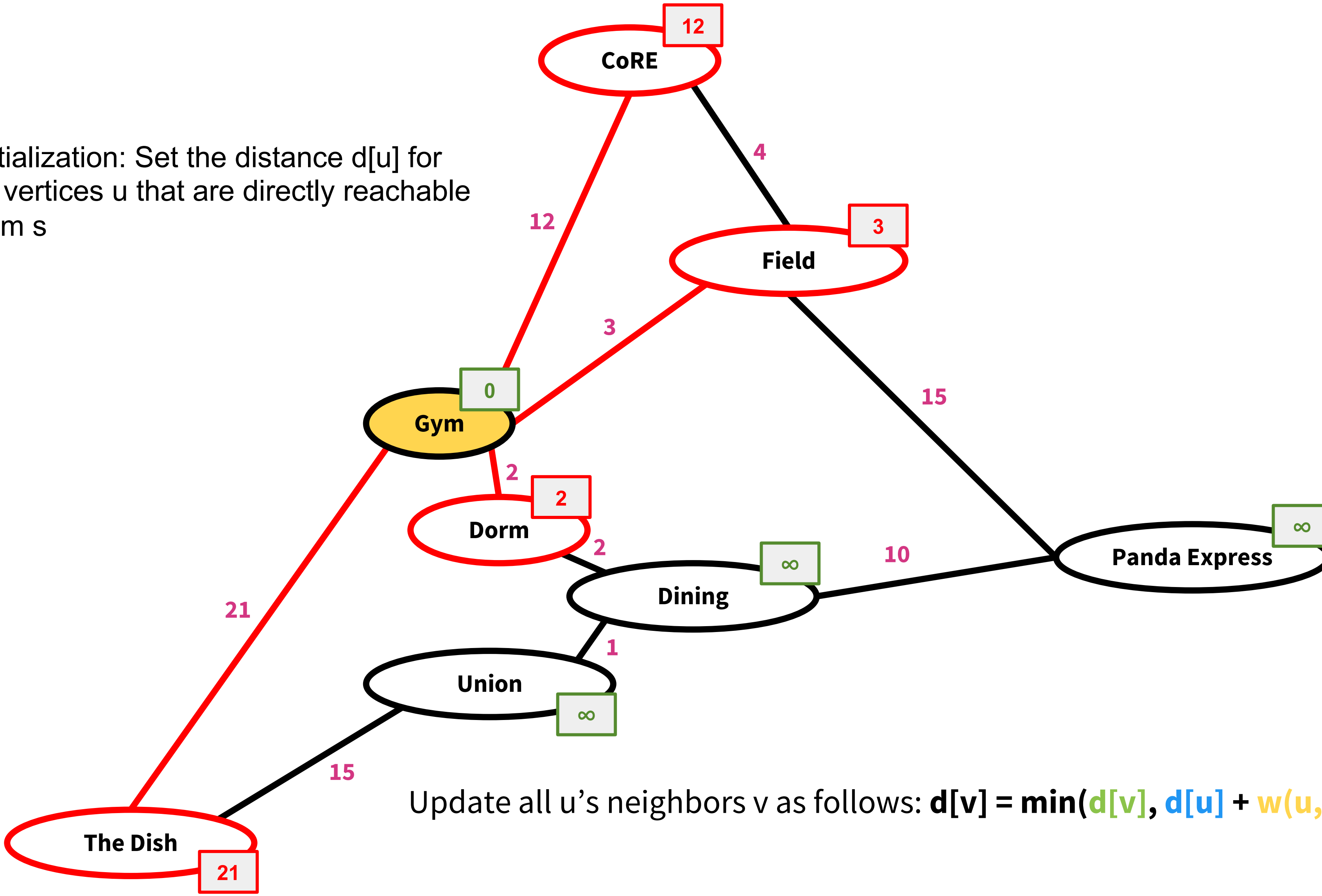
Every node u has a d[u] value, which means the currently known shortest path from source node s to this node u



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

Initialization: Set the distance d[u] for all vertices u that are directly reachable from s



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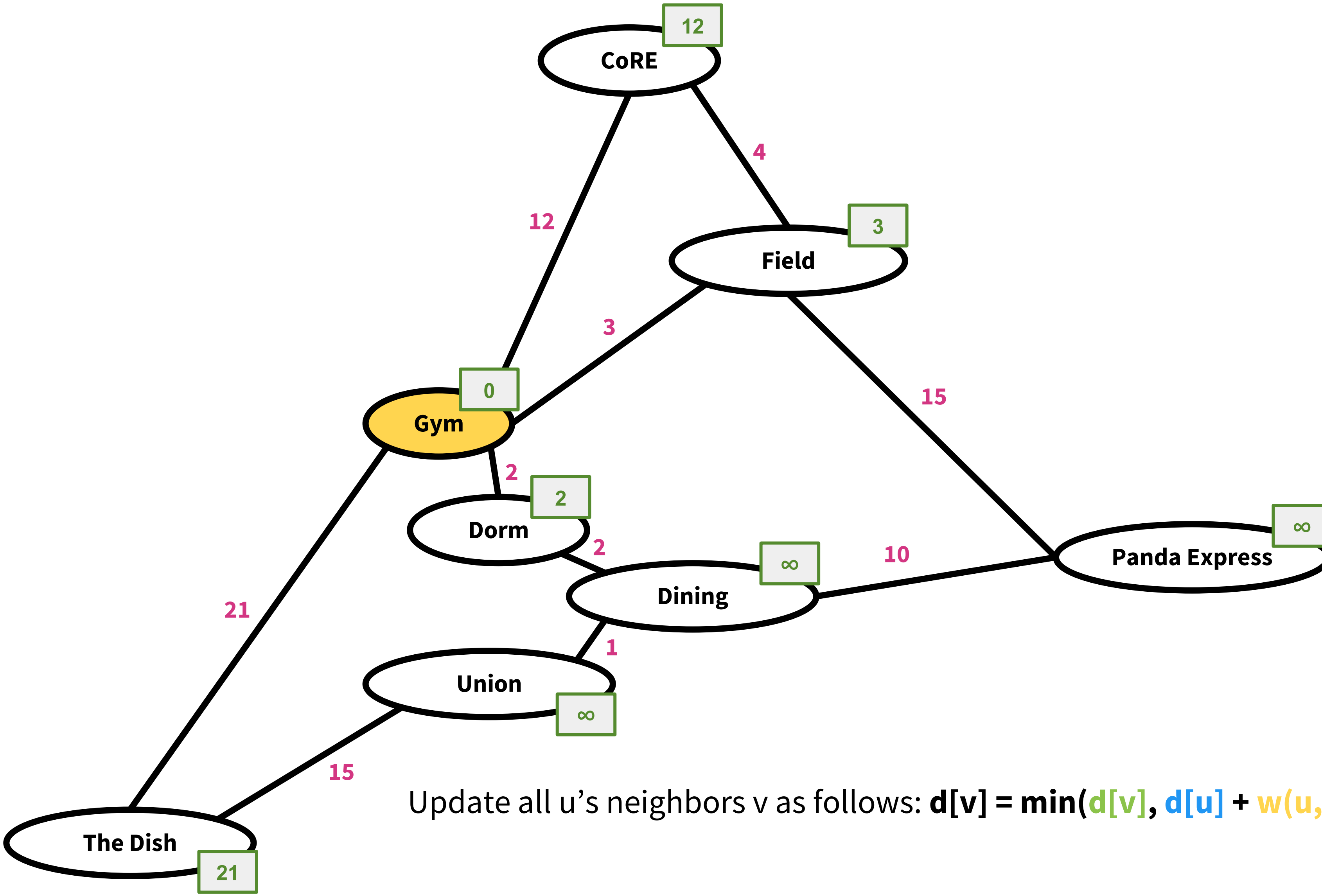
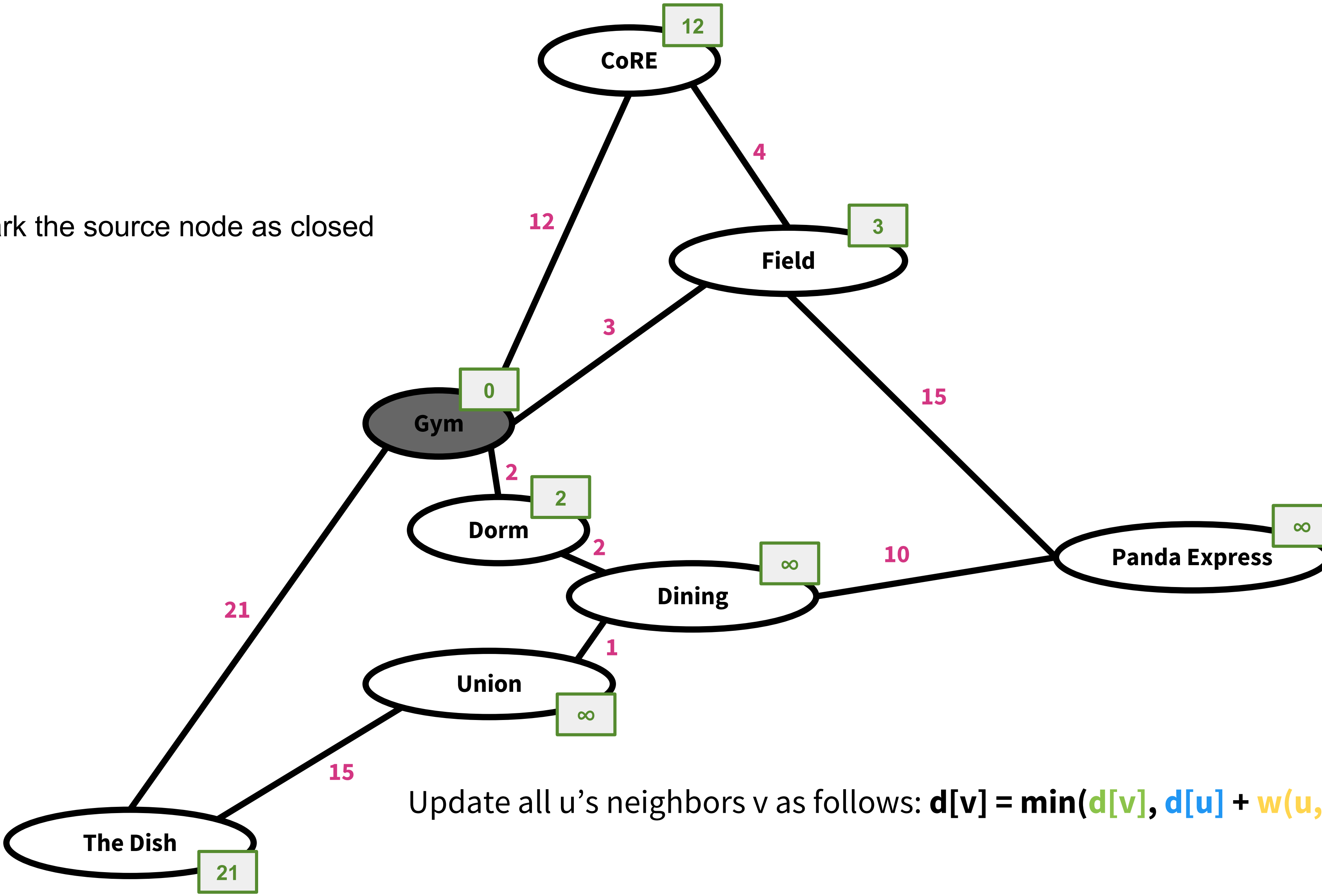
Dijkstra's Algorithm



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

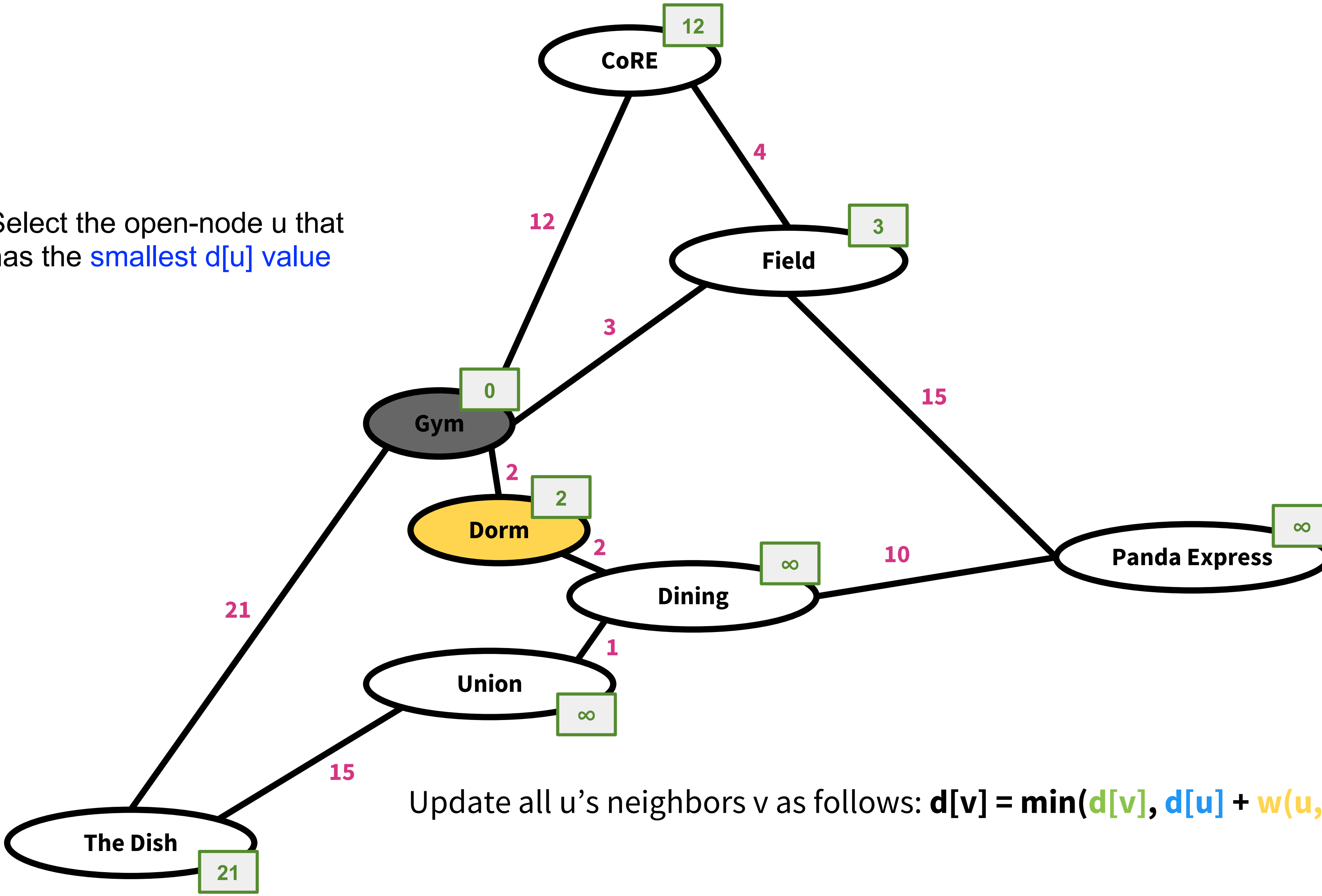
Mark the source node as closed



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

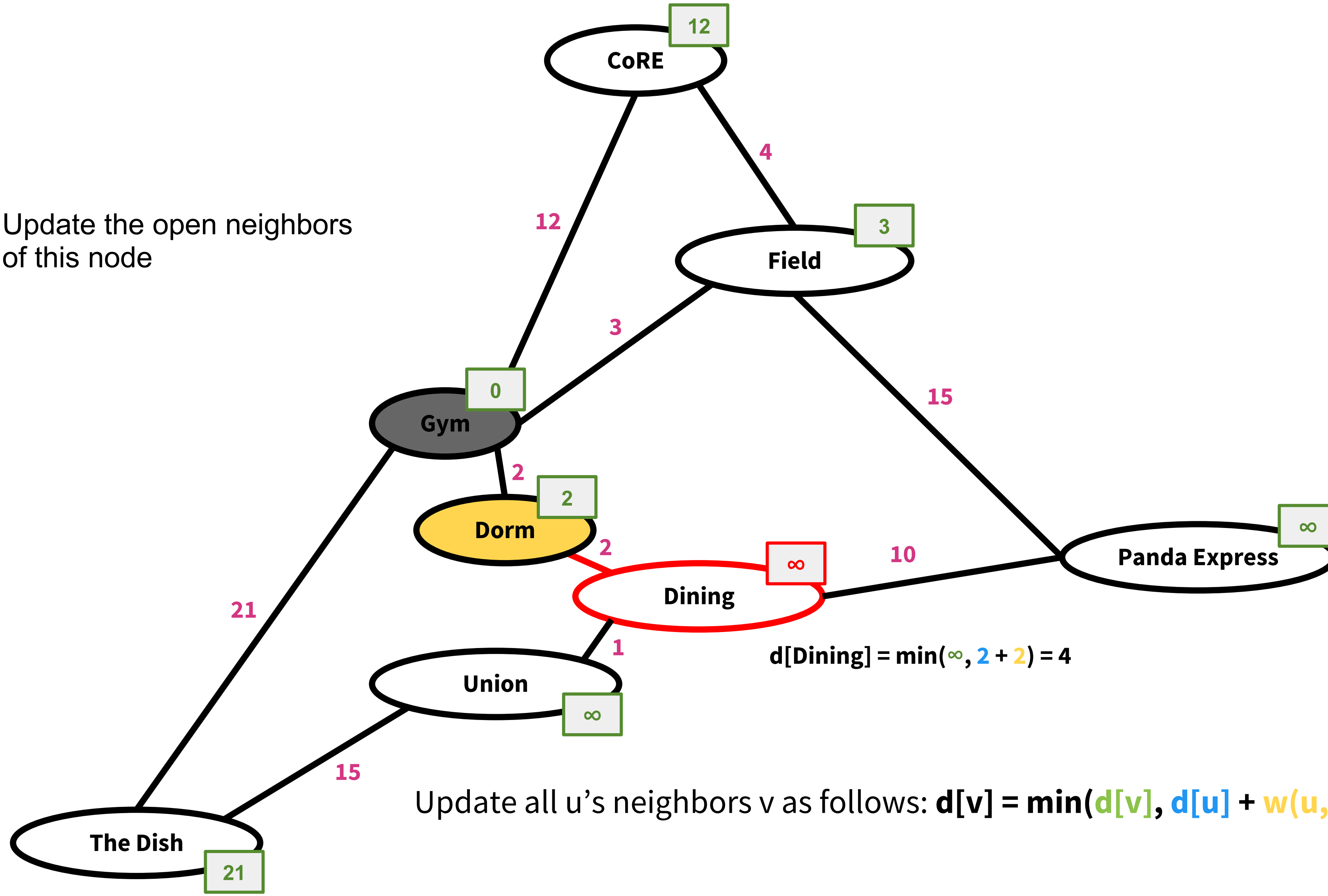
Select the open-node u that has the smallest d[u] value



