Graph Algorithms II

Outline for Today

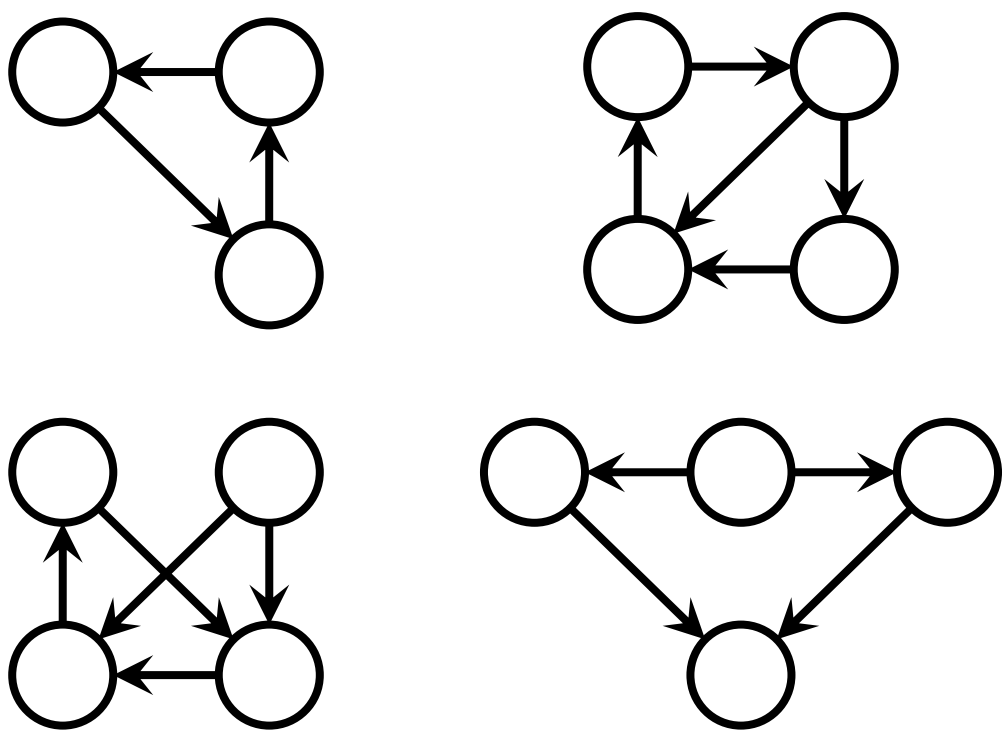
Graph algorithms

- Kosaraju's Algorithm for finding strongly connected components
- Karger's Algorithm for finding global minimum cuts

Kosaraju’s Algorithm

Strongly Connected Components

A directed graph G = (V, E) is strongly connected if,
for all pairs of vertices u and v, there's a path from u to v
and a path from v to u.



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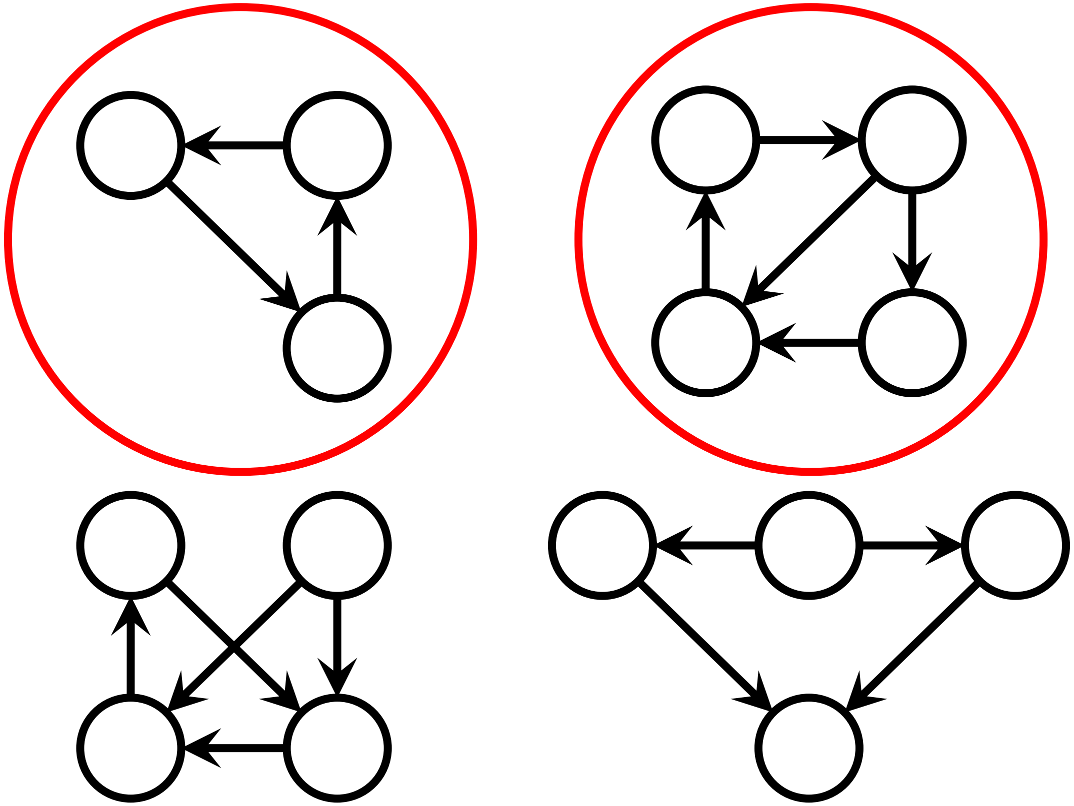
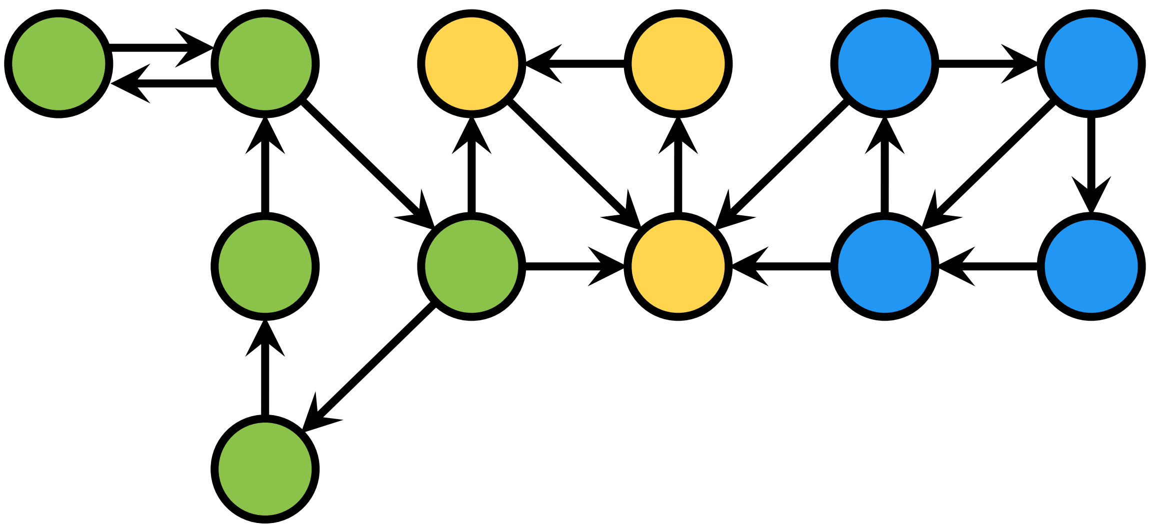


Strongly Connected Components

We can decompose a graph into its strongly connected components (SCCs).



Strongly Connected Components

Why do we care about SCCs?

SCCs provide information about communities.

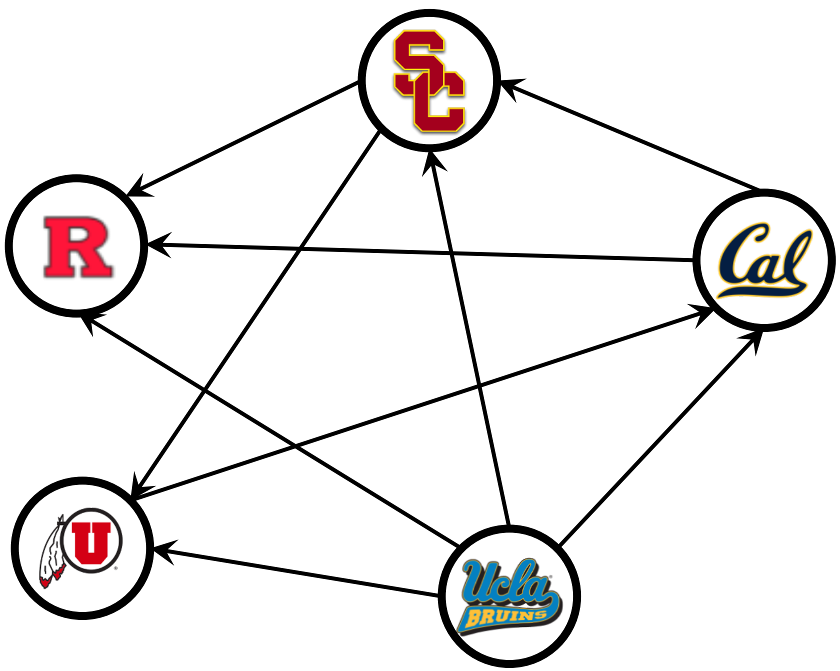
A computer scientist might want to decompose the Internet into SCCs to find connected areas.

An economist might want to decompose labor market data (e.g. employer-employee graph) into SCCs to analyze labor status.

A football executive might want to determine which schools should play in NCAA.

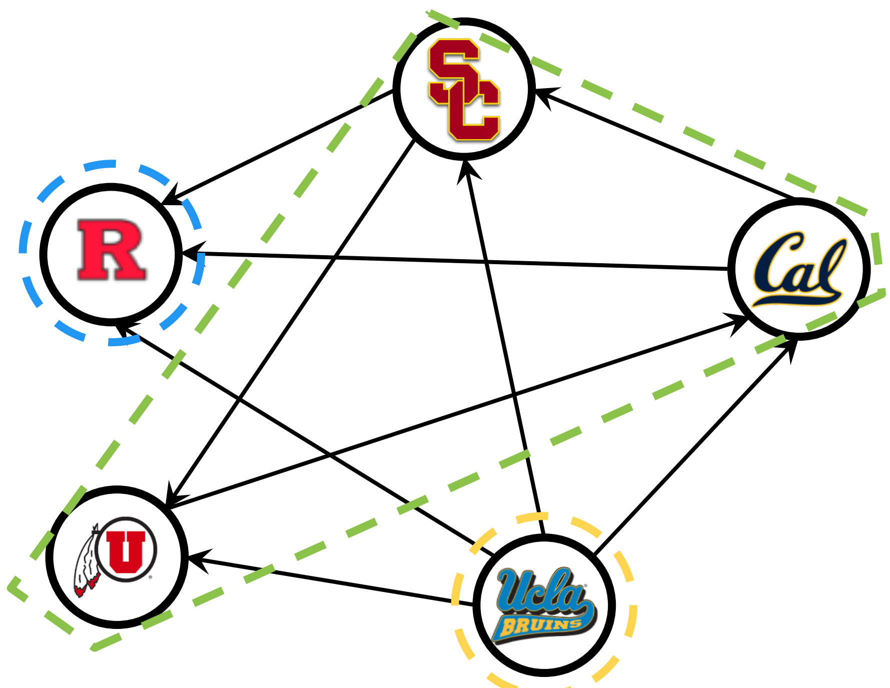
Strongly Connected Components

How many SCCs are in this graph? 🤔



Strongly Connected Components

How many SCCs are in this graph? 🤔 3; let's find them!



Kosaraju's Algorithm

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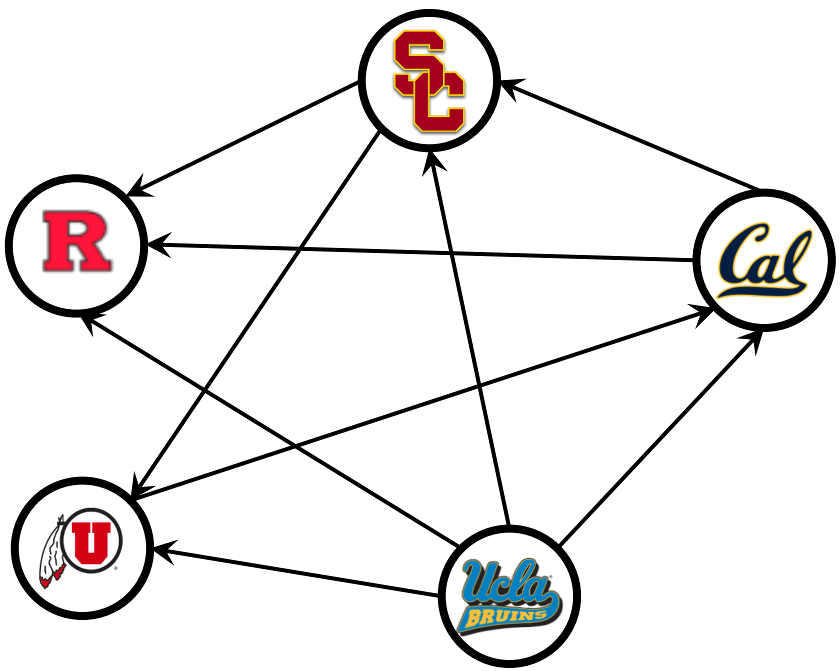

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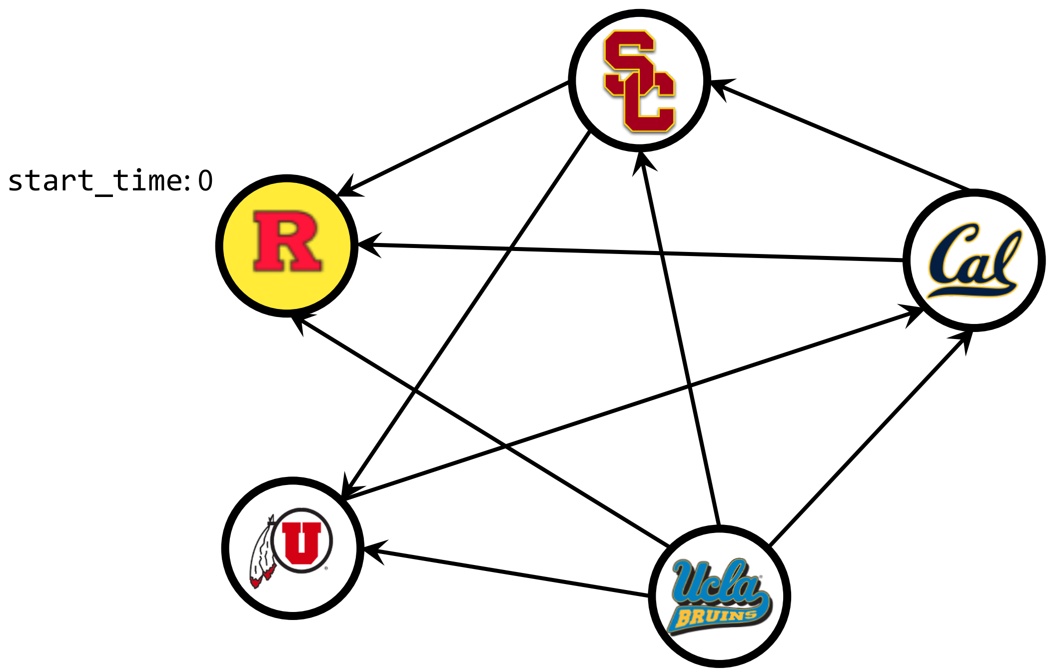

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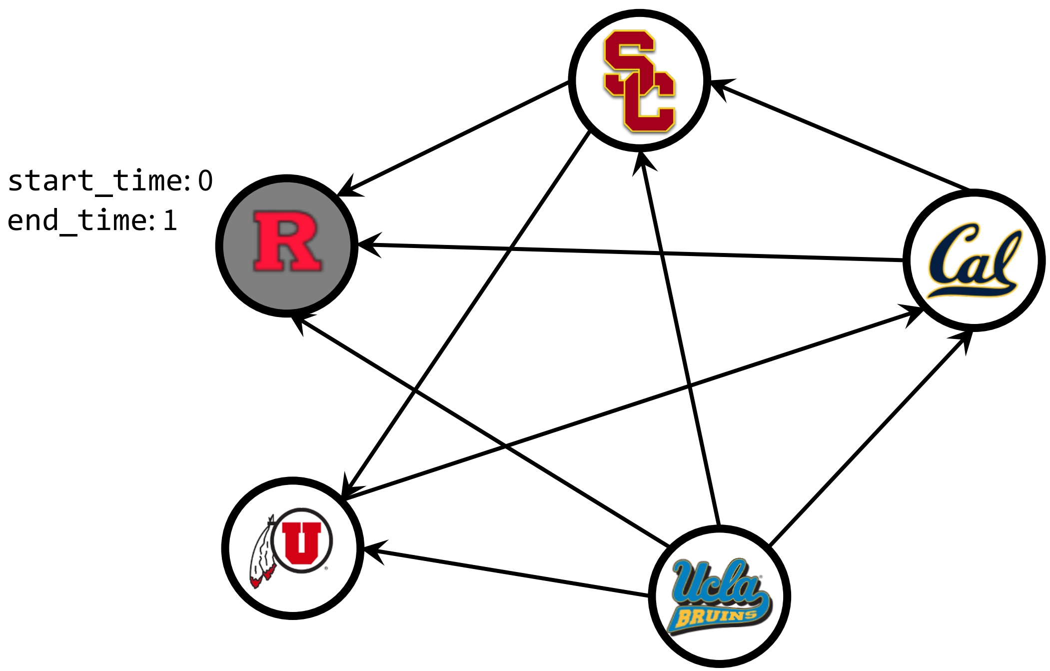

