Kev Sharma
Professor Ames
Fall 2020

1. The assumption I make with this problem is that an outcome is only valid if the last 3 tosses are heads. I.e an outcome cannot be TTT as we still have yet to flip the coin until 3 consecutive heads are reached.

a.

1a. 3 → HHH
4 → THHH
5 → TTHHH, HTHHH
6 → TTTHHH, THTHHH, HTTHHH, HHTHHH

b. The total number of outcomes above is 8, the sample space is defined as Omega.
$P(A_1) = 1/8$ {There is only one outcome out of the total that is HHH}
$P(A_2) = 1/2$ {There are 4 outcomes out of the total that start with T}
$P(A_3) = 1/4$ {There are 2 outcomes that start with HT}
$P(A_4) = 1/8$ {There is only one outcome out of the total that starts with HHT}.

c.

$E(N) = P(N) \cdot N$
$E(x) = E(A_1 + A_2 + A_3 + A_4)$
$\quad = E(A_1) + E(A_2) + E(A_3) + E(A_4)$
$E(x) = P(A_1) \cdot A_1 + P(A_2) A_2 + P(A_3) A_3 + P(A_4) A_4$
if x = three consecutive heads, then $A_2 = T*$, that means $A_2 = 1 + x$ {tails first + expected # rolls b4 three cons. heads}
$E(x) = \frac{1}{8}(3) + \frac{1}{2}(1+x) + \frac{1}{4}(2+x) + \frac{1}{8}(3+x)$

$E(x) = \frac{3}{8} + \frac{4}{8}(1+x) + \frac{2}{8}(2+x) + \frac{1}{8}(3+x) \Rightarrow$
$8E(x) = 3 + 4 + 4x + 4 + 2x + 3 + x$
$8x = 7x + 14$ {defined x earlier}
$\boxed{x = 14}$

2. a.

$x \cdot y = 2^n (x_L y_L) + 2^{n/2} (x_L y_R + x_R y_L) + (x_R y_R)$

$\rightarrow T(n) = O(n^2)$ {an n bit mult. takes $O(n^2)$ time}.

$\therefore$ if $k = \frac{n}{2}$, then $T(k) = O(k^2) \rightarrow T(\frac{n}{2}) = O((n/2)^2) = \underline{O(n^2/4)}$ *

$\rightarrow$ Shifting a number n times takes $O(n)$ time.

$\therefore$ if $k = n/2$, then shifting by k takes $O(k)$ time $= O(n/2)$ time.

$x \cdot y = 2^n(x_L y_L) + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$

$O(n^2) = 2^n(n/2 \text{ bit mult}) + 2^{n/2}(\text{two } n/2 \text{ bit mult}) + n/2 \text{ bit mult}.$

$O(n^2) =$ four $n/2$ bit multiplications & two shifts & 3 adds.

$= 4 \cdot O((n/2)^2) + \underbrace{O(n) + O(n/2) + 3 \text{ add operations}}.$

$= 4 \cdot O(n^2/4) + O(n)$ {adding & shifting is $O(n)$, ~~because~~ given in problem description}

$O(n^2) = 4 \cdot O(n^2/4) + O(n)$

$\boxed{T(n) = 4 \cdot T(n/2) + O(n)}$ {given/stated $\underline{\text{above}}$*}.

$\rightarrow T(n) = aT(n/b) + O(n^d)$ here $a = 4, b = 2, d = 1$.

By master theorem: $T(n) = \begin{cases} O(n^{\log_b a}) & \text{if } d < \log_b a \\ O(n^d \log(n)) & \text{if } d = \log_b a \\ O(n^d) & \text{if } d > \log_b a \end{cases}$

$d = 1$; $\log_b a = \log_2 4 = 2$.

$d < 2$, $\therefore T(n) = O(n^{\log_b a})$

$T(n) = O(n^2)$. ✓

$\therefore$ By master theorem $O(\cdot) = O(n^2)$

$$x \cdot y = 2^n(x_L y_L) + 2^{n/2}\underbrace{(x_L y_R + x_R y_L)} + x_R y_R$$

$$x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - (x_L y_L) - (x_R y_R)$$

$$x \cdot y = 2^n(x_L y_L) + 2^{n/2}\left[(x_L + x_R)(y_L + y_R) - (x_L y_L) - (x_R y_R)\right] + x_R y_R$$

note we have duplicate multiplications.