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

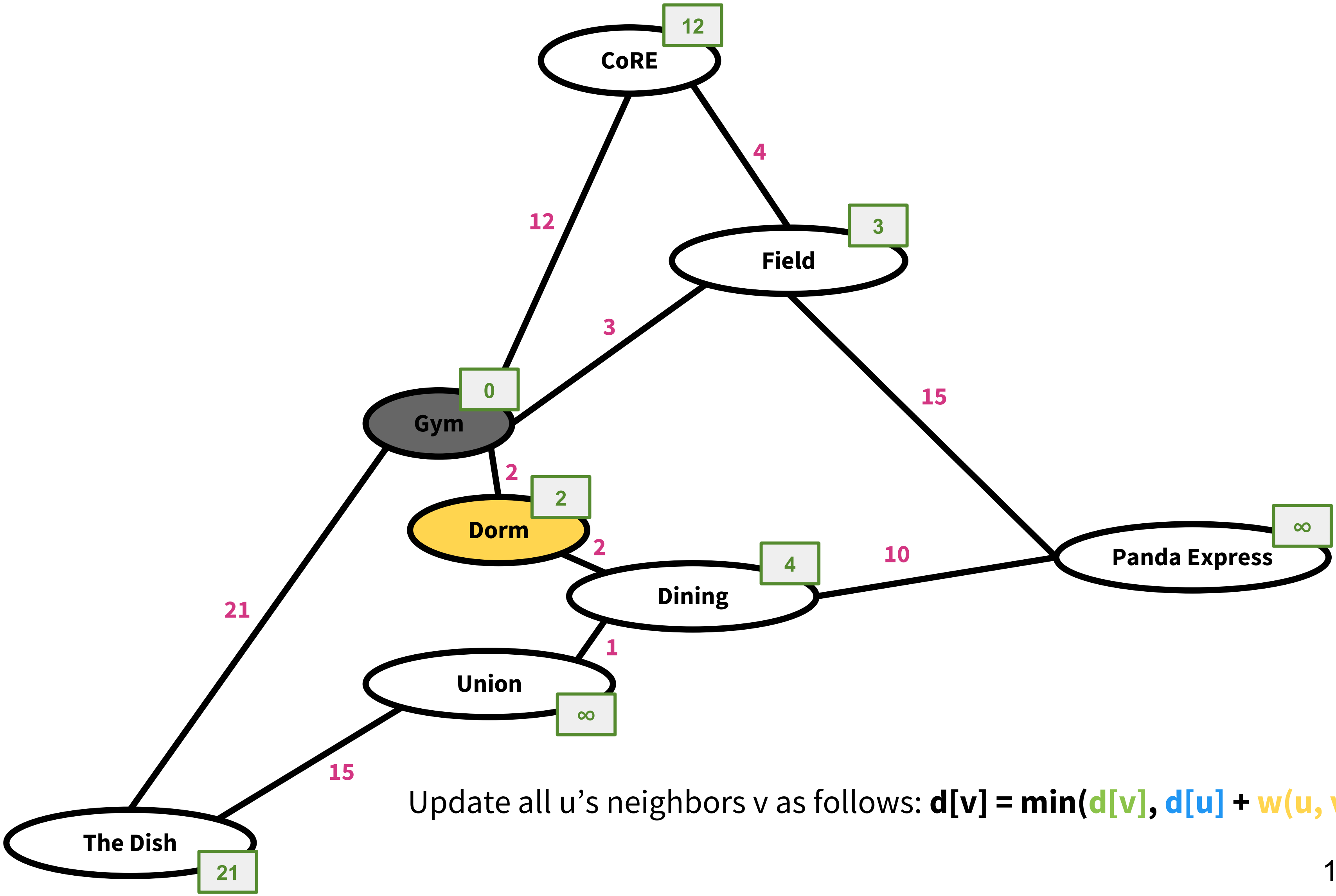
Dijkstra's Algorithm

Update the open neighbors
of this node



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

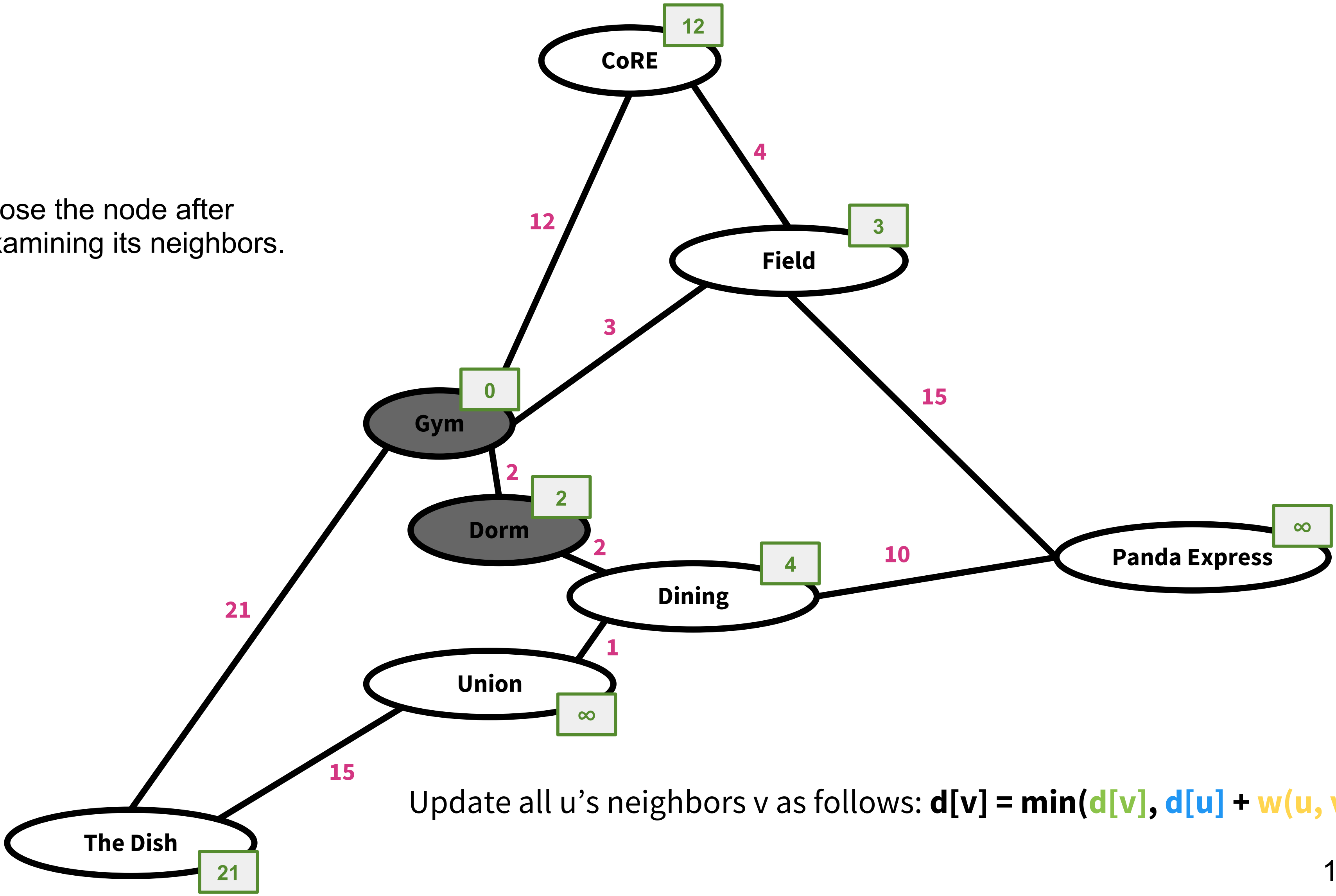
Dijkstra's Algorithm



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

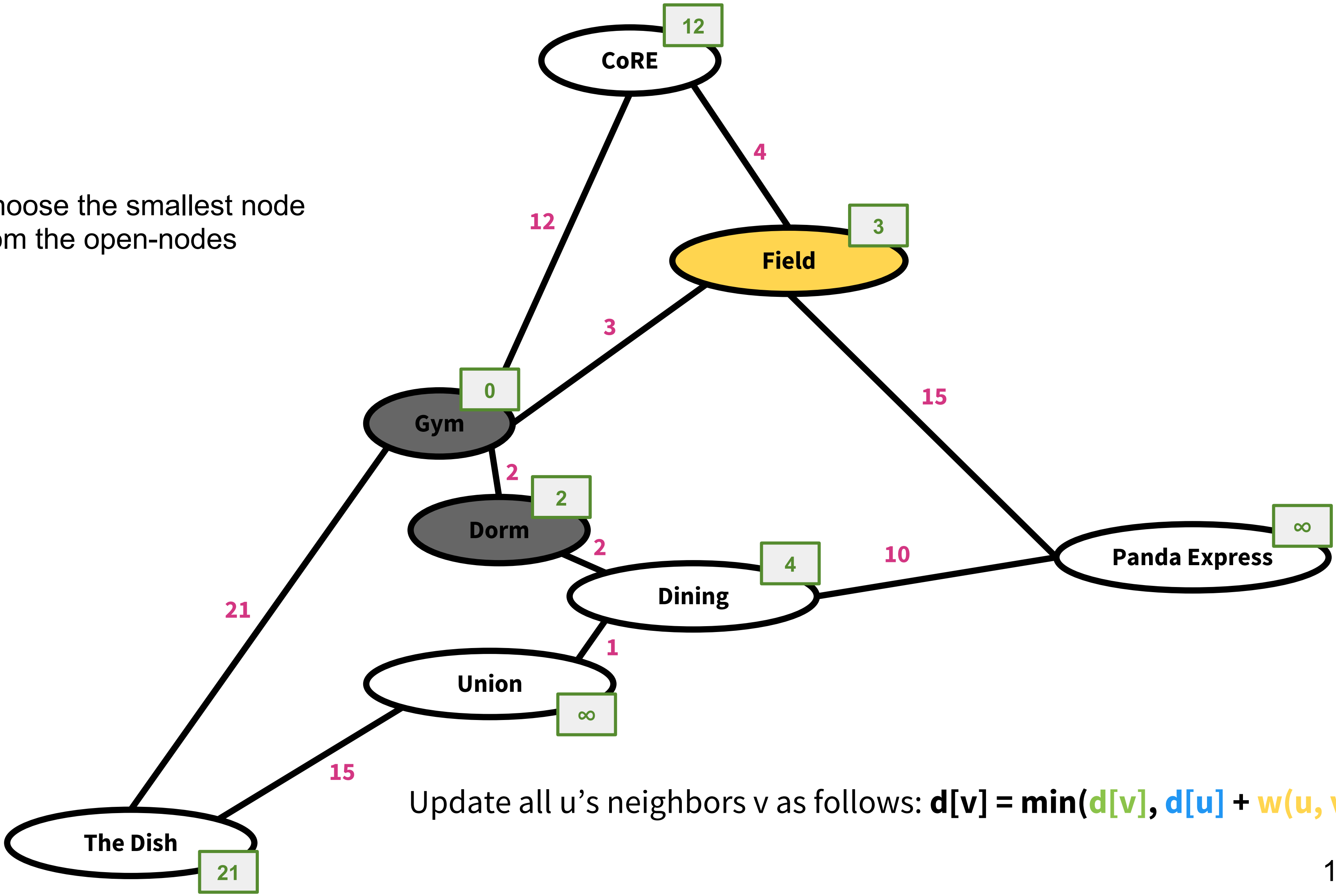
Close the node after examining its neighbors.



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

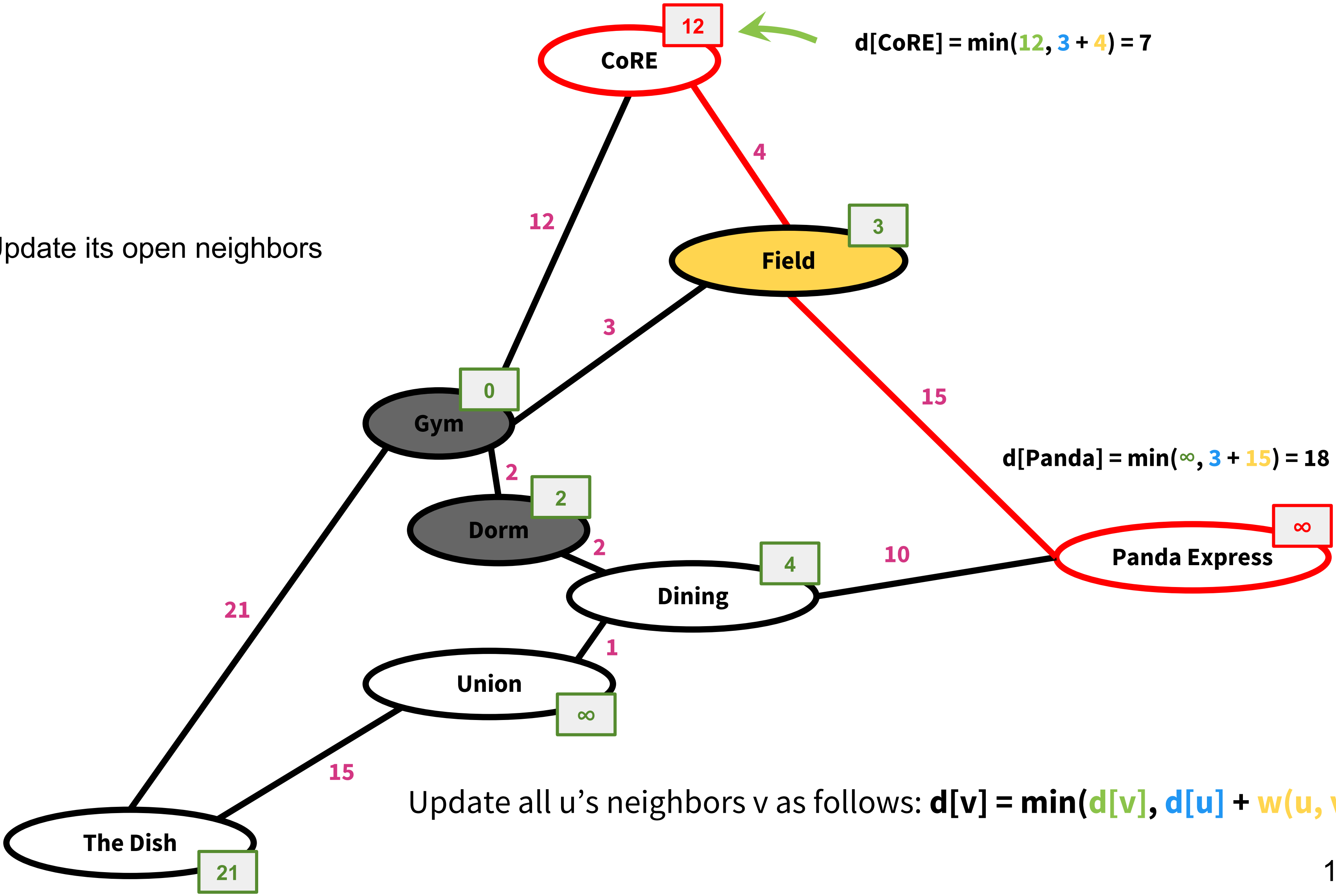
Choose the smallest node from the open-nodes