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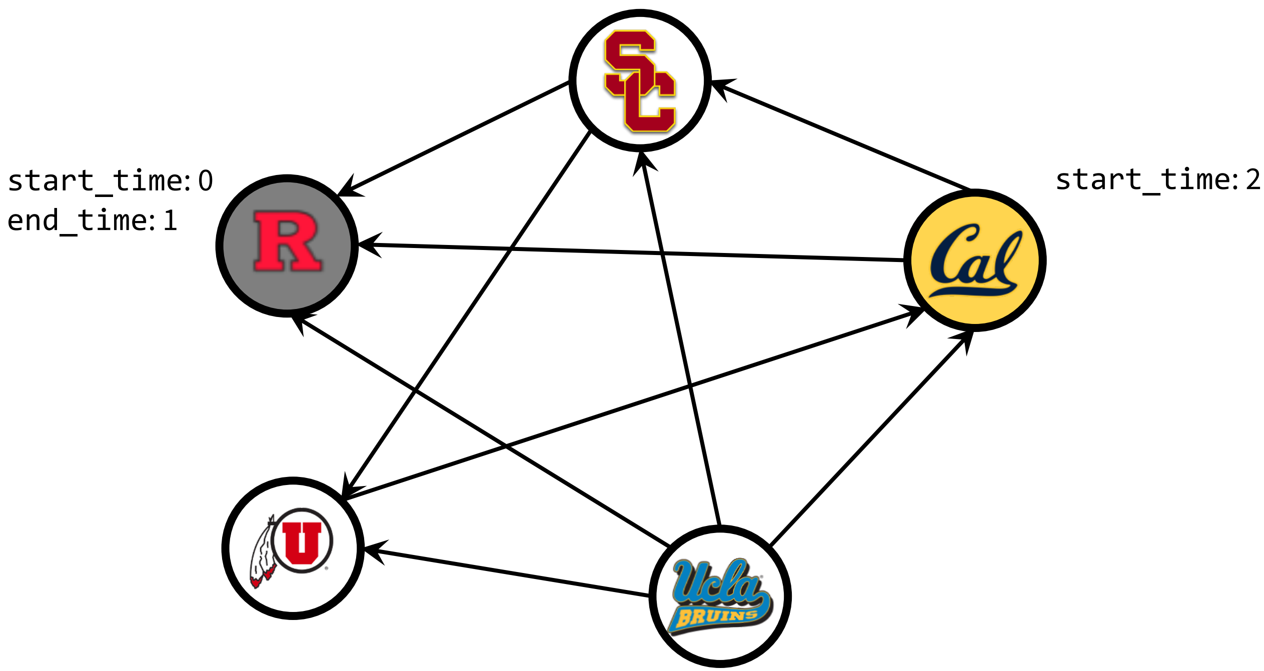

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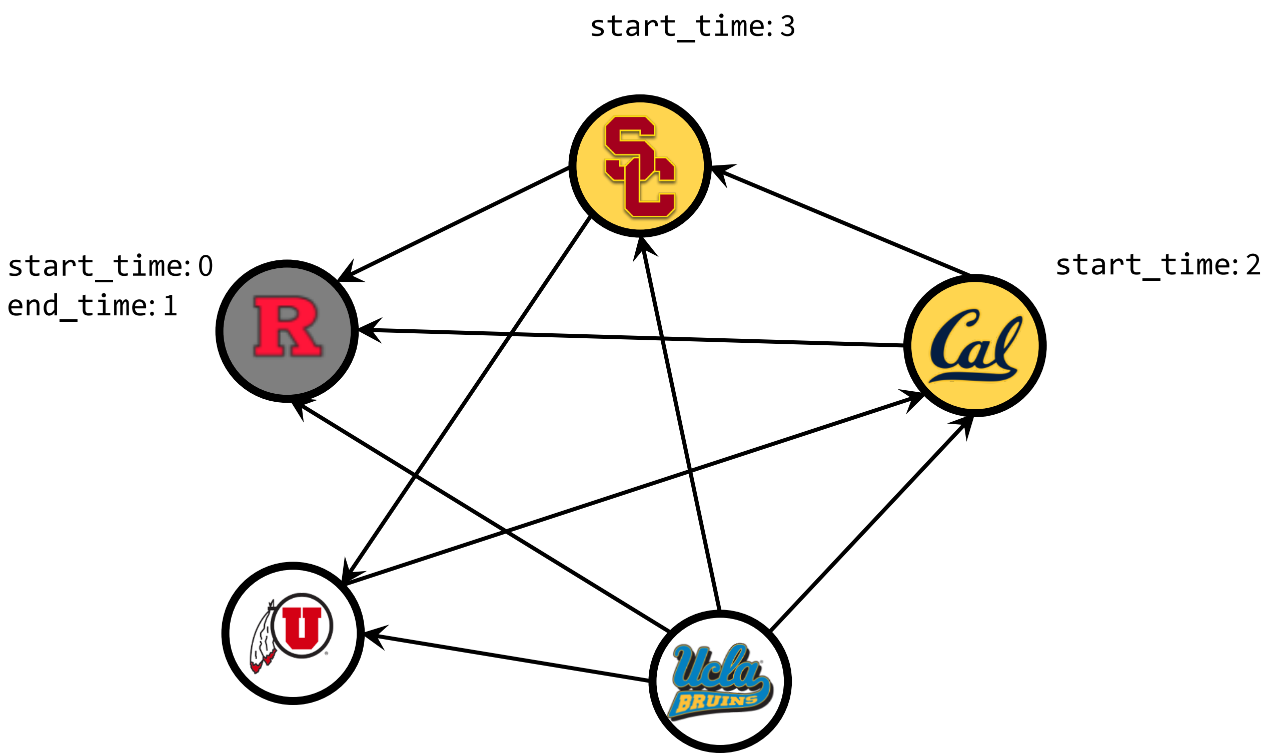

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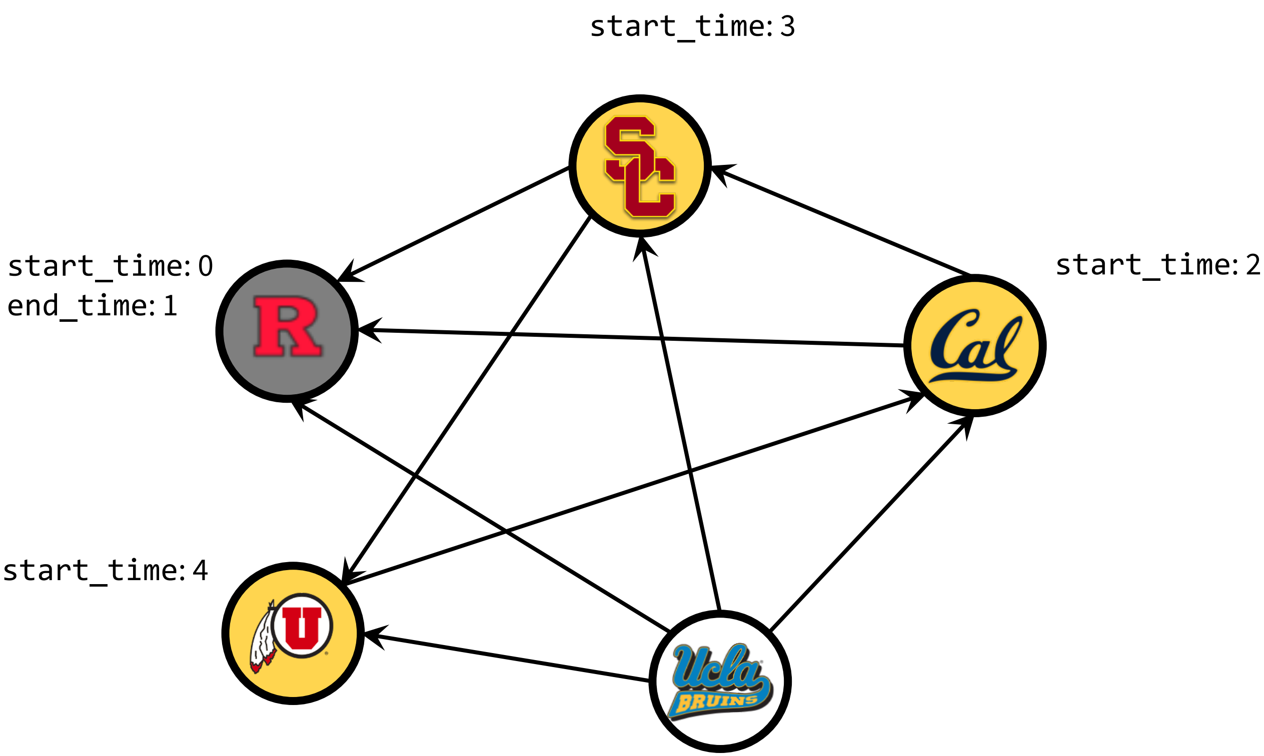

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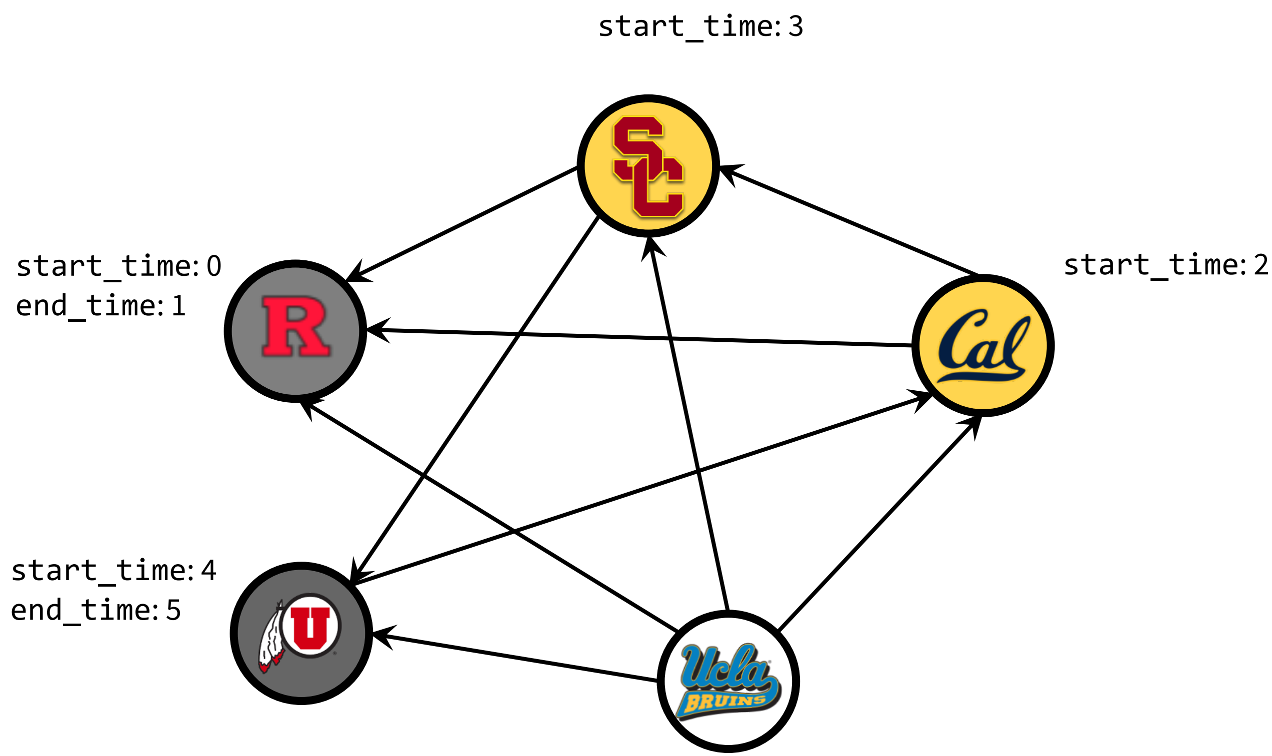

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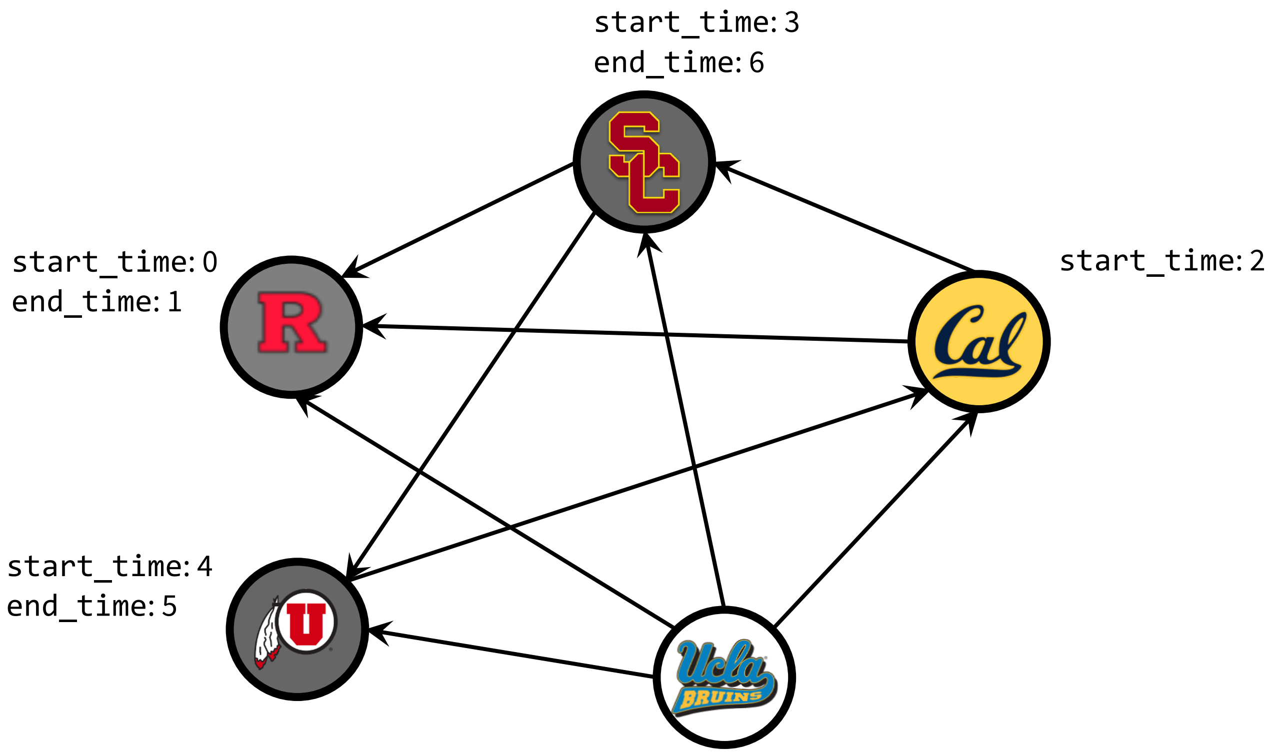

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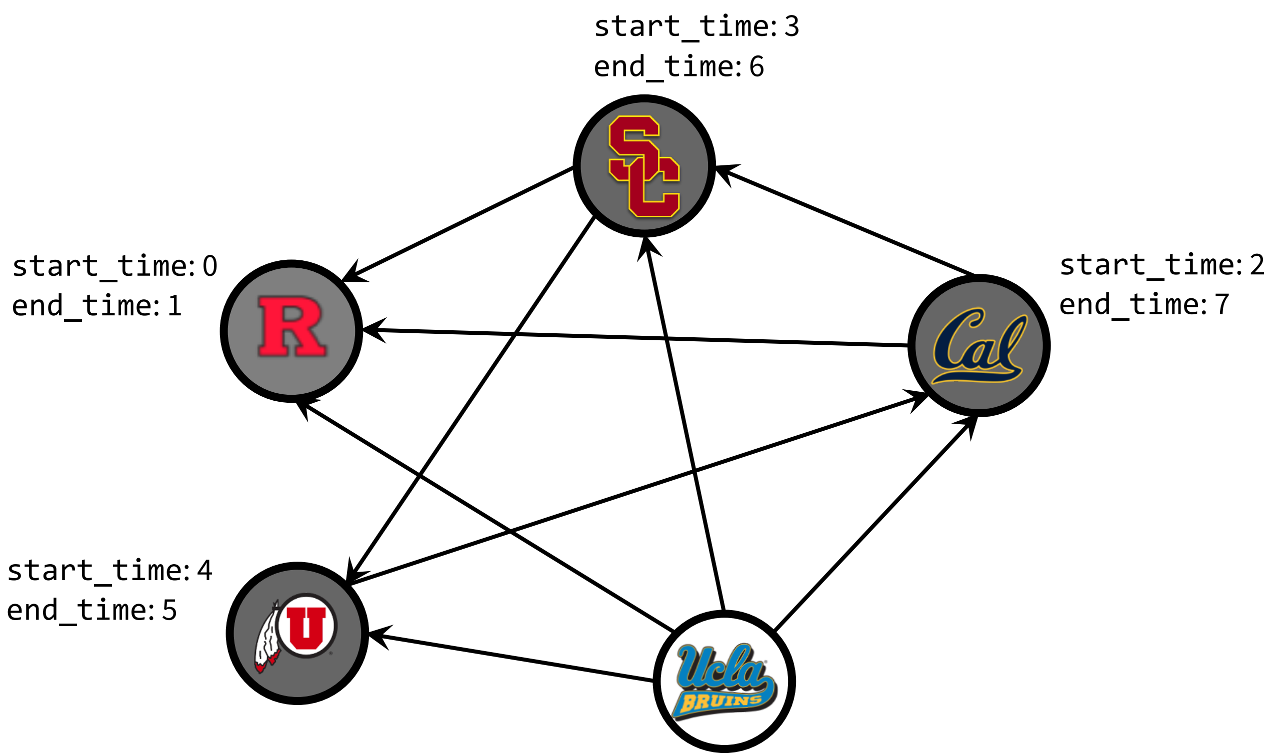