$x_L y_L$ and $x_R y_R$ need only be computed once & store for later use.

$$\therefore x \cdot y = 2^n O(n^2/4) + 2^{n/2}\left[O(n^2/4) + \text{reuse}\right] + O(n^2/4).$$

→ Now there are only three multiplications. Shifts & adds take a total of $O(n)$ {given}.

$$x \cdot y = 3 \cdot O(n^2/4) + O(n).$$

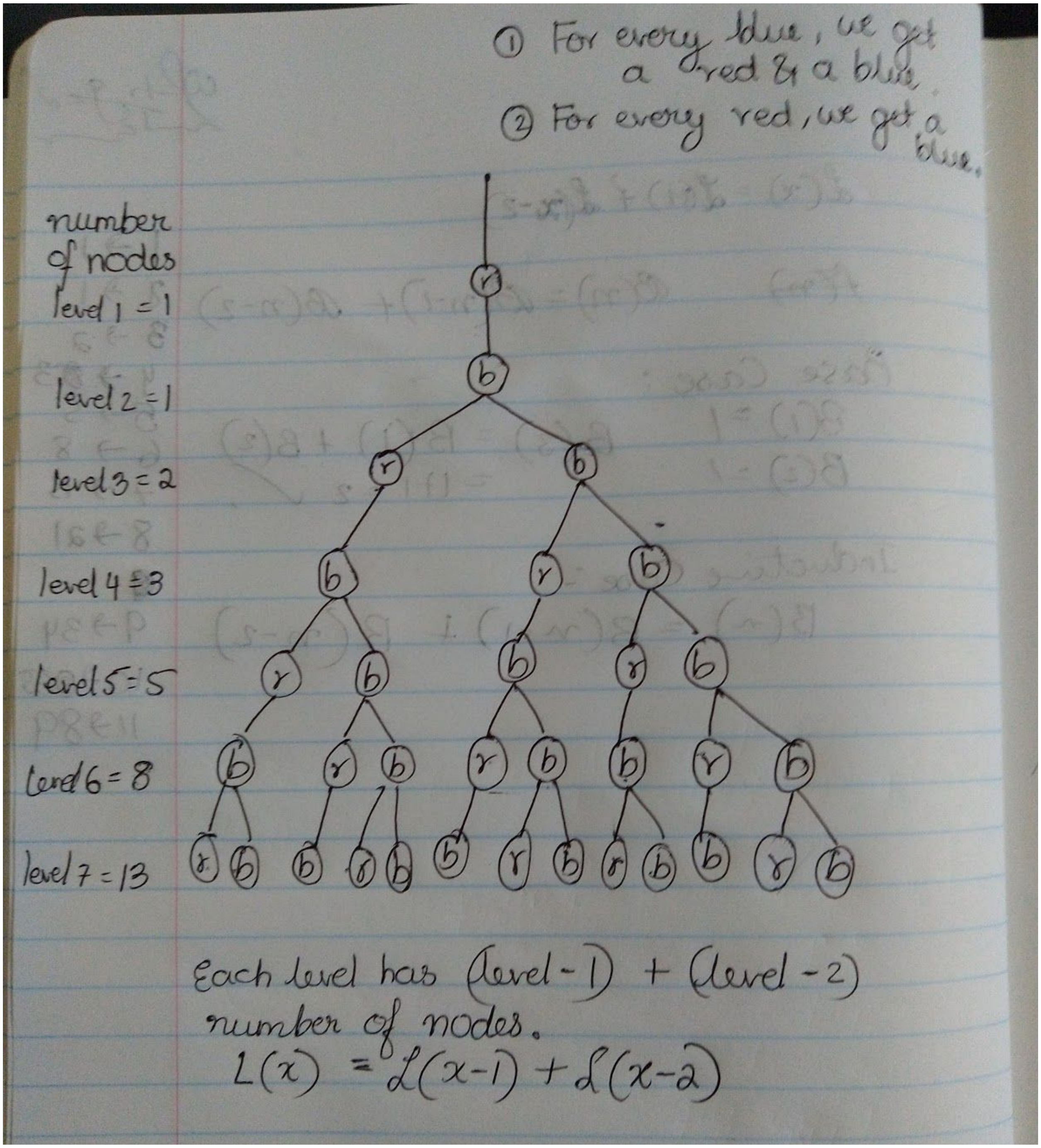
$$\boxed{T(n) = 3T(n/2) + O(n).}$$

$a=3, b=2, d=1$. $d < \log_b a$ since $d=1$ & $\log_b a = \log_2 3 = 1.58496$.