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

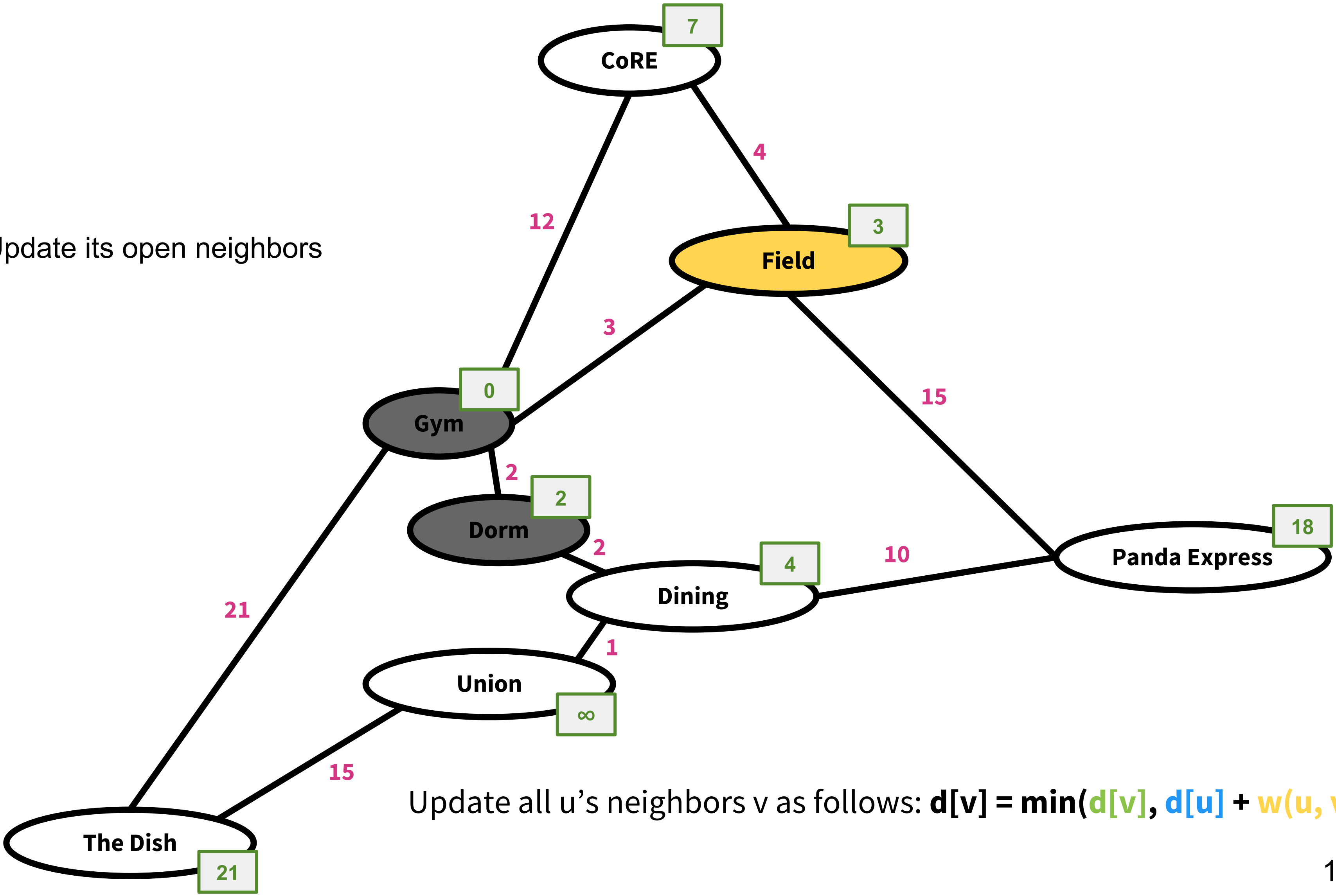
Update its open neighbors



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

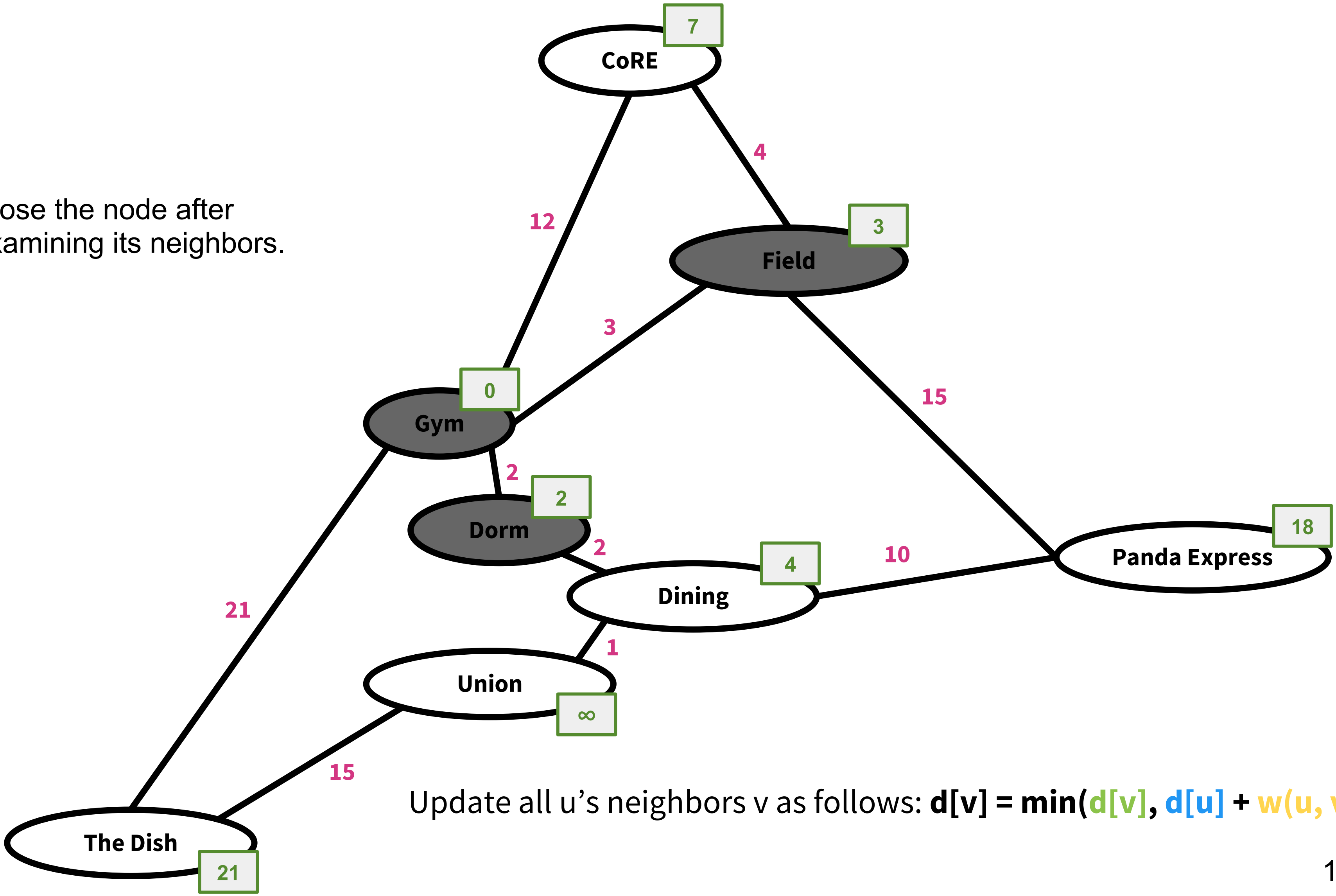
Update its open neighbors



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra’s Algorithm

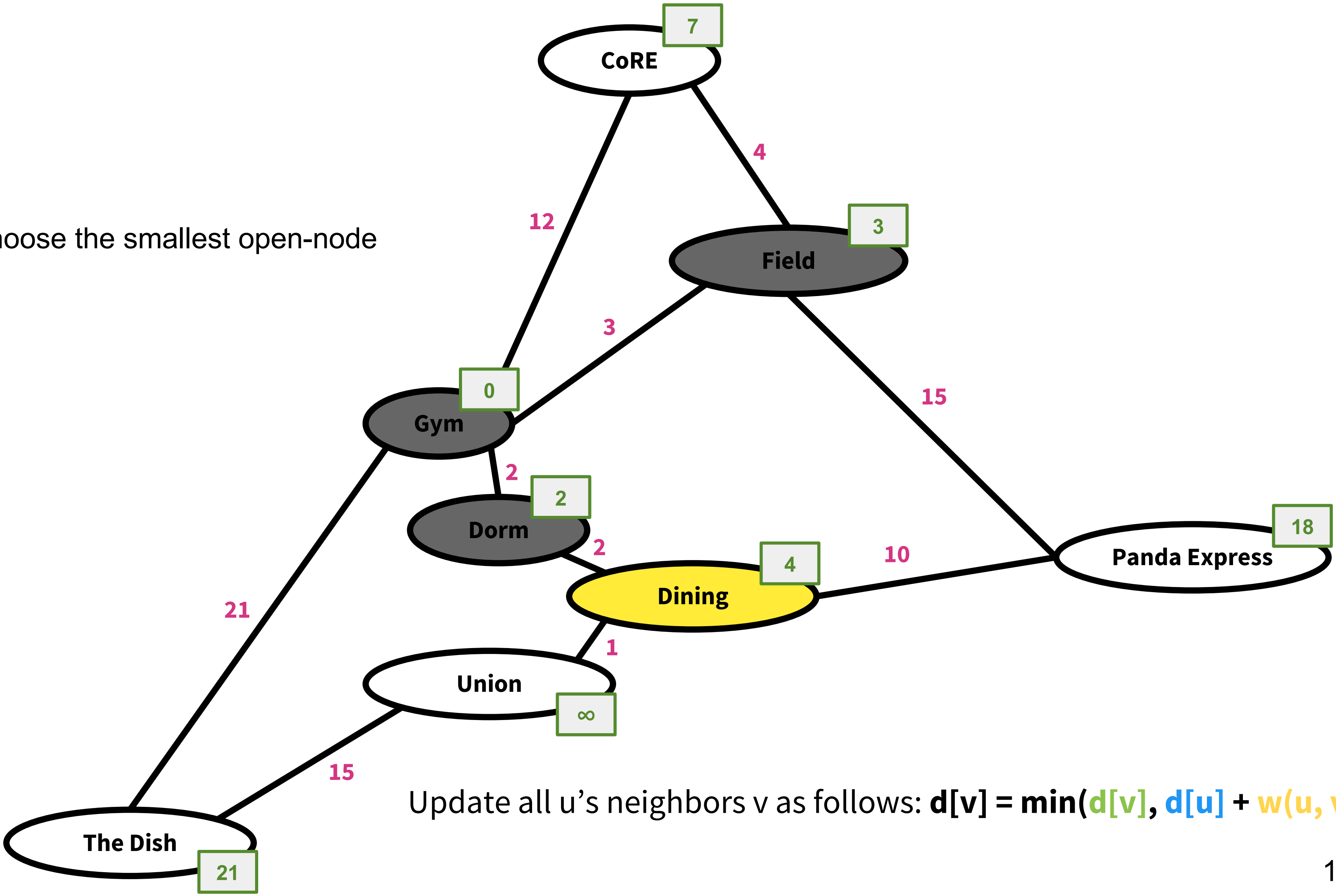
Close the node after examining its neighbors.



Update all u’s neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

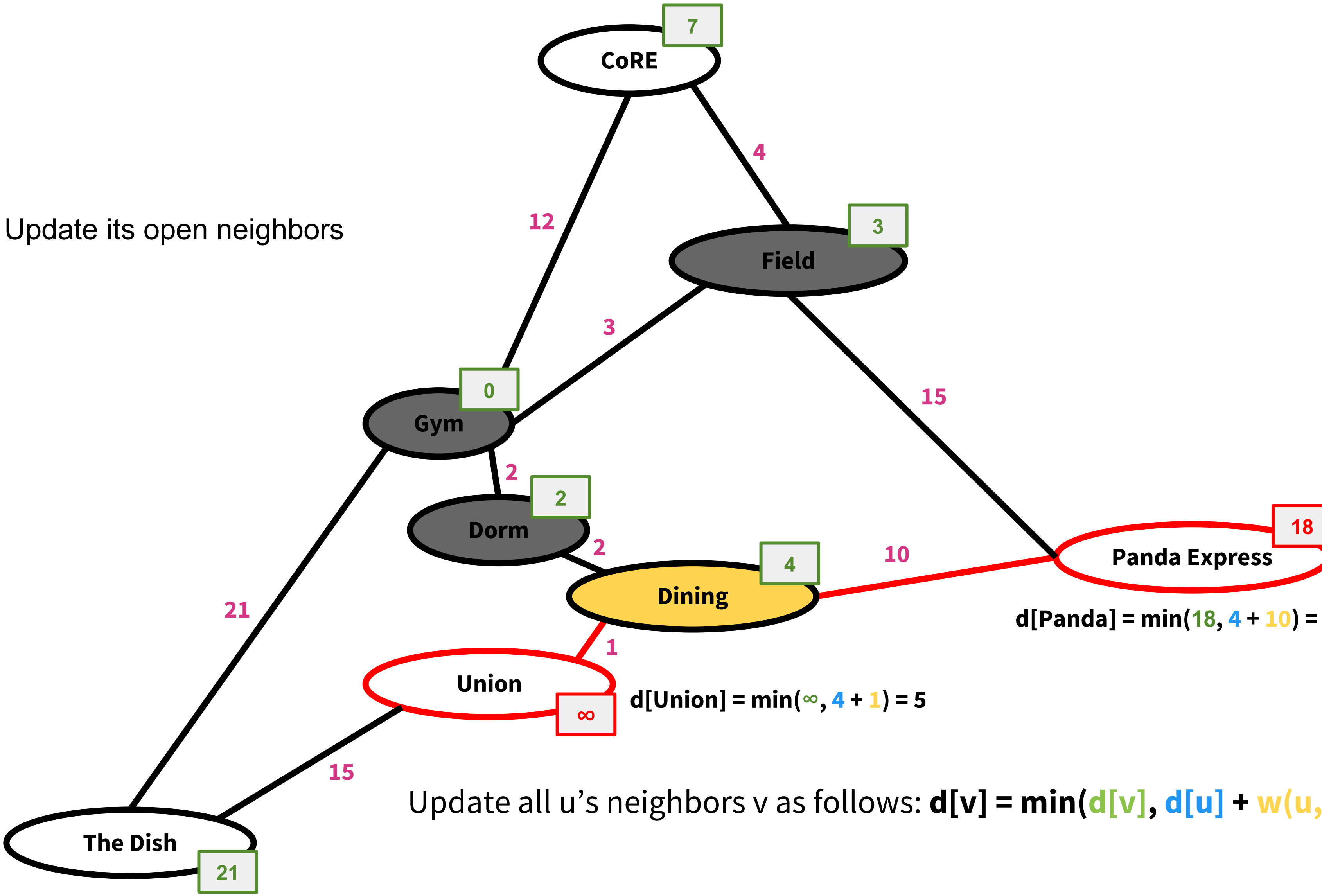
Choose the smallest open-node



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

Update its open neighbors



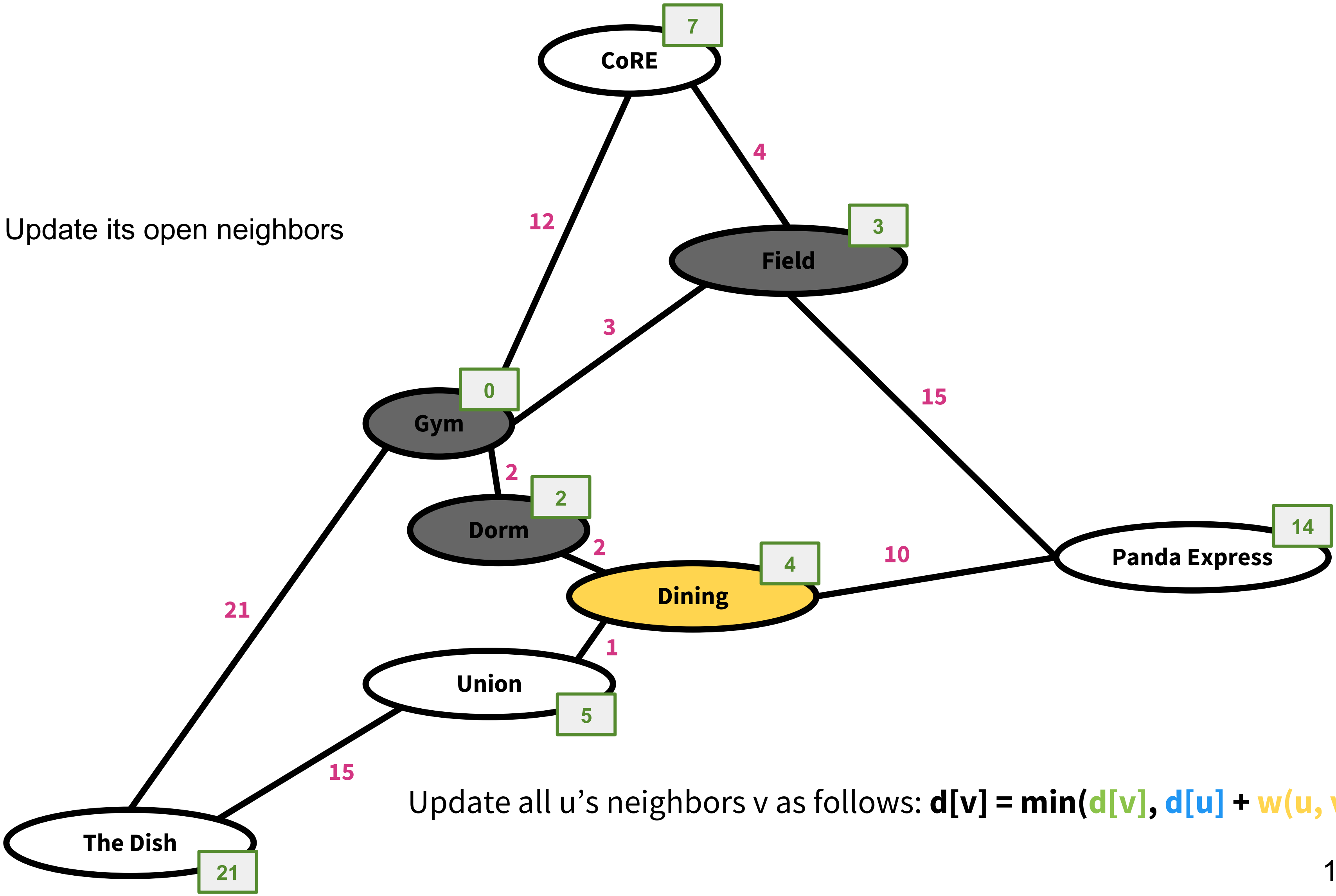
d[Panda] = min(18, 4 + 10) = 14

d[Union] = min(∞, 4 + 1) = 5

Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

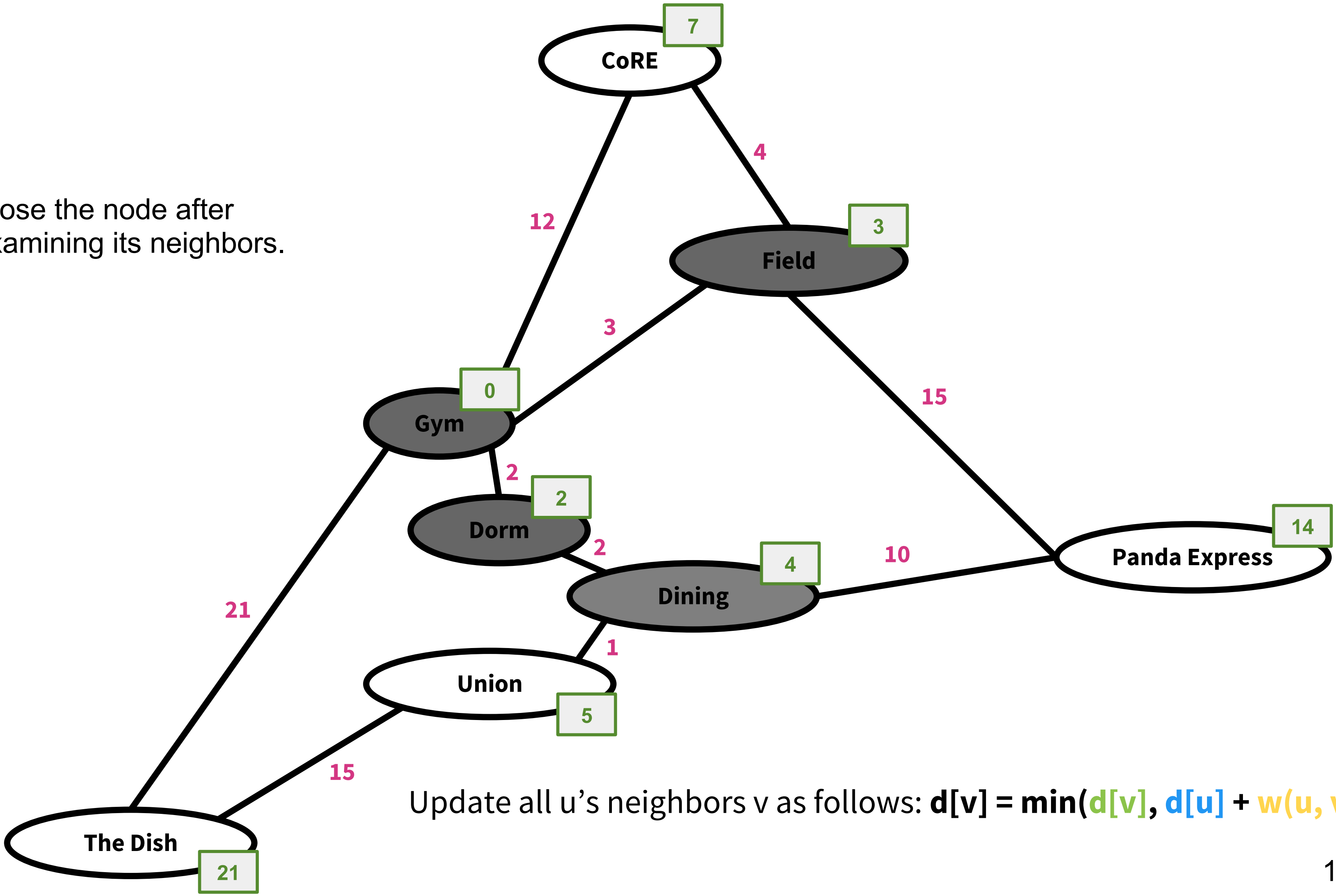
Update its open neighbors



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra’s Algorithm

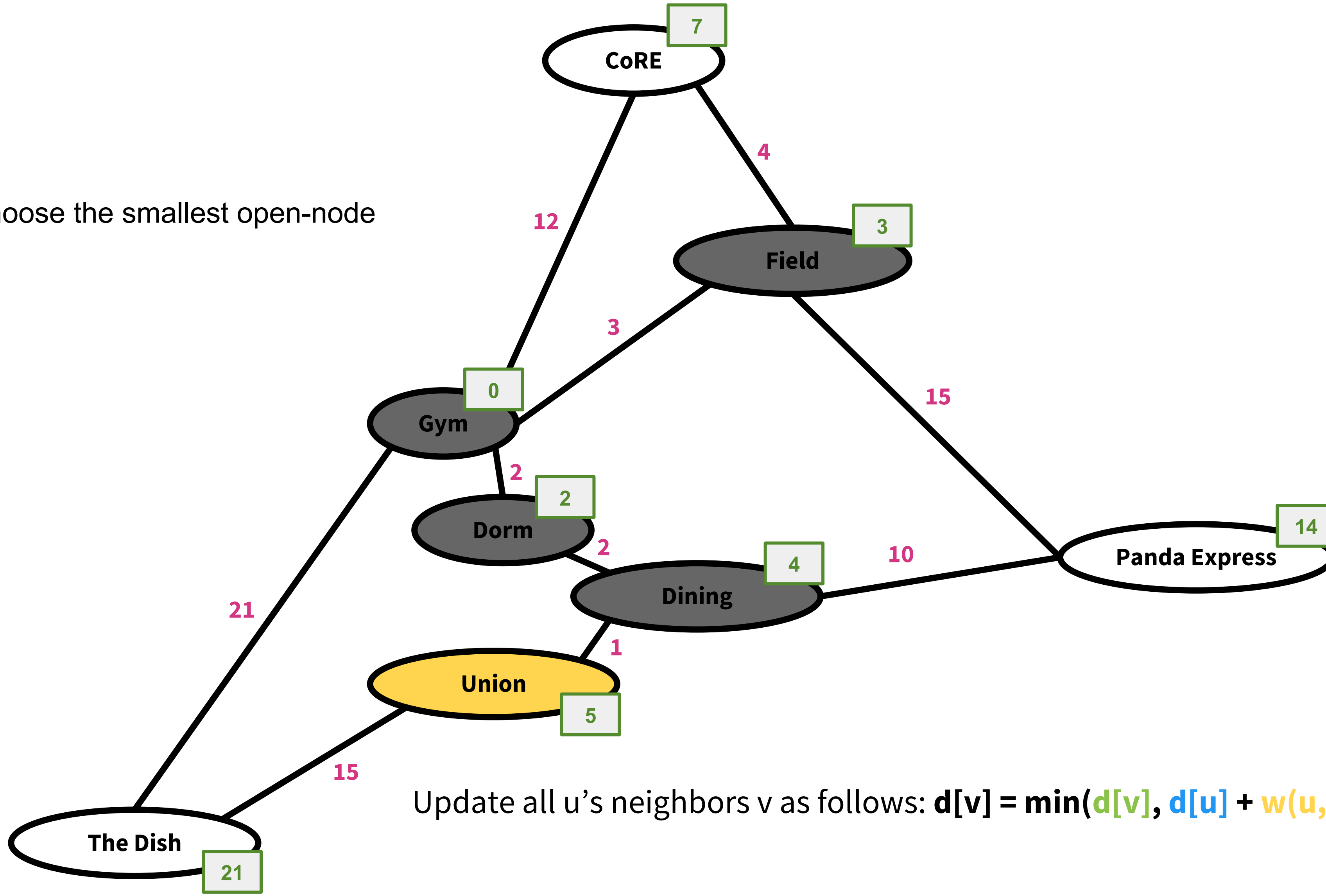
Close the node after examining its neighbors.



