Greedy Algorithms II

Outline for Today

- Greedy algorithms
  - Greedy graph algorithms
    - Minimum Spanning Trees
    - Prim’s Algorithm
    - Kruskal’s Algorithm

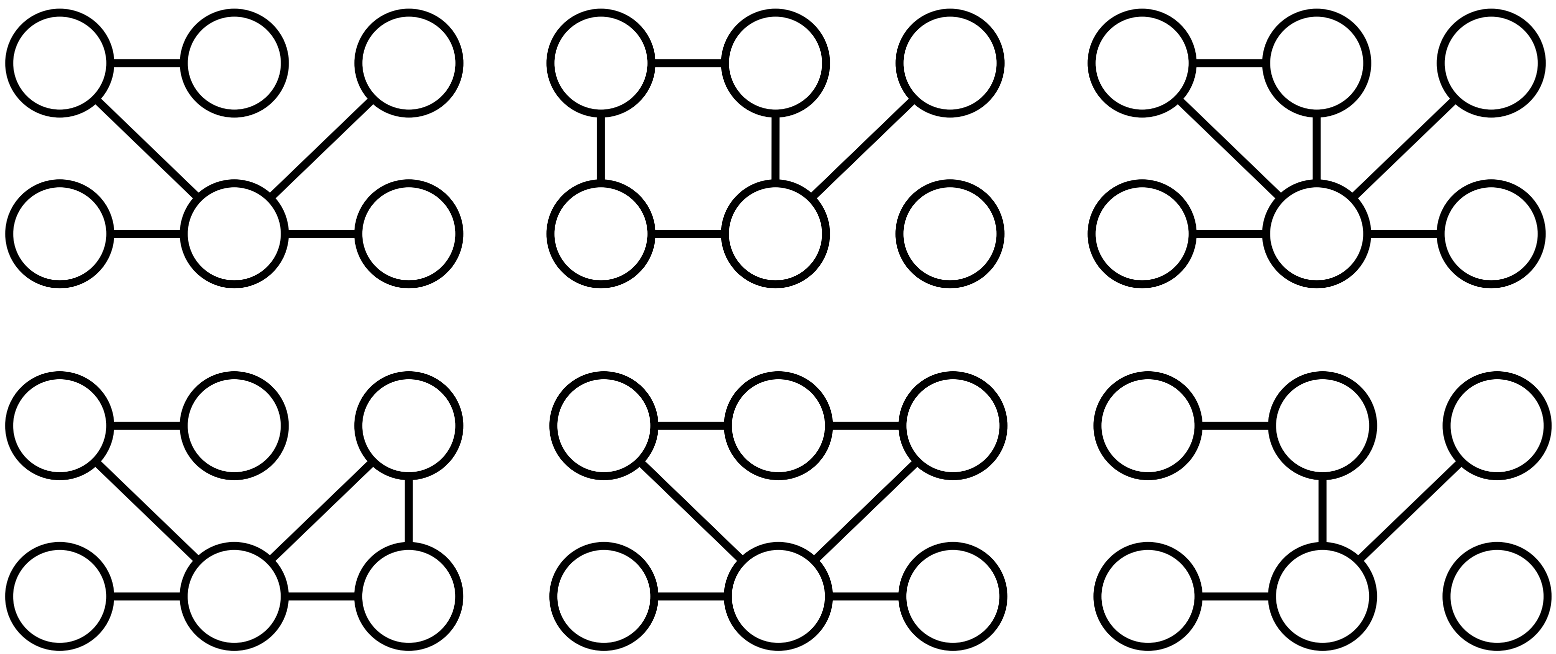
Minimum Spanning Trees

Tree

In Lecture 3, we studied trees with directed edges from parent to children vertices. In this lecture, edges will be undirected.

A tree is an undirected, acyclic, connected graph.

Which of these graphs contain connected components that are trees?



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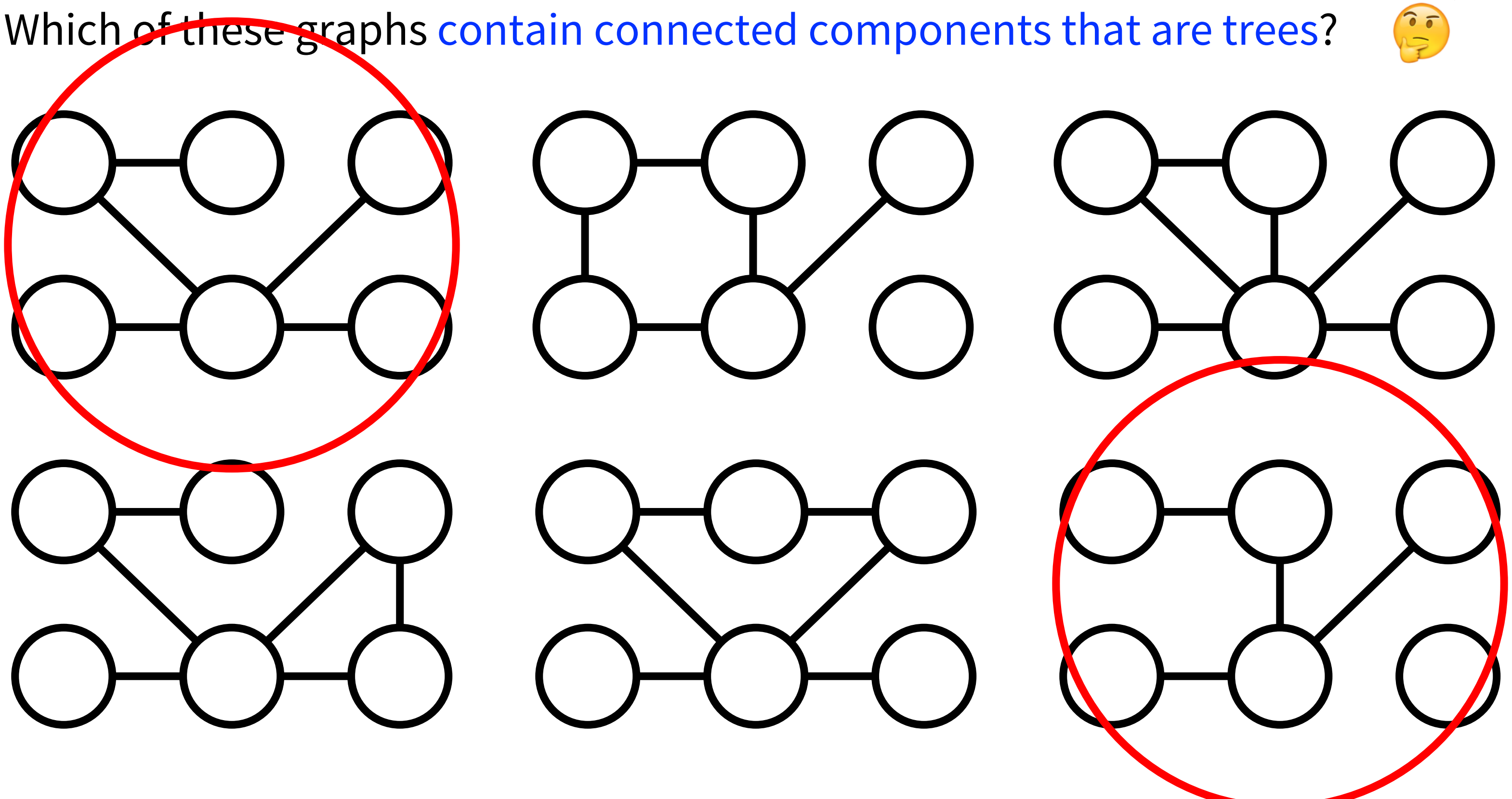
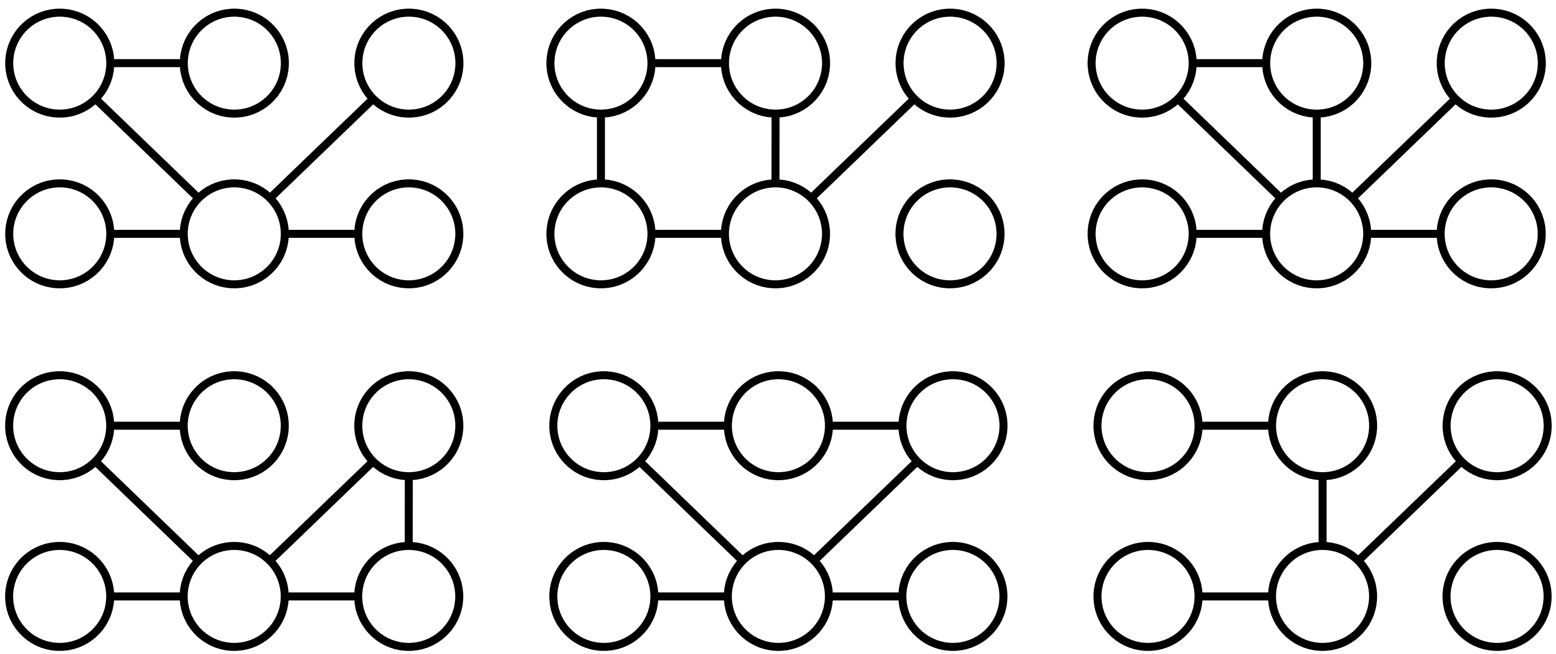


Spanning Tree

A spanning tree is a tree that connects all of the vertices.

Which of these graphs are spanning trees? 🤔



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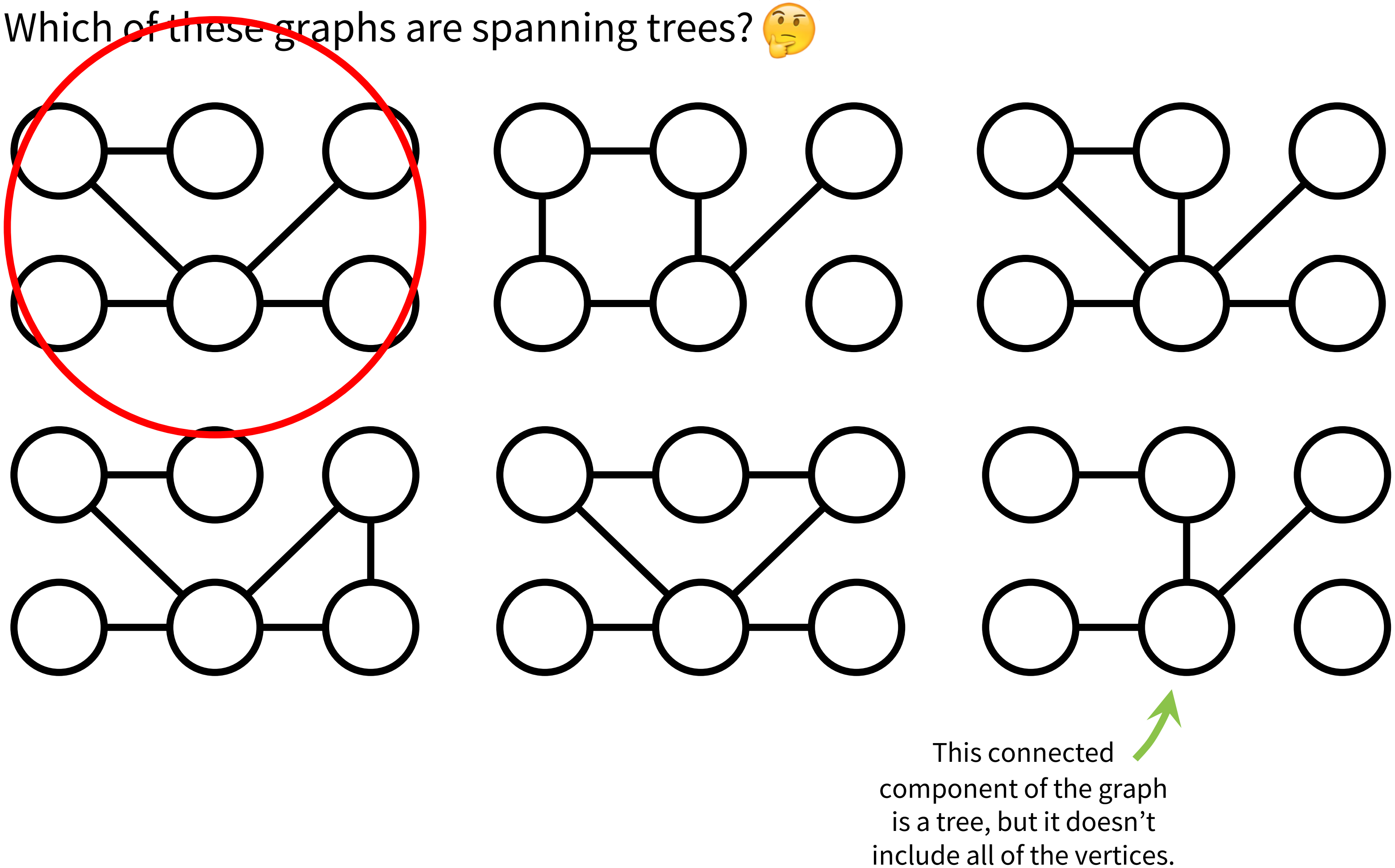

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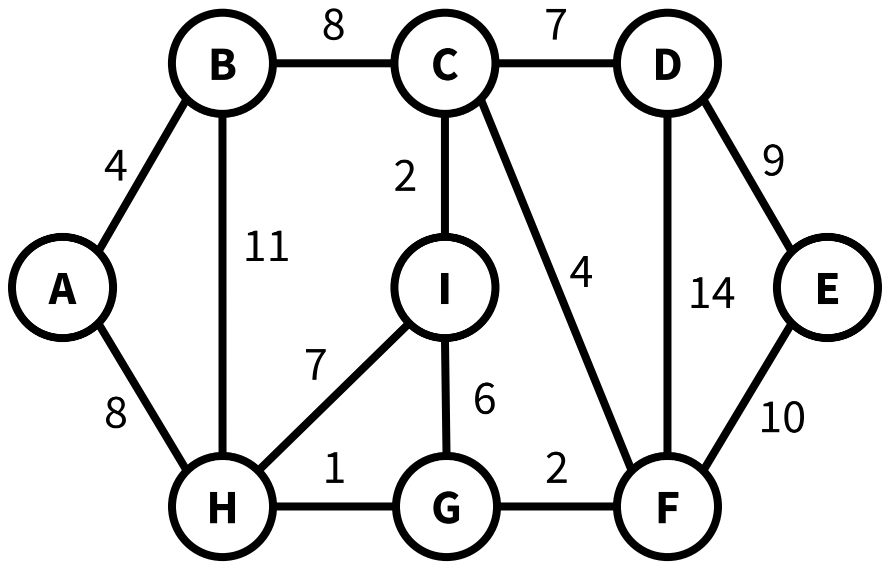
The cost of a spanning tree is the sum of the weights on the edges.