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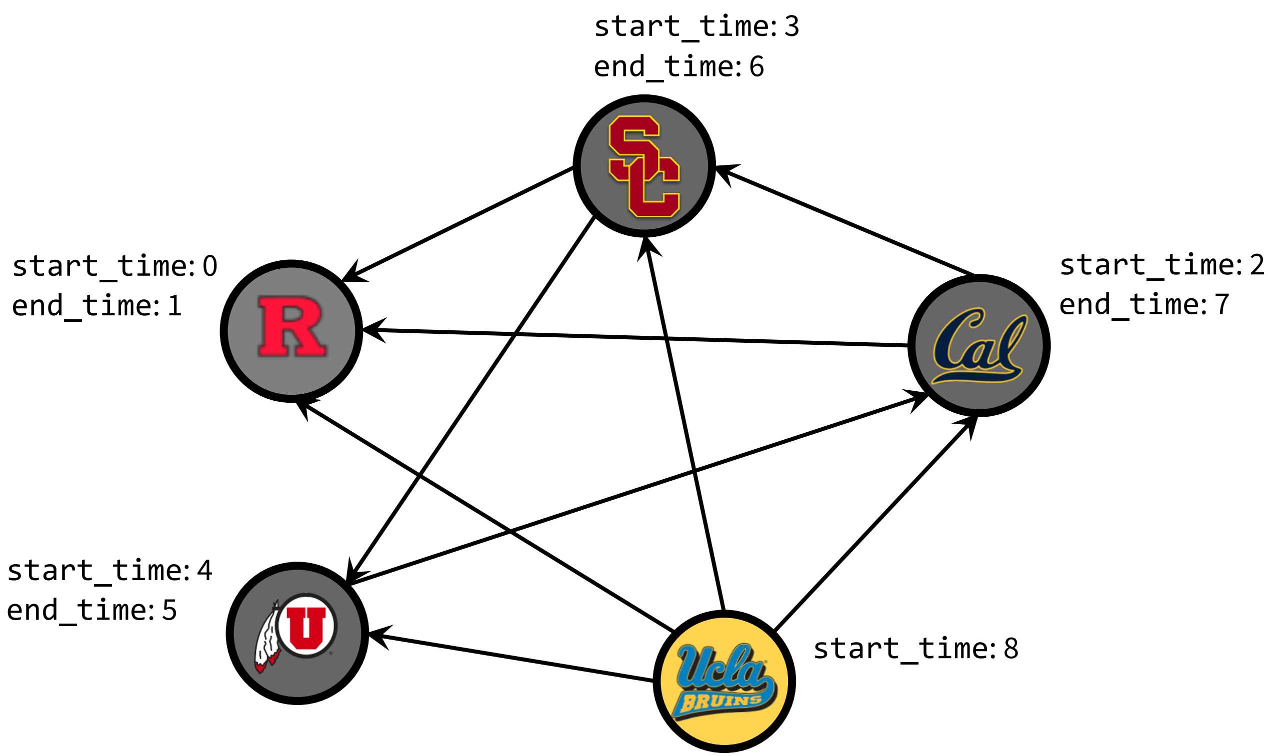

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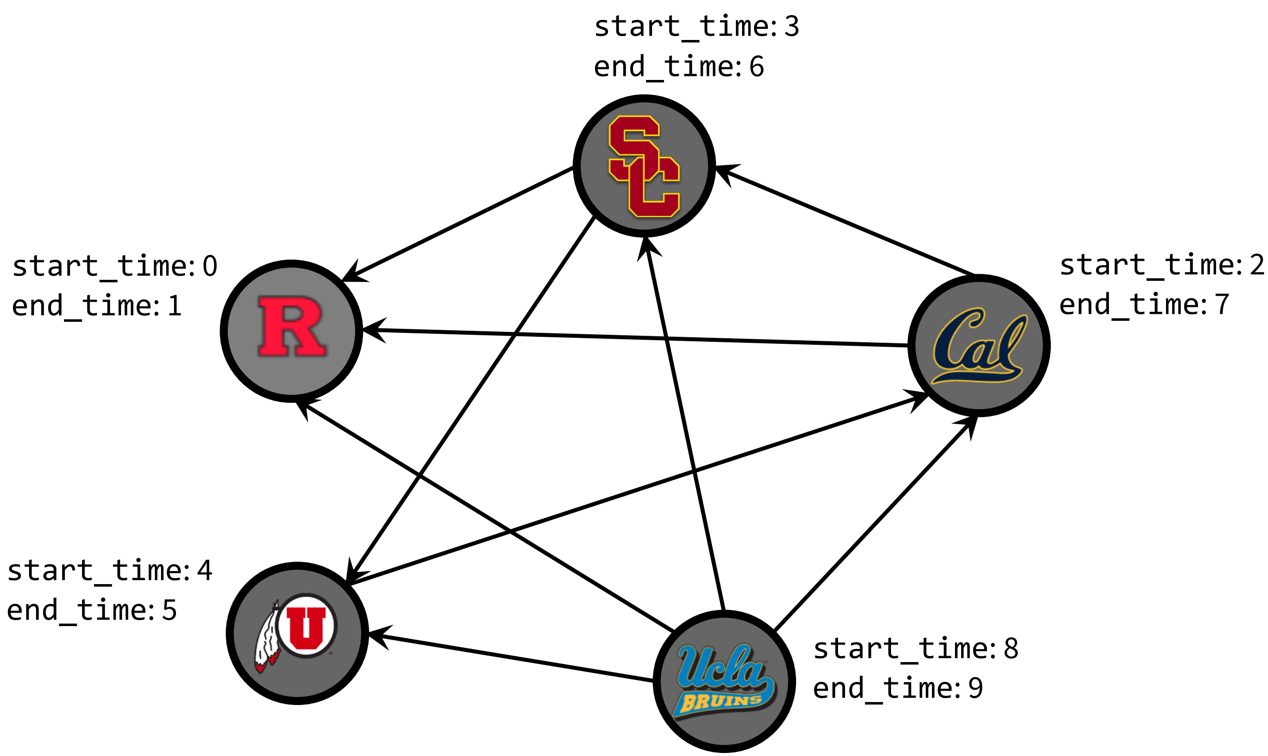
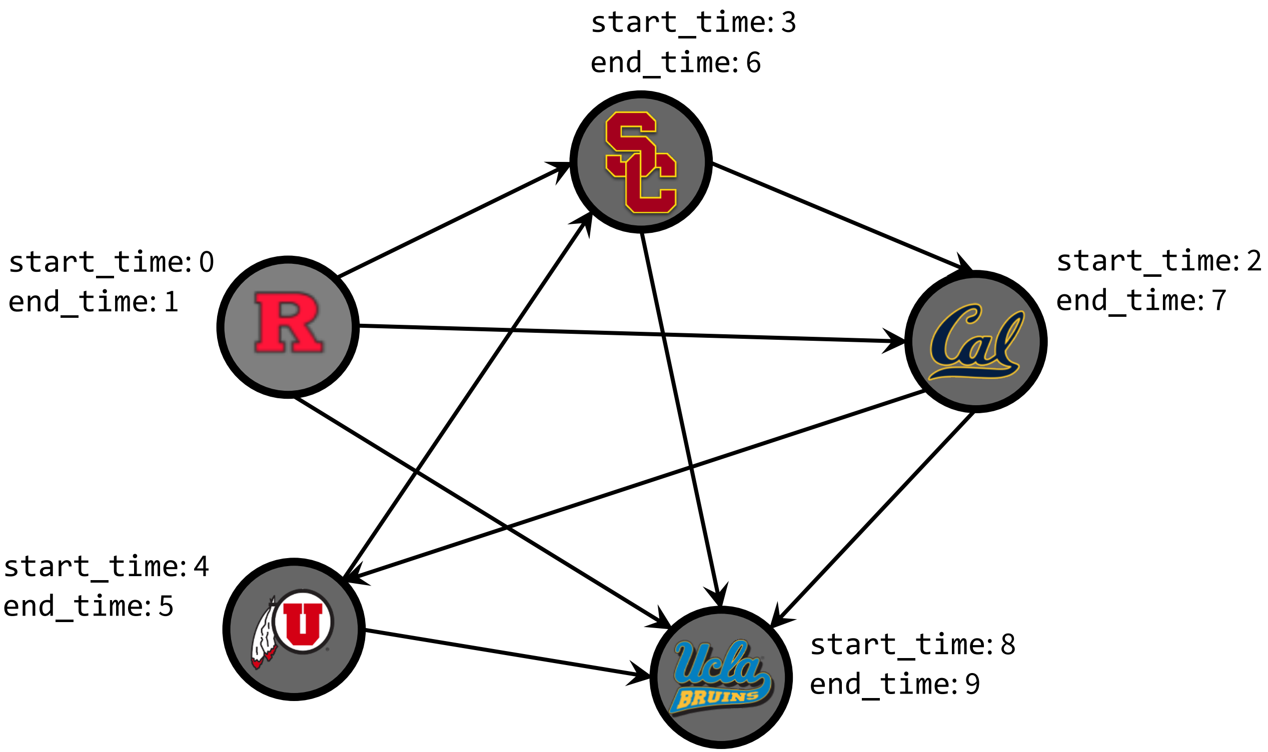


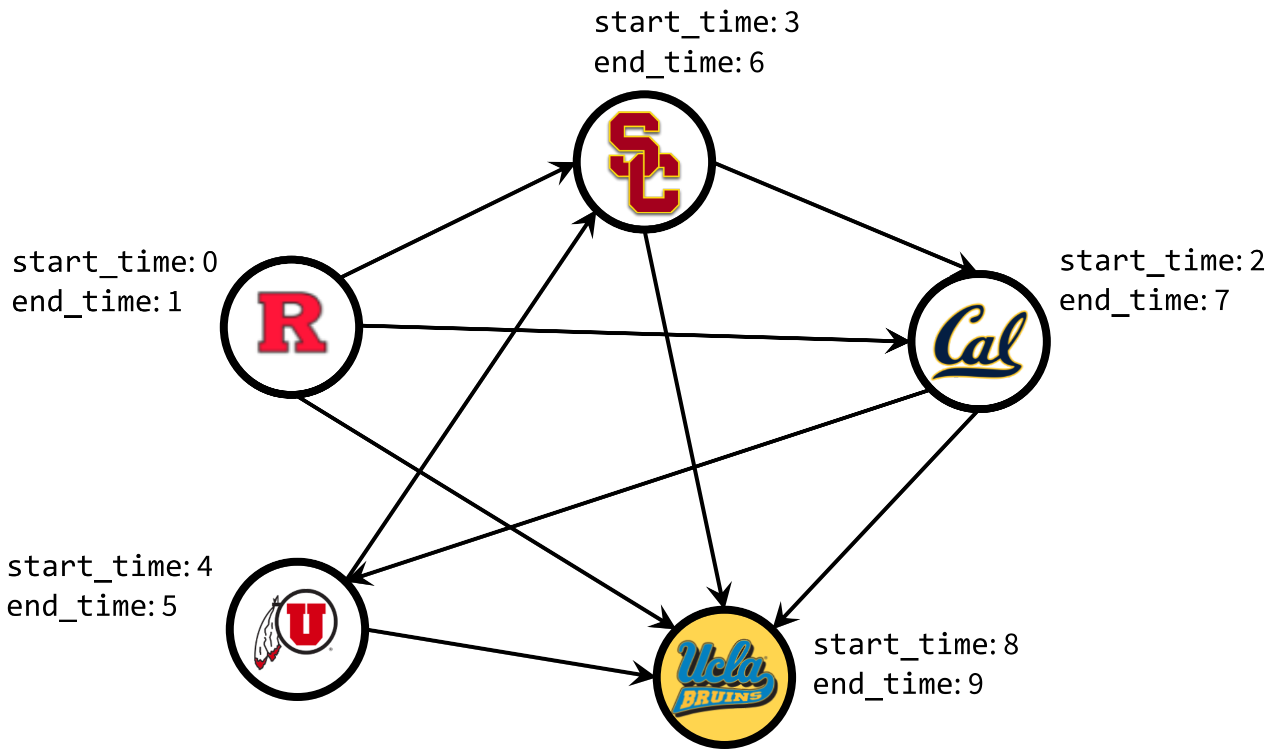
Kosaraju's Algorithm

2. Reverse all of the edges.



Kosaraju's Algorithm

3. Repeat `dfs` again, starting with vertices with the largest `end_time`.



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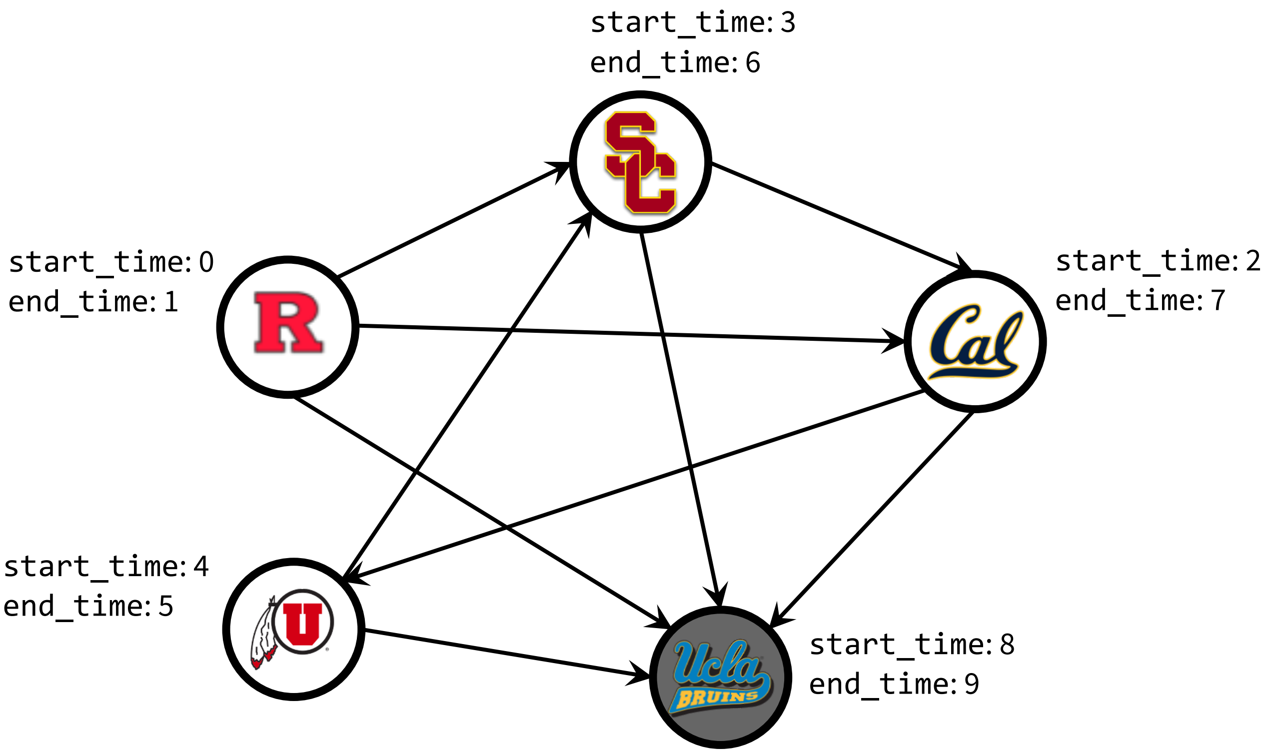