Update all u’s neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

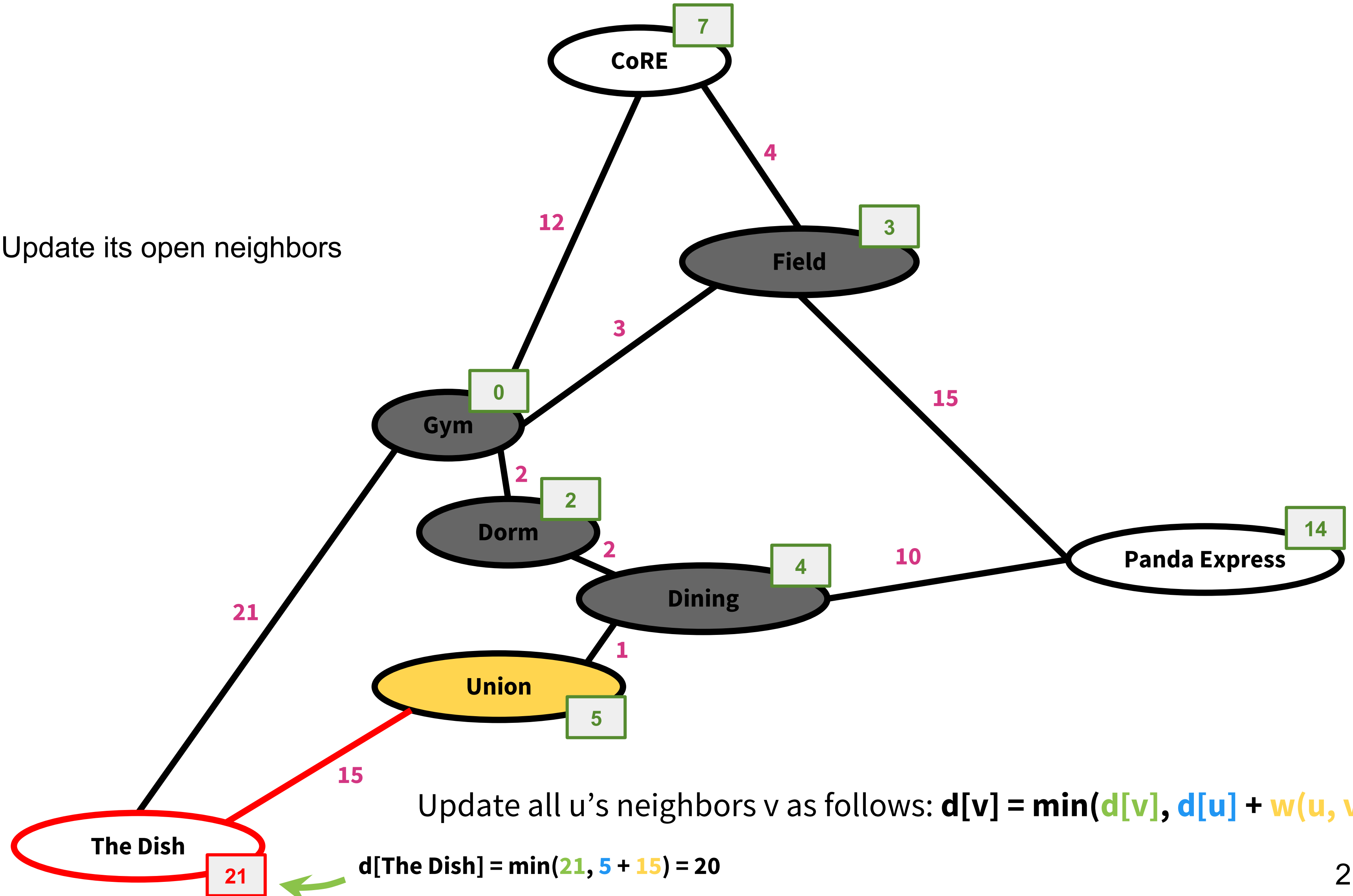
Choose the smallest open-node



Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

Update its open neighbors

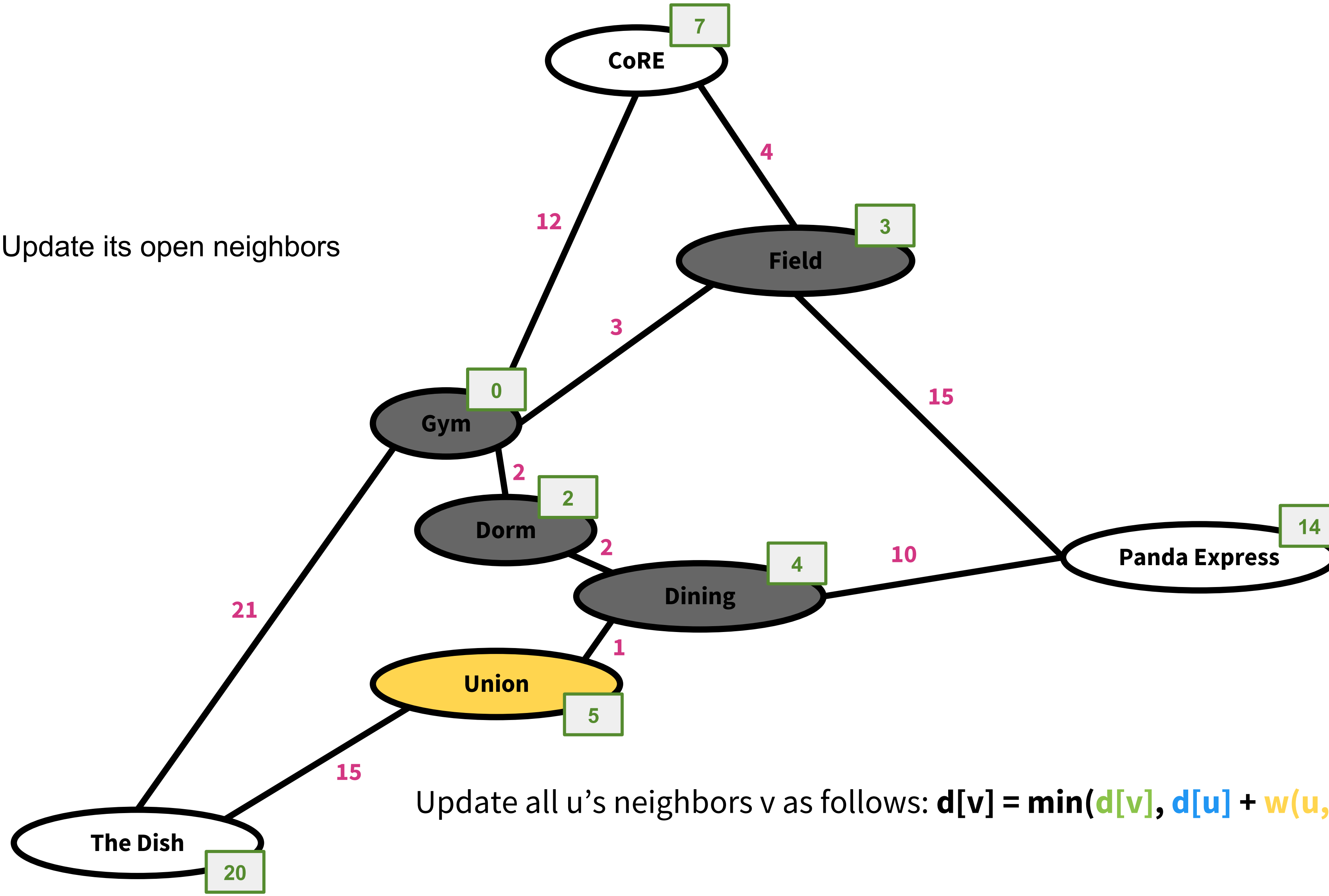


Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

d[The Dish] = min(21, 5 + 15) = 20

Dijkstra’s Algorithm

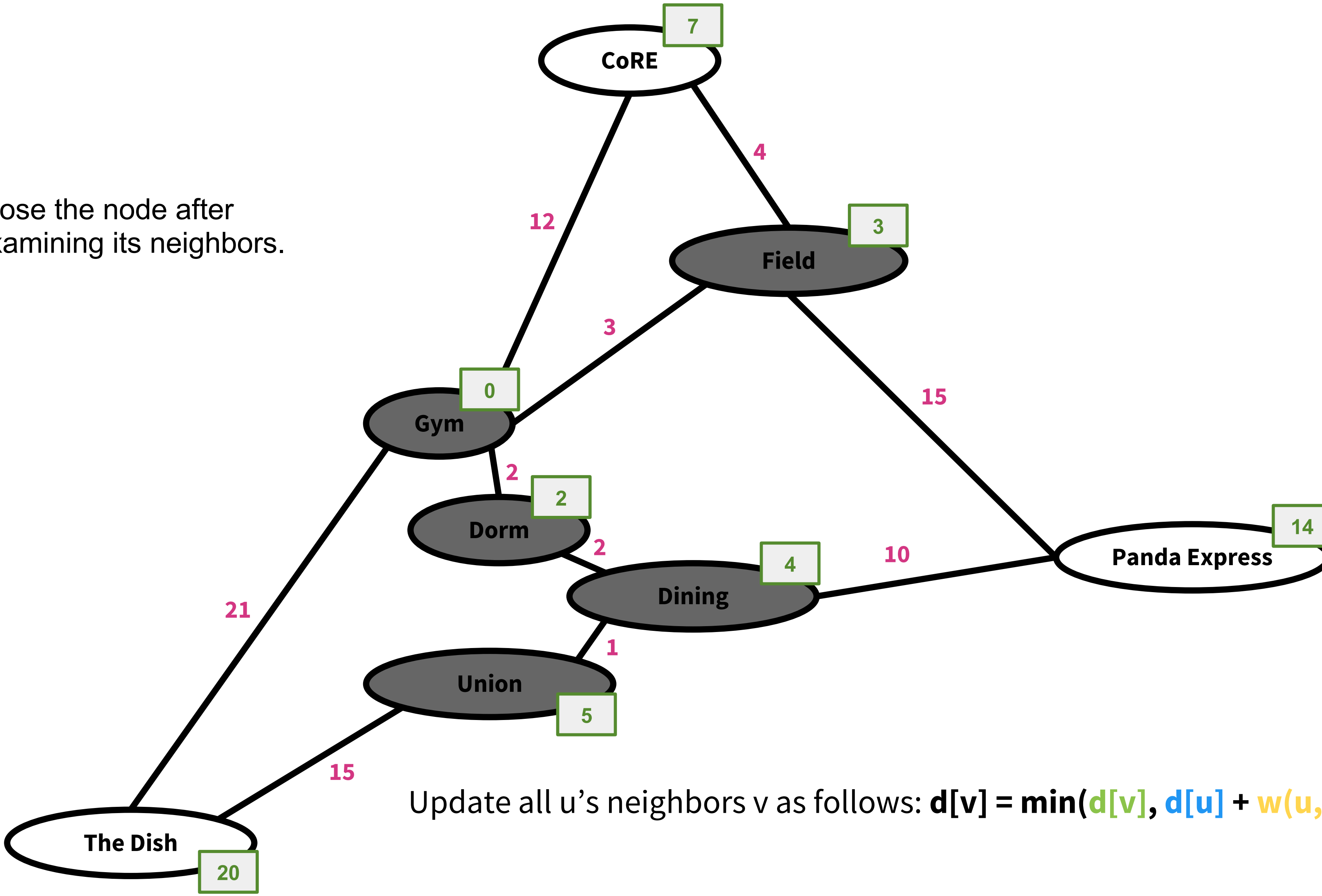
Update its open neighbors



Update all u’s neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

Close the node after examining its neighbors.



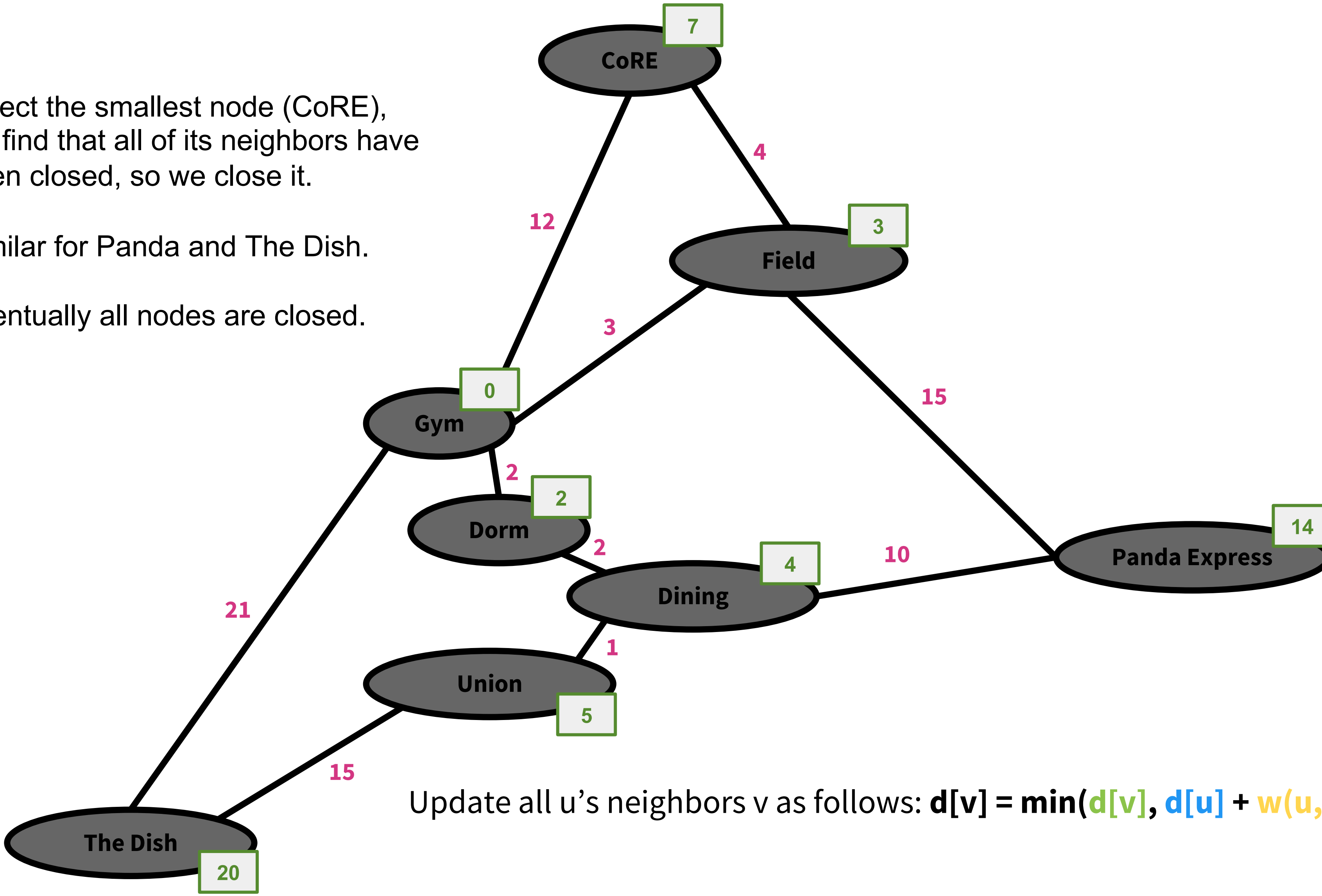
Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Dijkstra's Algorithm

Select the smallest node (CoRE), we find that all of its neighbors have been closed, so we close it.

Similar for Panda and The Dish.

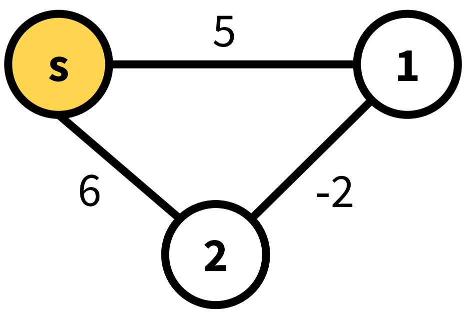
Eventually all nodes are closed.



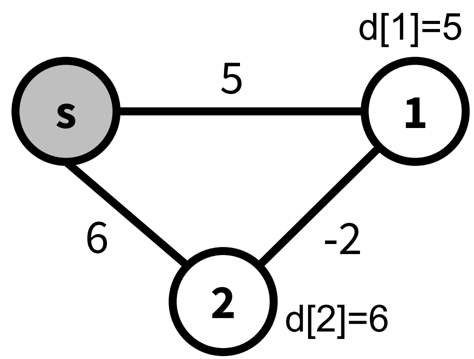
Update all u's neighbors v as follows: **d[v] = min(d[v], d[u] + w(u, v))**

Problem of Dijkstra’s Algorithm

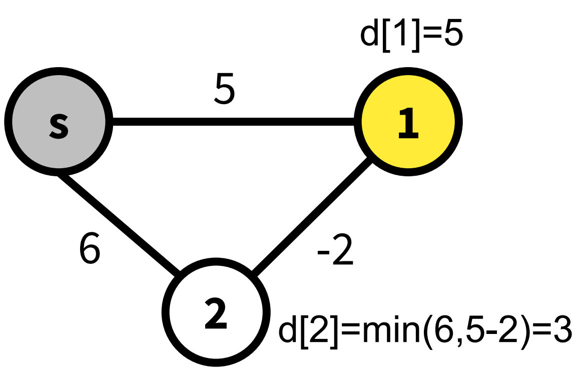
Can not handle negative edge weights properly.