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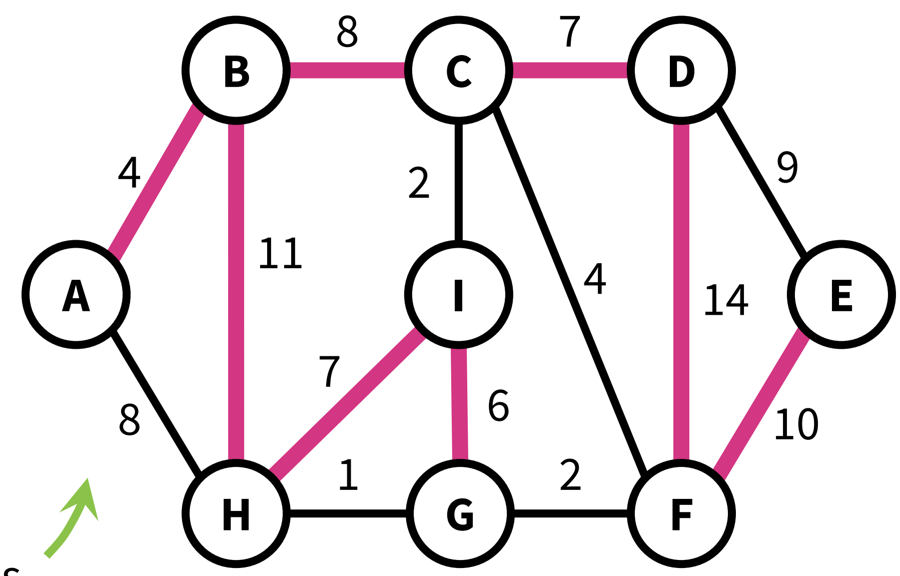

This spanning tree has a cost of 67.

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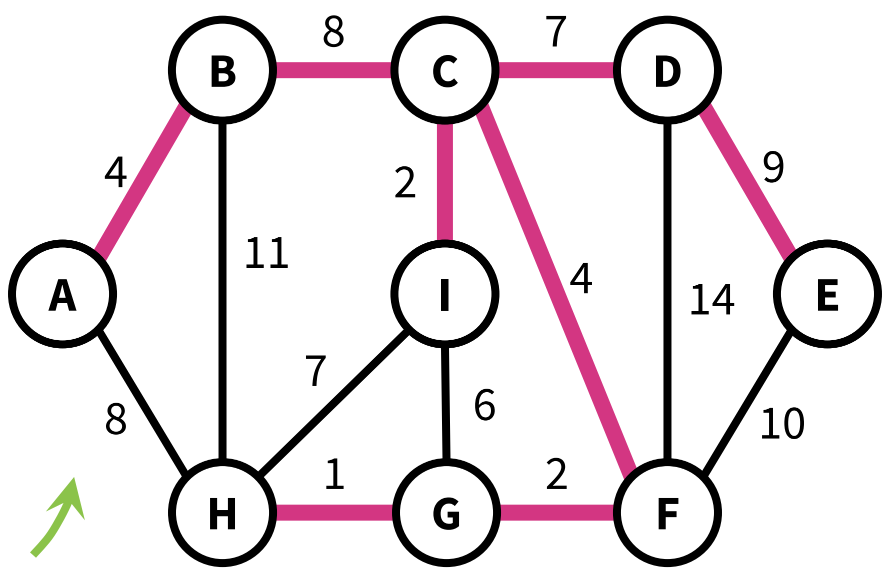

This spanning tree has a cost of 37.

Minimum Spanning Tree

mininmum **of minimal cost**

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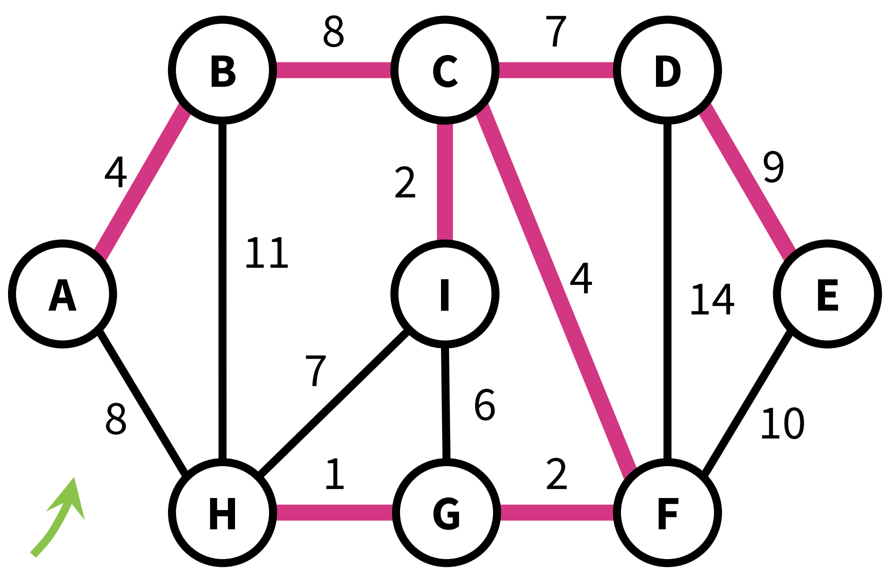
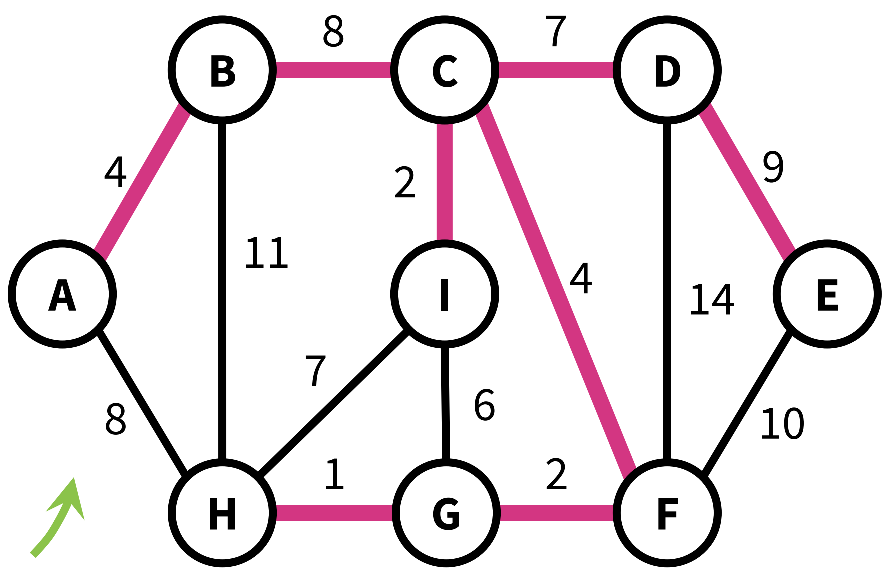


This spanning tree has a cost of 37. This is a minimum spanning tree.

Minimum Spanning Tree

Finding the MST is very useful in many problems.

E.g., Find the MST to connect all telephones/computers with shortest total cable length.



Edge weight is the distance between two telephones/computes.

Minimum Spanning Tree

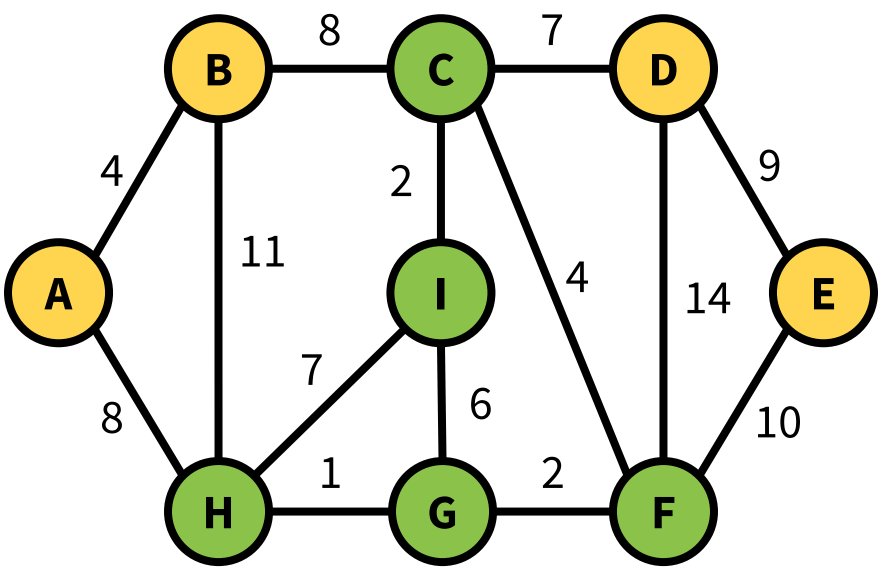
How might we find a MST?

Today, we'll see two greedy algorithms that find a MST.

Review: Graph Cut

Recall from Lecture 7, a **cut** is a partition of the vertices into two nonempty parts.

e.g. This is the cut "{A, B, D, E} and {C, I, F, G, H}".



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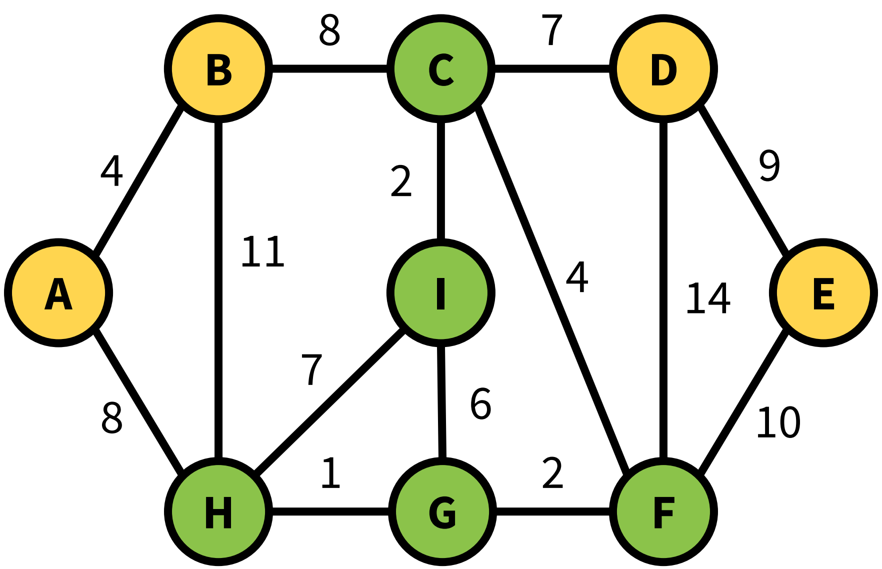

A cut **respects** a set of edges if no edges in the set cross the cut.

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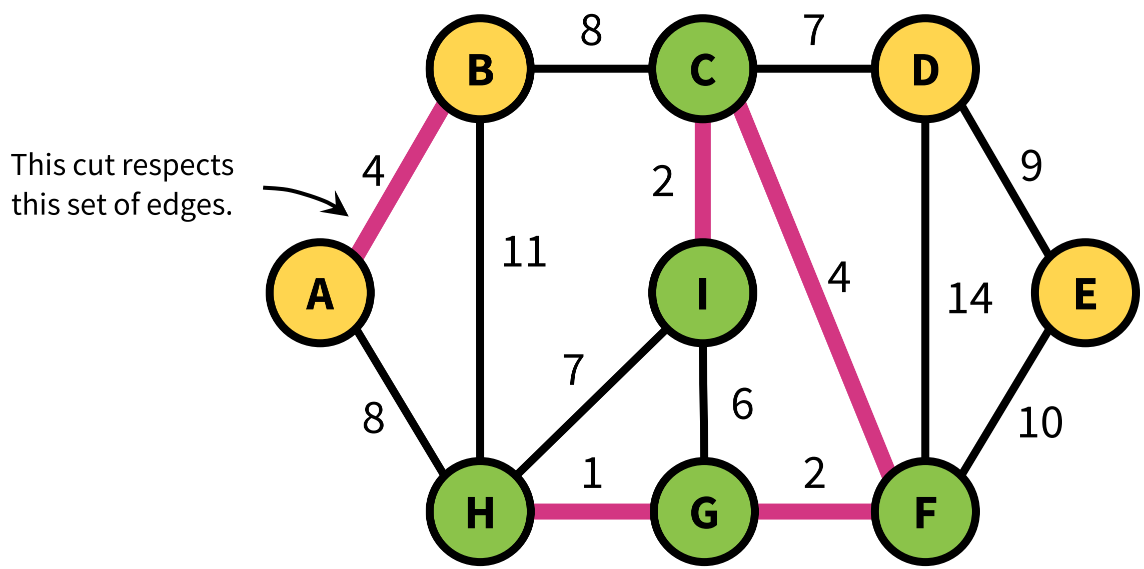

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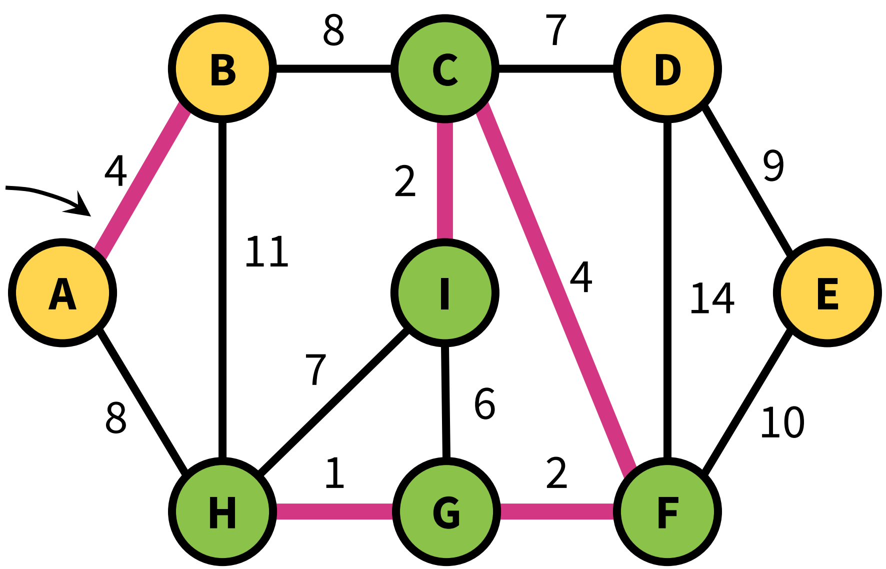
This cut respects this set of edges.



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An edge is **light** if it has the smallest weight of any edge crossing the cut.

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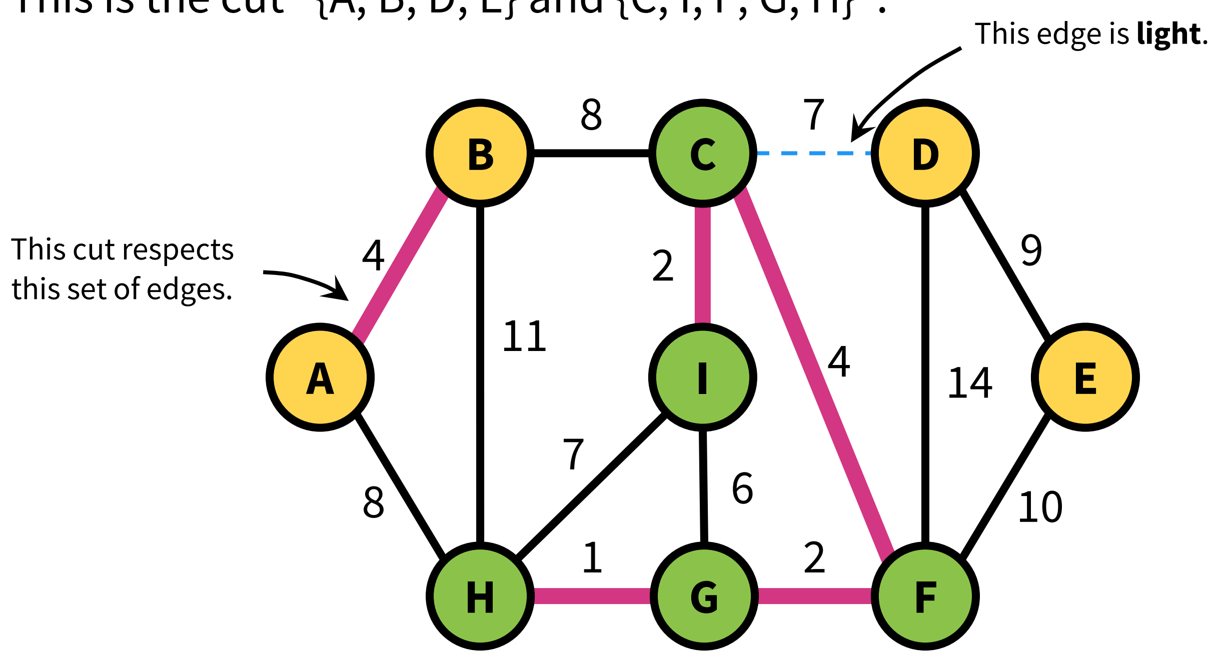

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Lemma

Consider a cut that respects a set of edges **A**.