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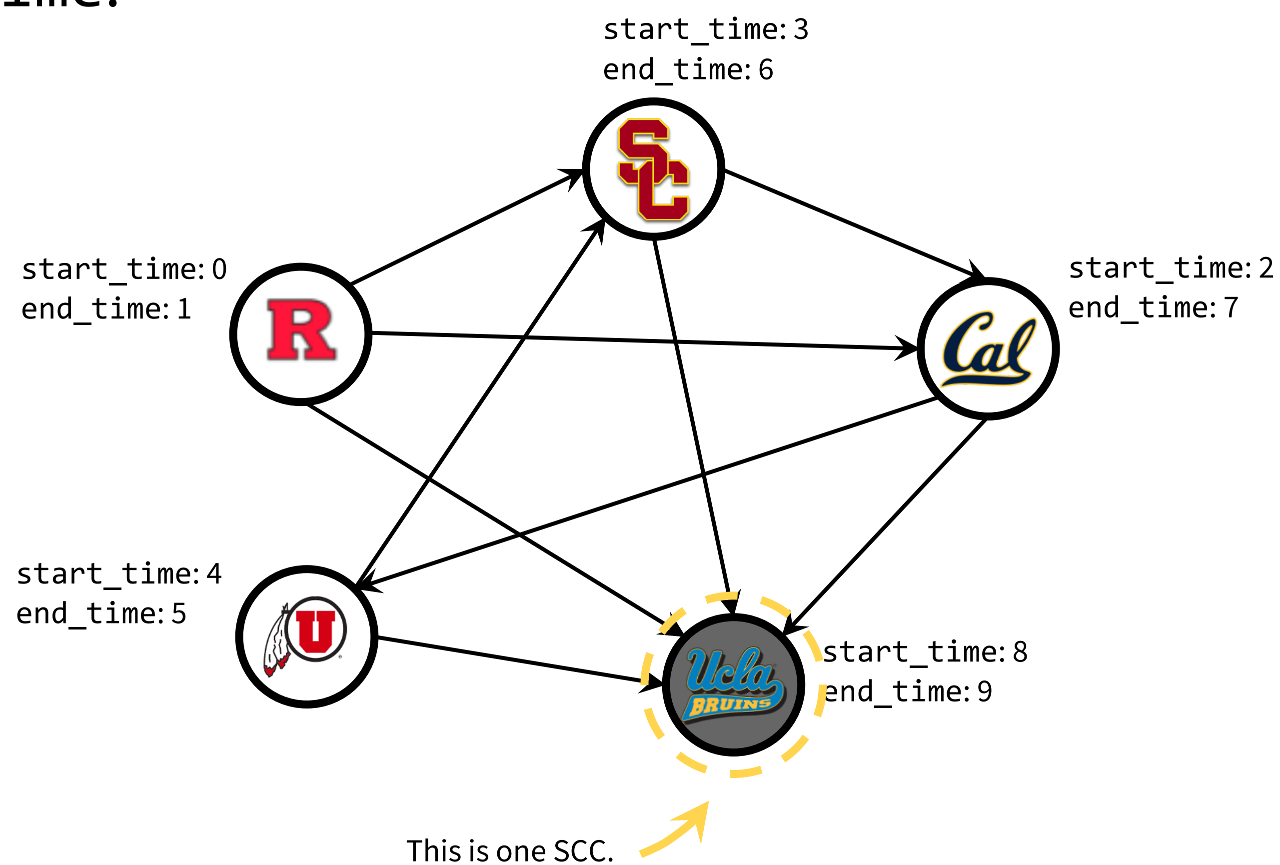

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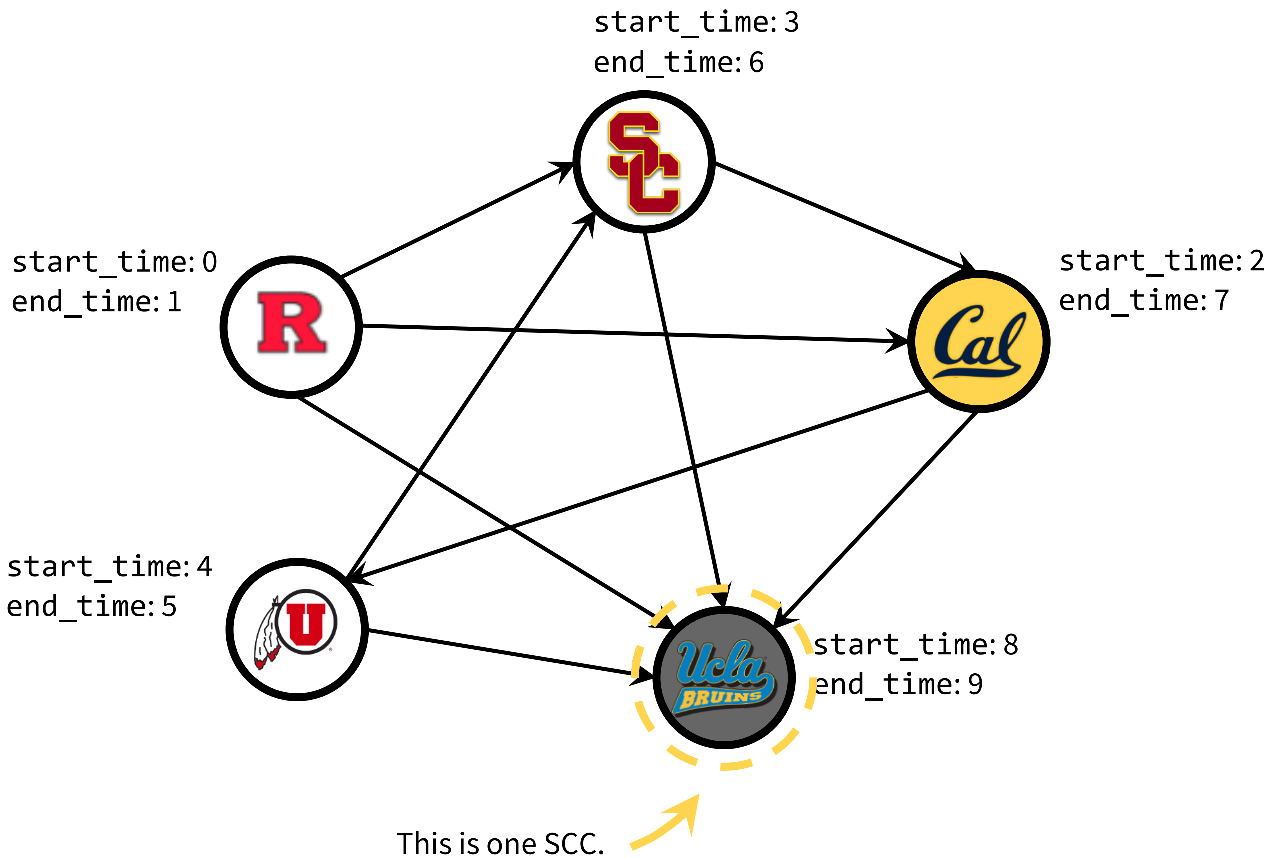

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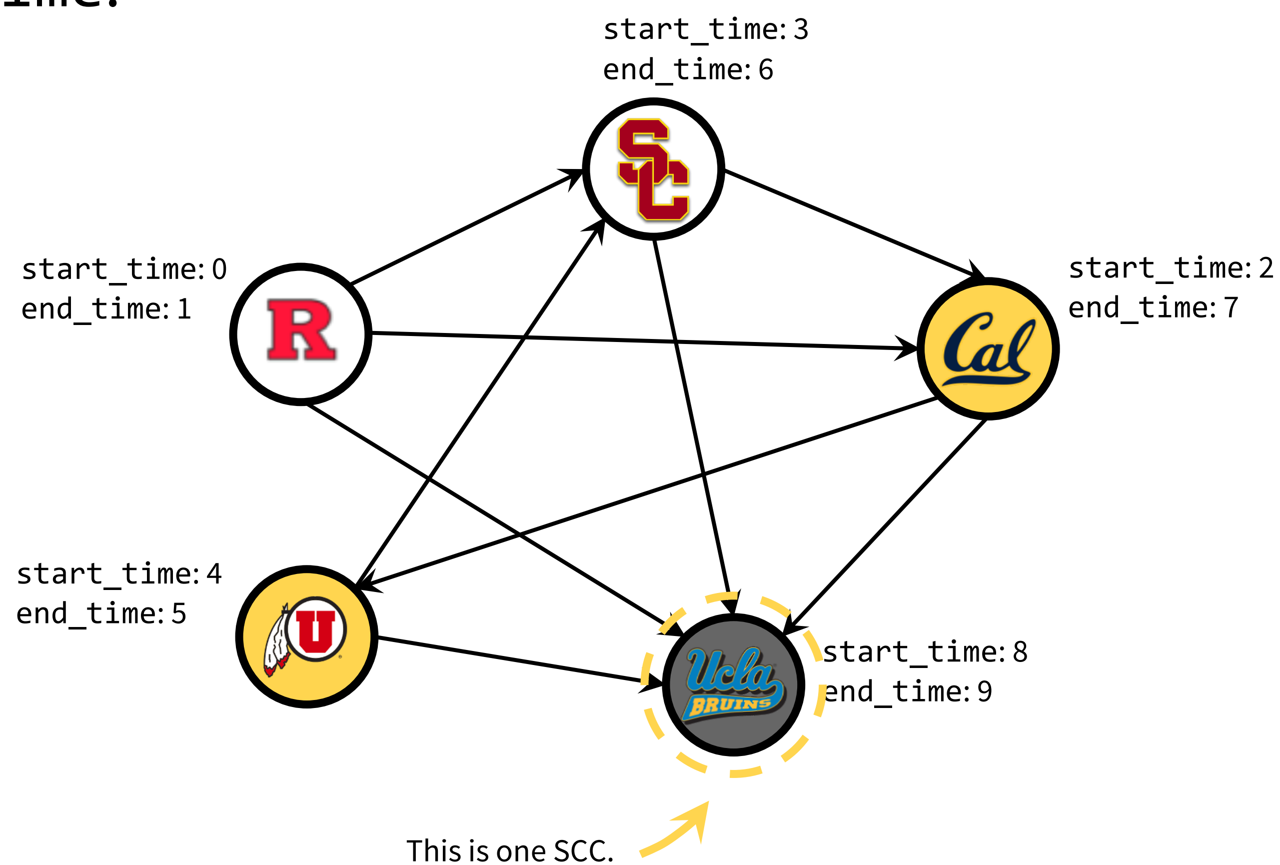

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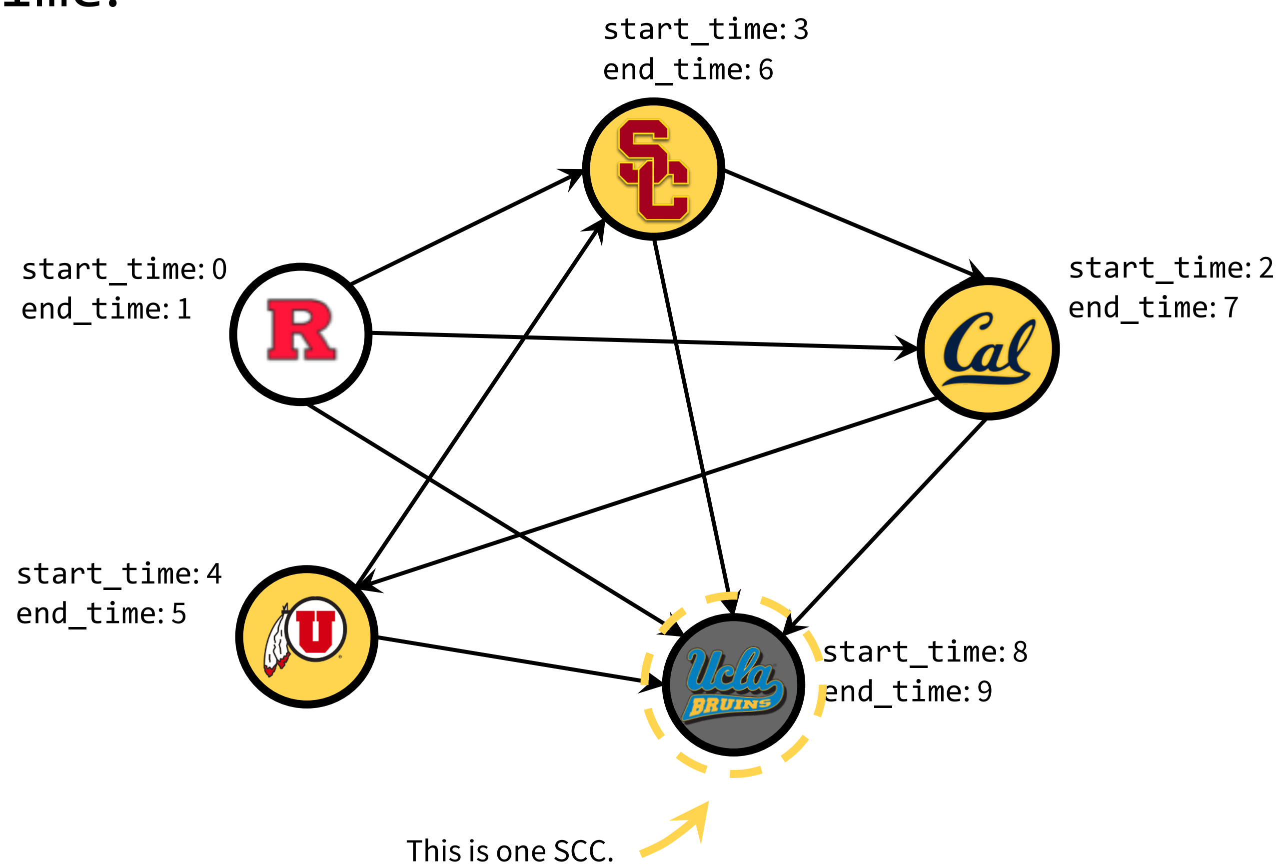

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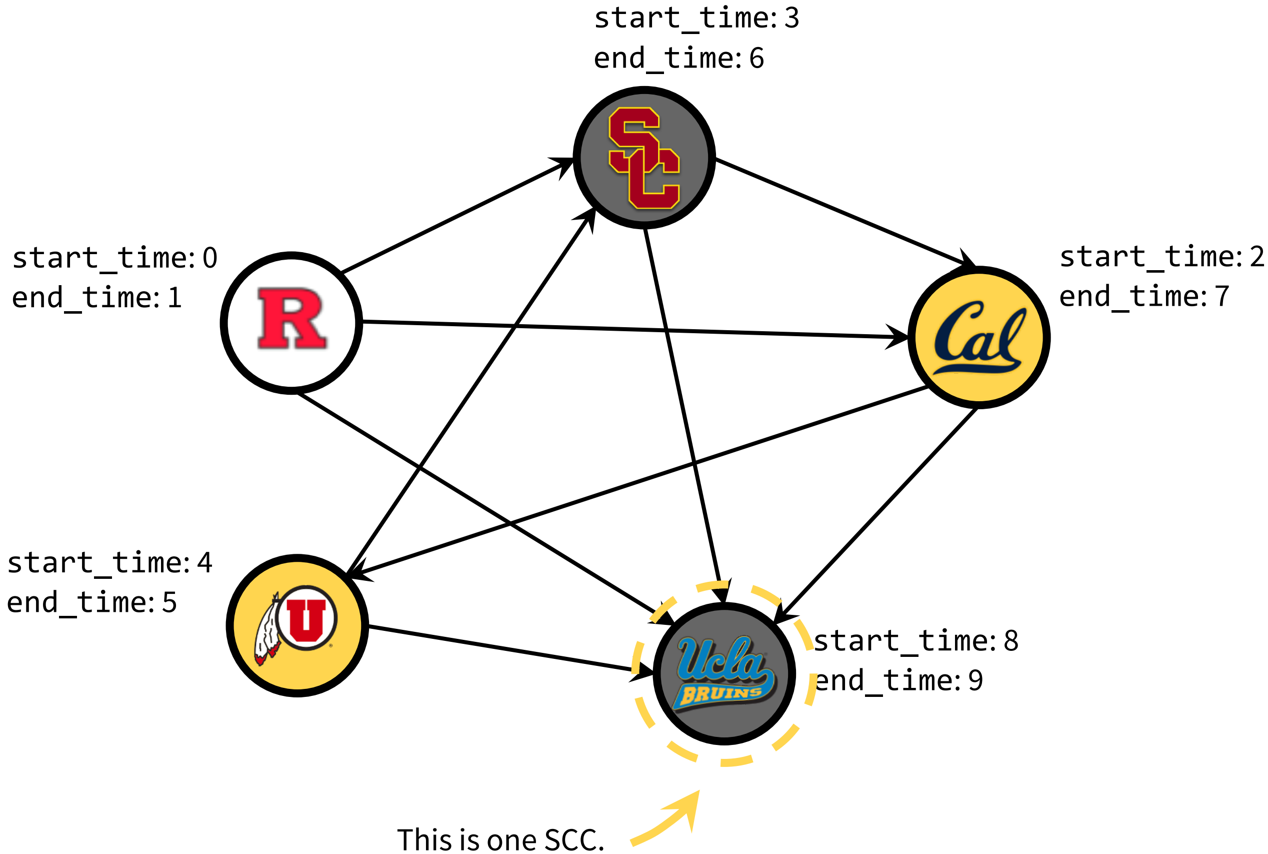