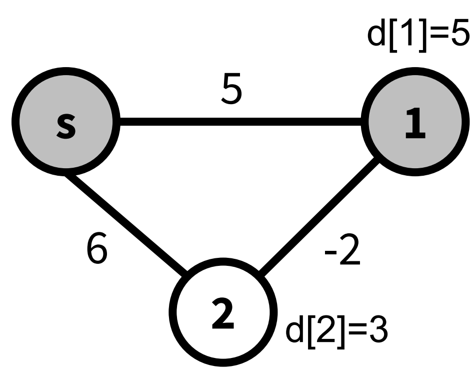
0. Original graph



1. Initialization



3. Select the smallest open node and update its open neighbors



Once a node is closed, its distance to source will never be updated anymore.

4. Close the node, however, the shortest path from s to 1 is through 2, and the distance is 4 not 5.

Bellman-Ford

Bellman-Ford Algorithm

Dijkstra's algorithm solves the single-source shortest path (SSSP) problem in weighted graphs.

- Sometimes it works on graphs with negative edge weights, but sometimes it doesn't work.

Bellman-Ford also solves the SSSP problem in weighted graphs.

- Always works on graphs with negative edge weights (when a solution exists).

When does a solution do not even exist?

Bellman-Ford Algorithm

Dijkstra's algorithm solves the single-source shortest path (SSSP) problem in weighted graphs.

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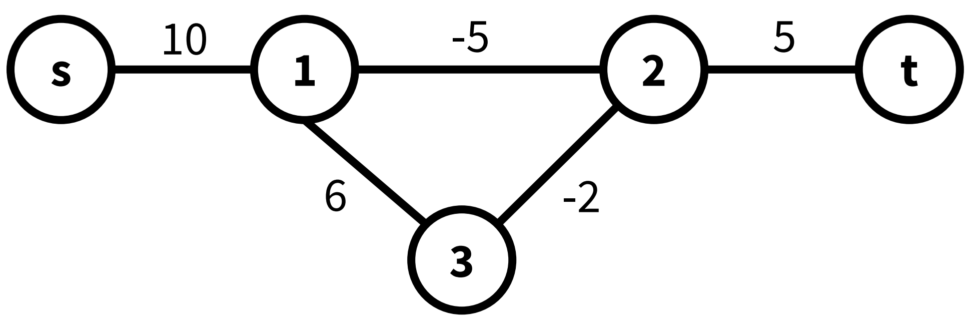
When does a solution do not even exist? When there exist negative cycles in the graph.



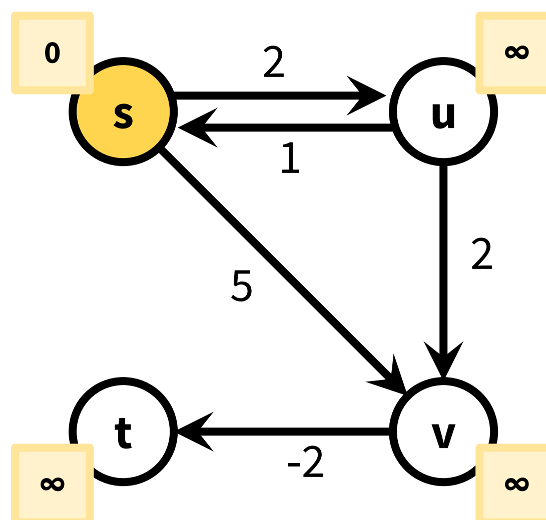
Bellman-Ford Algorithm

We maintain a vector $d^{(k)}$ of length |V| for each k = 0, 1, …, |V|-1.

1. Each dimension of the vector stands for a node in the graph.
2. $d^{(k)}[b]$ is the cost of the shortest path from s to b with at most k edges.
3. Why k = 0, 1, …, |V|-1? Because the longest # of steps to reach another node in a graph of |V| nodes is |V|-1.

We know k = 0, i.e. shortest paths to each vertex with at most 0 edges in it.

	s	**u**	**v**	**t**
$\mathbf{d^{(0)}}$	**0**	**∞**	**∞**	**∞**
$\mathbf{d^{(1)}}$				
$\mathbf{d^{(2)}}$				
$\mathbf{d^{(3)}}$				



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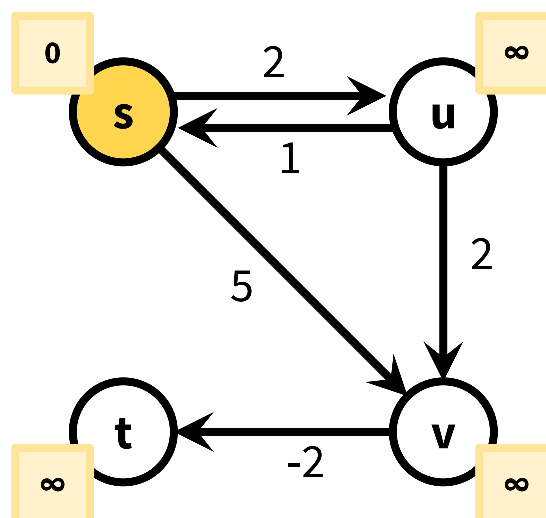
	s	**u**	**v**	**t**
$\mathbf{d^{(0)}}$	**0**	**∞**	**∞**	**∞**
$\mathbf{d^{(1)}}$				
$\mathbf{d^{(2)}}$				
$\mathbf{d^{(3)}}$				

We will use table $d^{(0)}$ to fill in $d^{(1)}$. More generally, we will use table $d^{(k-1)}$ to fill in $d^{(k)}$.

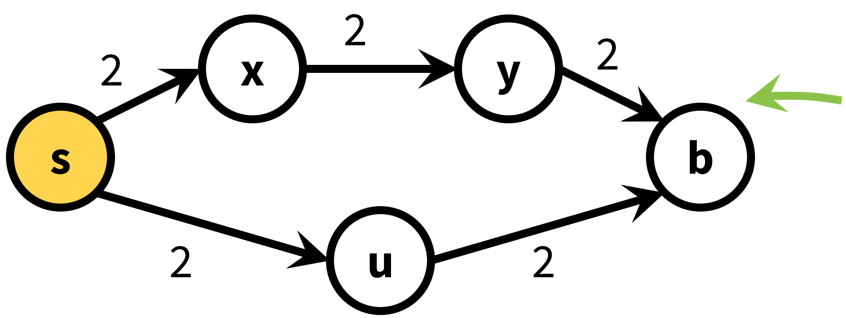


Bellman-Ford Algorithm

How do we use $d^{(k-1)}$ to fill in $d^{(k)}[b]$?

Recall $d^{(k)}[b]$ is the cost of the shortest path from s to b with at most k edges.

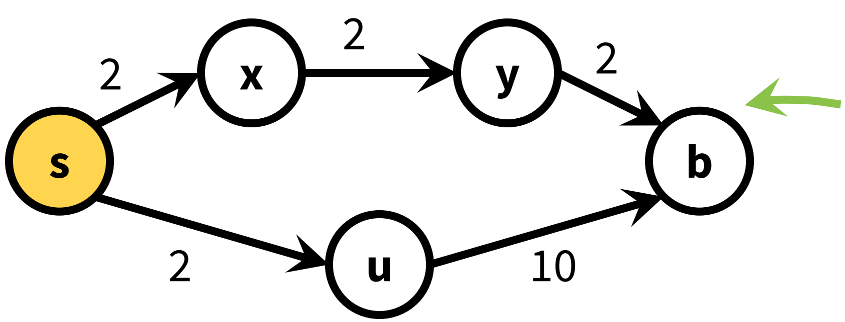
Case 1: the shortest path from s to b with at most k edges actually has at most k - 1 edges.



Suppose k = 3.

$d^{(k)}[b] = d^{(k-1)}[b]$, i.e. $d^{(3)}[b] = d^{(2)}[b]$, which means that the shortest path of at most k - 1 edges is at least as short as any path of at most k edges.

Case 2: the shortest path from s to b with at most k edges really has k edges.



Suppose k = 3.

$d^{(k)}[b] = \min_a\{d^{(k-1)}[a] + w(a, b)\}$
i.e. the shortest path of at most k edges is shorter than any path of at most k - 1 edges.

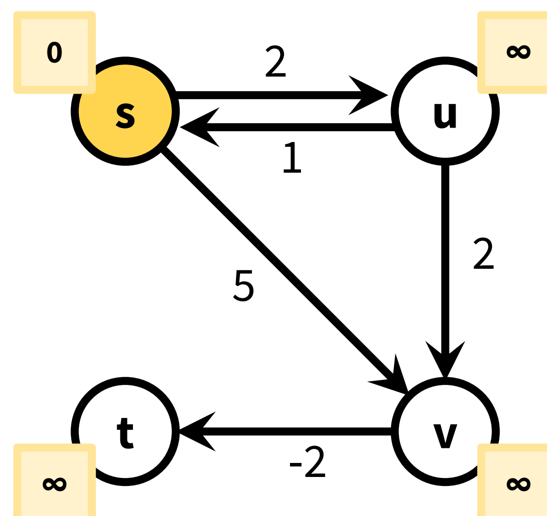
Bellman-Ford Algorithm

We maintain a list $d^{(k)}$ of length $|V|$ for each $k = 0, 1, \dots, |V|-1$.

```
for k = 1 to |V|-1:
  for b in V:
    d^(k)[b] = min{d^(k-1)[b], min_a{d^(k-1)[a] + w(a,b)} }
```

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$				
$d^{(2)}$				
$d^{(3)}$				

We will use table $d^{(0)}$ to fill in $d^{(1)}$. More generally, we will use table $d^{(k-1)}$ to fill in $d^{(k)}$.



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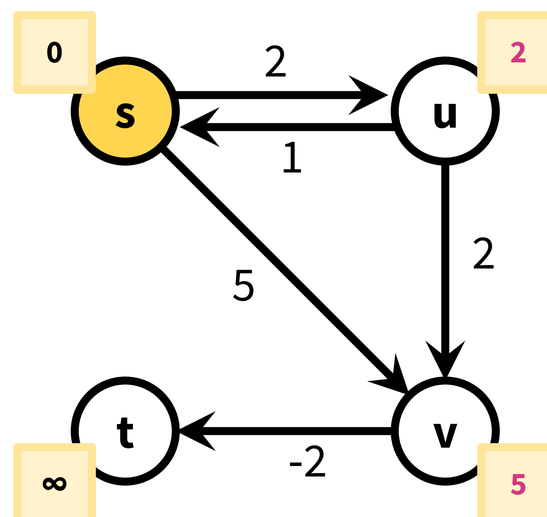
    d^(k)[b] = min{d^(k-1)[b], min_a{d^(k-1)[a] + w(a,b)} }
```

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$				
$d^{(3)}$				

We will use table $d^{(0)}$ to fill in $d^{(1)}$. More generally, we will use table $d^{(k-1)}$ to fill in $d^{(k)}$.



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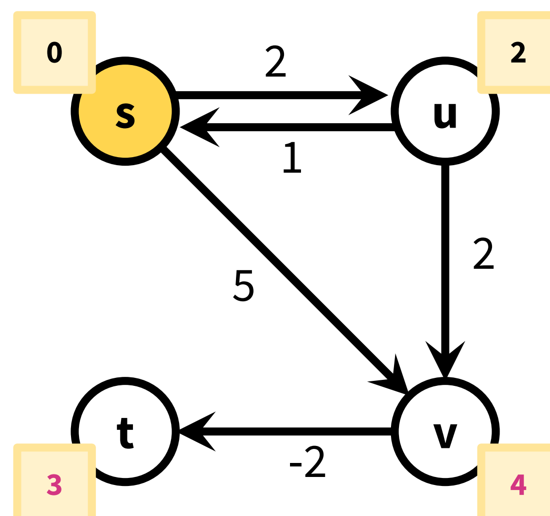
    d^(k)[b] = min{d^(k-1)[b], min_a{d^(k-1)[a] + w(a,b)} }
```

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
$d^{(3)}$				



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for k = 1 to |V|-1:
  for b in V:
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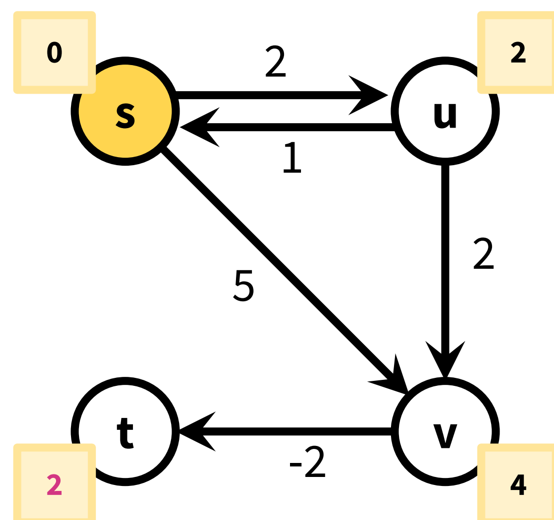
```

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
$d^{(3)}$	0	2	4	2



Bellman-Ford Algorithm

We maintain a list $d^{(k)}$ of length $|V|$ for each $k = 0, 1, \ldots, |V|-1$.

Recall $d^{(k)}[b]$ is the cost of the shortest path from s to b with at most k edges.

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
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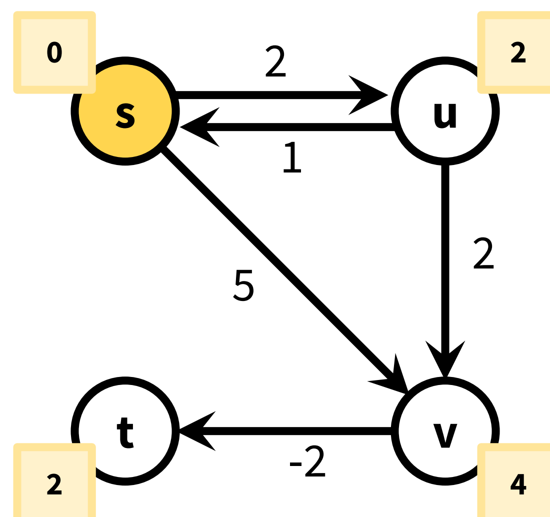


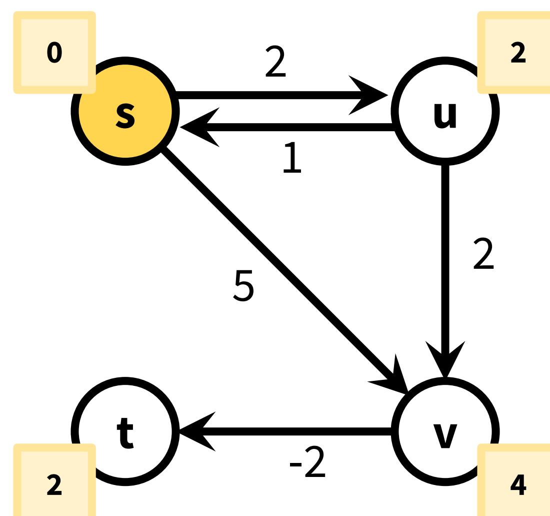
Bellman-Ford Algorithm

We maintain a list $d^{(k)}$ of length $|V|$ for each $k = 0, 1, \ldots, |V|-1$.

Recall $d^{(k)}[b]$ is the cost of the shortest path from s to b with at most k edges.

The shortest path from s to t with at most 1 edge has cost ∞ (no path exists).

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
$d^{(3)}$	0	2	4	2



Bellman-Ford Algorithm

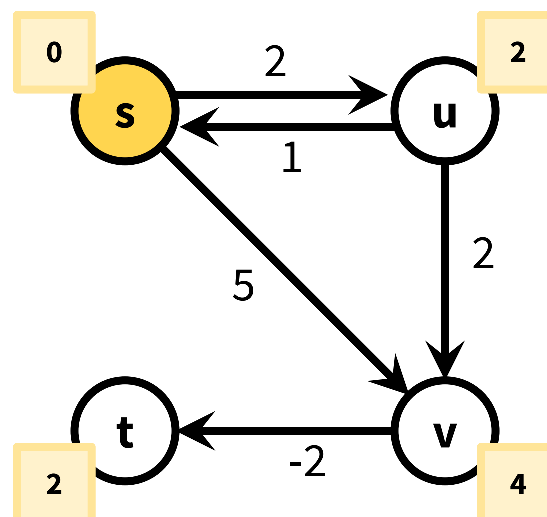
We maintain a list $d^{(k)}$ of length $|V|$ for each $k = 0, 1, \ldots, |V|-1$.

Recall $d^{(k)}[b]$ is the cost of the shortest path from s to b with at most k edges.

The shortest path from s to t with at most 1 edge has cost **∞** (no path exists).

The shortest path from s to t with at most 2 edges has cost **3** (s-v-t).

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
$d^{(3)}$	0	2	4	2



Bellman-Ford Algorithm

We maintain a list $d^{(k)}$ of length |V| for each k = 0, 1, …, |V|-1.

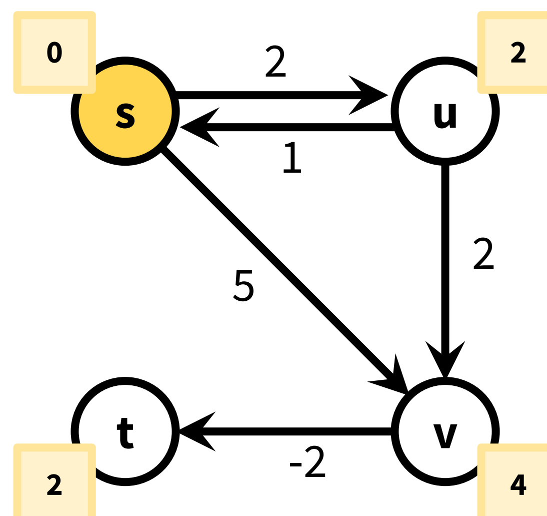
Recall $d^{(k)}[b]$ is the cost of the shortest path from s to b with at most k edges.

The shortest path from s to t with at most 1 edge has cost **∞** (no path exists).

The shortest path from s to t with at most 2 edges has cost **3** (s-v-t).

The shortest path from s to t with at most 3 edges has cost **2** (s-u-v-t).

	s	**u**	**v**	**t**
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
$d^{(3)}$	0	2	4	2



The shortest path from s to t with any number of edges is min{∞, 3, 2} = **2** and the path has 3 edges (s-u-v-t).

Actually this value will always be the value at the right-bottom corner, because it's at most |V|-1 edges, which includes cases that has fewer than |V|-1 edges.

Bellman-Ford Algorithm