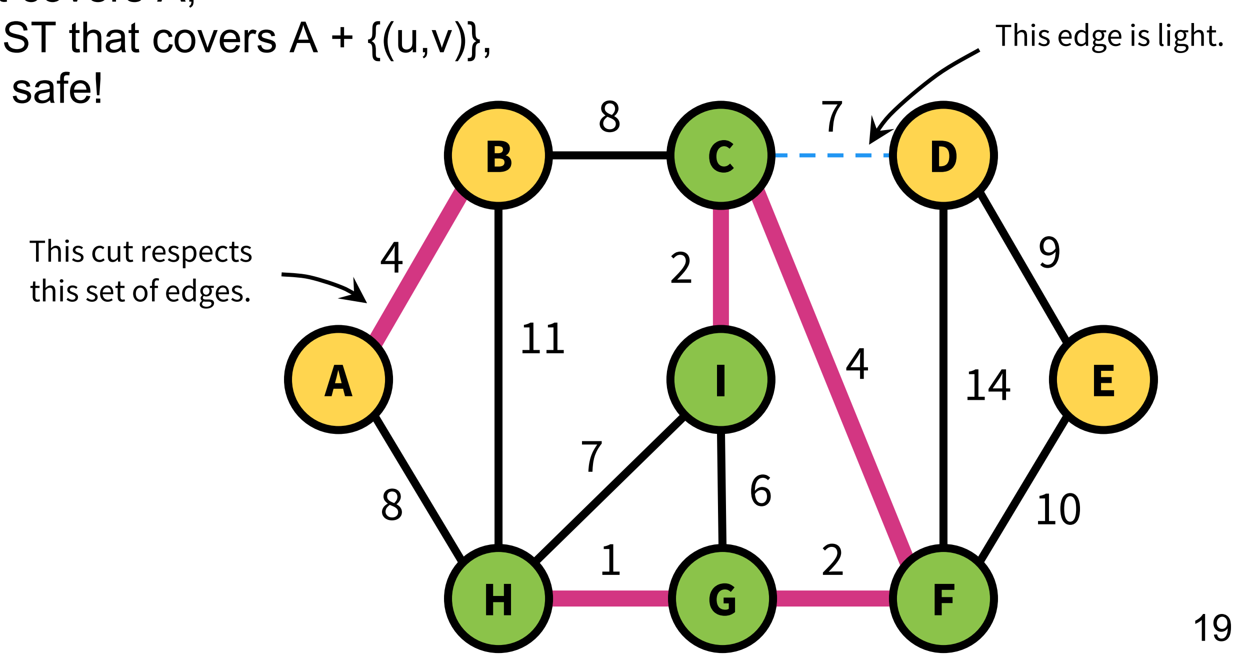
Suppose there exists an MST containing **A**.

Let (u, v) be a light edge.

Then there exists an MST containing **A** ∪ {(u, v)}.

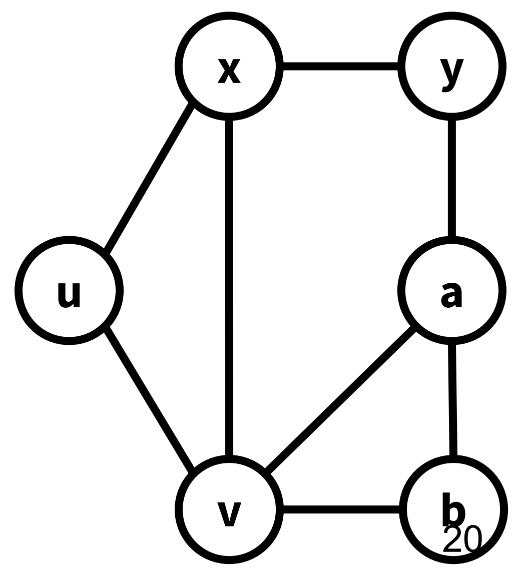
This is precisely the sort of statement we need for a greedy algorithm: If we haven't ruled out the possibility of success so far, then adding a light edge won't rule it out.

If we can find a MST that covers A,
then we can still find a MST that covers A + {(u,v)},
so adding (u,v) would be safe!



Proof of Lemma

Consider a graph with …



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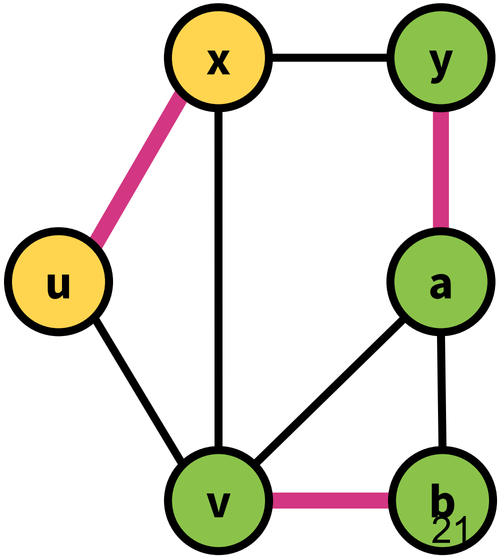

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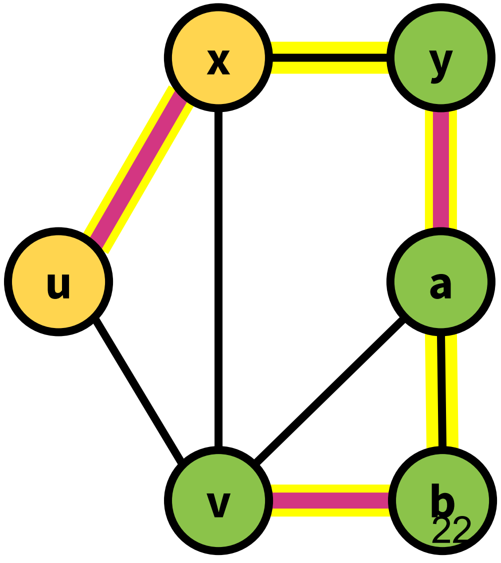
A cut that respects a set of edges **A**, if there's an MST **T** containing **A**,



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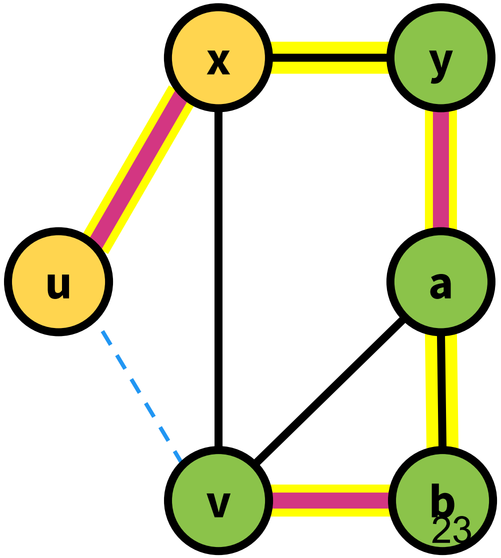
A cut that respects a set of edges **A**, if there's an MST **T** containing **A**, and if there is a light edge (u, v) not in **T**.



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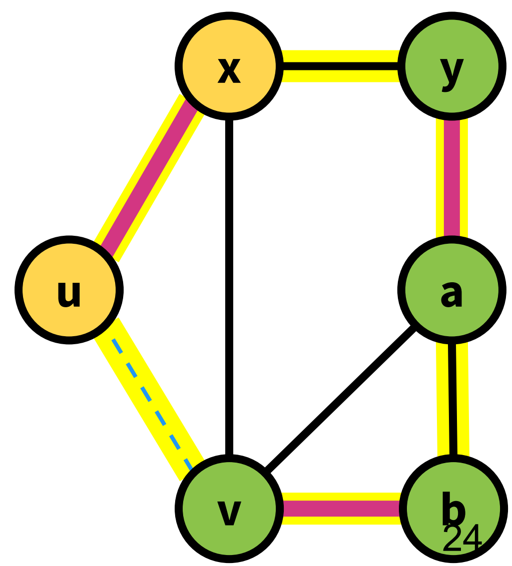
Adding (u, v) to **T** will make a cycle. Because it has more than n-1 edges.



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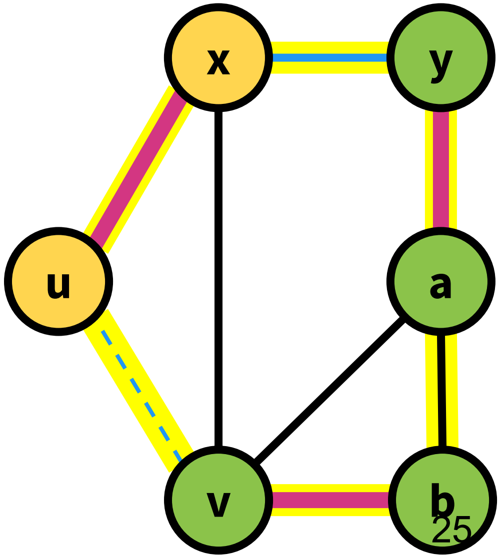
There must be another edge in this cycle crossing this cut.
Let's call this edge (x, y).



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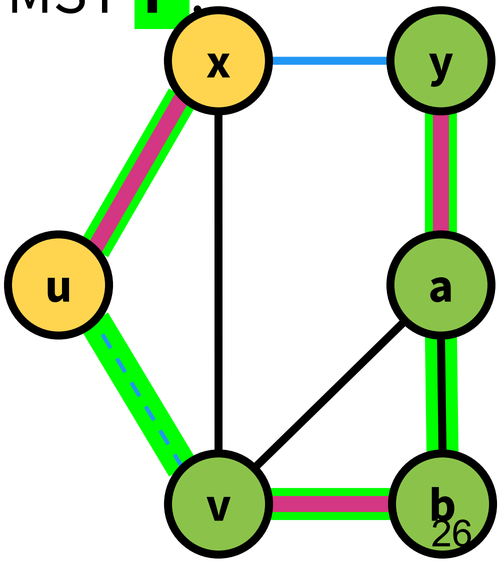

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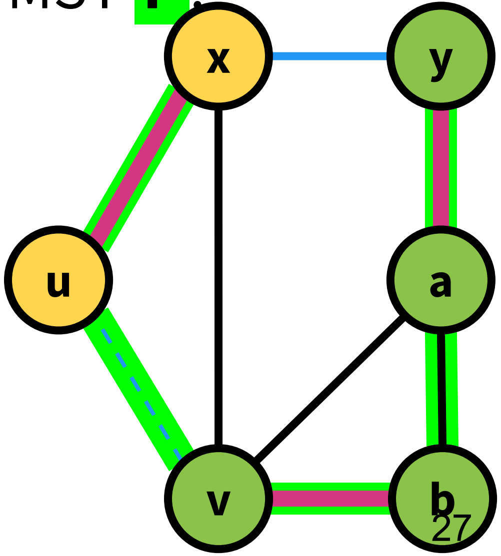
Claim: **T'** is still an MST.

Since we deleted (x, y), **T'** is still a tree.

Since (u, v) is light, **T'** has cost at most that of **T**.



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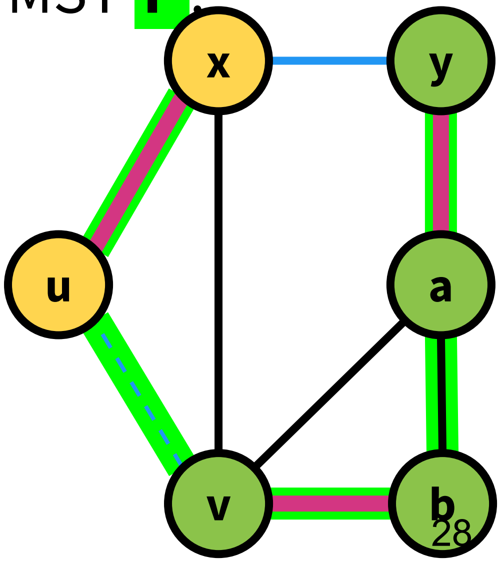
Thus, there exists a MST containing **A** ∪ {(u, v)}.



Prim’s Algorithm

Any Ideas?

Recall our lemma:

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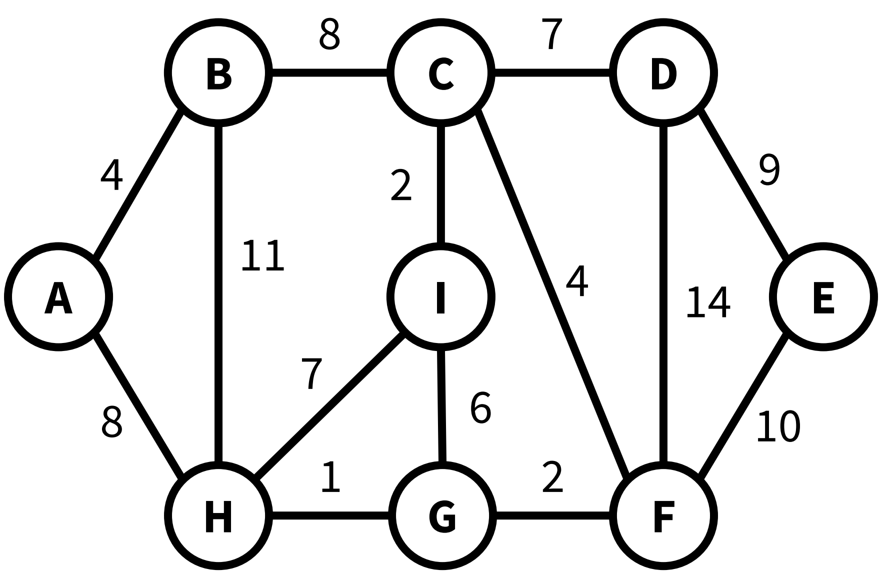
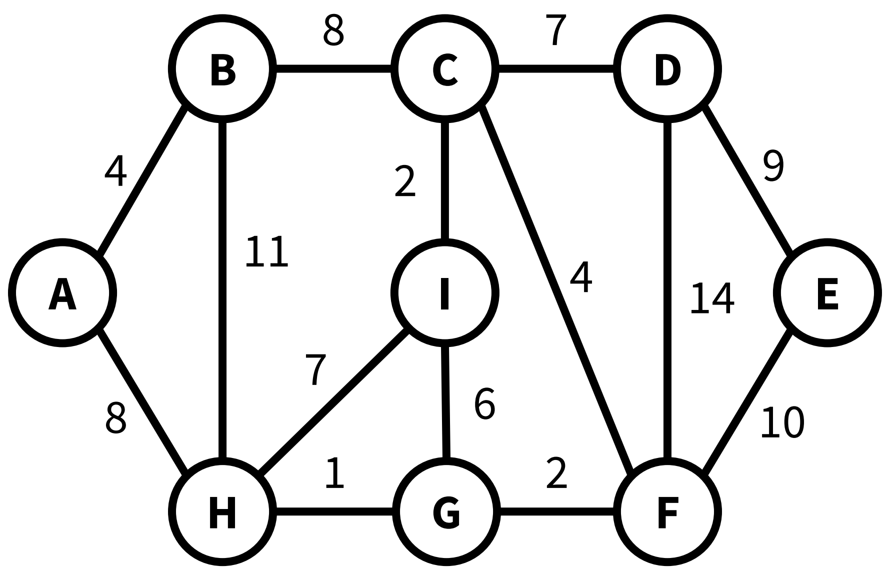
Lemma: There exists an MST containing **A** ∪ {(u, v)}.