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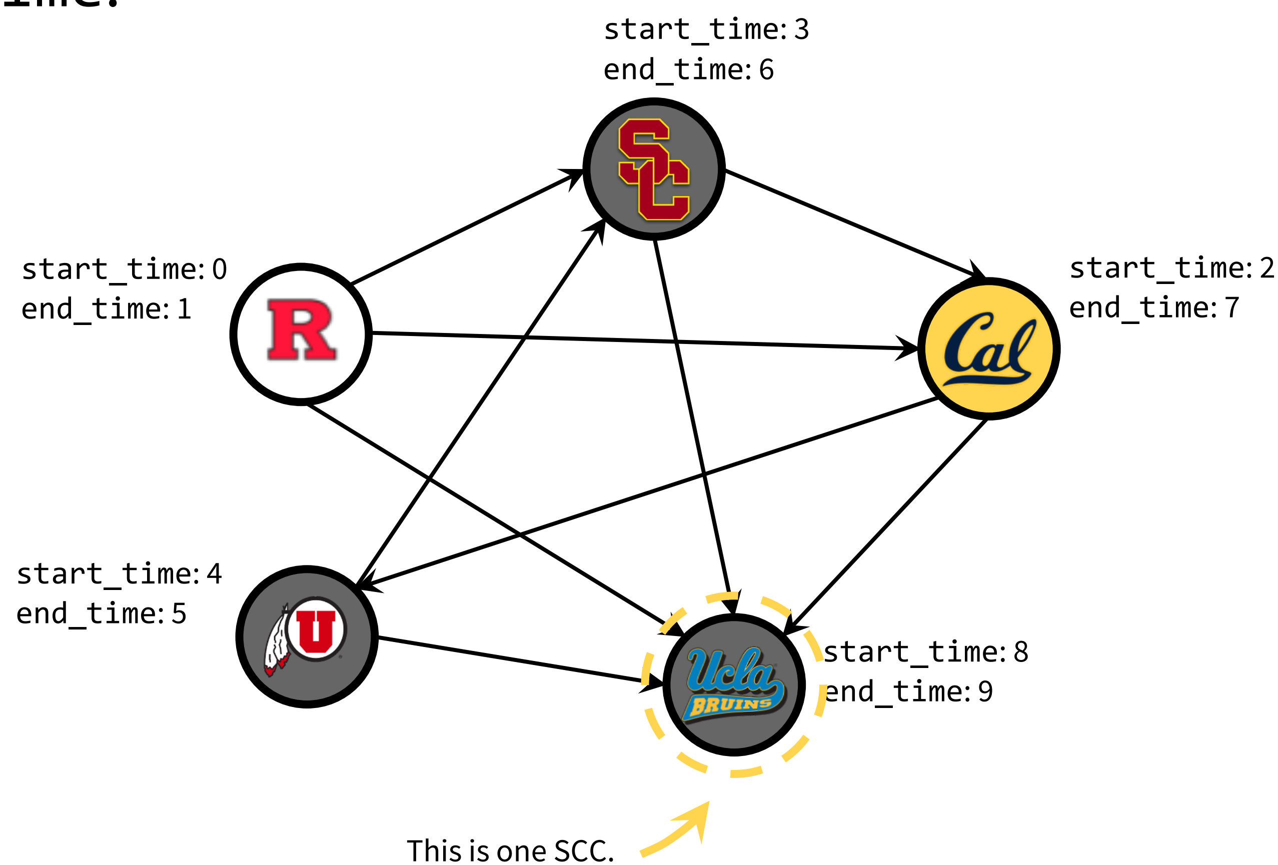

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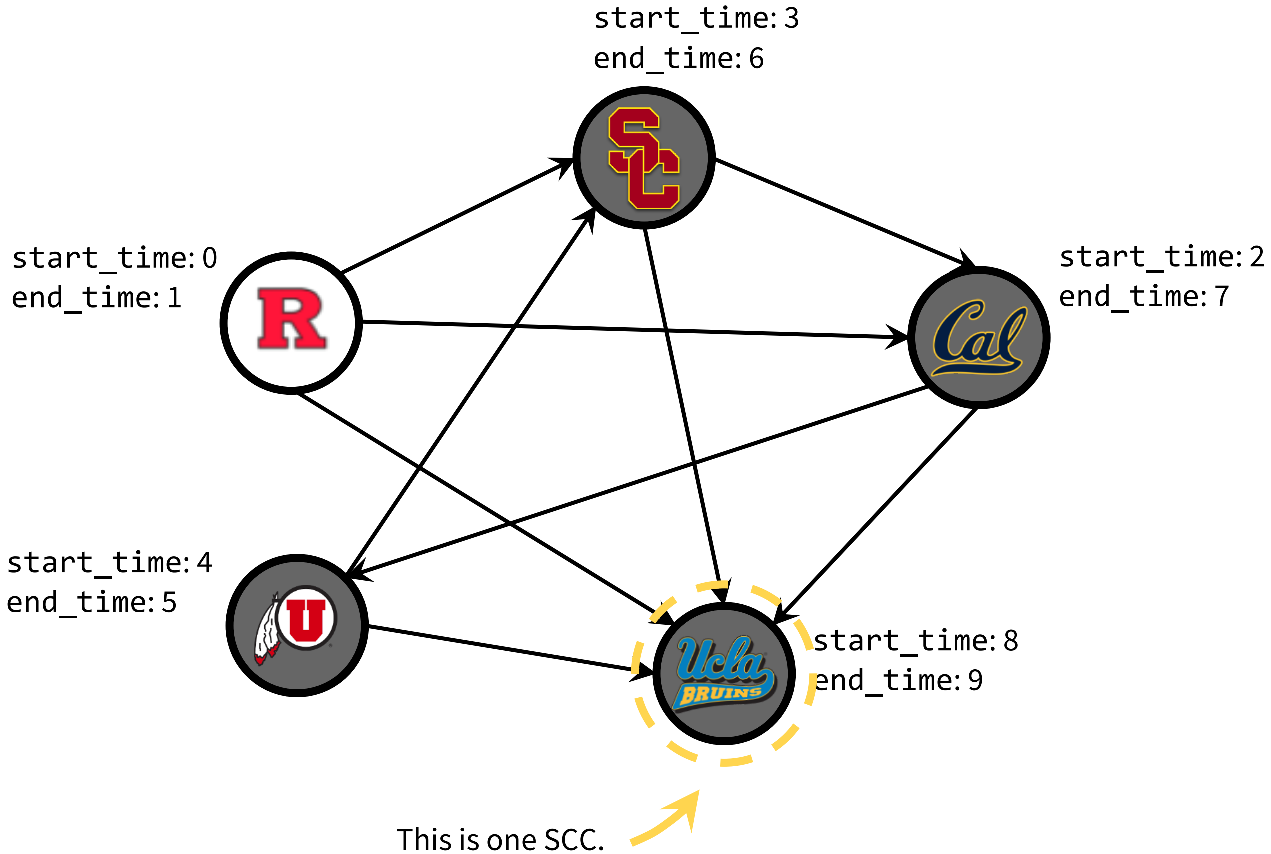

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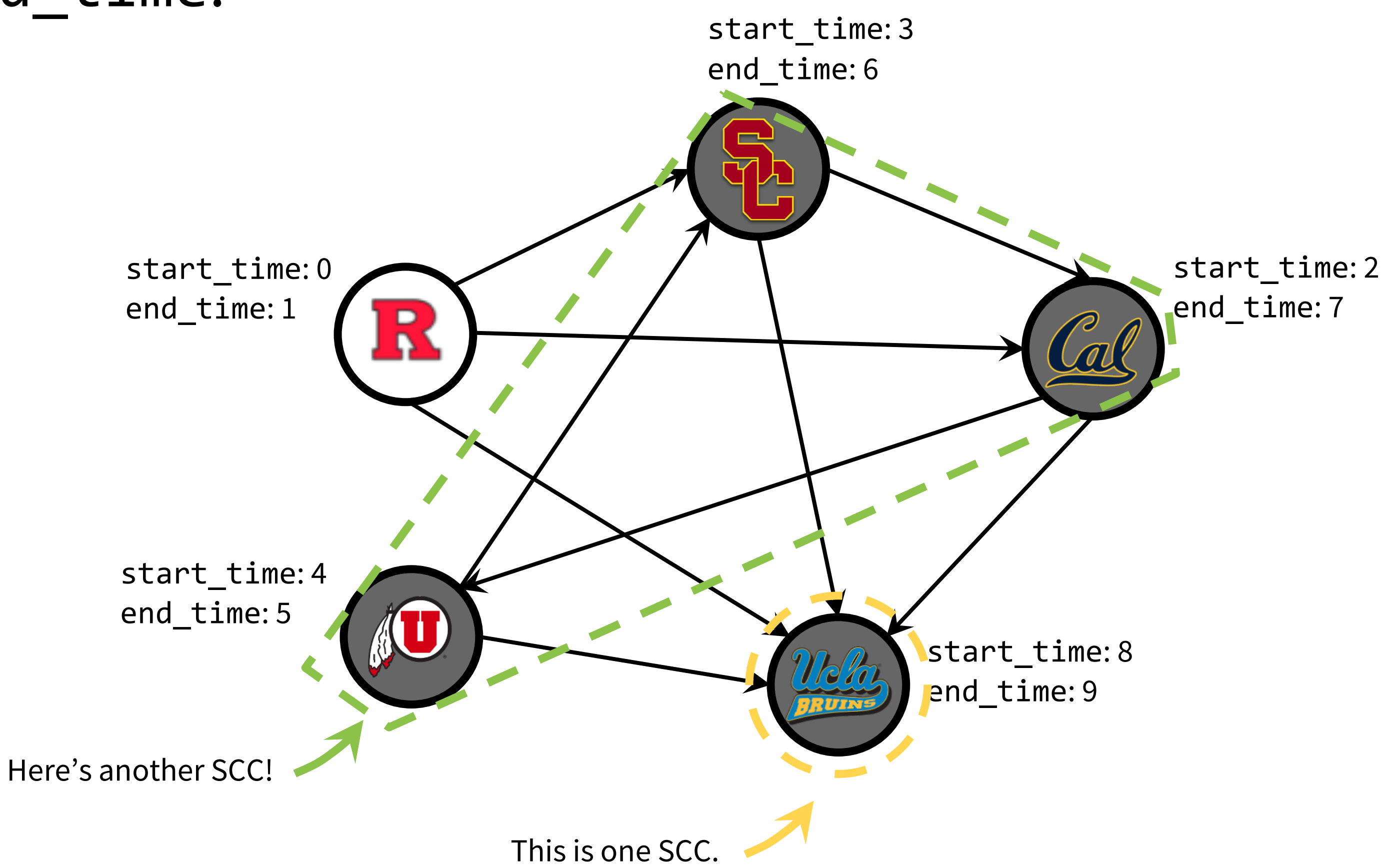

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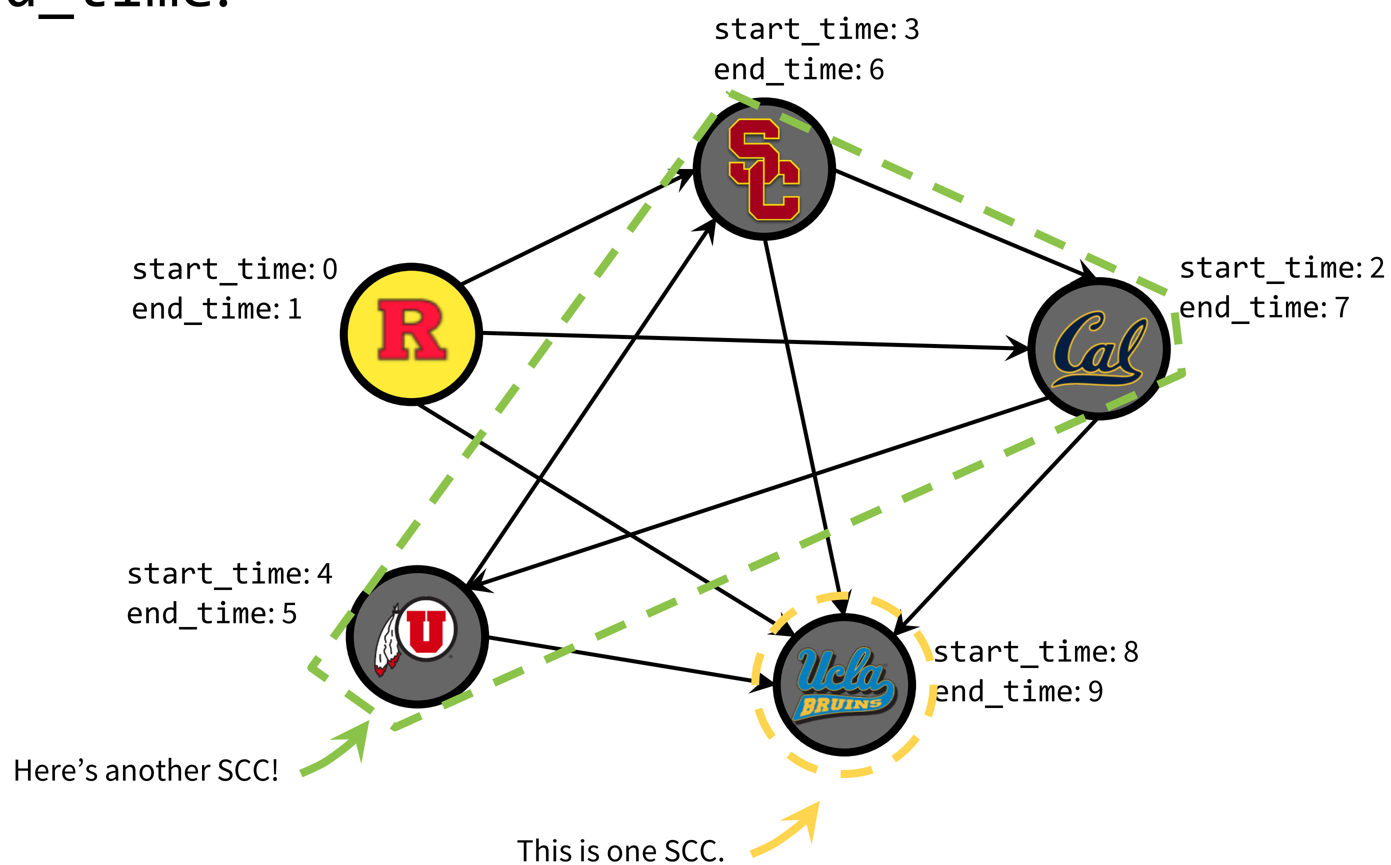

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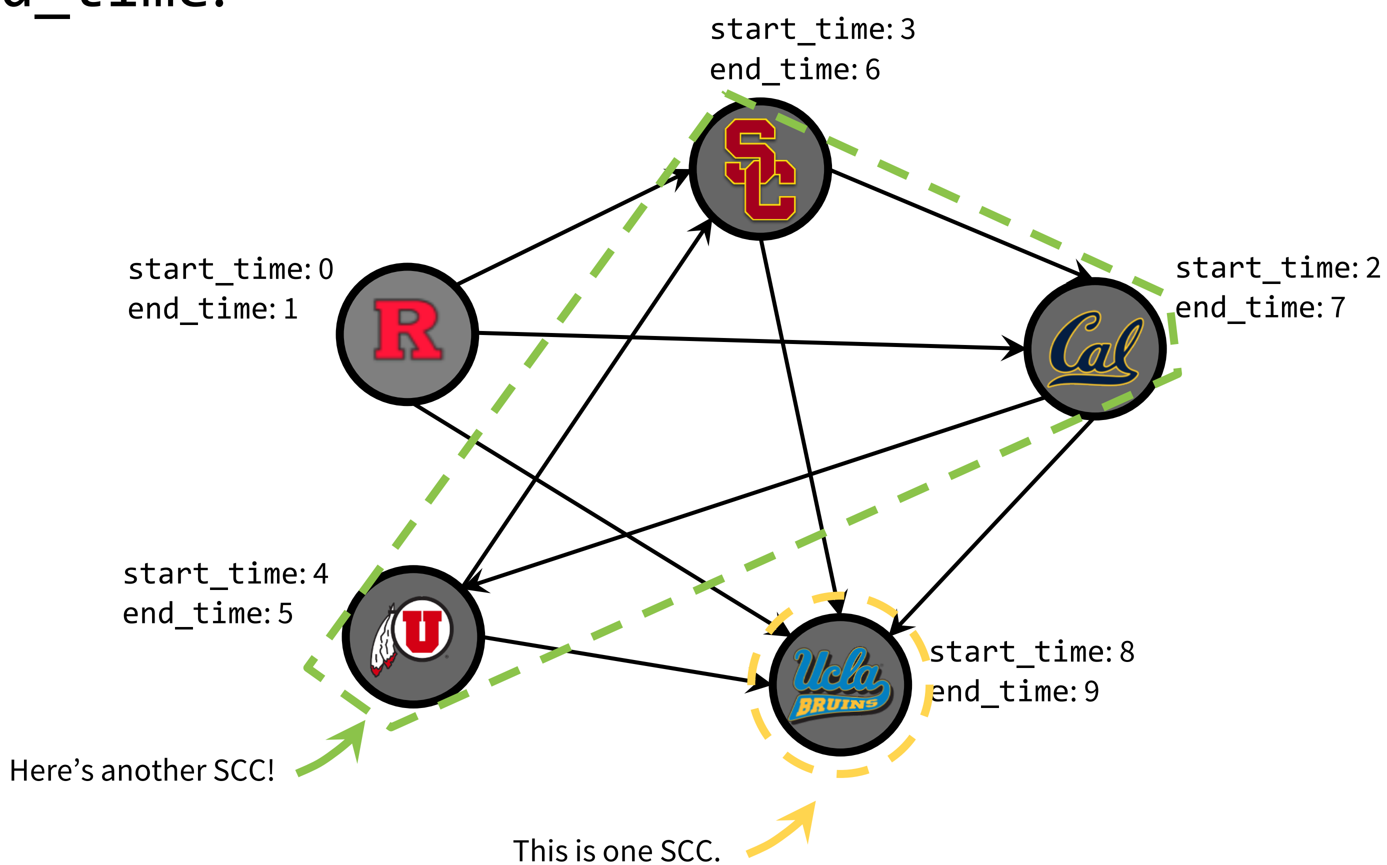