```
algorithm bellman_ford(G):
  d⁽ᵏ⁾ = [] for k = 0 to |V|-1
  d⁽⁰⁾[v] = ∞ for all v ≠ s
  d⁽⁰⁾[s] = 0
  for k = 1 to |V|-1:
    for b in V:
      d⁽ᵏ⁾[b] = min{d⁽ᵏ⁻¹⁾[b], minₐ{d⁽ᵏ⁻¹⁾[a] + w(a,b)} }
  return d⁽|V|-1⁾
```

Case 1: $d^{(k-1)}[b]$

Case 2: $\min_a\{d^{(k-1)}[a] + w(a,b)\}$

This is a simplification to make the pseudocode nice. In reality, we'd only keep two of them at a time.

Minimum over all a such that $(a, b) \in E$.

Runtime: $O(|V||E|)$

Slower than Dijkstra's $O(|E| + |V|\log(|V|))$

Bellman-Ford Proof of Correctness

BF Proof of Correctness

We need to prove our main argument.

$d^{(|V|-1)}[b]$ is the cost of the shortest path from s to b with at most $|V|$-1 edges.

BF Proof of Correctness

Lemma: $d^{(|V|-1)}[b]$ is the cost of the shortest path from s to b with at most |V|-1 edges.

Proof: We proceed by induction on k, the number of iterations completed by the algorithm.

For our base case, at the start of iteration $k = 1$, the shortest path from s to s with 0 edges has cost 0. The path from s to all vertices $v \neq s$ contains at least 1 edge; there doesn't exist a path from s to v with 0 edges, and this path costs ∞. Therefore, $d^{(0)}$ is correct.

For our inductive step, assume that at the start of iteration k, $d^{(k-1)}[b]$ is the cost of the shortest path from s to b with at most k - 1 edges. We consider two cases:

Case 1: $d^{(k-1)}[b] < \min_a\{d^{(k-1)}[a] + w(a, b)\}$. This corresponds to the case in which the shortest path contains fewer than k edges. Then our algorithm correctly sets $d^{(k)}[b] = d^{(k-1)}[b]$.

Case 2: $d^{(k-1)}[b] \geq \min_a\{d^{(k-1)}[a] + w(a, b)\}$. This corresponds to the case in which the shortest path contains exactly k edges. Then our algorithm correctly sets $d^{(k)}[b] = \min_a\{d^{(k-1)}[a] + w(a, b)\}$, which minimizes the sum of the shortest path with at most k-1 edges to an in-neighbor of b and the weight from a to b.

At the start of iteration $k = |V|$, the algorithm terminates and $d^{(|V|-1)}$ is correct.

BF Proof of Correctness

We need to prove our main argument.

$d^{(|V|-1)}[b]$ is the cost of the shortest path from s to b with at most |V|-1 edges. 👌

What else to do? 🤔

BF Proof of Correctness

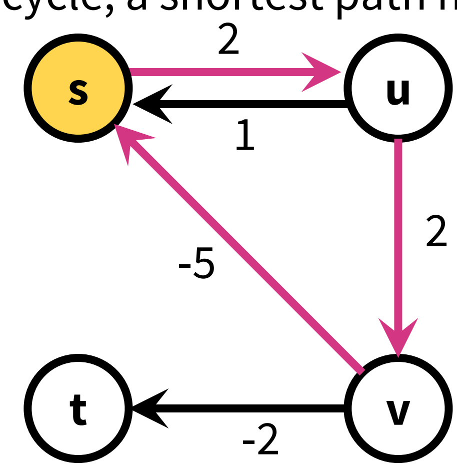
We need to prove our main argument.

- $d^{(|V|-1)}[b]$ is the cost of the shortest path from s to b with at most |V|-1 edges.👌

What else to do? 🤔

Why can we stop by the end of iteration |V|-1

- We still need to prove that this argument implies `bellman_ford` is correct i.e. $d^{(|V|-1)}[a]$ = distance(s, a).
- To show this, we'll prove that the shortest path with at most |V|-1 edges is the shortest path with any number of edges (if a shortest path exists).
- If the graph has a negative cycle, a shortest path might not exist!

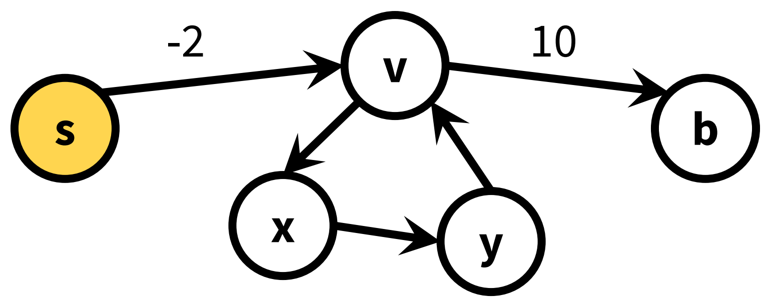


BF Proof of Correctness

But if there's no negative cycle.

There's always a simple shortest path.

A simple path has no cycles.



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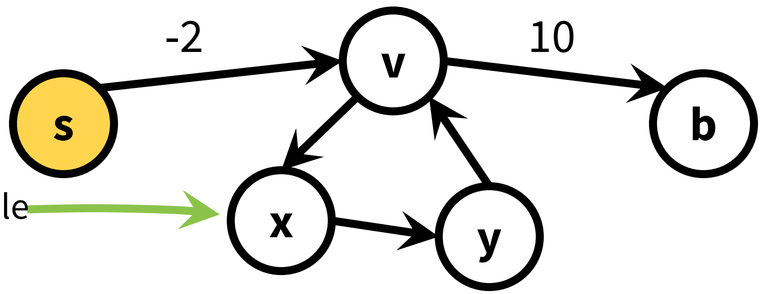
How do we know this cycle doesn't help? 🤔



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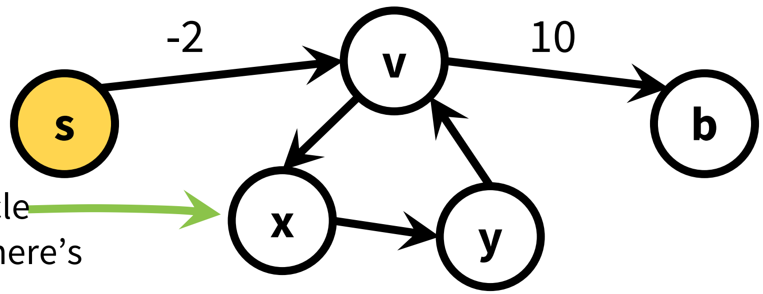

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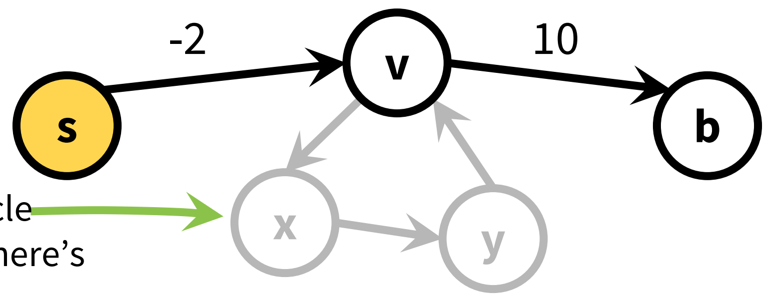

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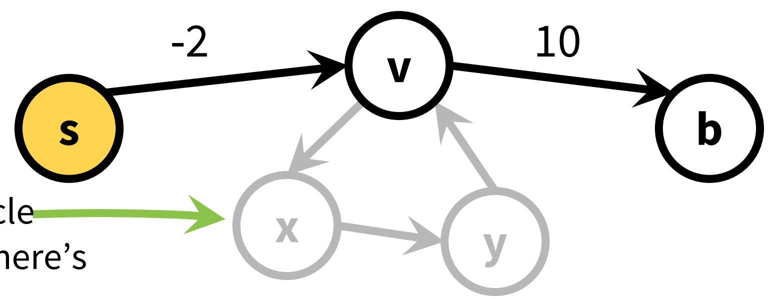
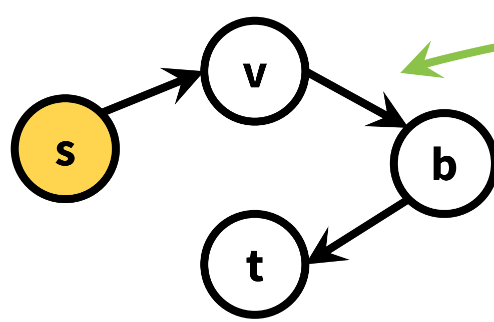


How do we know this cycle doesn't help? 🤔 Since there's no negative cycles!

A simple path in a graph with |V| vertices has at most |V|-1 edges in it.



We can't add another edge to this s-t path without making a cycle (an edge from s to b wouldn't be along the path).

BF Proof of Correctness

Theorem: `bellman_ford` is correct as long as the graph has no negative cycles.

Proof:

By our lemma, $d^{(|V|-1)}[b]$ contains the cost of the shortest path from s to b with at most |V|-1 edges. If there are no negative cycles, then the shortest path must be simple, and all simple paths have at most |V|-1 edges. Therefore, the value the algorithm returns, $d^{(|V|-1)}[b]$, is also the cost of the shortest path from s to b with any number of edges.

$$\left.\begin{array}{l}\text{shortest path -> simple -> at most } |V|\text{-1 edges} \\ \\ d^{(|V|-1)}[b] \text{ is the shortest path of at most } |V|\text{-1 edges}\end{array}\right\} d^{(|V|-1)}[b] \text{ must be the shortest path}$$

Use Bellman-Ford to detect Negative Cycle

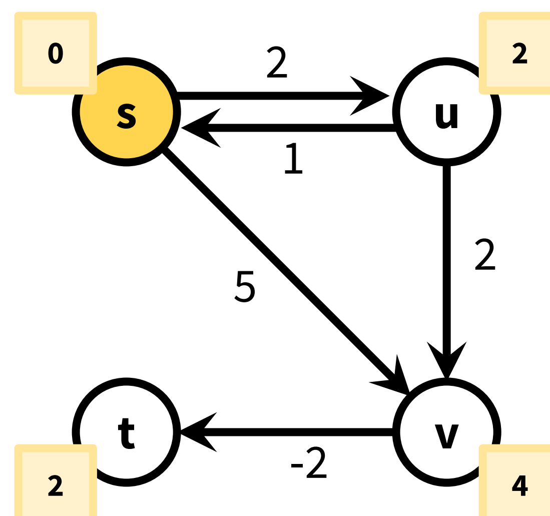
Use BF to Detect Negative Cycle

Basic idea: perform an extra iteration.

If there is no negative cycle, then BF algorithm only needs |V|-1 iterations.

If the |V|-th iteration changed the shortest distance of a vertex, then there must exist a negative cycle.

	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
$d^{(3)}$	0	2	4	2
$d^{(4)}$	0	2	4	2



Use BF to Detect Negative Cycle

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If there is no negative cycle, then BF algorithm only needs |V|-1 iterations.

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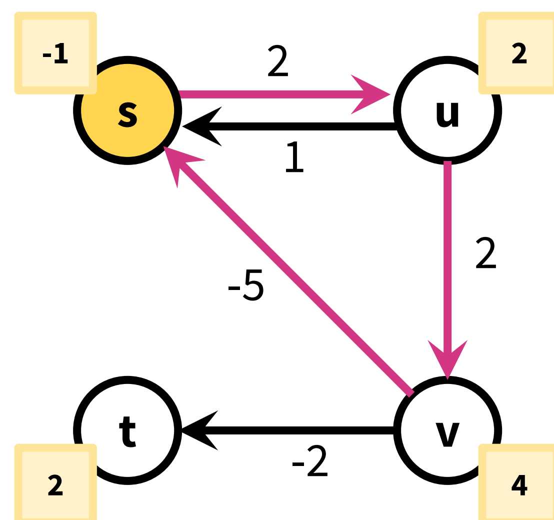
	s	u	v	t
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	∞	∞
$d^{(2)}$	0	2	4	∞
$d^{(3)}$	-1	2	4	2
$d^{(4)}$	-1	1	4	2



Bellman-Ford Algorithm

Bellman-Ford gets used in practice.

e.g. Updating Routing Information in computer networks uses it. Each router keeps a table of shortest distances to every other router. Periodically, we do a Bellman-Ford to update the distance information.

Dynamic Programming

Dynamic Programming

Bellman-Ford is an example of **dynamic programming**!

Dynamic programming is an algorithm design paradigm.

- Often it's used to solve optimization problems e.g. **shortest** path.

Dynamic Programming

Elements of dynamic programming

Large problems break up into small problems.

e.g. shortest path with at most k edges.

Optimal substructure the optimal solution of a problem can be expressed in terms of optimal solutions of smaller sub-problems.

e.g. $d^{(k)}[b] = \min\{d^{(k-1)}[b], \min_a\{d^{(k-1)}[a] + w(a,b)\}\}$

Overlapping sub-problems the sub-problems overlap a lot.

e.g. Lots of different entries of $d^{(k)}$ ask for $d^{(k-1)}[a]$.

This means we're saving time by solving a sub-problem once and caching the answer. Answer of the sub-problems can be reused for multiple times.

Dynamic Programming

Comparing Dynamic Programming and Divide-and-Conquer

DP can be seen as a special case of D-n-C

Divide-and-Conquer solves the problem “batch-by-batch” (e.g., usually half-by-half)

Dynamic Programming solves the problem “step-by-step”

The reason is that some problem are too difficult that we can not reduce the problem size very efficiently as in D-n-C (e.g., half-to-half), we have to begin from very small problem sizes first (e.g., shortest path with 0 steps in the BF algorithm) and increase the problem size by a small increment at each time (e.g., using shortest paths of at most k-1 steps to calculate shortest paths of at most k steps, by increase one step at each time).

Dynamic Programming

Two approaches for DP: bottom-up and top-down.

Bottom-up iterates through problems by size and solves the small problems first (Bellman-Ford solves $d^{(0)}$ then $d^{(1)}$ then $d^{(2)}$, etc.)

Top-down recurses to solve smaller problems, which recurse to solve even smaller problems.

How is this different than divide and conquer? **Memoization**, which keeps track of the small problems you've already solved to prevent resolving the same problem more than once.

Divide-and-conquer usually solve a sub-problem for once and use it also for only once. E.g., the merge-sort algorithm.

Dynamic programming solve a sub-problem for once and use it for multiple times. E.g., Bellman-Ford algorithm.

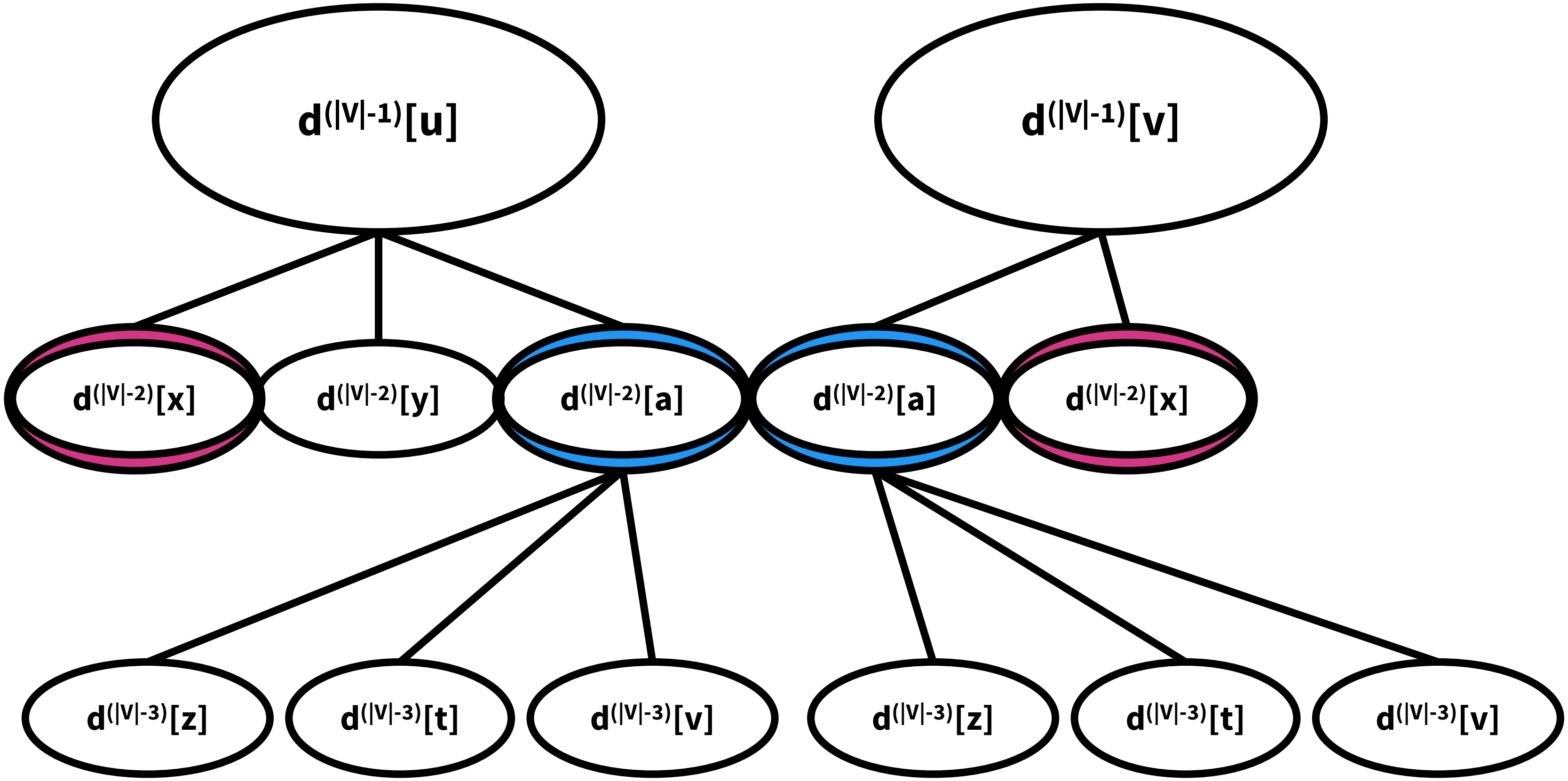
Top-Down BF Algorithm

```
algorithm recursive_bellman_ford(G):
  d⁽ᵏ⁾ = [None] * |V| for k = 0 to |V|-1
  d⁽⁰⁾[v] = ∞ for all v ≠ s
  d⁽⁰⁾[s] = 0
  for b in V:
    recursive_bf_helper(G, b, |V|-1)

algorithm recursive_bf_helper(G, b, k):
  A = {a such that (a, b) in E} ∪ {b}
  for a in A:
    if d⁽ᵏ⁻¹⁾[a] is None:
      d⁽ᵏ⁻¹⁾[a] = recursive_bf_helper(G, a, k-1)
  return min{d⁽ᵏ⁻¹⁾[b], minₐ{d⁽ᵏ⁻¹⁾[a] + w(a, b)} }
```

Runtime: $O(|V||E|)$

Visualization of Top-Down



$d^{(|V|-2)}[x]$ and $d^{(|V|-2)}[a]$ are reused by $d^{(|V|-1)}[v]$ after they have been calculated for $d^{(|V|-1)}[u]$
Similar in the second layer

Floyd-Warshall Algorithm

(Advanced Topic)

Floyd-Warshall Algorithm

Another example of a graph DP algorithm!

The algorithm solves the all-pairs shortest path (**APSP**) problem.

A simple solution