Any ideas about what to greedily choose?



Prim's Algorithm

Main idea: Extend a single tree of visited vertices by greedily adding the closest vertex.



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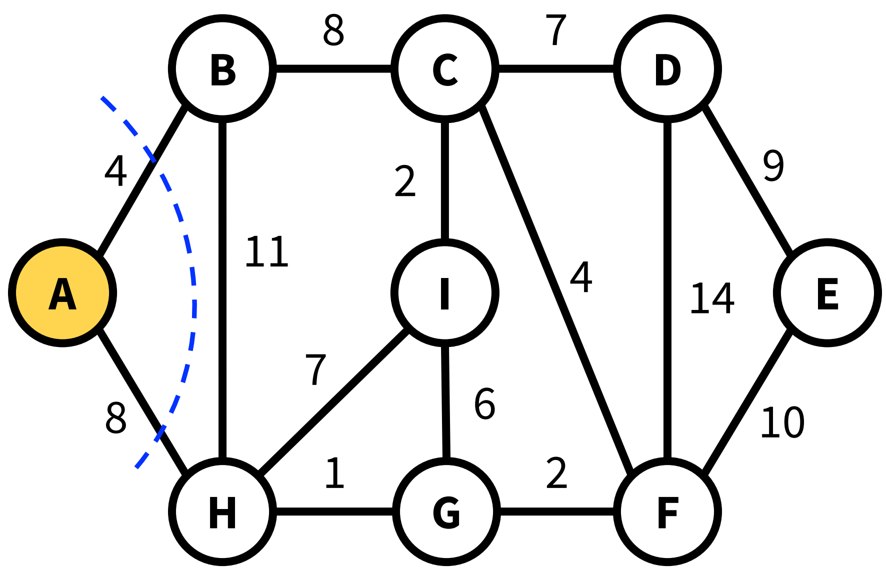

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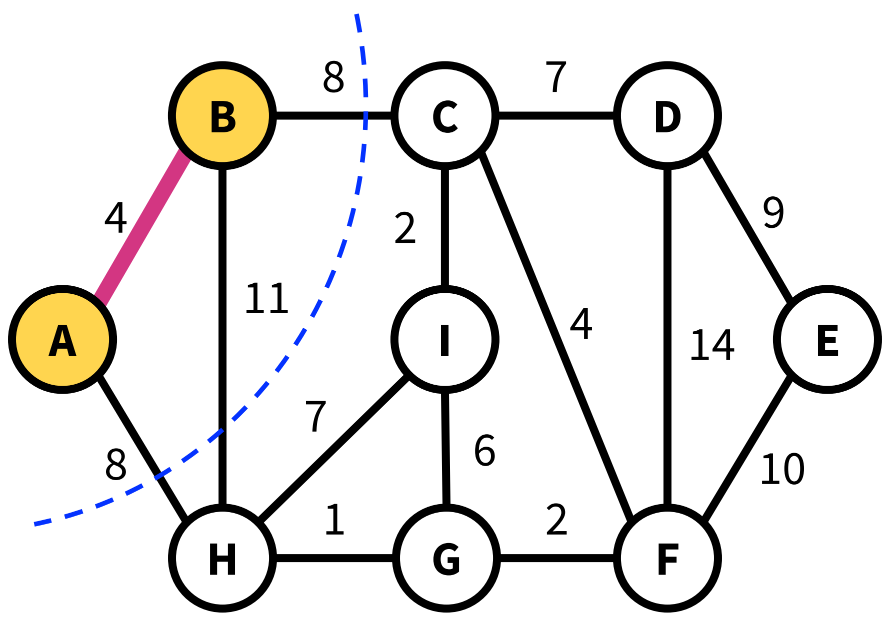

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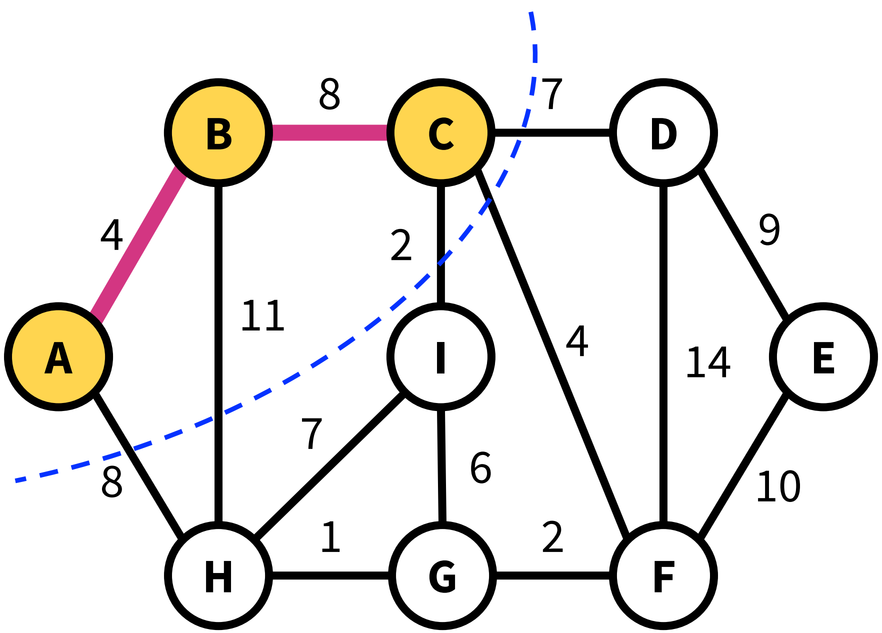

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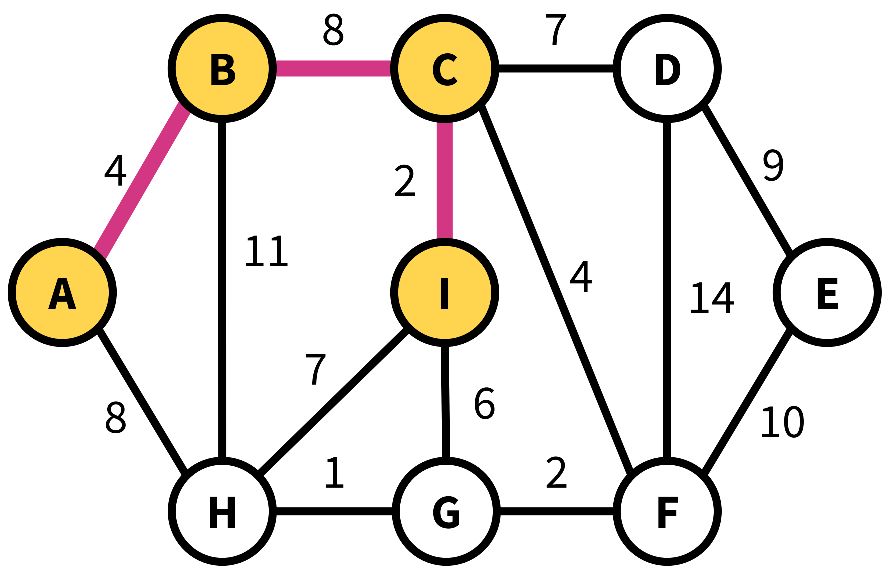

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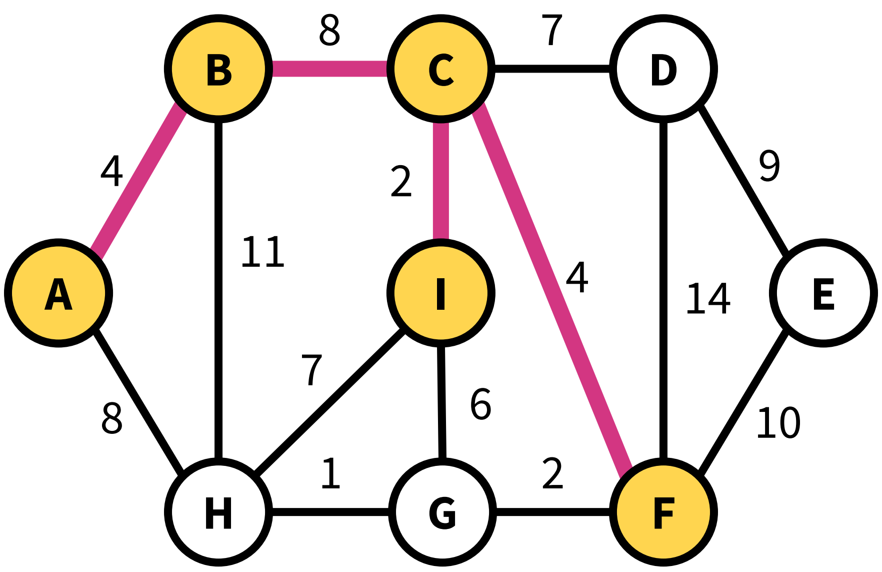

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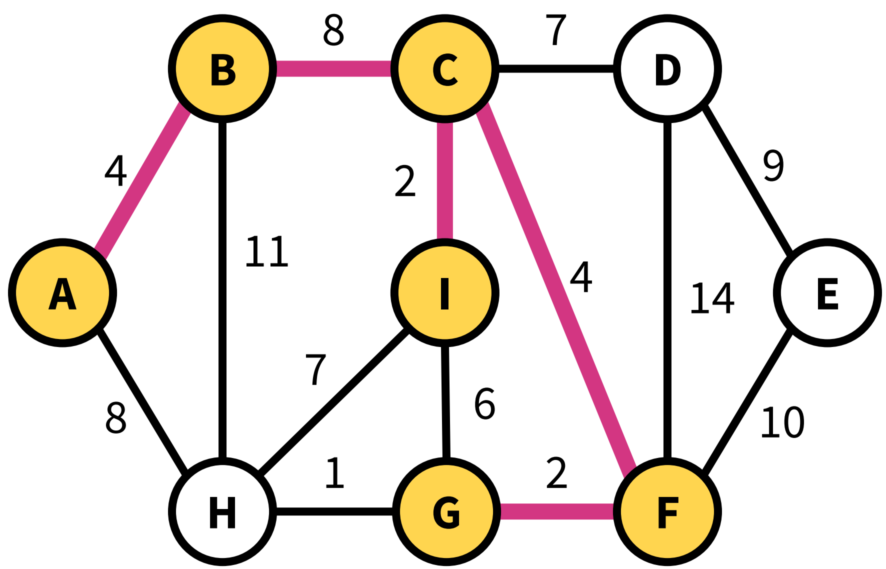

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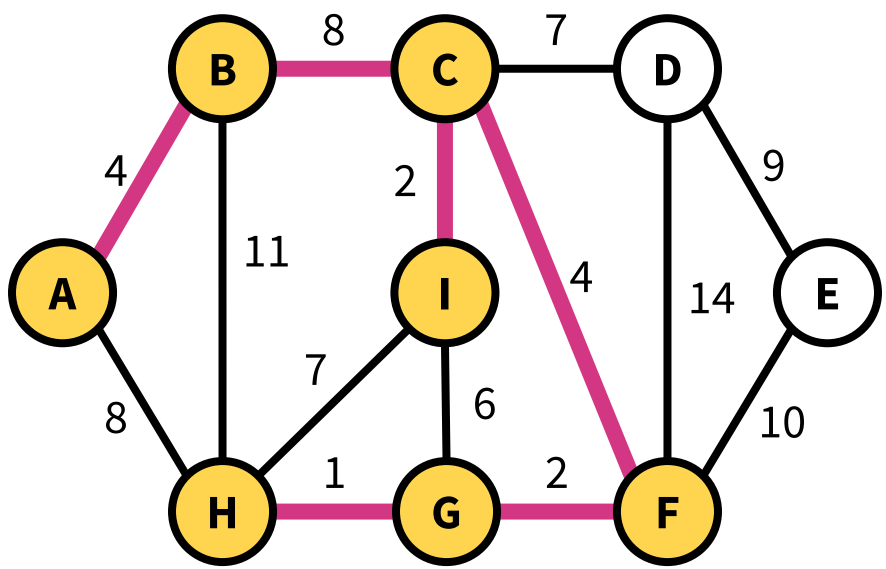

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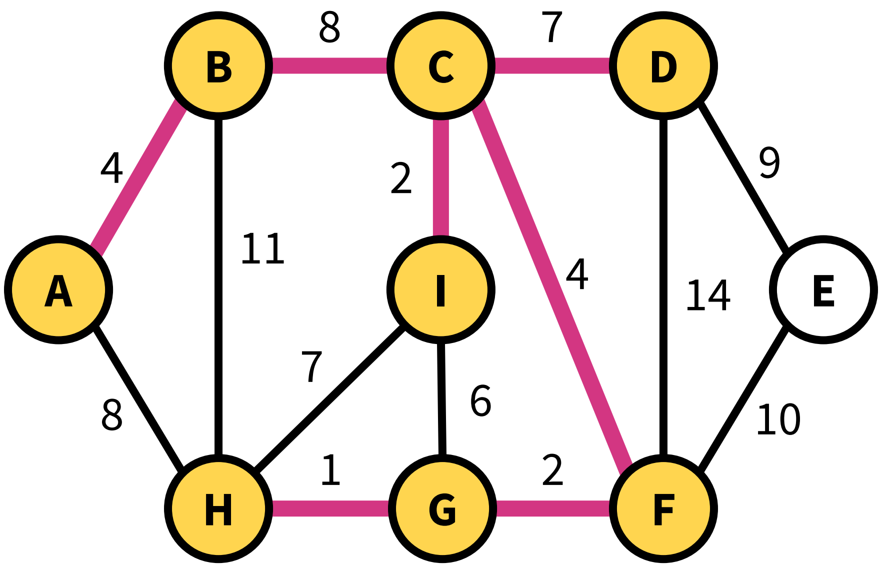

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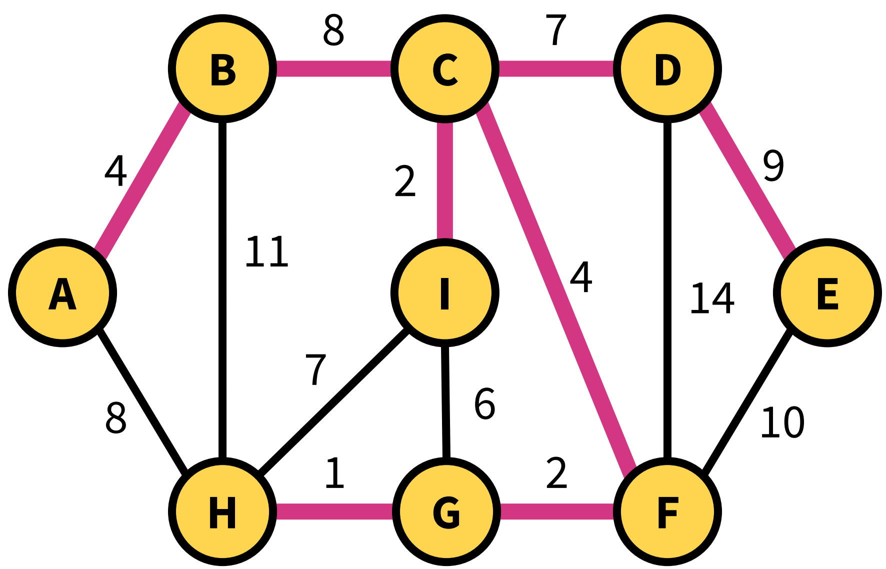


Prim's Algorithm

```
algorithm slow_prim(G):
  s = random vertex in G
  MST = {}
  visited_vertices = {s}
  while |visited_vertices| < |V|:
    (x, v) = lightest_edge(G, visited_vertices)
    MST.add((x, v))
    visited_vertices.add(v)
  return MST
```

aka while we haven't visited all of the vertices

Finds the lightest edge (x, v) in E such that x is in visited_vertices and v is not.

Runtime: O(|V|•|E|)

For each of the |V| iterations of the while loop, might need to iterate through all edges.

Proving Feasibility

Theorem: `prim` finds a feasible spanning tree.

Proof:

To prove this statement, we prove the loop invariant: `MST` contains edges of a spanning tree of the vertices in `visited_vertices`.

At the start of the first iteration, `MST` contains no edges, which corresponds to a spanning tree of one vertex.