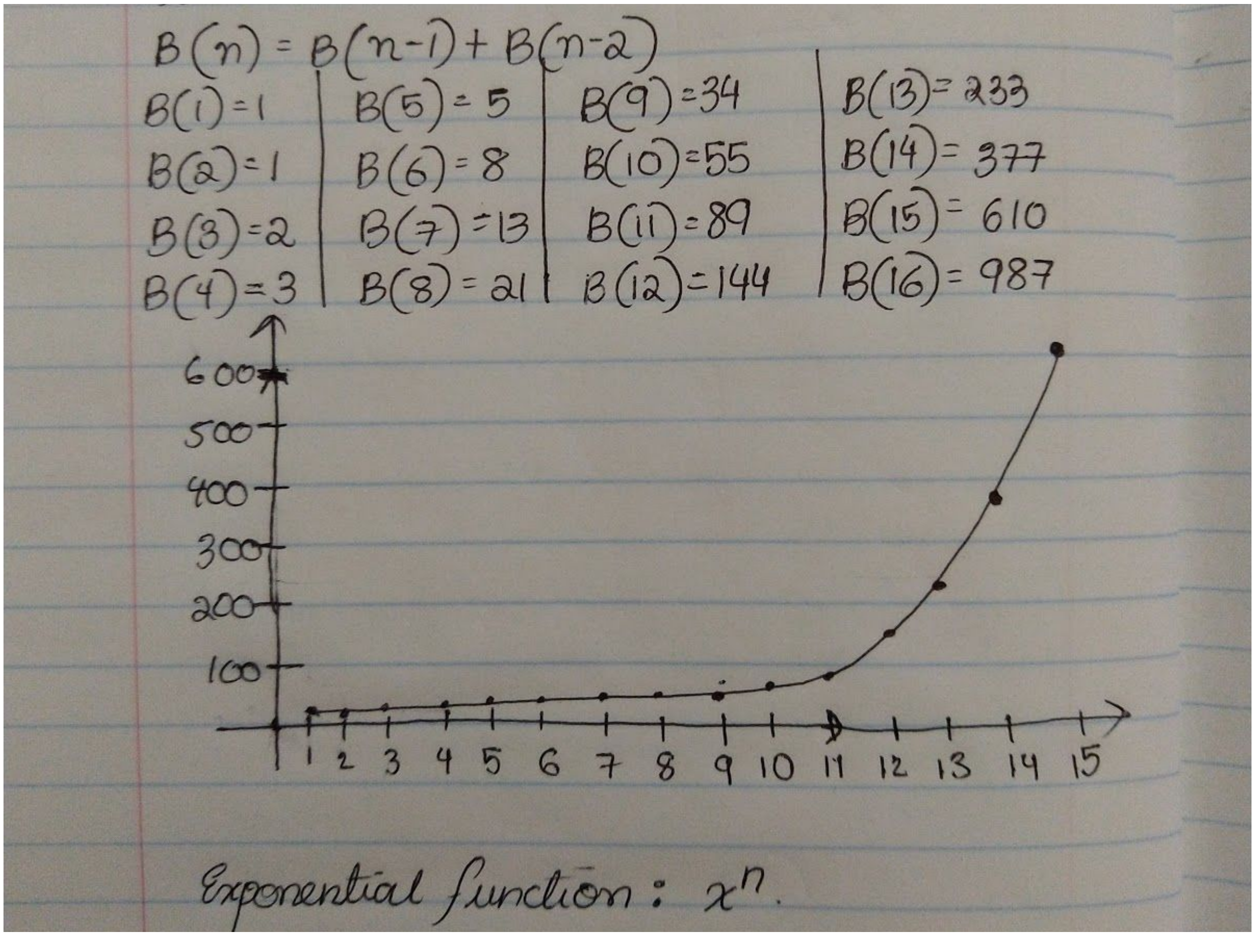
$\therefore T(n) = O(n^{\log_b a}) = O(n^{1.58496})$

This gives us a faster algorithm.