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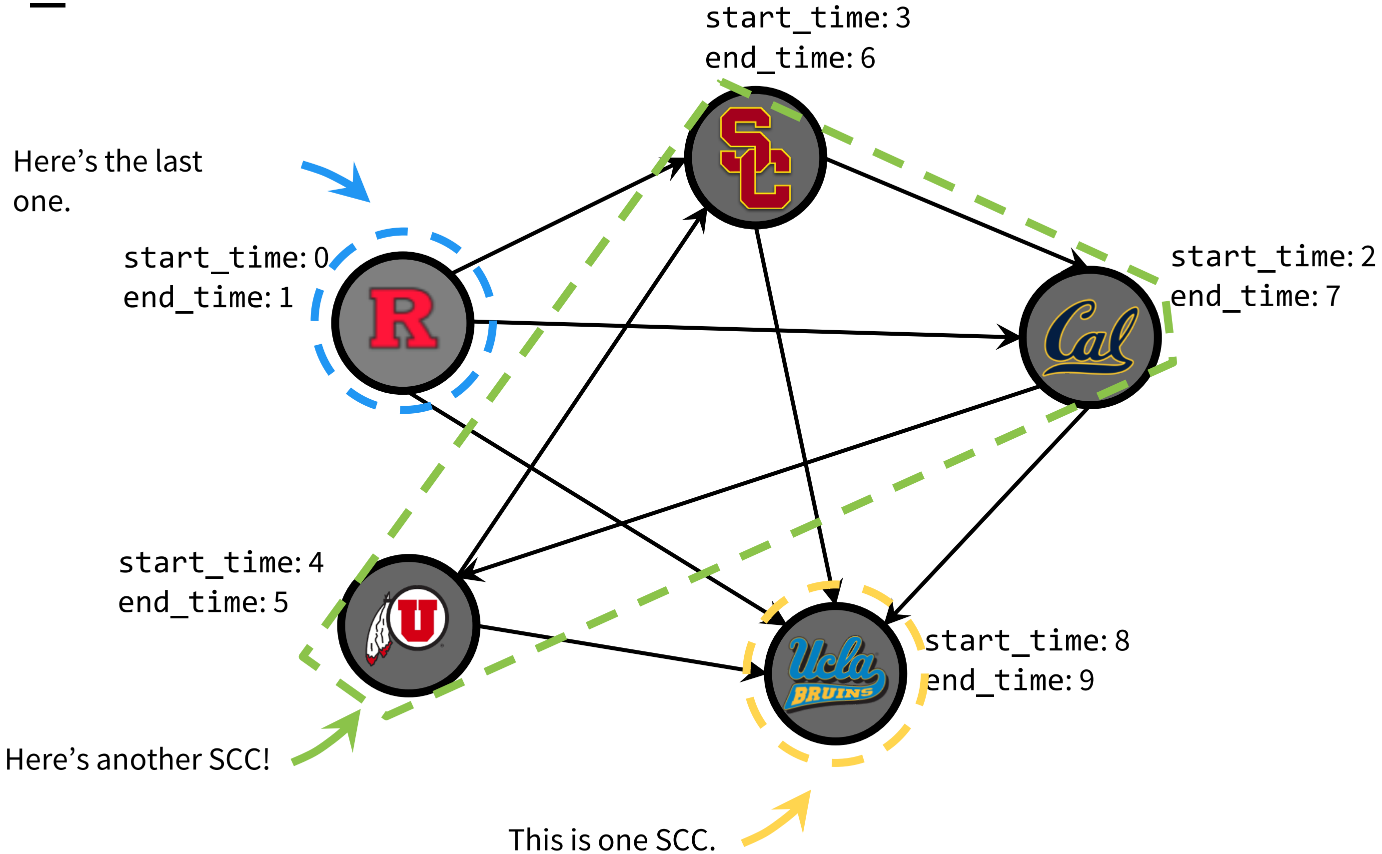


Why Kosaraju's Algorithm Works

Kosaraju’s Algorithm

Whoa. How did that work?

We explain by answering two questions:

1) Why do we use Depth First Search

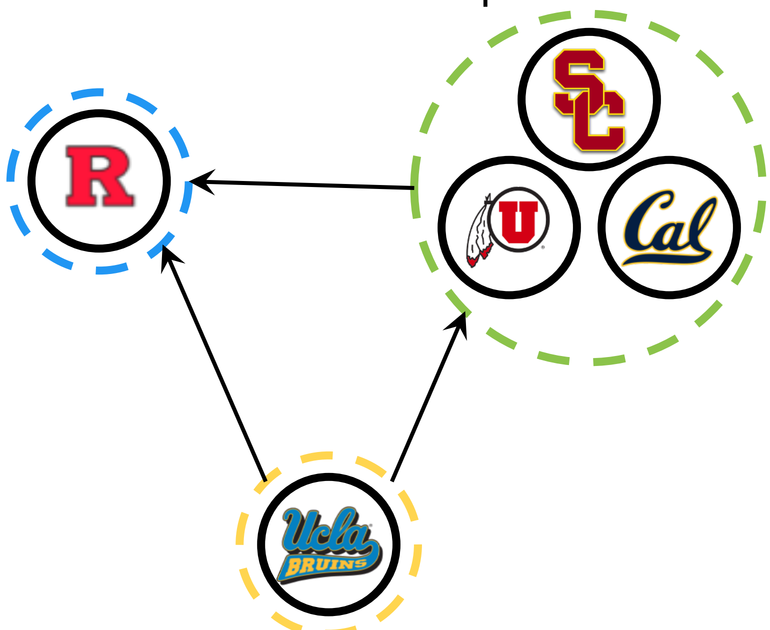
2) In the second DFS, why do we start from vertex with largest end_time in each round.

Kosaraju’s Algorithm

Why do we use Depth First Search

Lemma 1: The SCC metagraph is a directed acyclic graph (DAG).

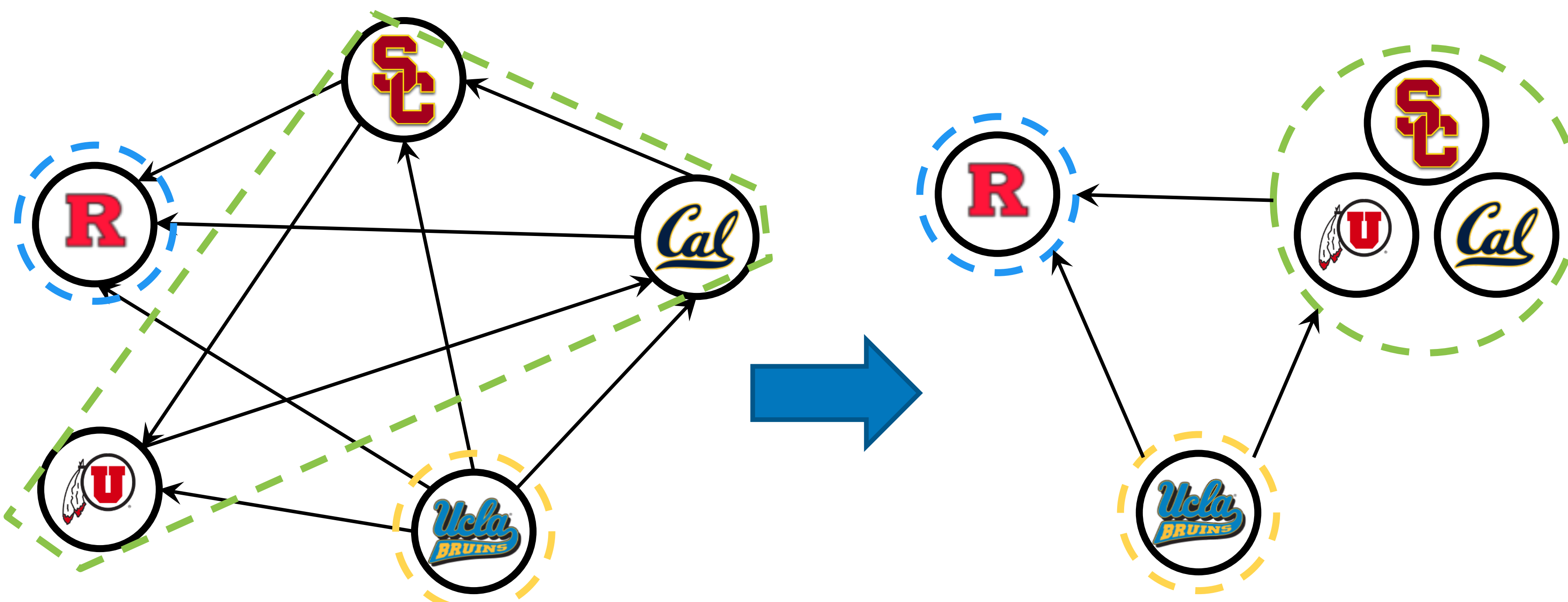
Intuition: If not, then two SCCs would collapse into one.



Kosaraju’s Algorithm

Why do we use Depth First Search

Review: DAG, a directed graph with no directed cycles.
Metagraph: connect a pair of meta vertex as long as there exists an edge between any vertex pair from each meta vertex.



Original Graph SCC MetaGraph

Kosaraju’s Algorithm

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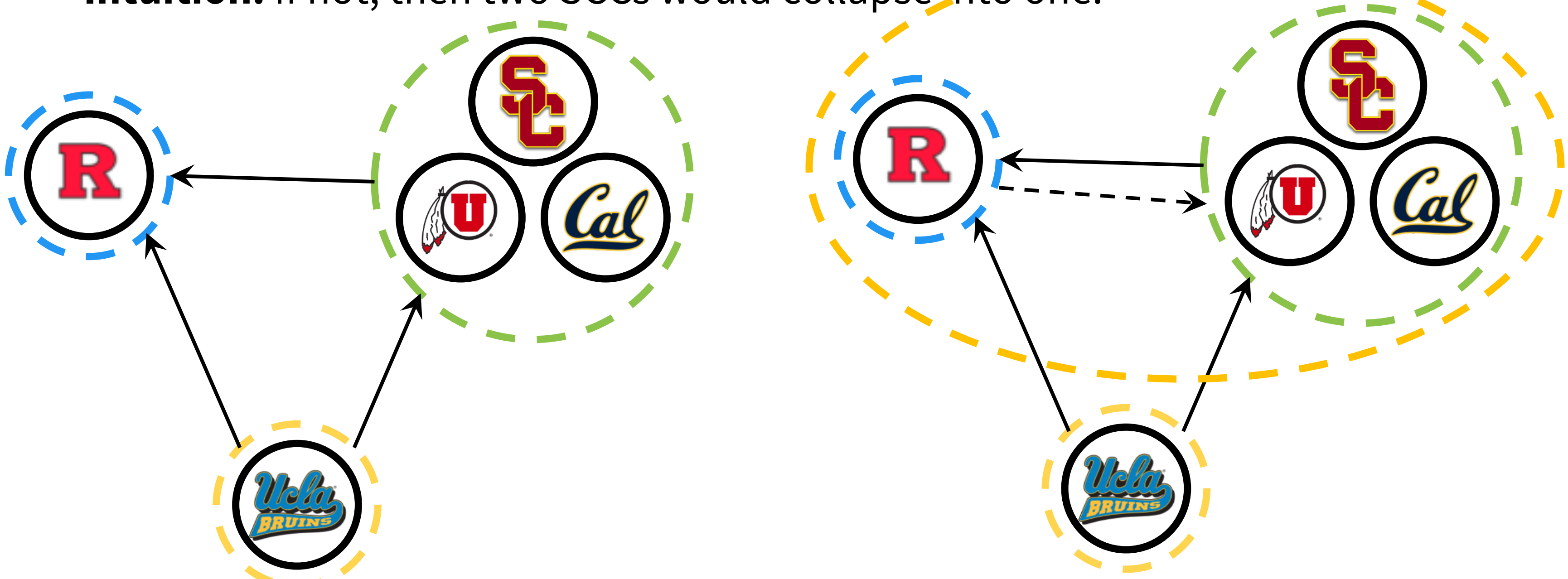


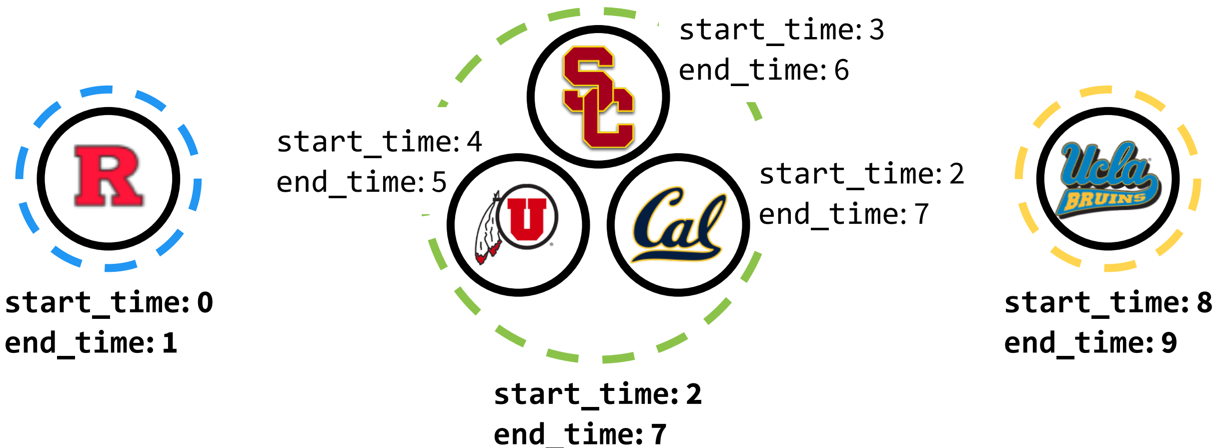
Proof: assume there exist a cycle path between two meta vertex, then they can be merged into a bigger metavertex.

Kosaraju's Algorithm

Why do we use Depth First Search

Let the **end time** of a SCC be the largest end time of any element of that SCC.

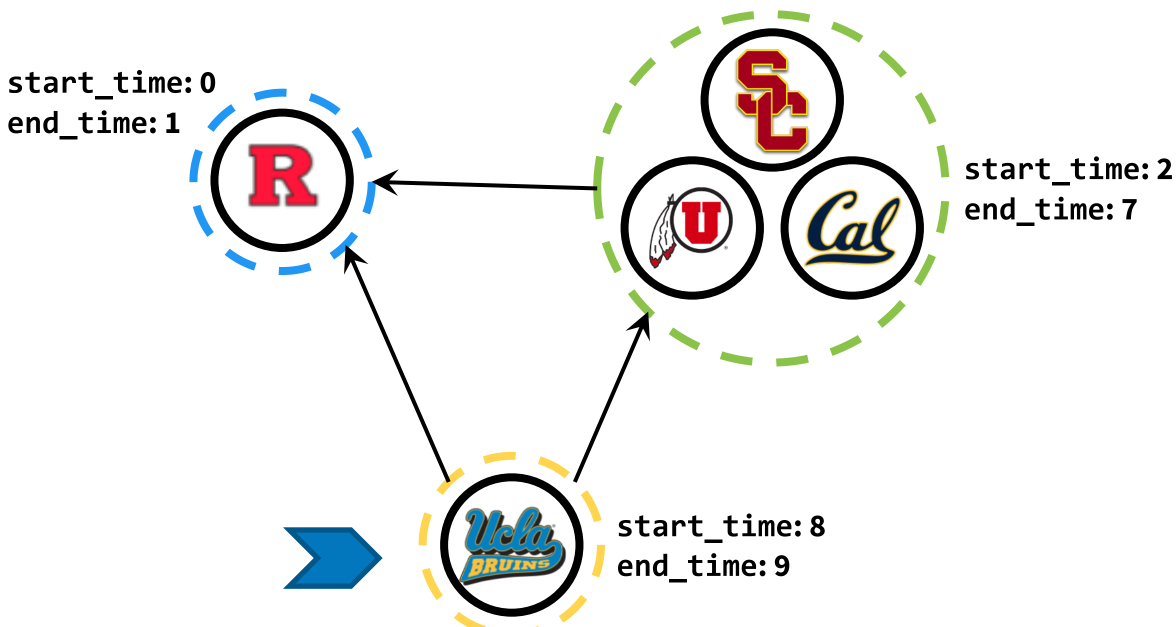
Let the **starting time** of a SCC be the smallest starting time of any element of that SCC.



Kosaraju's Algorithm

Why do we use Depth First Search

The main idea leverages the fact that vertex in the **SCC metagraph** with the largest `end_time` has no incoming edges.