Now, we prove the inductive step. Suppose that the invariant holds at the start of iteration i, so the edges in `MST` are (1) acyclic and (2) connect all vertices in `visited_vertices`. Then prim adds an edge (x, v) to `MST` and vertex v to `visited_vertices`. By construction, v has not been visited yet, so the edges in `MST` must still be acyclic. Furthermore, v connects to x, which connects to the rest of the vertices in `visited_vertices`; therefore, the edges in `MST` must still connect all vertices in `visited_vertices`, completing the induction.

At the termination of the loop, `visited_vertices` contains all of the vertices, so `MST` contains a spanning tree over the entire graph. ■

Proving Optimality

Recall our lemma:

Consider a cut that respects a set of edges **A**, such that there's an MST **T** containing **A**, and a light edge (u, v) not in **T**.

Lemma: There exists an MST containing **A** ∪ {(u, v)}.

Theorem: `slow_prim` returns a minimum spanning tree.

Proof:

At the start of the first iteration of the while loop, there exists a minimum spanning tree with the edges in `MST`. This trivially holds since we initialize `MST` to the empty set.

Consider the cut of visited vertices and unvisited vertices; MST is respected by this cut. By our lemma, there exists a minimum spanning tree containing `MST` ∪ {(x, v)}.

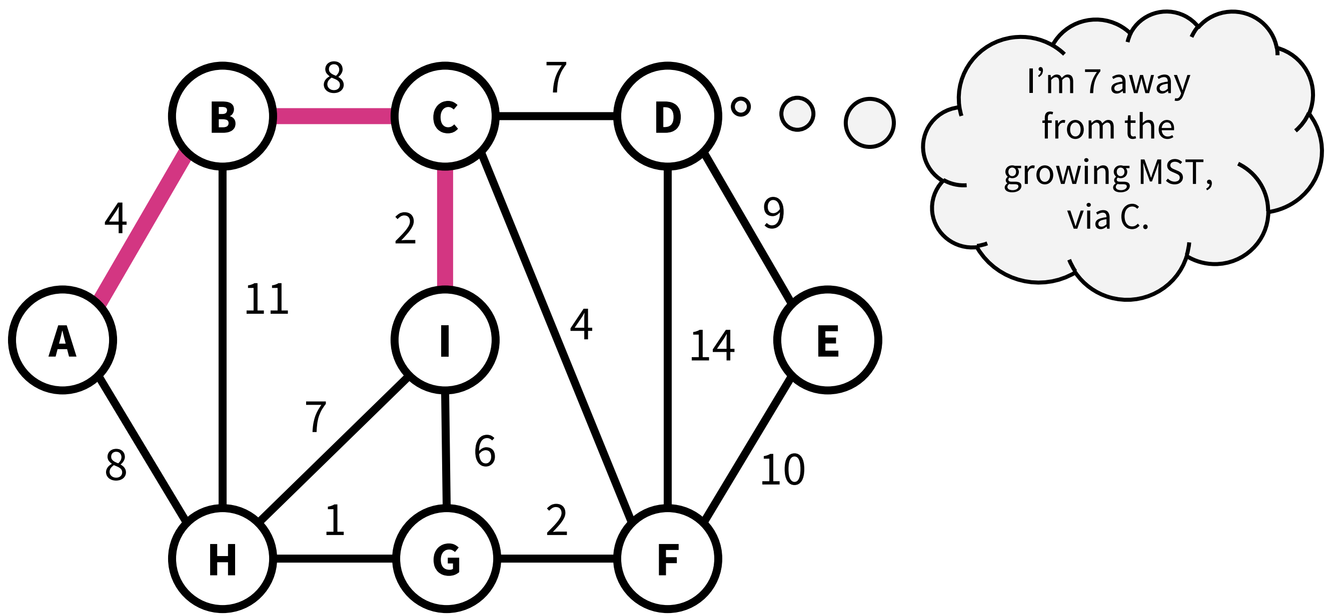
Recall, we proved our lemma with an exchange argument!

After adding the $(n\text{-}1)^{st}$ edge, we have a spanning tree; because each time we add an edge we always add the light edge, therefore, `MST`contains a minimum spanning tree. ■

Prim's Algorithm

We called the algorithm `slow_prim`. There's a more efficient implementation.

Main idea: vertices maintain information about the distance from itself to the growing spanning tree (**if one edge away**) and how to get there.



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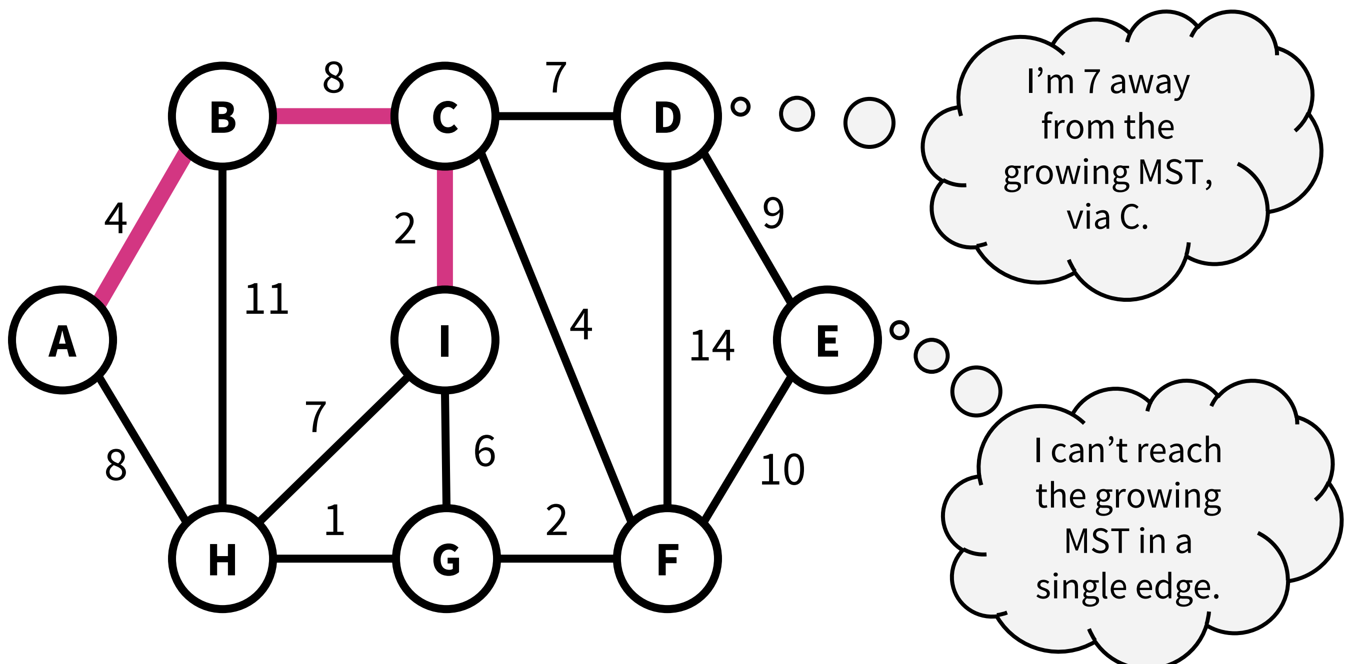

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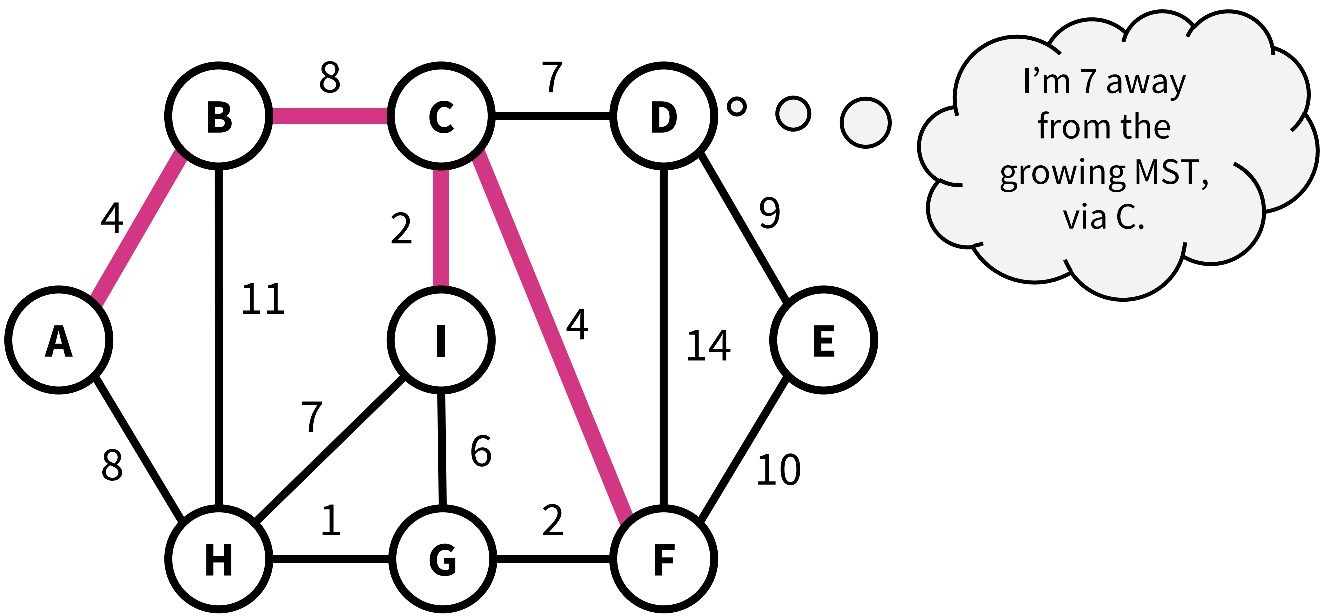

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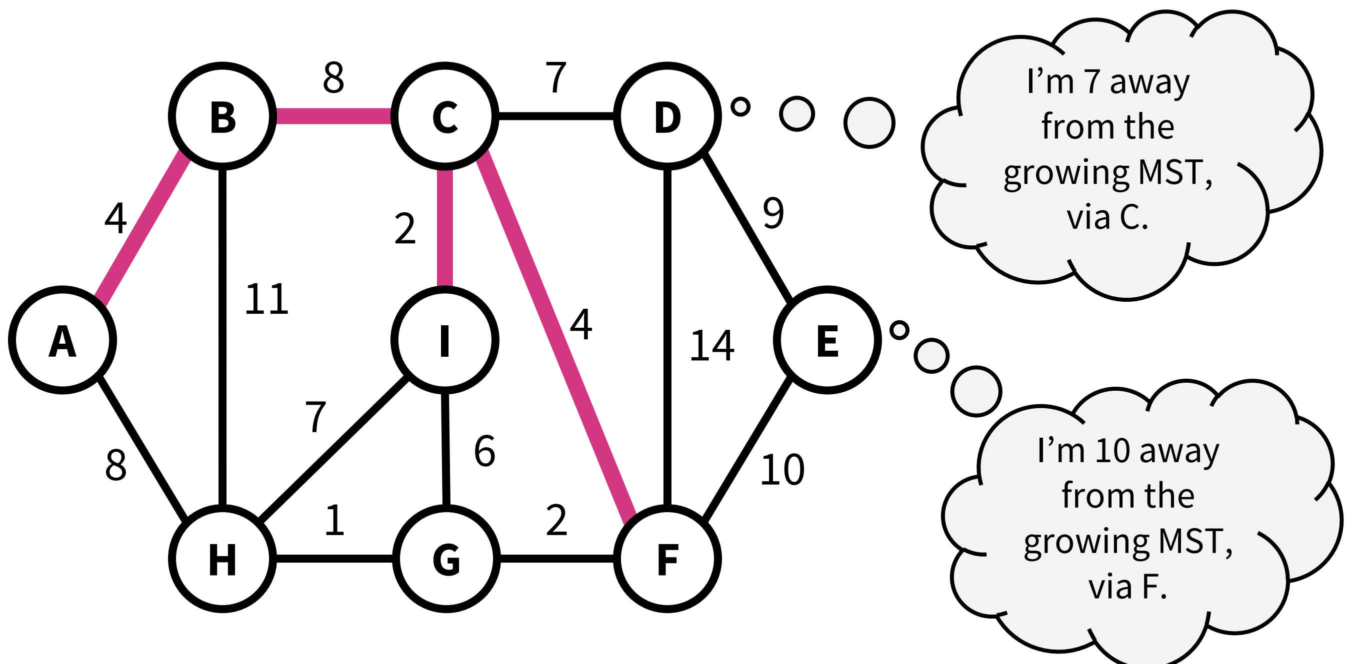

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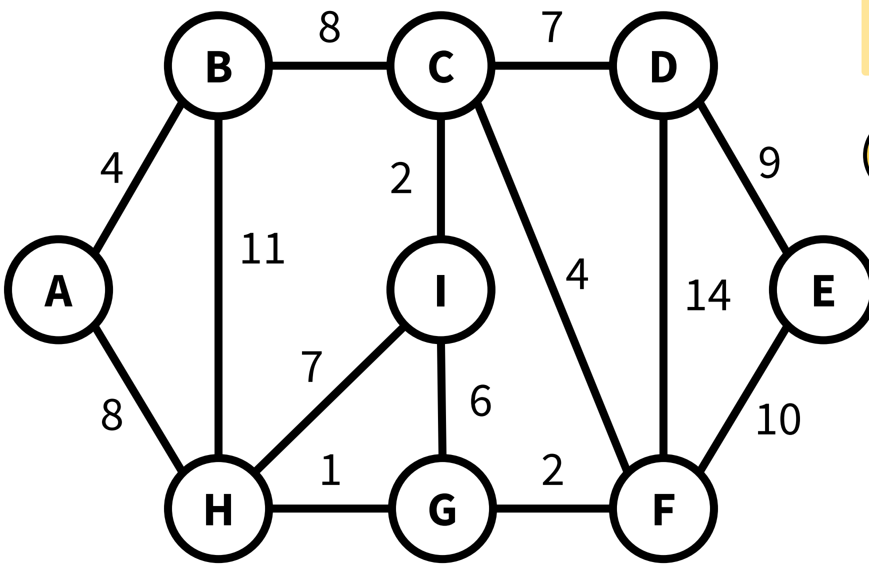
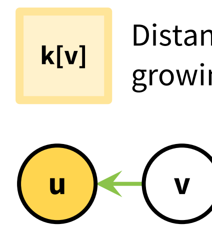




Distance from the growing MST.

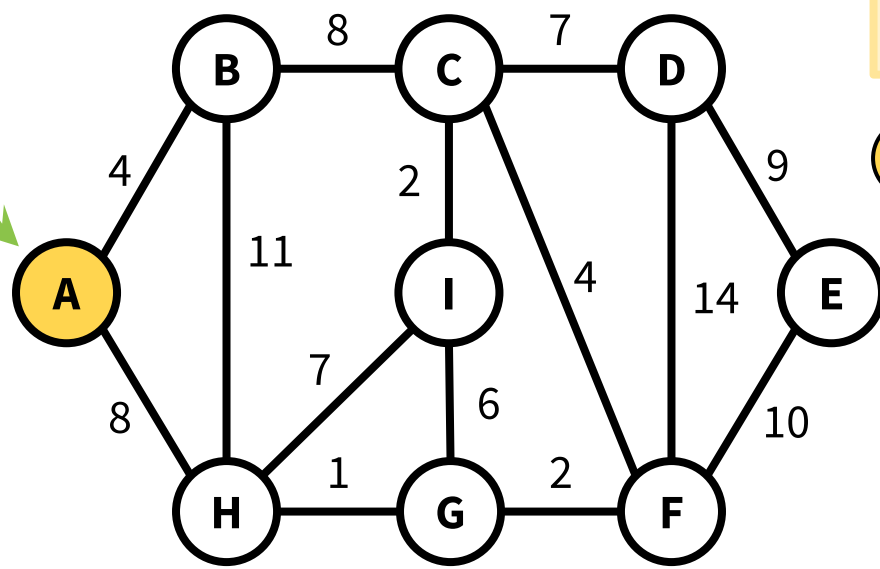
u is the vertex in the growing MST that’s k[v] away.

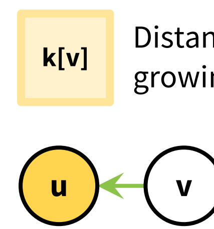
Prim's Algorithm

Main idea: vertices maintain information about the distance from the growing spanning tree and how to get there.