```
for s in V:
  run bellman_ford starting at s
```

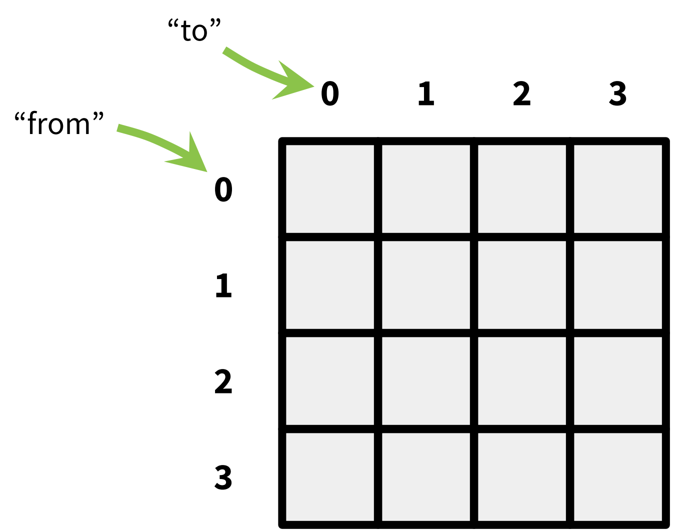
Runtime $O(|V|^2|E|)$

Can we do better?

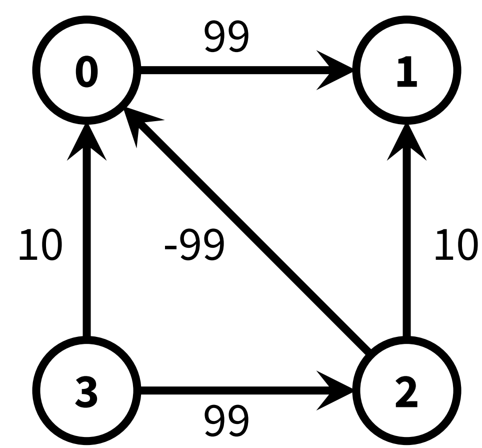
Floyd-Warshall Algorithm

We maintain a |V|×|V| matrix $D^{(k)}$ for each k = 0, 1, …, |V|.

$D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices {0, …, k-1}.



$D^{(0)}[u, v]$



Basic idea: we iterate for |V|+1 times, at iteration k (k=0,1,…,|V|), any path between any pair of nodes are only allowed to path though the first k nodes {0, 1, …, k-1}.
After the last iteration k=|V|, any path are allowed to path through the nodes {0,1,…,|V|-1}, which are all the nodes, so the final results should be the correct answer.

Floyd-Warshall Algorithm

We maintain a |V|×|V| matrix $D^{(k)}$ for each $k = 0, 1, \ldots, |V|$.

$D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices $\{0, \ldots, k\text{-}1\}$.

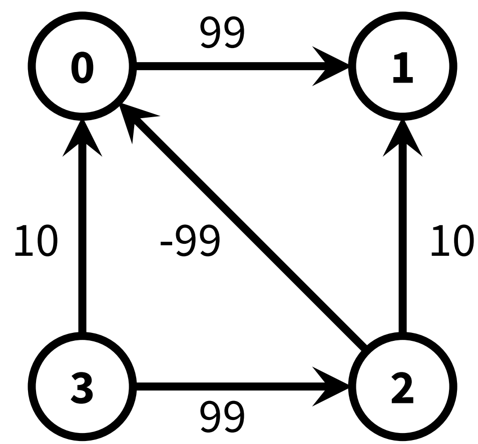
"to"

"from"

	0	**1**	**2**	**3**
0	0			
1				
2				
3				

This is the cost of the shortest path from 0 to 0, such that all of the internal vertices on the path are in the set of vertices {0,…,-1} i.e. the cost of the shortest path from 0 to 0 that passes through no other vertices.

$D^{(0)}[u, v]$



Floyd-Warshall Algorithm

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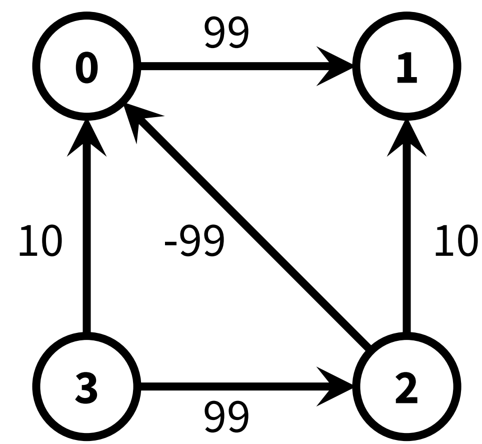
“to”

“from”

	0	**1**	**2**	**3**
0	**0**			
1		**0**		
2			**0**	
3				**0**

$D^{(0)}[u, v]$



Floyd-Warshall Algorithm

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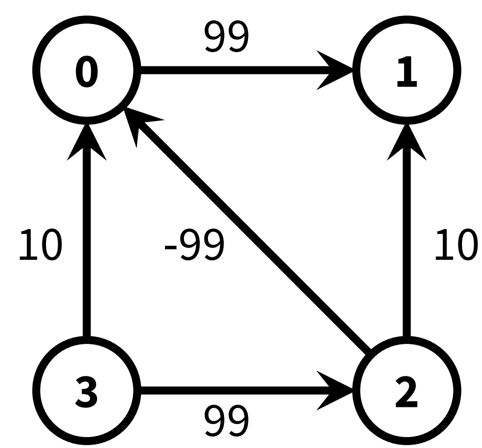
“to”

“from”

	0	1	2	3
0	0			
1		0		
2			0	
3				0

What should this value be? 🤔

$D^{(0)}[u, v]$



Floyd-Warshall Algorithm

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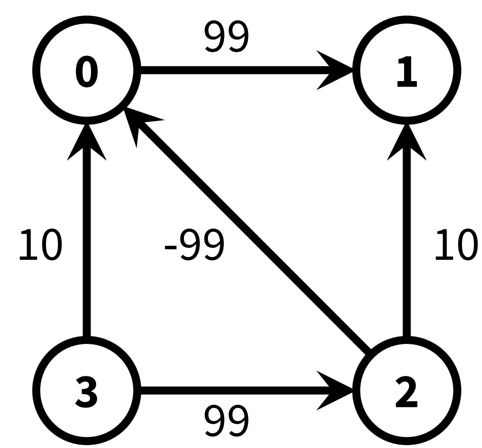
“to”

“from”

	0	1	2	3
0	0			
1		0		
2	-99		0	
3				0

What should this value be? 🤔 -99, since the shortest path from 2 to 0, passing through no other vertices has weight -99.

$D^{(0)}[u, v]$



Floyd-Warshall Algorithm

“to”

“from”

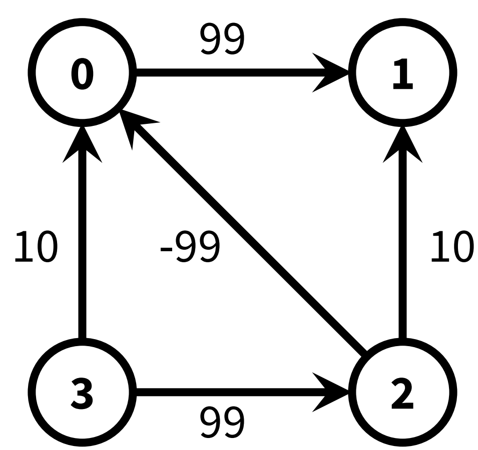
	0	1	2	3
0	0	99	∞	∞
1	∞	0	∞	∞
2	-99	10	0	∞
3	10	∞	99	0

$D^{(0)}[u, v]$

	0	1	2	3
0				
1				
2				
3				

$D^{(1)}[u, v]$

$D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices {0, …, k-1}.



Since k = 1, shortest paths are allowed to pass through vertices {0} now. So the we can compare the current cost to the cost of path 3-0-1. $D^{(0)}$ tells us the cost of 3-0 is **10** and the cost of 0-1 is **99**.

Floyd-Warshall Algorithm

"to"

"from"

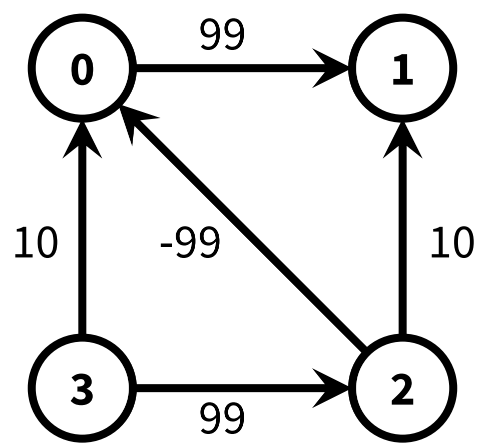
	0	1	2	3
0	0	99	∞	∞
1	∞	0	∞	∞
2	-99	10	0	∞
3	10	∞	99	0

$D^{(0)}[u, v]$

	0	1	2	3
0				
1				
2				
3		109		

$D^{(1)}[u, v]$

$D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices {0, …, k-1}.



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Floyd-Warshall Algorithm

“to”

“from”

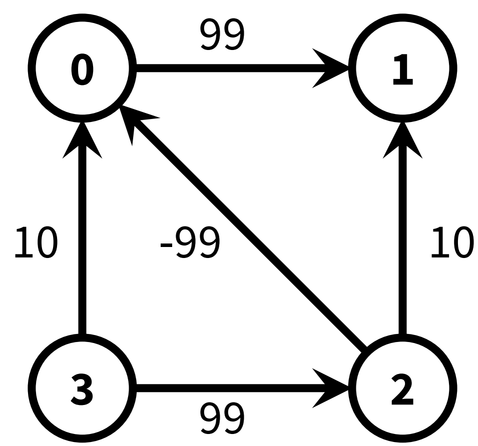
	0	1	2	3
0	0	99	∞	∞
1	∞	0	∞	∞
2	-99	10	0	∞
3	10	∞	99	0

$D^{(0)}[u, v]$

	0	1	2	3
0				
1				
2		0		
3		109		

$D^{(1)}[u, v]$

$D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices {0, …, k-1}.



Floyd-Warshall Algorithm

“to”

“from”

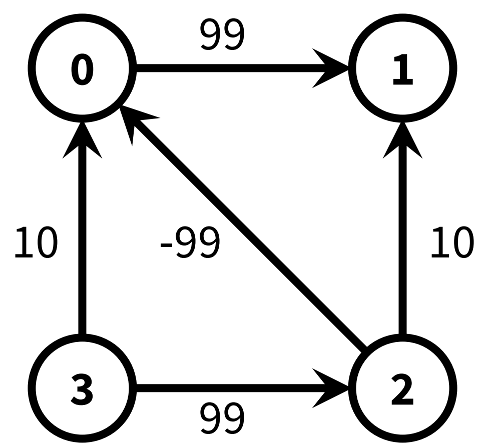
	0	1	2	3
0	0	99	∞	∞
1	∞	0	∞	∞
2	-99	10	0	∞
3	10	∞	99	0

$D^{(0)}[u, v]$

	0	1	2	3
0	0	99	∞	∞
1	∞	0	∞	∞
2	-99	0	0	∞
3	10	109	99	0

$D^{(1)}[u, v]$

$D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices {0, …, k-1}.

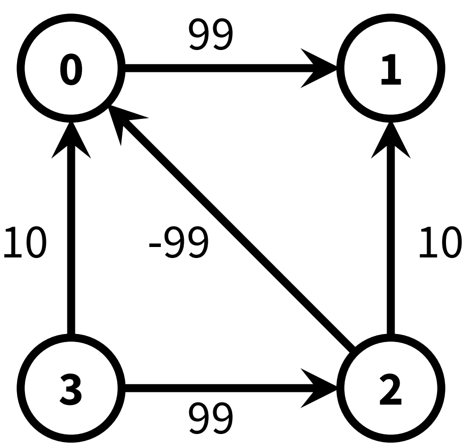


Floyd-Warshall Algorithm

O(|V|³)

0	99	∞	∞
∞	0	∞	∞
-99	10	0	∞
10	99	99	0

$D^{(4)}[u, v]$



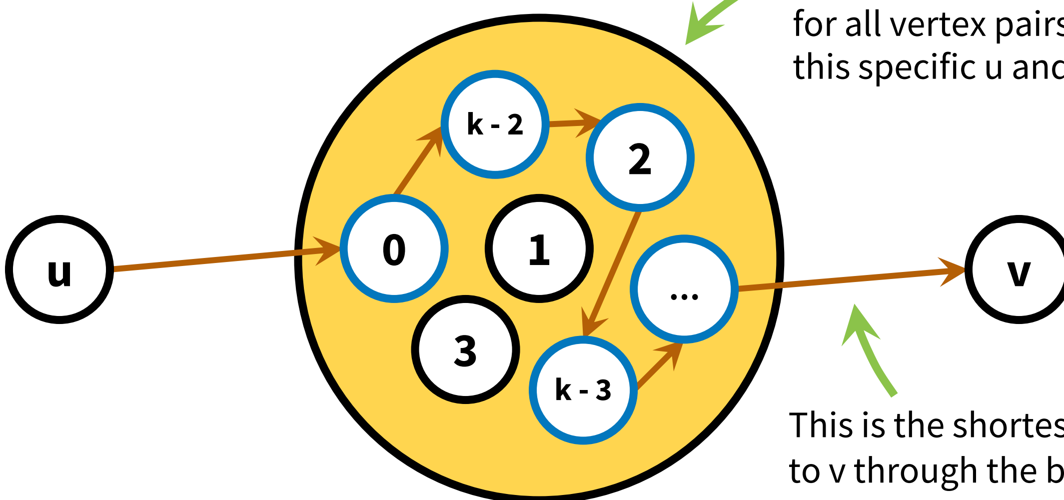
Floyd-Warshall Algorithm

We can represent it more graphically.

$D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices {0, …, k-1}.

How might we find $D^{(k)}[u, v]$ using $D^{(k-1)}$?