Kosaraju's Algorithm

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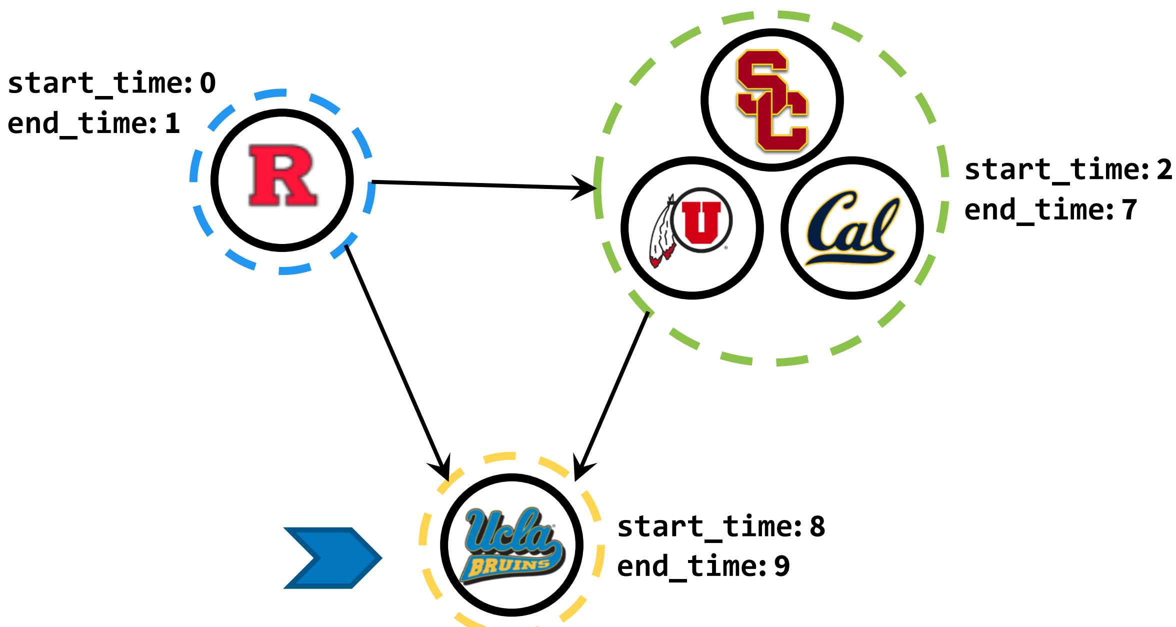

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Why do we use Depth First Search

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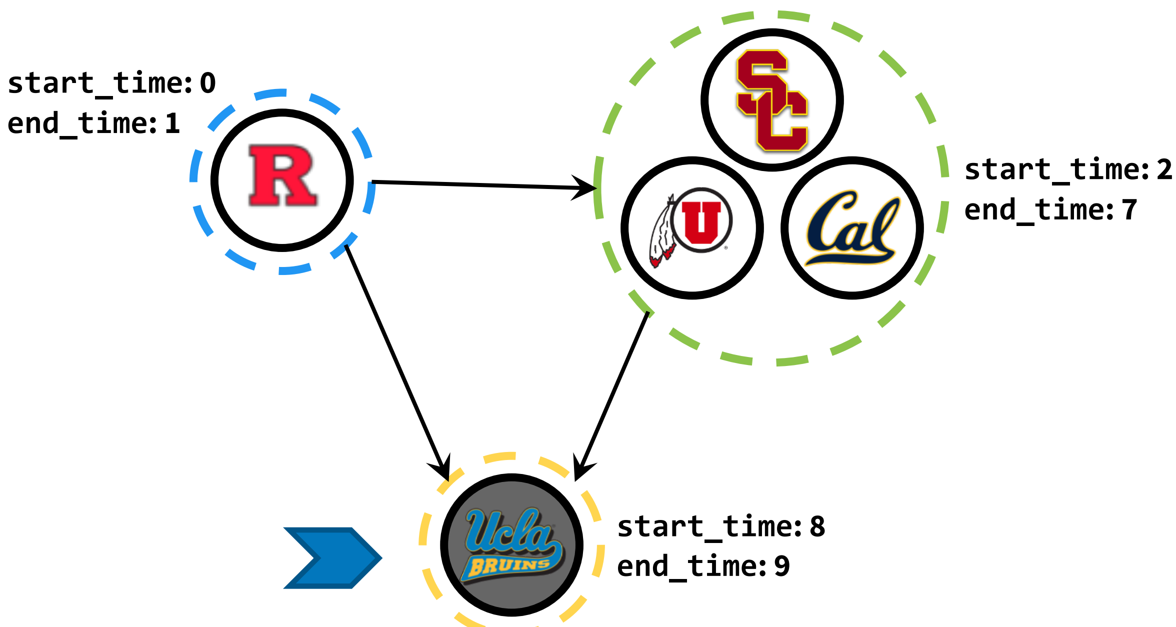
Running `dfs` on that vertex finds exactly that component.



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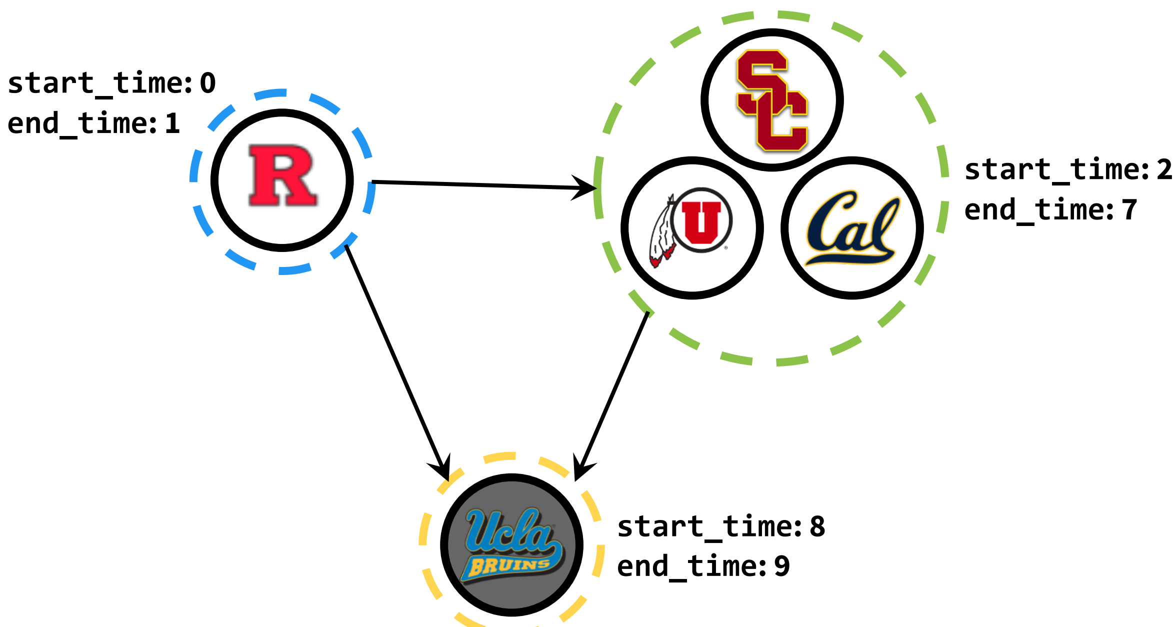
Same argument for the rest.



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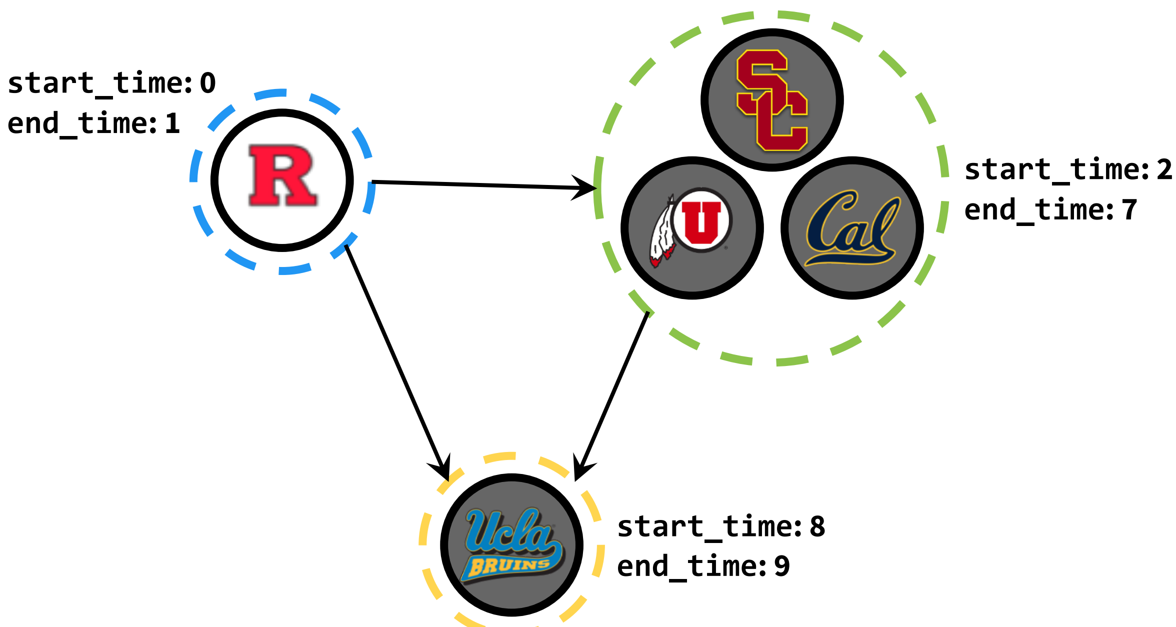

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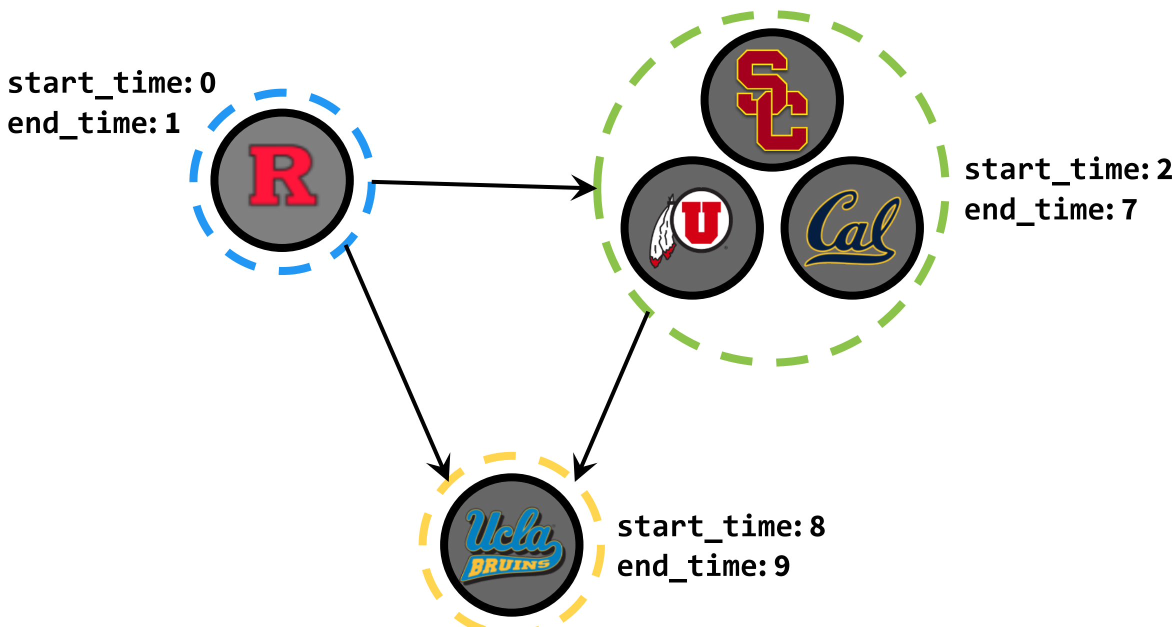
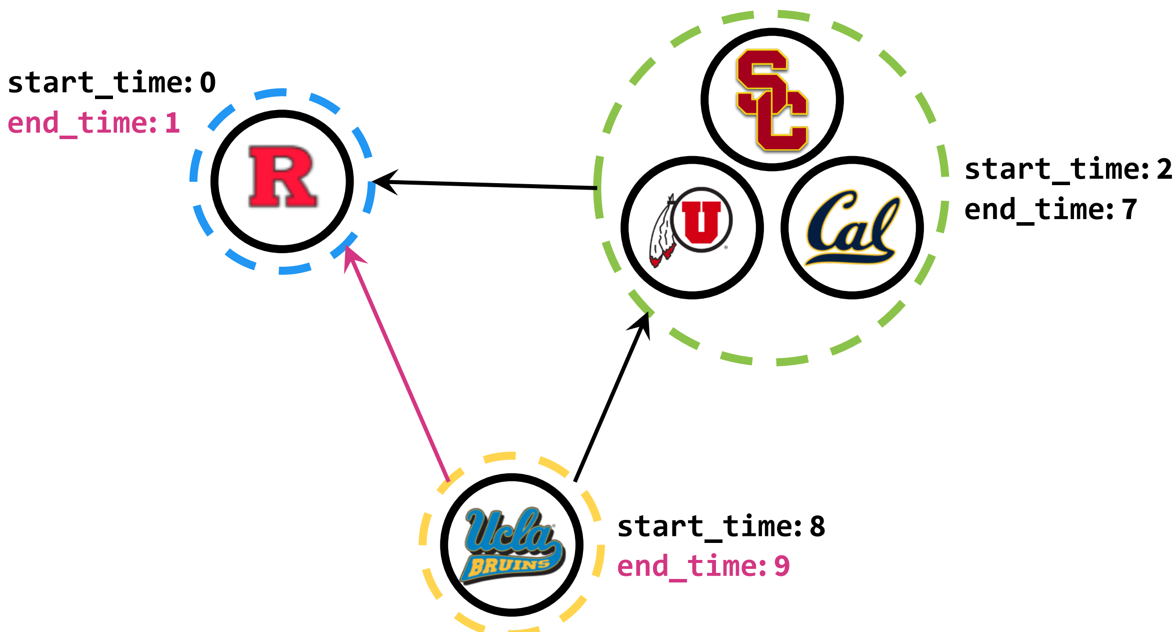


Kosaraju's Algorithm

Why do we use start with the largest end_time vertex

Claim: For each edge (u, v) in the SCC metagraph where $u \in C_1$ and $v \in C_2$, `end_time` of C_1 must be larger than `end_time` of C_2.



In this way, the second DFS will converge with an SCC before entering another SCC (because all edges are inversed in the second DFS).

Kosaraju's Algorithm

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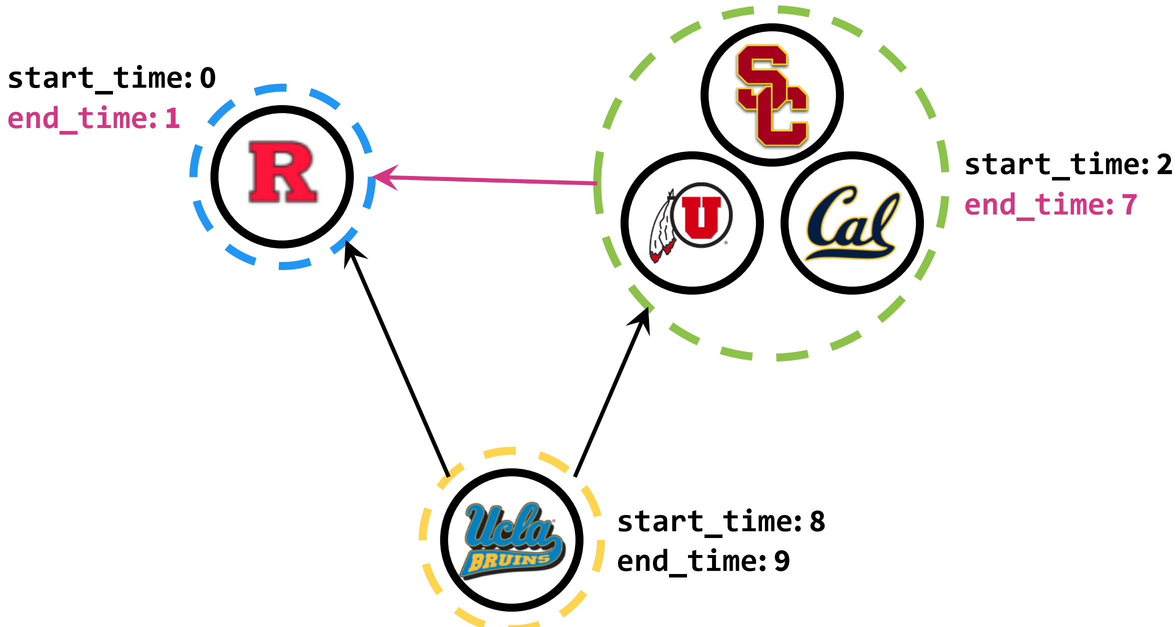

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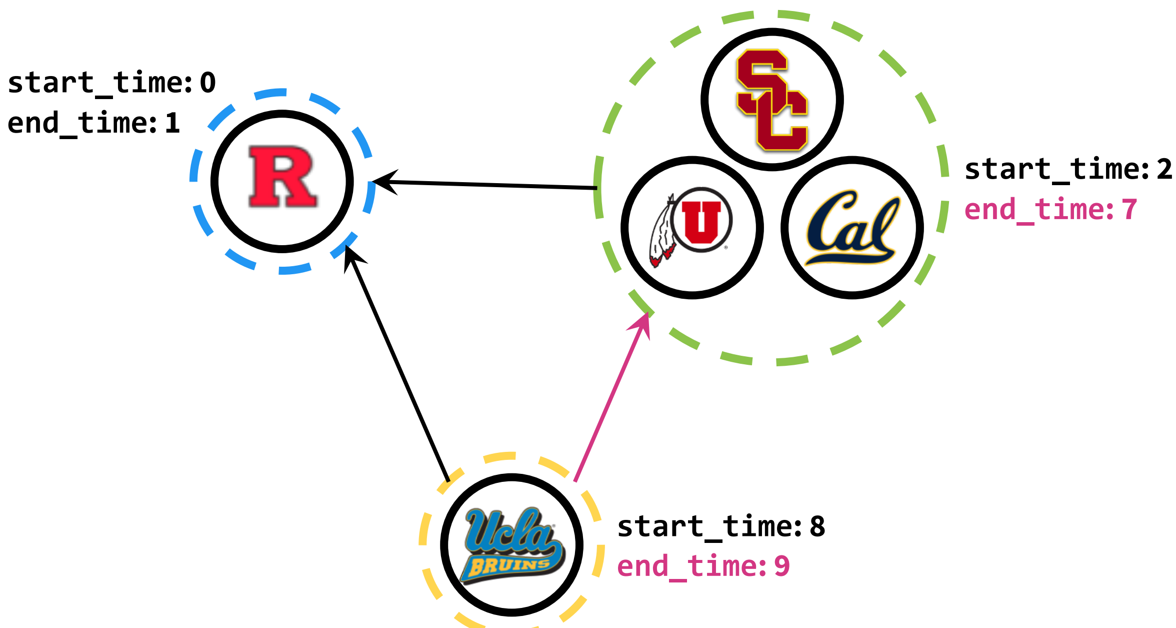

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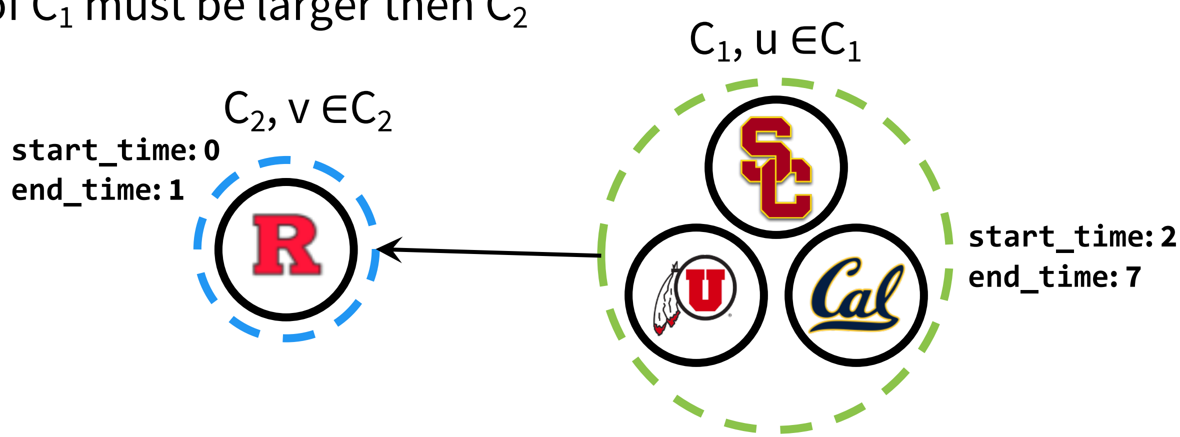
Kosaraju's Algorithm

Why do we use start with the largest end_time vertex

Claim: For each edge (u, v) in the SCC metagraph where u ∊ C_1 and v ∊ C_2, `end_time` of C_1 must be larger than `end_time` of C_2.