In addition to visiting A, update information of its neighbors with:

for v in u.neighbors:
k[v] = min(k[v], w(u, v))





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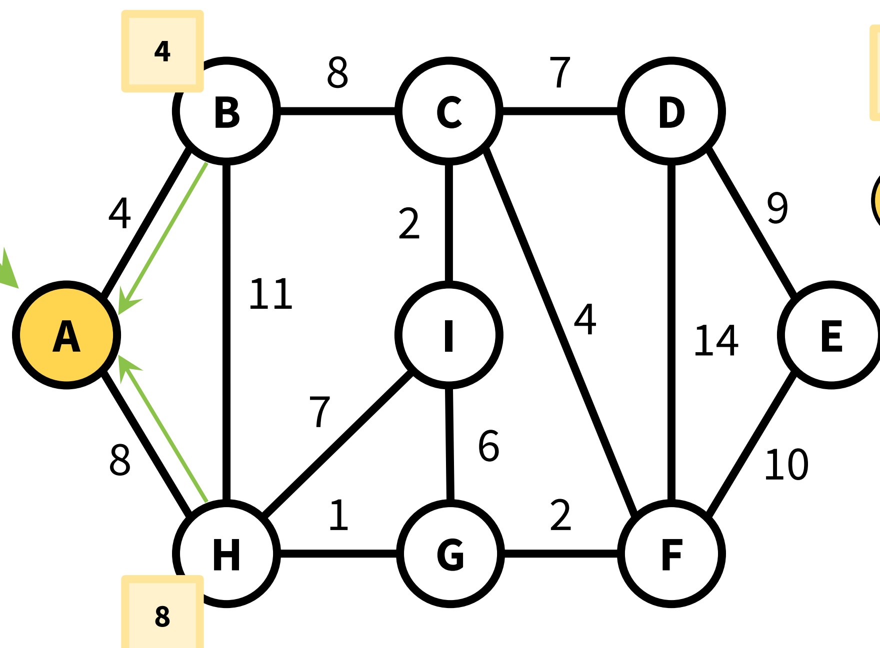

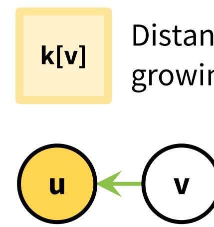

Distance from the growing MST.

u is the vertex in the growing MST that’s k[v] away.

Prim’s Algorithm

Main idea: vertices maintain information about the distance from the growing spanning tree and how to get there.

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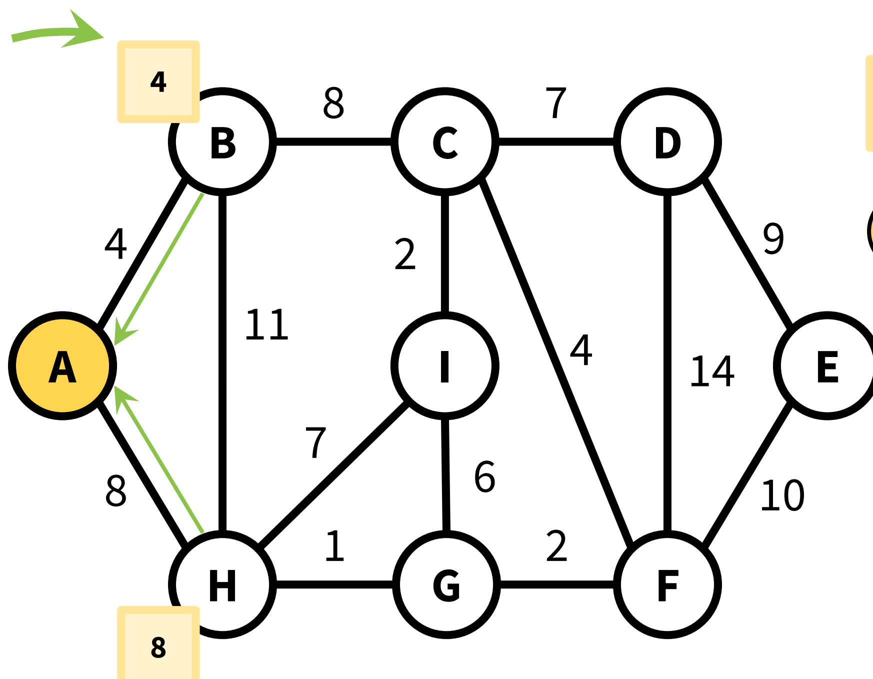

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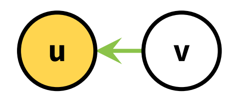

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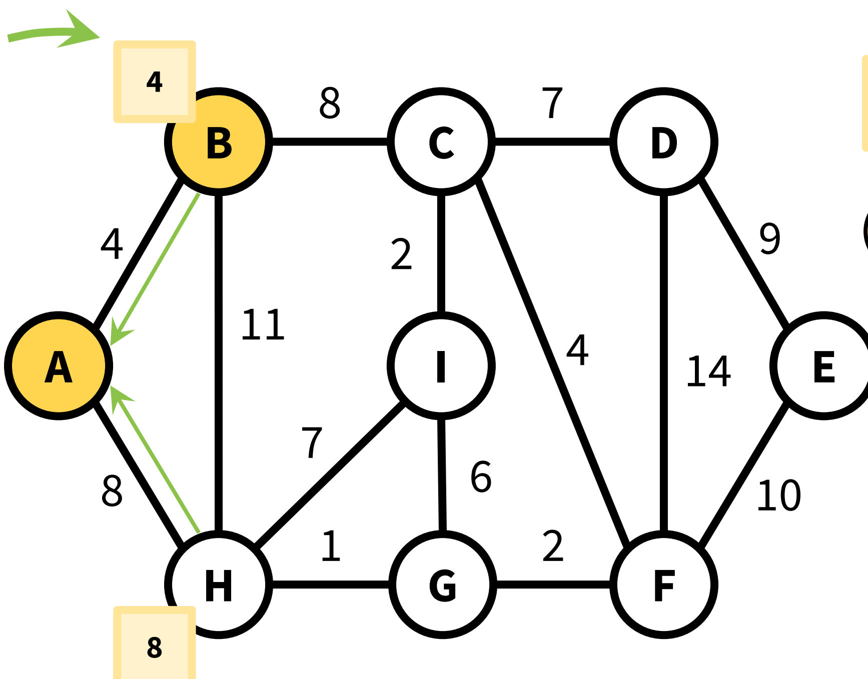

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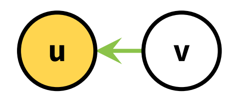

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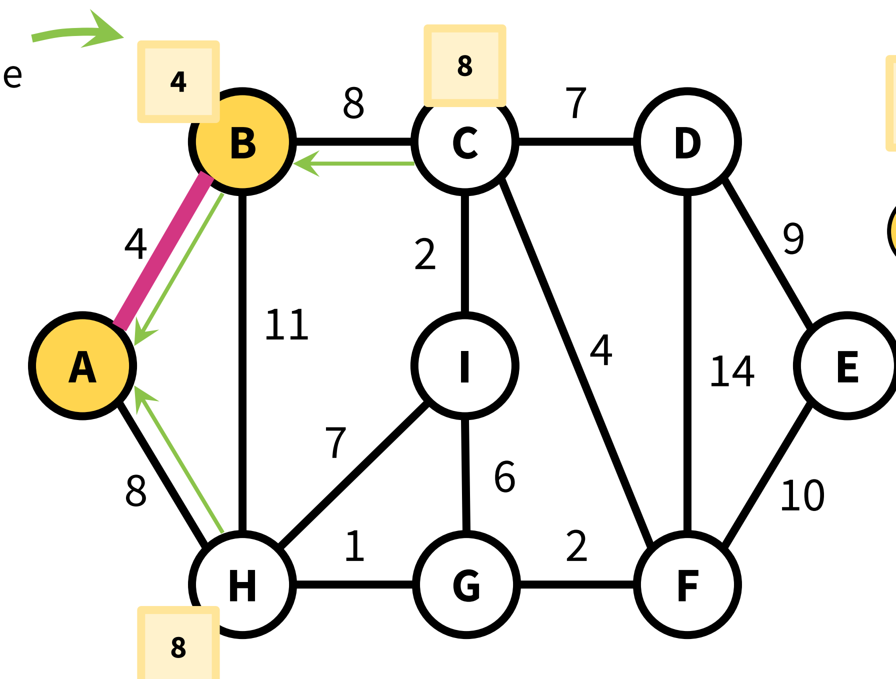

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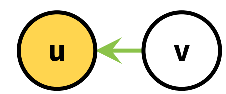

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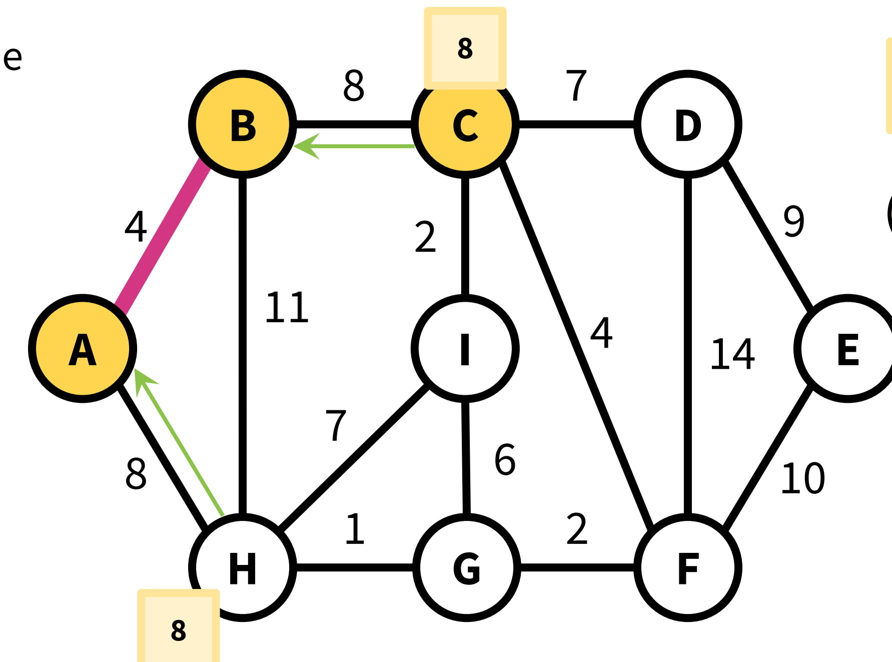

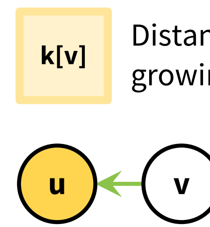

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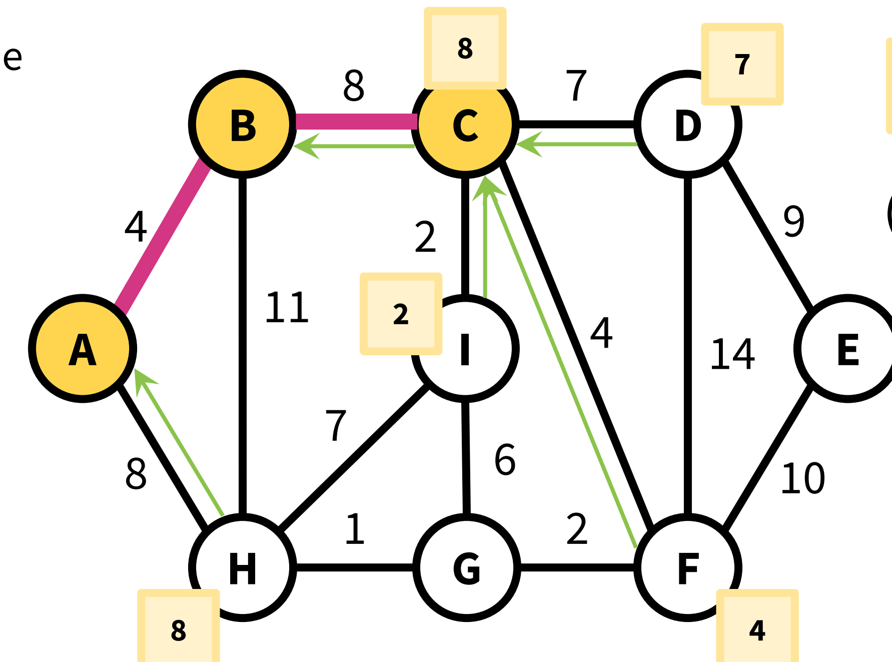

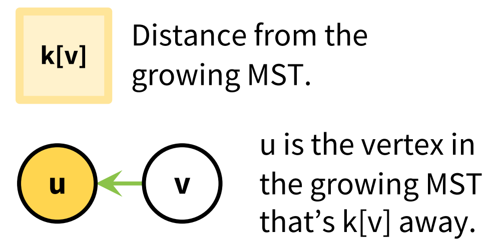

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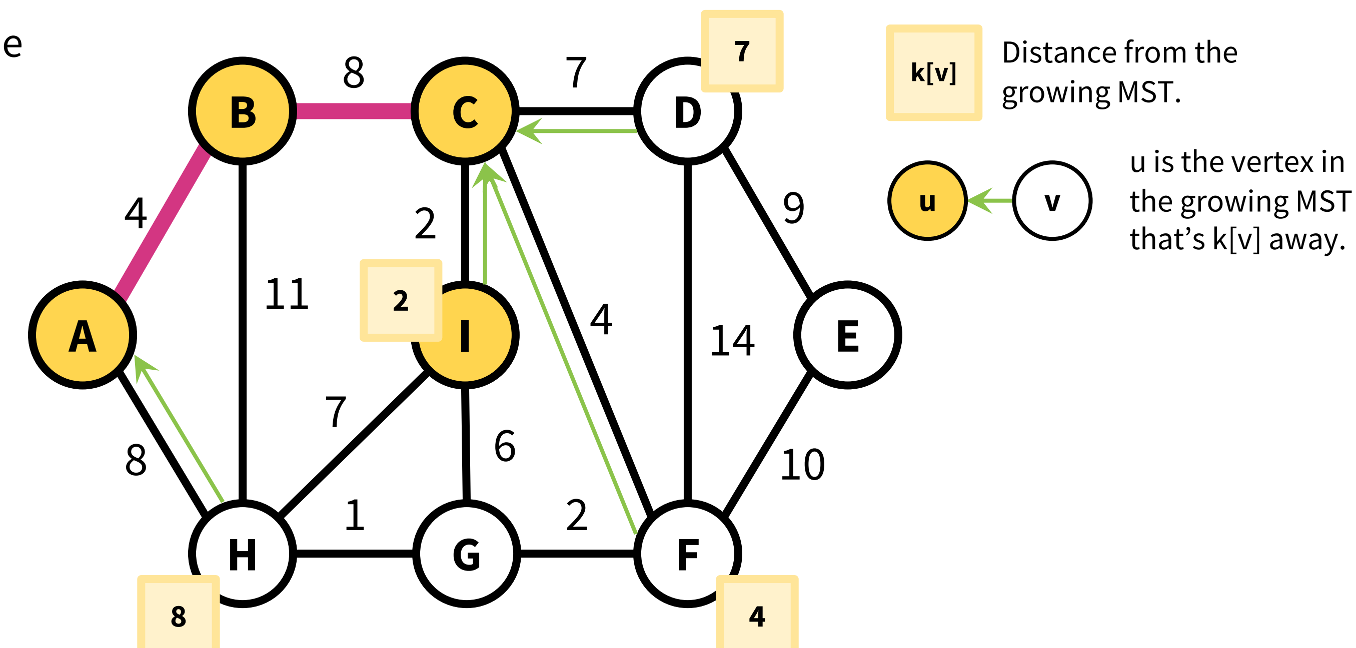