3. Let us first determine how many branches one particular level will have.



Since we have determined a recurrence relation for the number of branches at any level, let us attempt to model it in a graph to determine how best to approach finding a closed form solution for the recursive function B(n) = B(n-1) + B(n-2). I have written some values of the function and plotted them to observe the general trend. B(n) most closely resembles an exponential function given the graph.



Having found the general form of the function B(n) as x^n. We must still determine the exact closed form solution based on the homogeneous linear recurrence relation we found previously.

The first step is to solve for x.

We have 2 solutions for x and thus can craft a final solution to the recursive function by employing the homogeneous linear recurrence solution theorem presented in class.

We have been given 2 base cases and can solve for alpha and beta (in the equation derived from the theorem) using Gaussian Jordan Elimination on a matrix with a system of equations representing the two base cases.

$* f(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n ; g(n) = \left(\frac{1-\sqrt{5}}{2}\right)^2$

$B(n) = B(n-1) + B(n-2)$. Let $x^n \xRightarrow{\text{represent}} B(n)$

Then $B(n) = B(n-1) + B(n-2)$ is equal to.

$x^n = x^{n-1} + x^{n-2}$;

$x^{n-2} x^2 = x^{n-2} x + x^{n-2} 1 \Rightarrow x^2 = x+1$

$x^2 - x - 1 = 0$. Sols $= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$

Sols: $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

Recall: $B(n) = x^n$; Since $x = \frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$, we have two solutions for $B(n)$.

$*$ If $f(n)$ & $g(n)$ are solutions to a homogerous linear recurrence, then $h(n) = \alpha f(n)^* + \beta g(n)^*$ is also a solution (theorem provided).

$h(1) = 1 \; \{B(1) = 1\}$; $h(2) = 1 \; \{B(2) = 1\}$

$1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^1 + \beta\left(\frac{1-\sqrt{5}}{2}\right)^1 ; 1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^2 + \beta\left(\frac{1-\sqrt{5}}{2}\right)^2$

Augmented matrix: $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \text{RREF}\left[\begin{array}{cc|c} \left(\frac{1+\sqrt{5}}{2}\right) & \left(\frac{1-\sqrt{5}}{2}\right) & 1 \\ \left(\frac{1+\sqrt{5}}{2}\right)^2 & \left(\frac{1-\sqrt{5}}{2}\right)^2 & 1 \end{array}\right]$

$= \left[\begin{array}{cc|c} 1 & 0 & 0.44721 \\ 0 & 1 & -0.44721 \end{array}\right]$ $\alpha = 0.44721 = 1/\sqrt{5}$

$\beta = -0.44721 = -1/\sqrt{5}$

$\therefore h(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

$$\boxed{B(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}}$$

Having found alpha and beta we now have a closed form solution to our recurrence relation.

We must still verify that the closed form solution we have found exactly resembles the recurrence relation we have defined for this problem.

Recall that the number of branches on any given level (such that the level exceeds 2 is always the sum of the number of branches on the previous two levels).

A direct proof has been shown below to verify that the closed form solution we have above is correct.

We must verify this closed form using our recurrence relation:

$$B(n) = B(n-1) + B(n-2)$$

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}}{\sqrt{5}}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}$$

$$= \left[\left(\frac{1+\sqrt{5}}{2}\right)^n \left[\left(\frac{1+\sqrt{5}}{2}\right)^{-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{-2}\right] - \left(\frac{1-\sqrt{5}}{2}\right)^n \left[\left(\frac{1-\sqrt{5}}{2}\right)^{-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{-2}\right]\right] / \sqrt{5}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n [1] - \left(\frac{1-\sqrt{5}}{2}\right)^n [1]}{\sqrt{5}}$$

$$= B(n) \checkmark$$

Our closed form solution is in congruence with our recurrence relation.