$D^{(k-1)}$ describes the cost of the shortest path through internal vertices {0, …, k-2} for all vertex pairs, including this specific u and v.

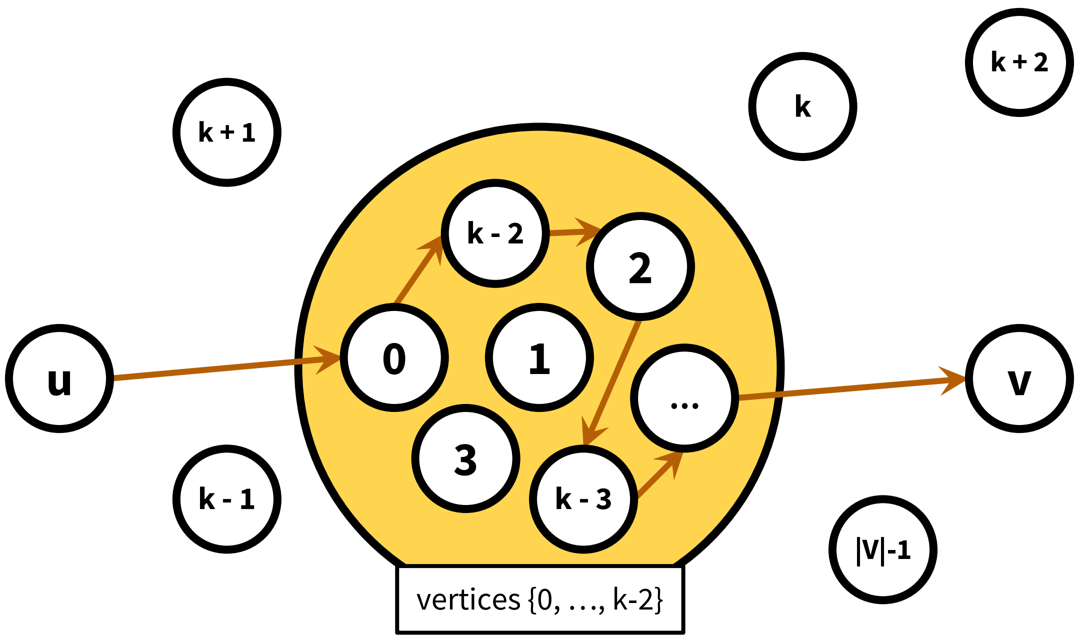


This is the shortest path from u to v through the blue set of vertices; it has weight $D^{(k-1)}[u, v]$.

Floyd-Warshall Algorithm

We can represent it more graphically.

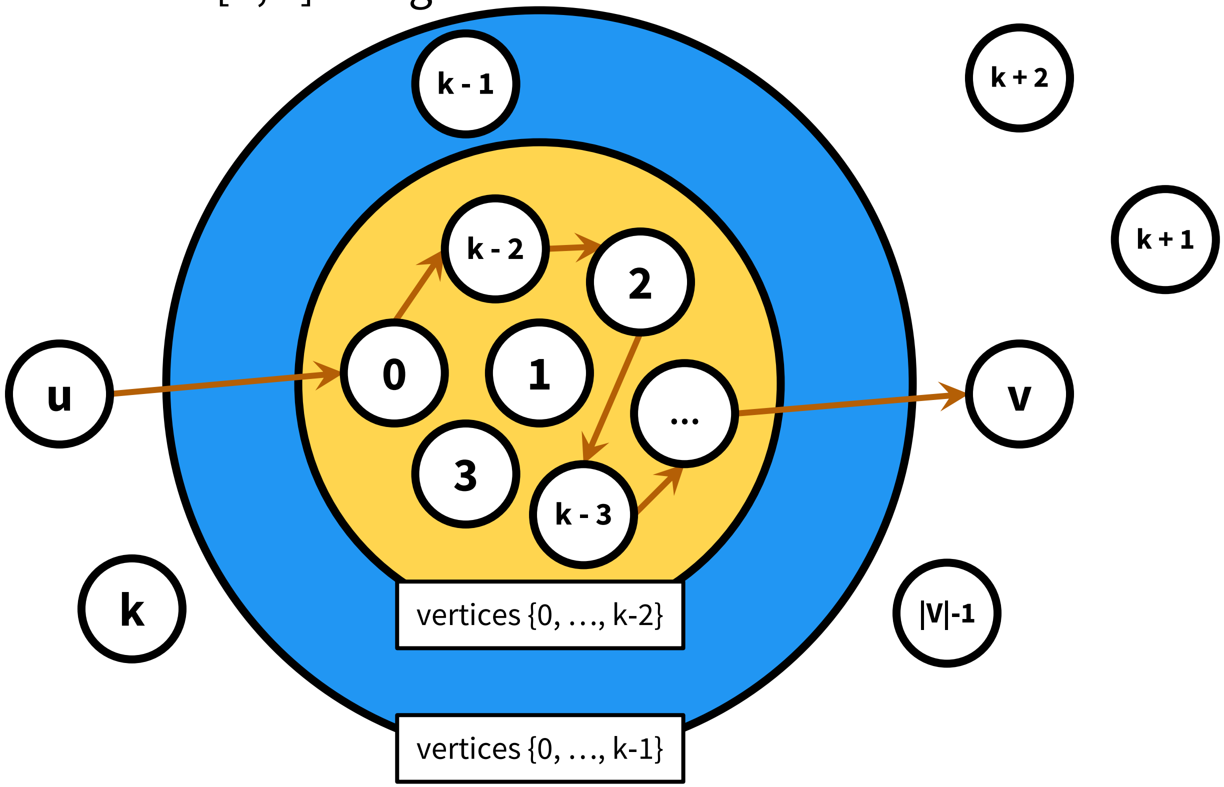
- $D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices {0, …, k-1}.
- How might we find $D^{(k)}[u, v]$ using $D^{(k-1)}$?



Floyd-Warshall Algorithm

We can represent it more graphically.

- $D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices {0, …, k-1}.
- How might we find $D^{(k)}[u, v]$ using $D^{(k-1)}$?



Floyd-Warshall Algorithm

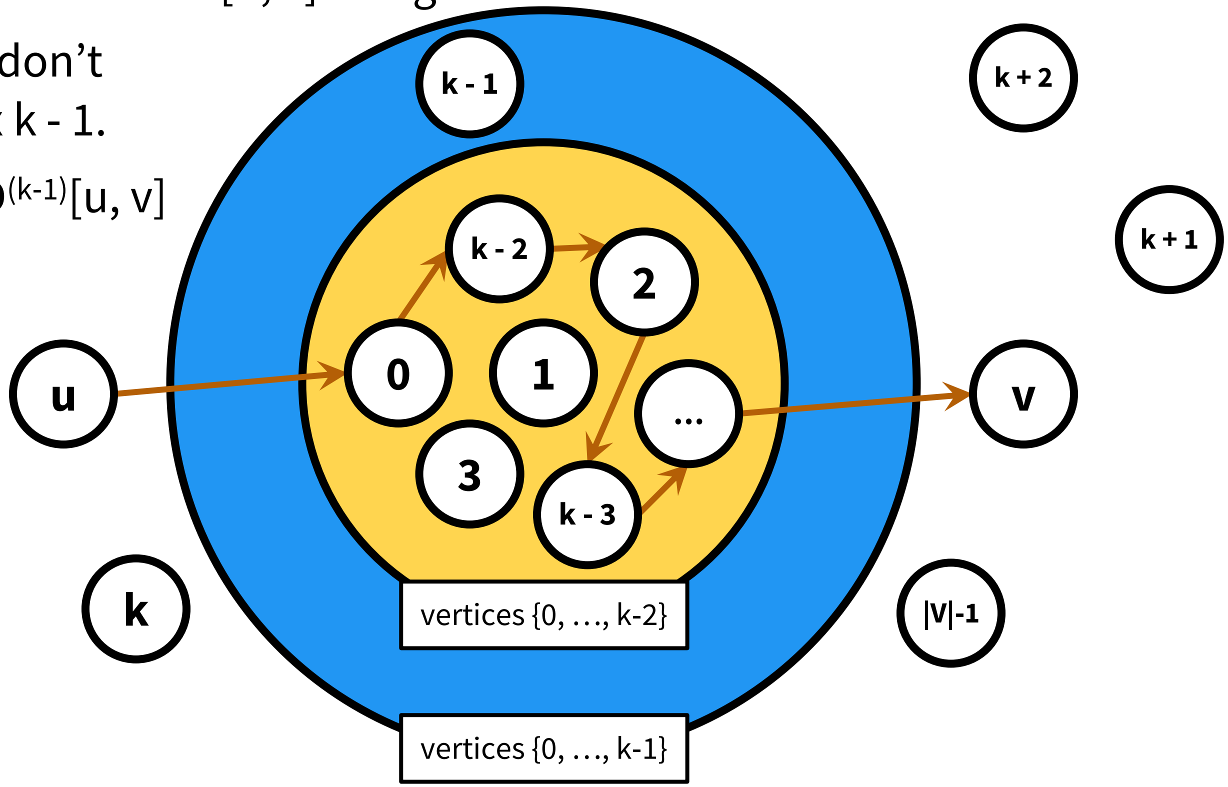
We can represent it more graphically.

$D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices {0, …, k-1}.

How might we find $D^{(k)}[u, v]$ using $D^{(k-1)}$?

Case 1: we don't need vertex k - 1.

$D^{(k)}[u, v] = D^{(k-1)}[u, v]$



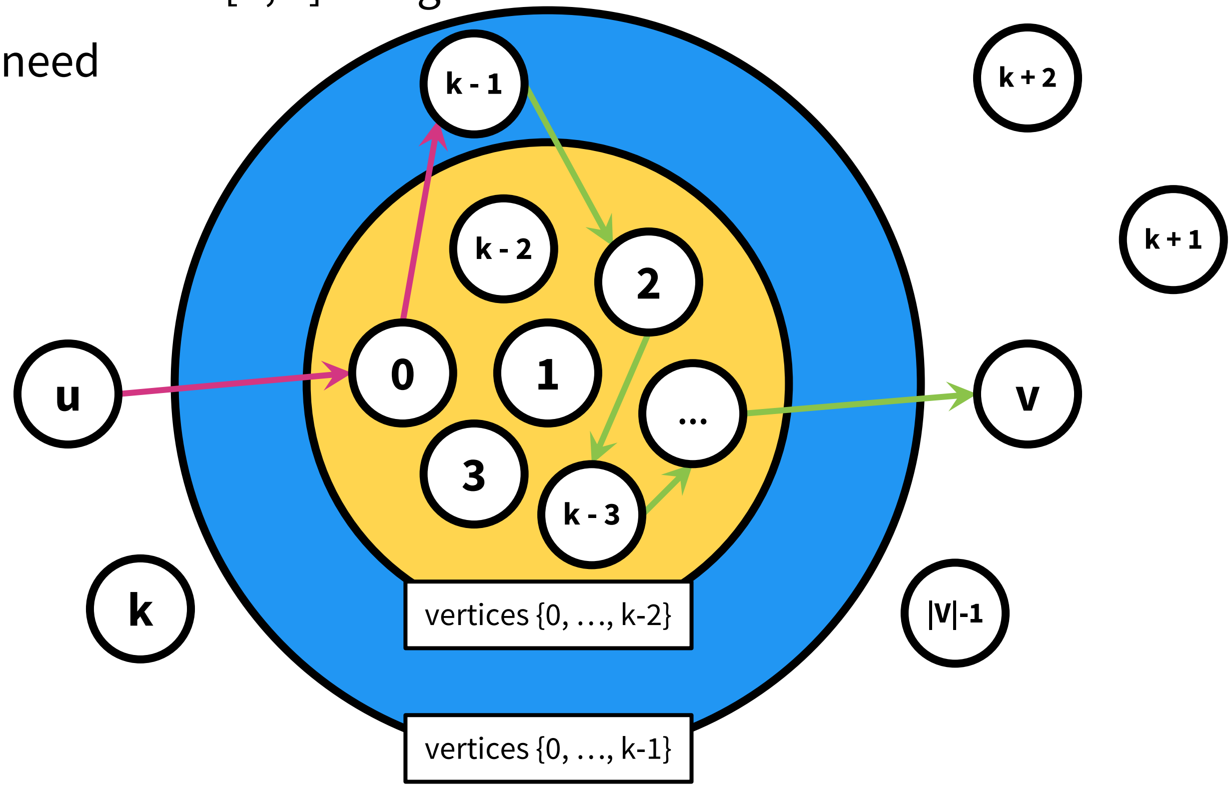
Floyd-Warshall Algorithm

We can represent it more graphically.

$D^{(k)}[u, v]$ is the cost of the shortest path from u to v, such that all of the internal vertices on the path are in the set of vertices {0, …, k-1}.

How might we find $D^{(k)}[u, v]$ using $D^{(k-1)}$?

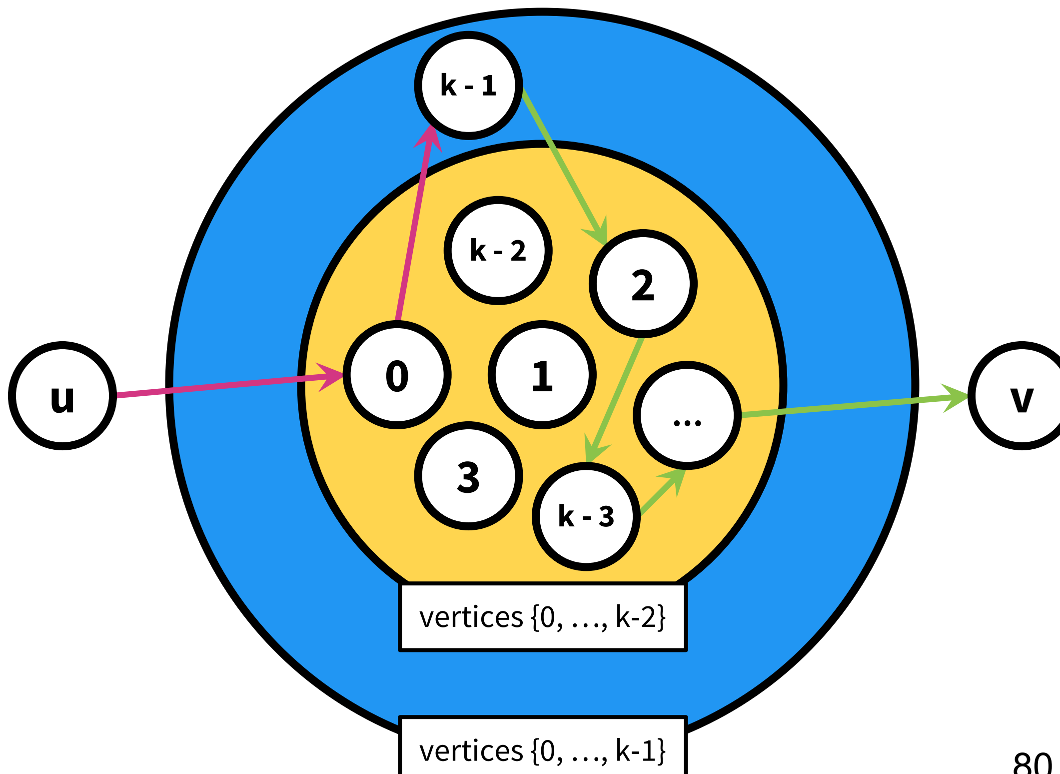
Case 2: we need vertex k - 1.



Floyd-Warshall Algorithm

Case 2, cont.: we need vertex k - 1.

If there are no negative cycles, then the shortest path from u to v through {0, …, k-1} is simple.



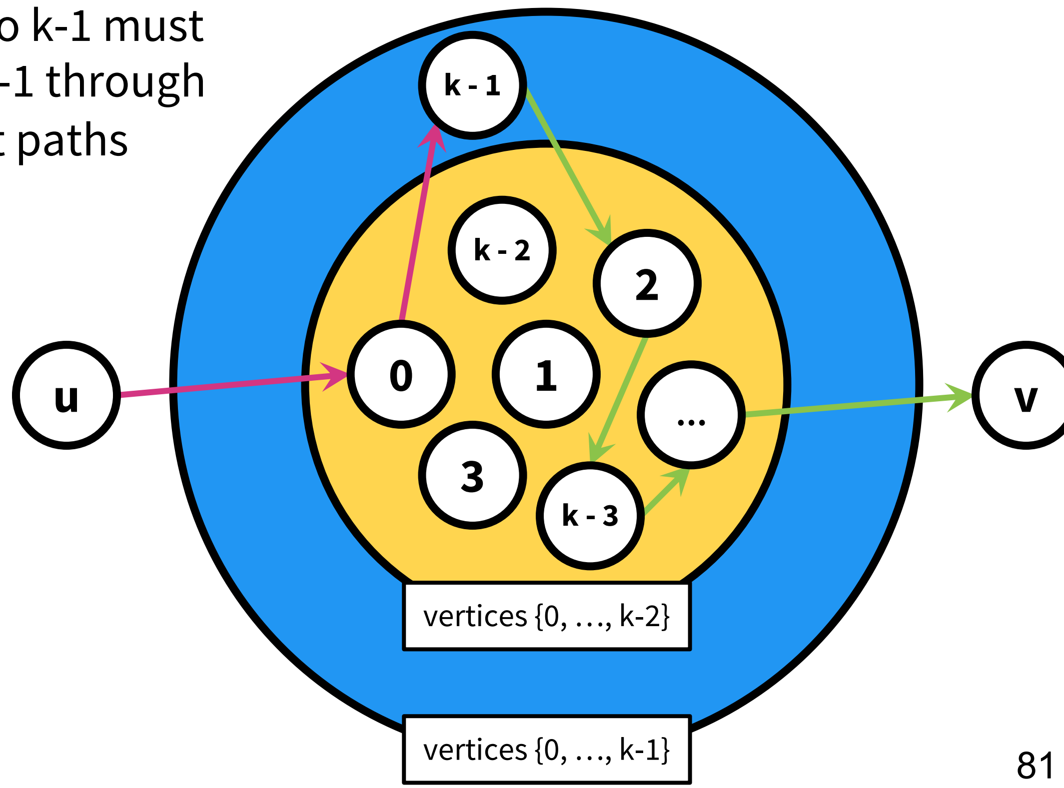
Floyd-Warshall Algorithm

Case 2, cont.: we need vertex k - 1.

If there are no negative cycles, then the shortest path from u to v through {0, …, k-1} is simple.

If the shortest path from u to v needs vertex k - 1, then **the subpath** from u to k-1 must be the shortest path from u to k-1 through {0, …, k-2} (subpaths of shortest paths are shortest paths).

This looks like our inductive hypothesis :)



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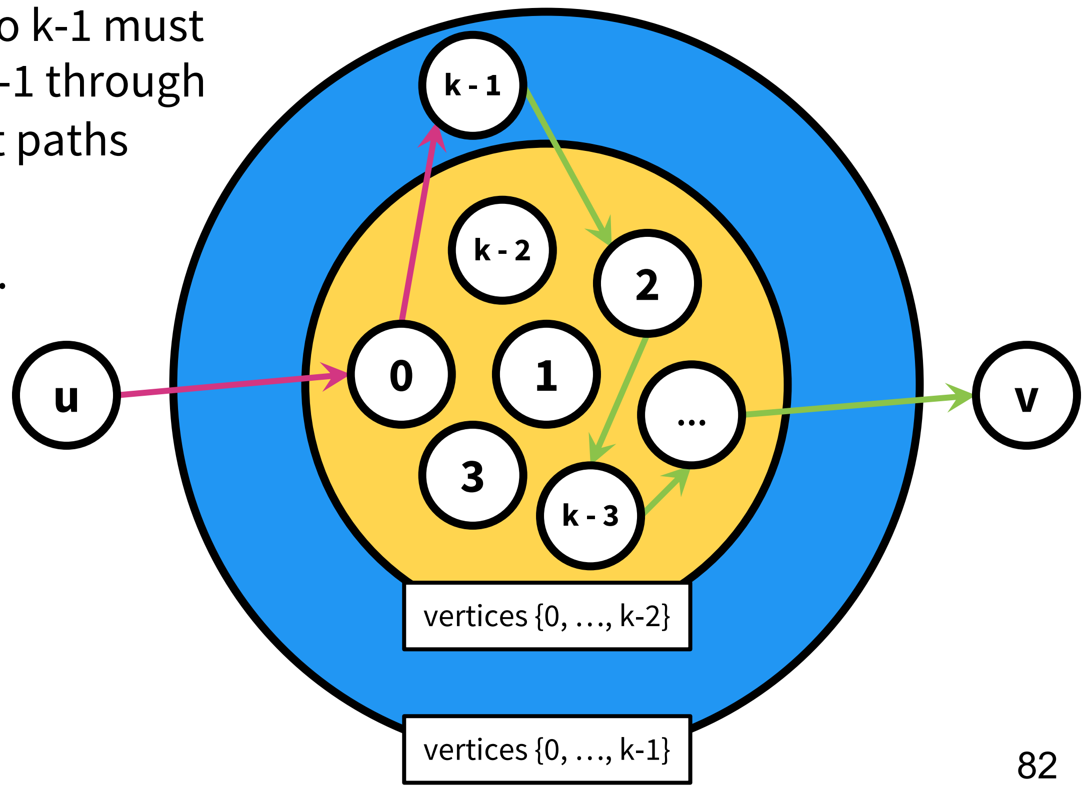
Same for **the path** from k-1 to v.

$D^{(k)}[u, v] =$
$D^{(k-1)}[u, k-1] + D^{(k-1)}[k-1, v]$



Floyd-Warshall Algorithm

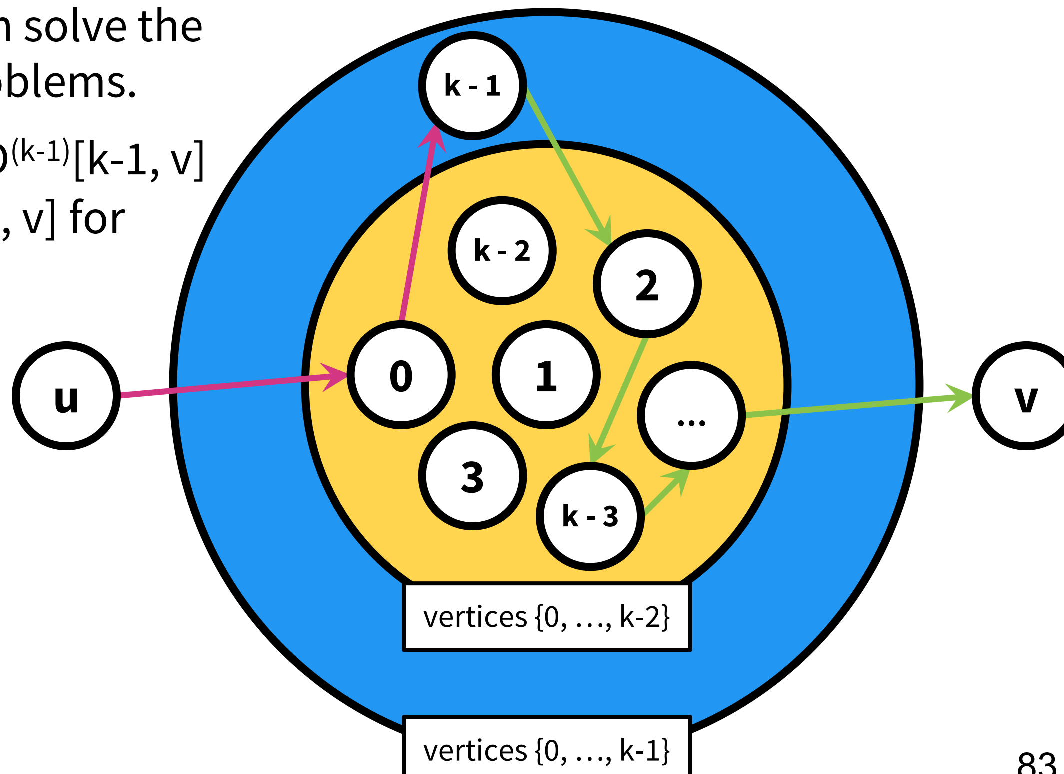
How might we find $D^{(k)}[u, v]$ using $D^{(k-1)}$?

$D^{(k)}[u, v] = \min\{D^{(k-1)}[u, v], D^{(k-1)}[u, k-1] + D^{(k-1)}[k-1, v]\}$

Case 1 Case 2

Optimal substructure We can solve the big problem using smaller problems.

Overlapping sub-problems $D^{(k-1)}[k-1, v]$ can be used to compute $D^{(k)}[u, v]$ for lots of different u's.



Floyd-Warshall Algorithm

Floyd-Warshall can detect negative cycles.

- If there's a negative cycle, then there's a path from v to v that has cost < 0.
- How do we check for this condition? 🤔

Floyd-Warshall Algorithm

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- How do we check for this condition? 🤔 We can just check $D^{(|V|)}[v, v] < 0$ at the end of the algorithm.

Graph Algorithms

	Dijkstra	Bellman-Ford	Floyd-Warshall
Problem	Single source shortest path (SSSP)	Single source shortest path (SSSP)	All pairs shortest path (APSP)
Runtime	`O(|E|+|V|log(|V|))` **worst-case** with an RB-tree	`O(|V||E|)` **worst-case**	`O(|V|³)` **worst case**
Strengths	---	Works on graphs with negative edge-weights; can detect if there exist negative cycles	Works on graphs with negative edge-weights; can detect if there exist negative cycles and where they are
Weaknesses	Might not work on graphs with negative edge-weights	---	---

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Acknowledgement: Part of the materials are adapted from Virginia Williams and David Eng's lectures on algorithms. We appreciate their contributions.