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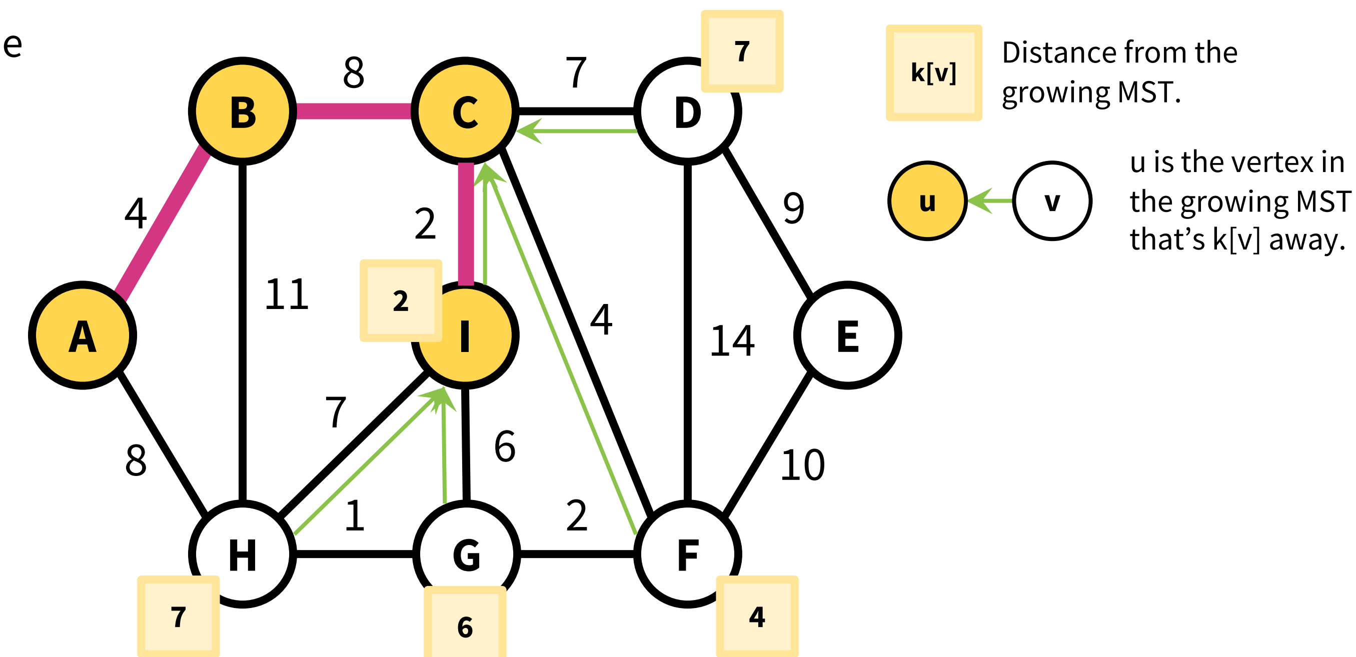

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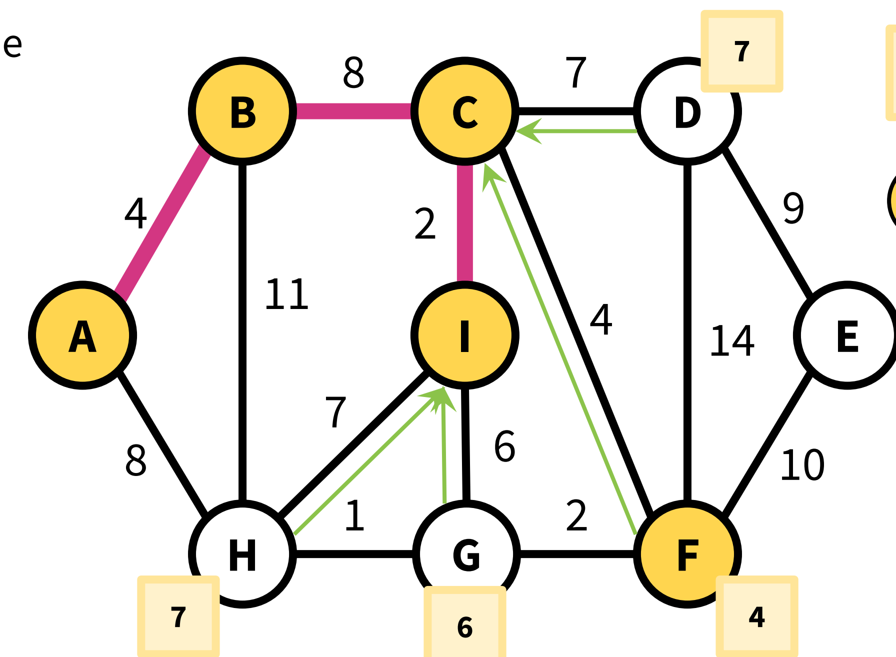

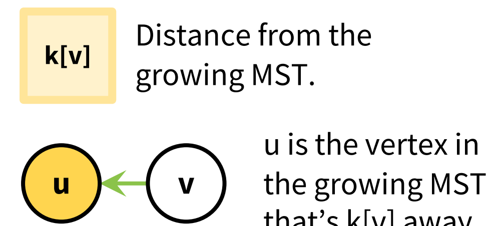

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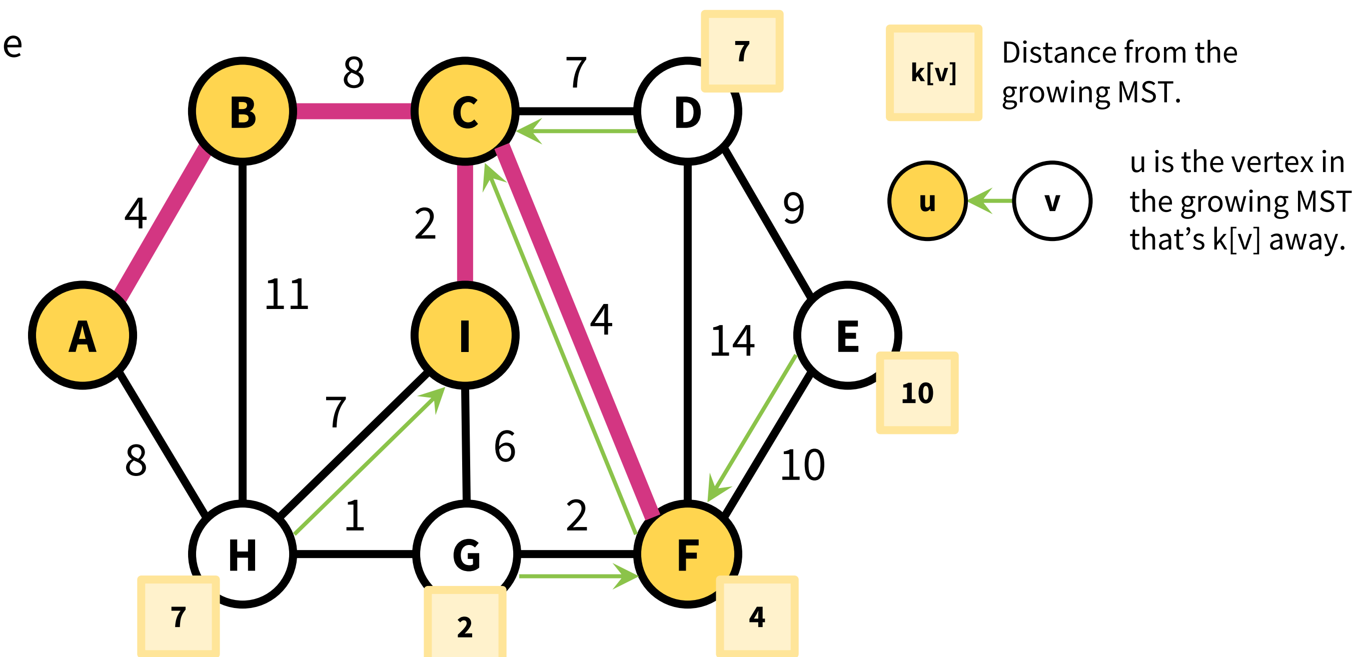

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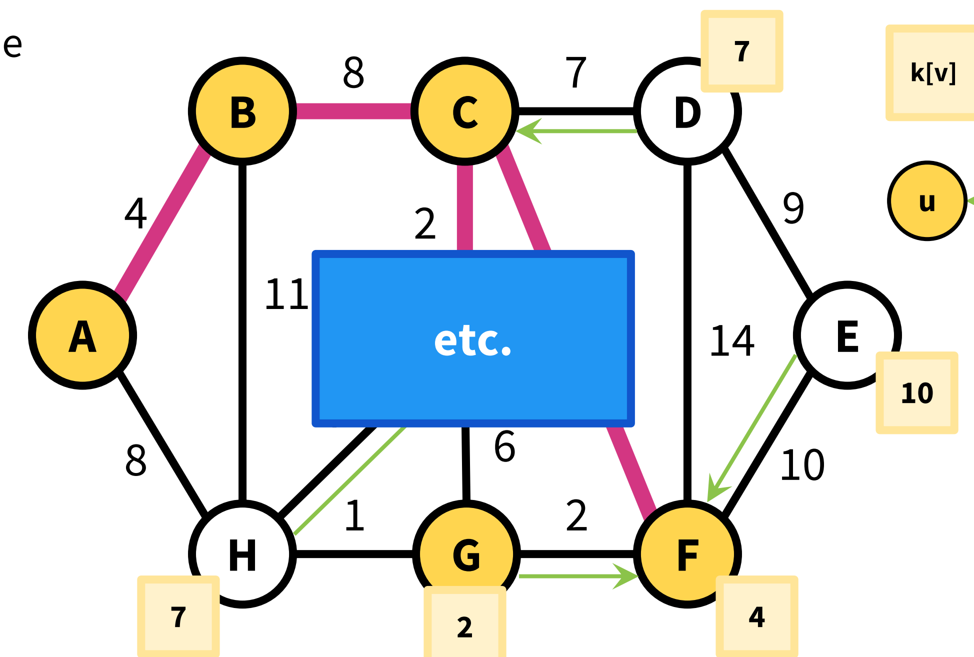

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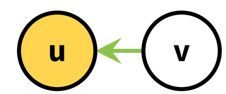


u is the vertex in the growing MST that's k[v] away.

Prim's Algorithm

```
algorithm prim(G):
  s = random vertex in G
  MST = {}
  visited_vertices = {s}
  update_info(G, s)
  while |visited_vertices| < |V|:
    (x, v) = lightest_edge(G, visited_vertices)
    MST.add((x, v))
    visited_vertices.add(v)
    update_info(G, v)
  return MST
```

Updates information about distance from the growing MST.

Runtime:

$O(|V|\log(|V|)+|E|)$

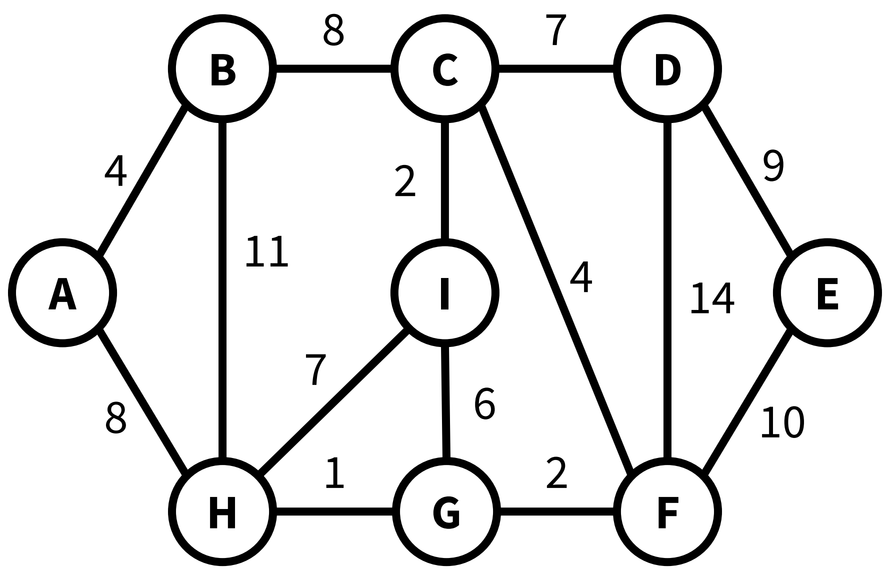
|V| is the number of while loops, log|V| comes from picking up the lightest edge using a priority queue implemented by RB-tree.

Comes from updating k values of nodes in update_info(G,v) subroutine. Eventually, each edge will be visited for one and only once.

Kruskal’s Algorithm

Kruskal's Algorithm

Main idea: Maintain a forest of trees of visited vertices by greedily adding the cheapest edge.



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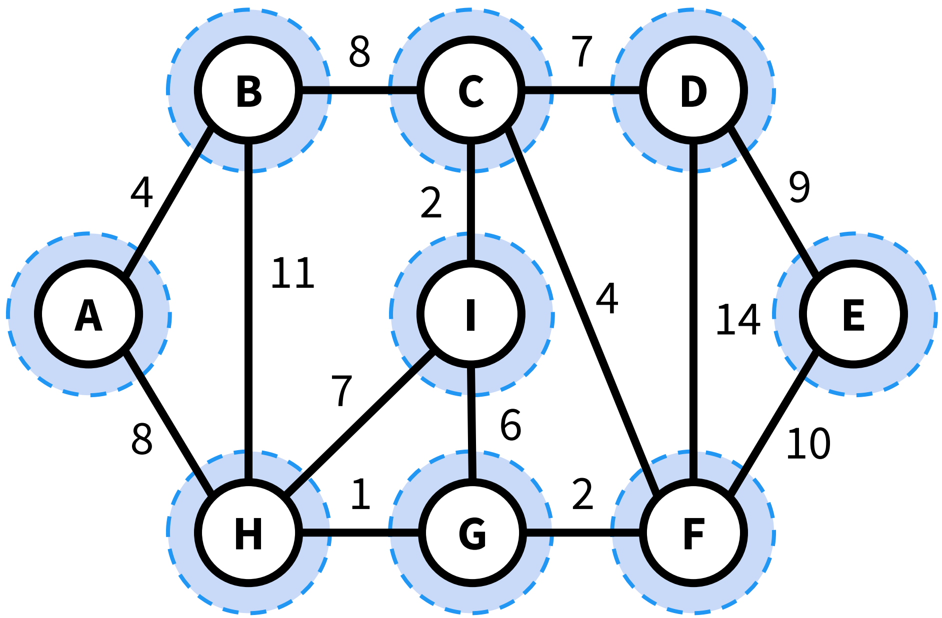

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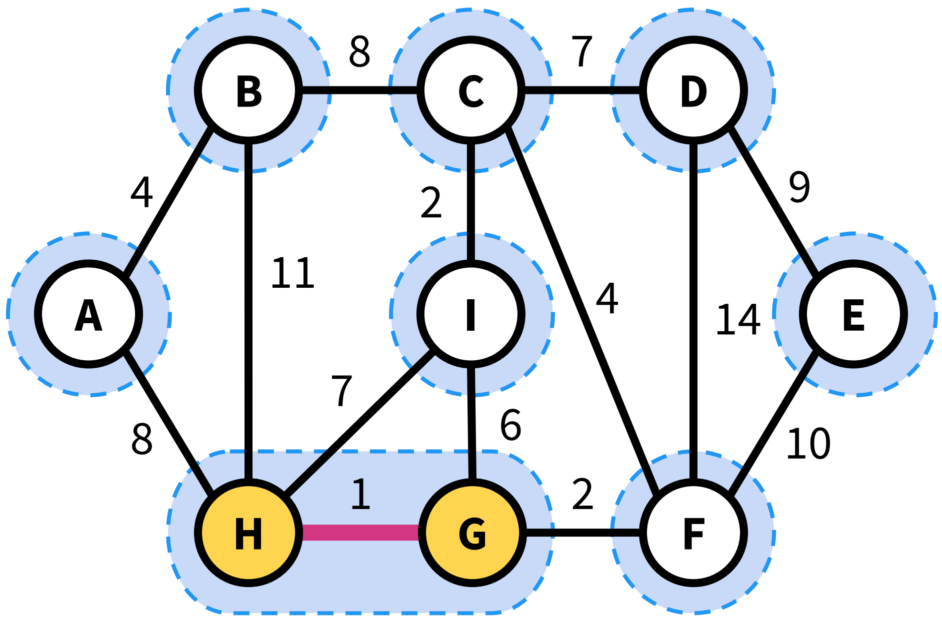

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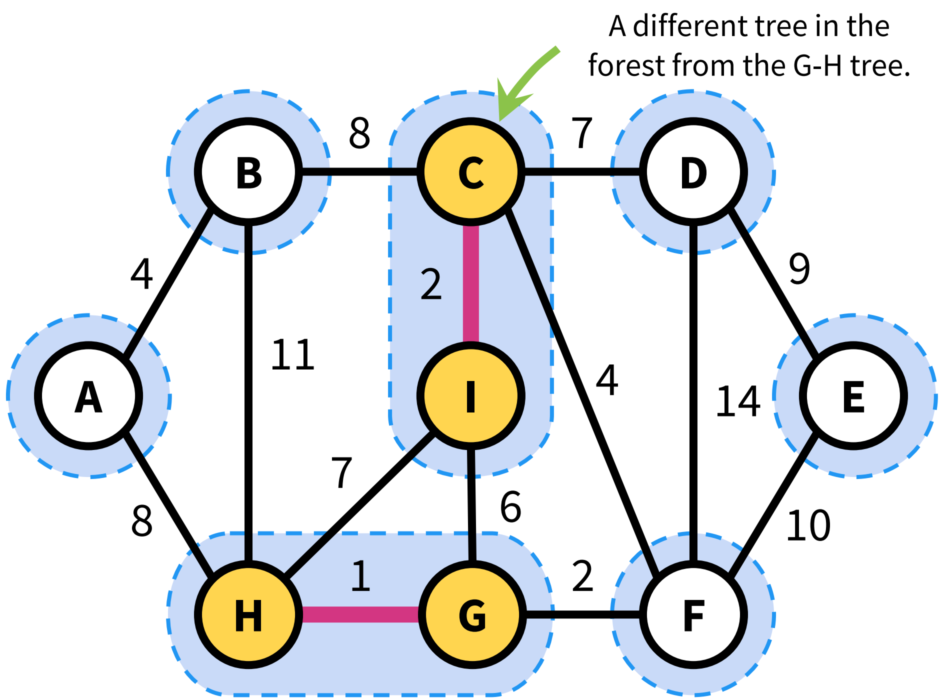

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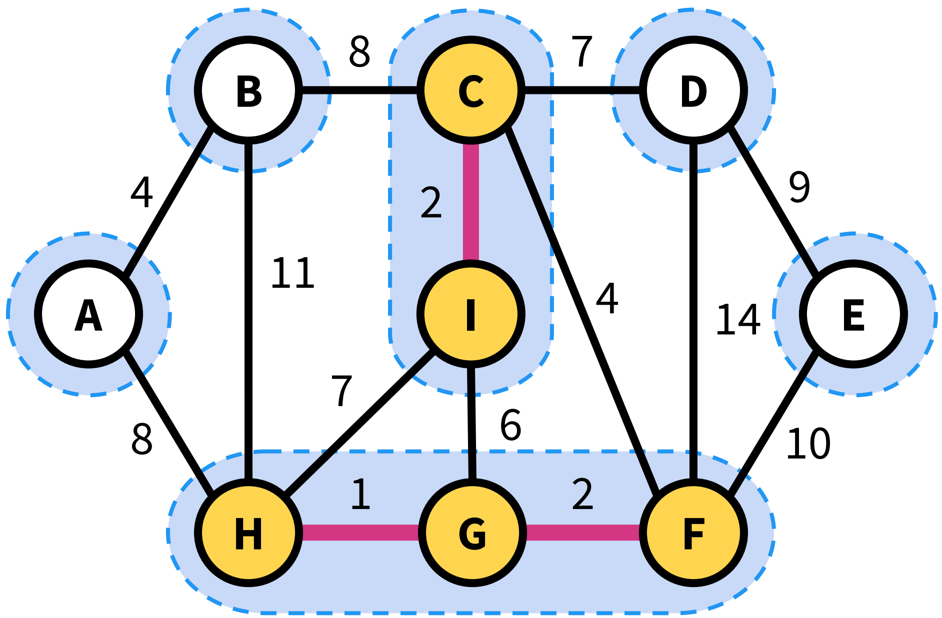

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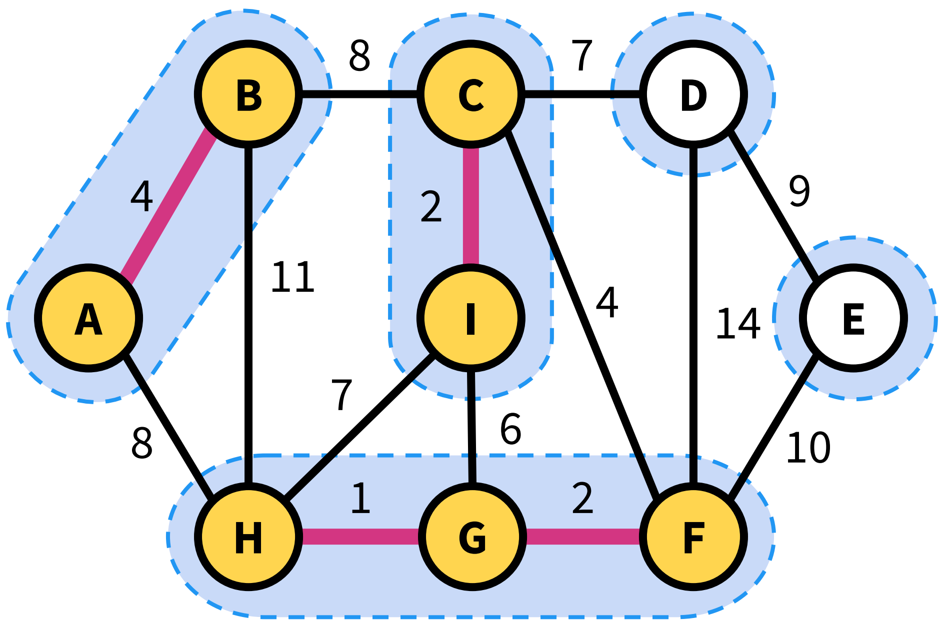

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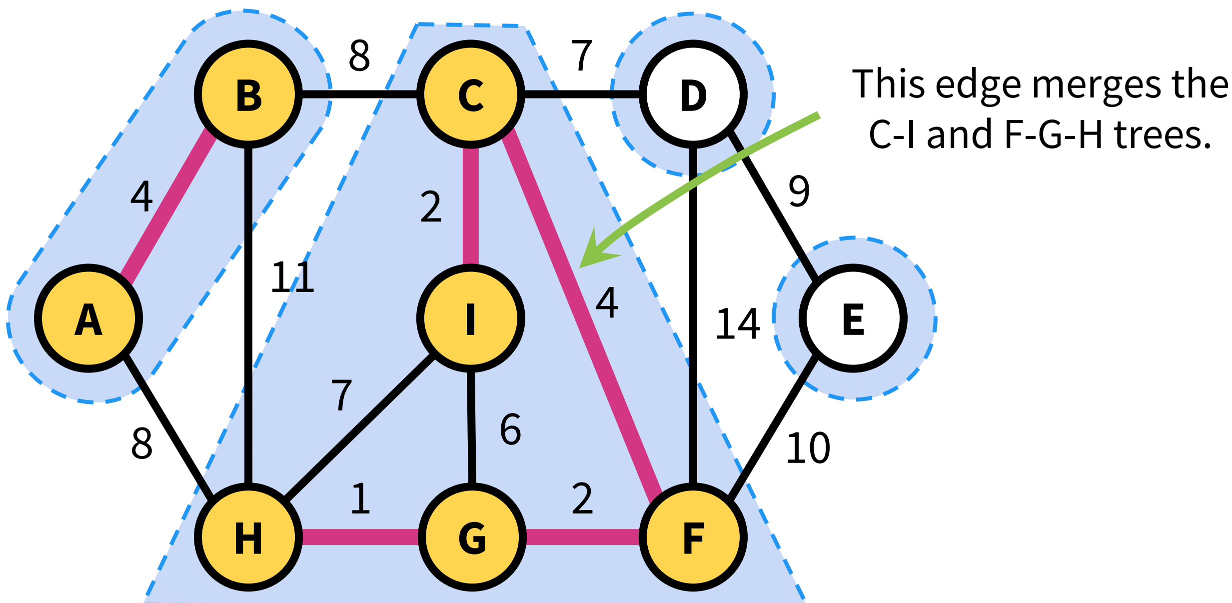

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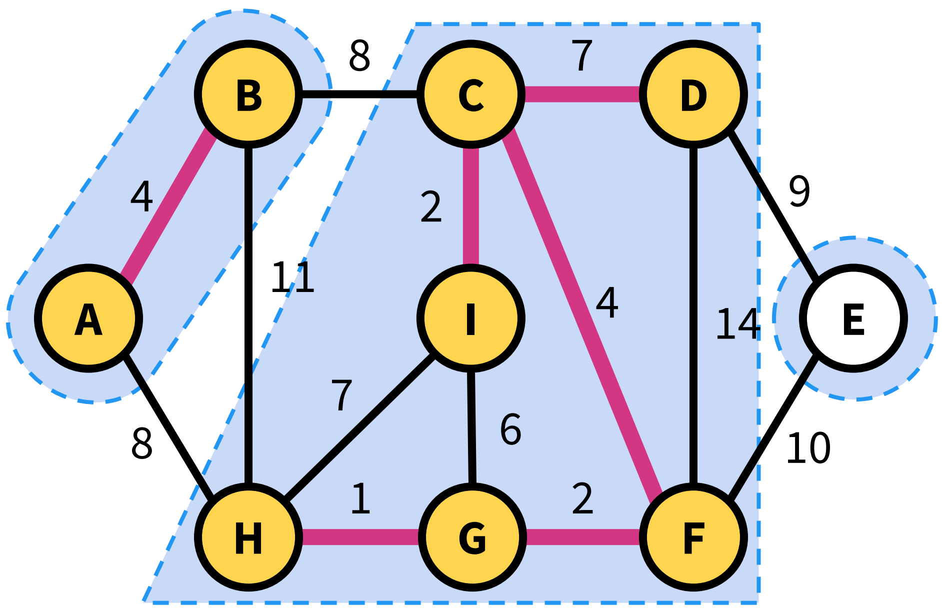

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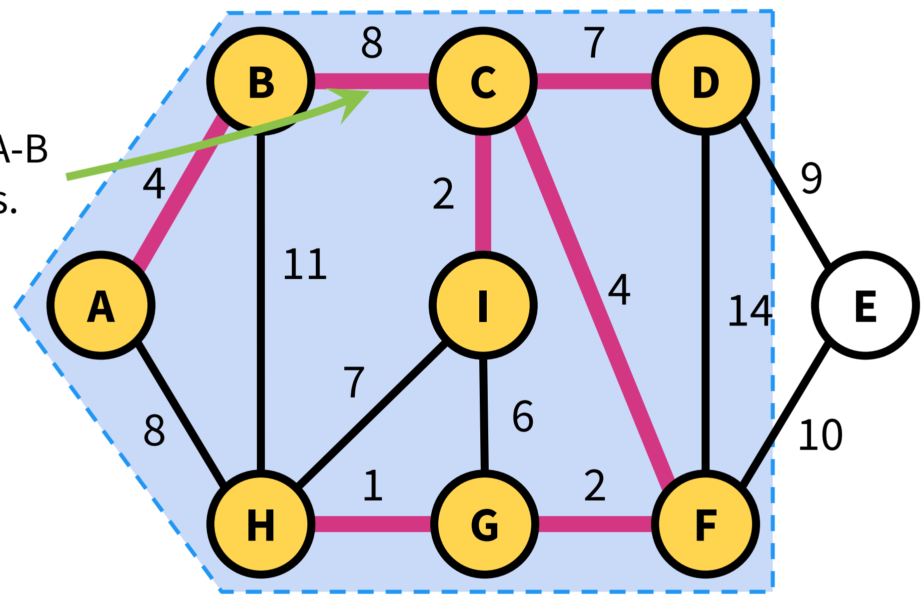
This edge merges the A-B and C-D-F-G-H-I trees.



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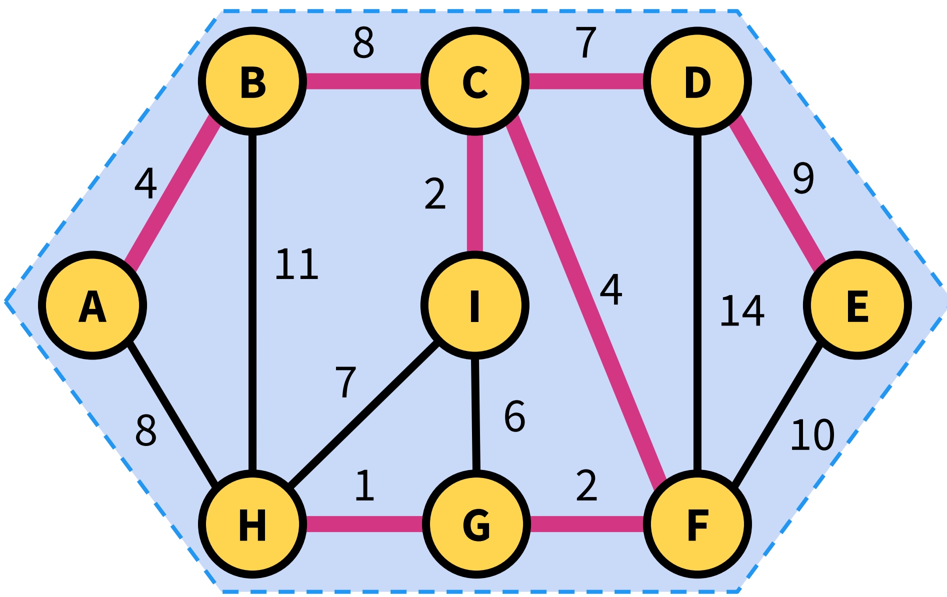

Kruskal's Algorithm

`kruskal` uses union-find data structure, which supports …

`make_set(u)`: create a set {u} in **`O(1)`**

`find(u)`: returns the set containing u in **`O(1)`**

`union(u,v)`: merges the sets containing u and v in **`O(1)`**

Technically, these operations all run in amortized-time **`α(|V|)`**; $\alpha(n) \leq 4$, provided n < # of atoms in the universe. We will discuss amortized analysis in greater detail later.

Kruskal's Algorithm

```
algorithm kruskal(G):
  E_sorted = sort the edges in E by non-decreasing weight
  MST = {}
  for v in V:
    make_set(v)  # put each vertex in its own tree
  for (u, v) in E_sorted:
    if find(u) != find(v):  # u and v in different trees
      MST.add((u, v))
      union(u, v)  # merge u's tree with v's tree
  return MST
```

Runtime:

$O(|E|\log(|E|))$ Using comparison-based sort.

$O(|E|)$

Using radix sort

Kruskal's Algorithm

Recall our lemma:

Consider a cut that respects a set of edges **A**, such that there's an MST **T** containing **A**, and a light edge (u, v) not in **T**.

Lemma: There exists an MST containing **A** ∪ {(u, v)}.

Theorem: `kruskal` returns a minimum spanning tree.

Proof:

At the start of the first iteration of the while loop, there exists a minimum spanning tree with the edges in `MST`. This trivially holds since we initialize MST to the empty set.

Recall, we proved our lemma with an exchange argument!

`kruskal` finds an edge (u, v) that merges two trees T_1 and T_2. Consider the cut $\{T_1, V - T_1\}$; `MST` respects this cut. By our lemma, there exists a minimum spanning tree containing MST ∪ {(u, v)}.

After adding the $(n-1)^{st}$ edge, we have a valid spanning tree; and we know there exists a MST containing this spanning tree, therefore, this is exactly a minimum spanning tree. ■

Prim's and Kruskal's

	Description	Runtime	Use-cases
Prim's	Grows a tree	$O(\lvert V\rvert \log(\lvert V\rvert) + \lvert E\rvert)$ **with red-black tree**	Better on dense graphs
Kruskal's	Grows a forest	$O(\lvert E\rvert \log(\lvert E\rvert))$ **with union-find** $O(\lvert E\rvert)$ **with union-find and radix sort**	Better on sparse graphs and if the edge weights can be radix sorted.

Beyond Prim's and Kruskal's

Karger-Klein-Tarjan (1995): Las Vegas randomized algorithm

$O(|E|)$ expected, $O(\min\{|E|\log(|V|), |V|^2\})$ worst-case

Chazelle (2000): $O(|E|\alpha(|V|))$ deterministic algorithm

Inverse Ackermann function

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Acknowledgement: Part of the materials are adapted from Virginia Williams and David Eng's lectures on algorithms. We appreciate their contributions.