Intuition: Suppose the first DFS started with C_2, it will only finish C_2 before entering C_1 (because there can be no edge from C_2 to C_1, or else C_2 and C_1 can be merged as a single SCC), so the `end_time` of C_1 must be larger then C_2.

Suppose the first DFS started with C_1, because of the existence of the edge (u,v), the algorithm will enter C_2 from u and reach v at some time during DFS; because C_2 is an SCC, all nodes in C_2 are reachable from v, so the DFS will visit all node in C_2; because there is no edge from C_2 to C_1 (same as above), so the DFS will only finish all nodes in C_2 and then trace back to C_1. As a result, the `end_time` of C_1 must be larger then C_2



Kosaraju's Algorithm

The runtime of Kosaraju's algorithm is **O(|V|+|E|)**.

- Runtime for the first DFS is **O(|V|+|E|)**.
- Runtime for reversing the graph is **O(|V|+|E|)**.
- Runtime for the second DFS is **O(|V|+|E|)**.

We can find connected components and SCCs in the same (asymptotic) runtime!

Karger’s Algorithm

Cuts

A **cut** is a partition of the vertices into two nonempty parts.

Cuts

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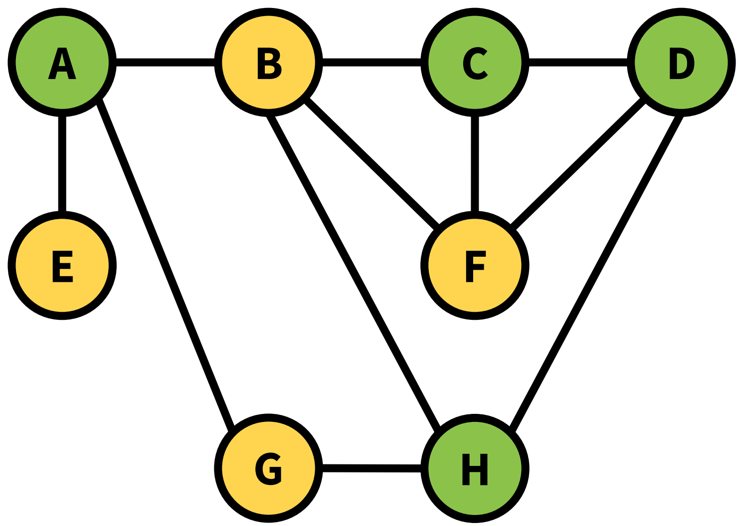
e.g. This is the cut "{A, C, D, H} and {B, E, F, G}".



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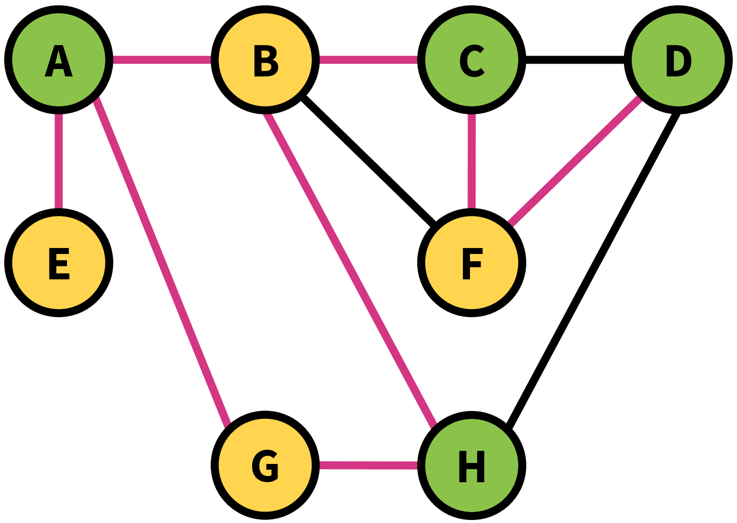


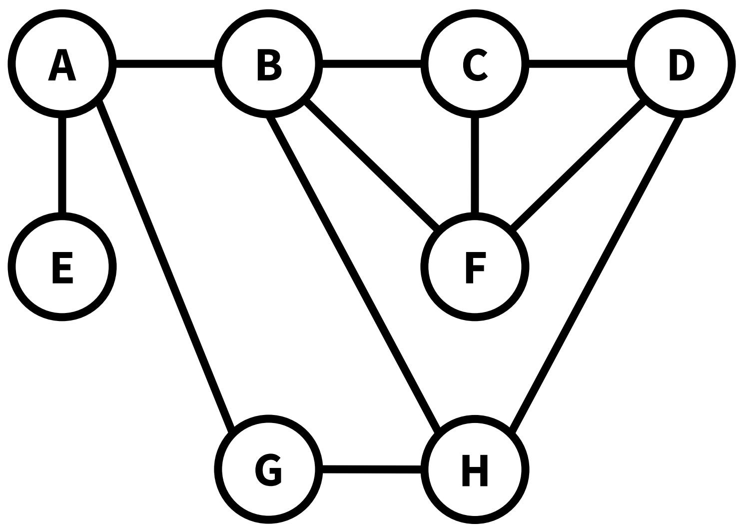
Edges that **cross the cut** go from one part to the other.

e.g. These edges cross the cut.

Explain on board: divide the vertices into two parts

Cuts

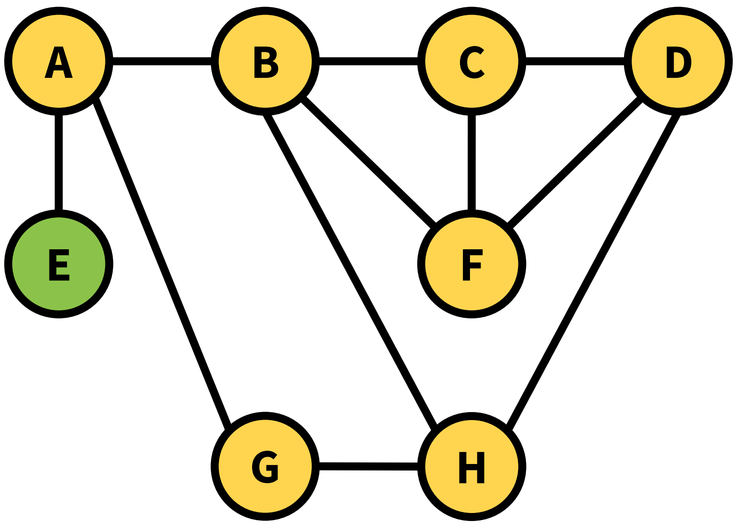
A **global minimum cut** is a cut that has the fewest edges possible crossing it.



Cuts

A **global minimum cut** is a cut that has the fewest edges possible crossing it.

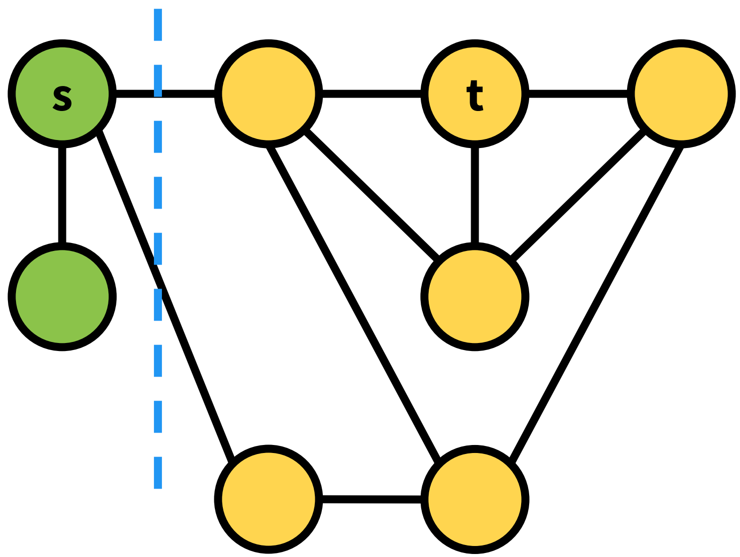
e.g. The global minimum cut is “{A, B, C, D, F, G, H} and {E}”.



Cuts

Later this semester, we'll talk about **minimum s-t cuts**, which separate specific vertices **s** and **t**.

e.g. The minimum s-t cut is this cut.



Karger's Algorithm

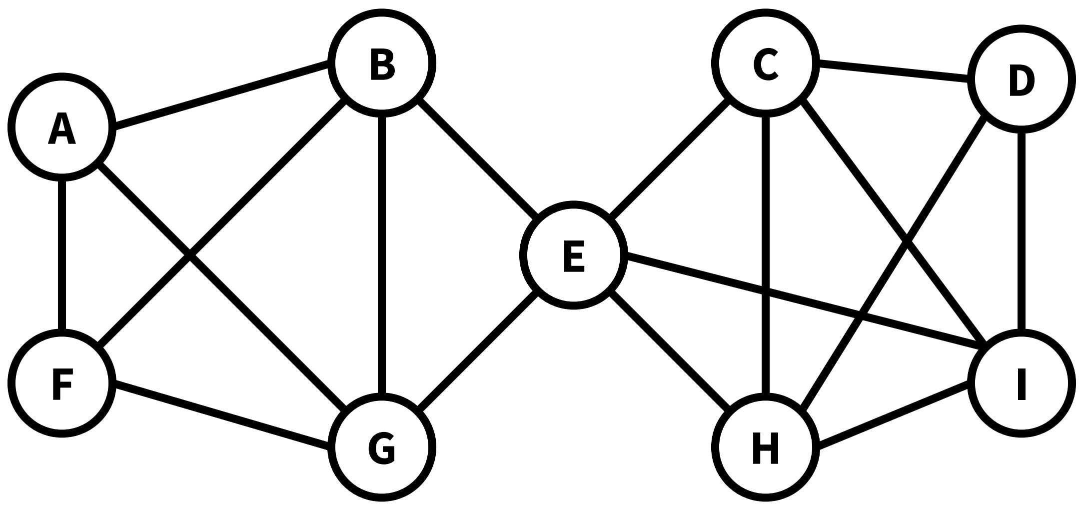
Karger's Algorithm finds global minimum cuts.

- It's a Monte Carlo randomized algorithm! Unlike `quicksort`, which is always correct but sometimes slow, Karger's algorithm is always fast but sometimes Incorrect.
  - For all inputs A, `quicksort` returns a sorted list. For all inputs A, with high probability over the choice of pivots, `quicksort` runs fast.
  - For all inputs G, `karger` runs fast. For all inputs G, with high probability over the randomness in the algorithm, `karger` returns a minimum cut.

Karger’s Algorithm

The general idea is to pick random edges to “contract” until there are a minimal number of vertices and edges left.

Karger's Algorithm



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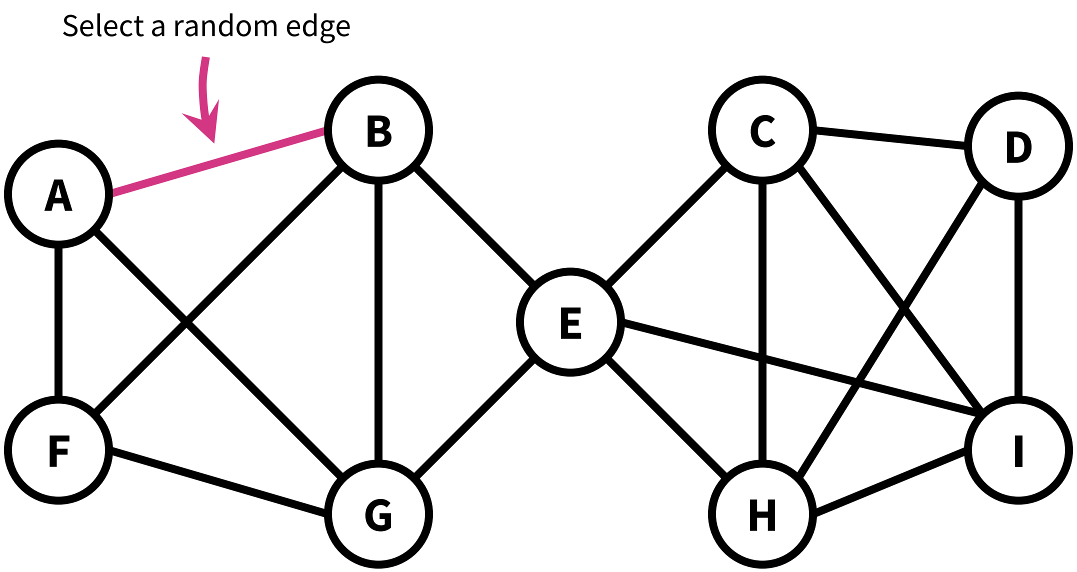

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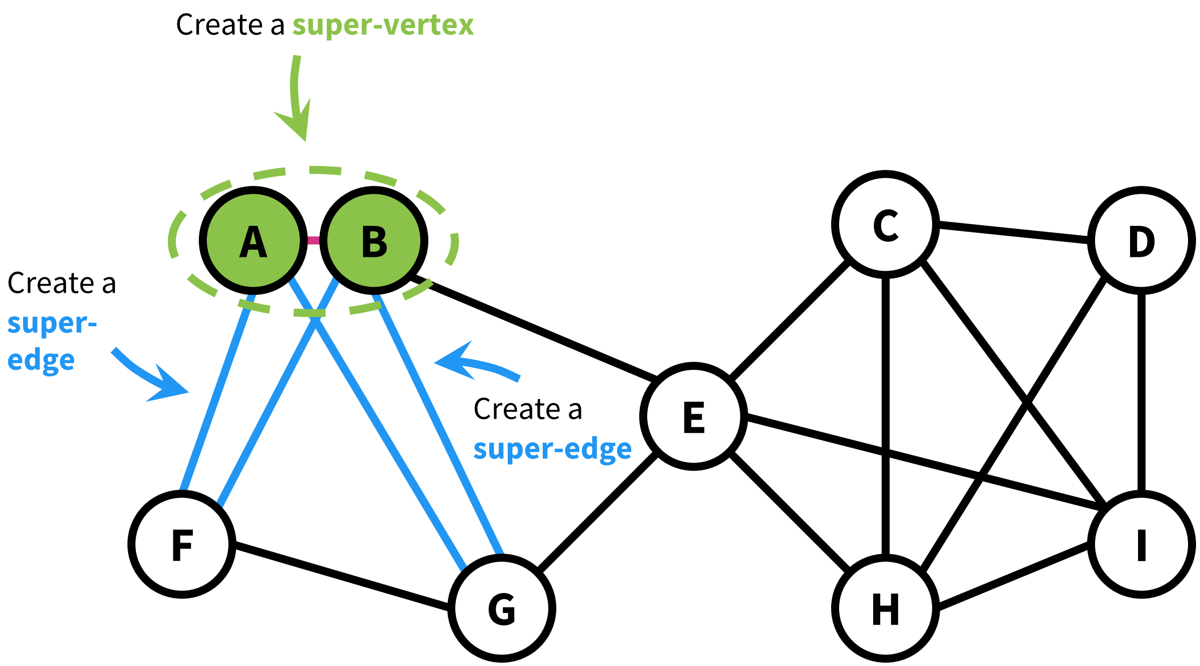

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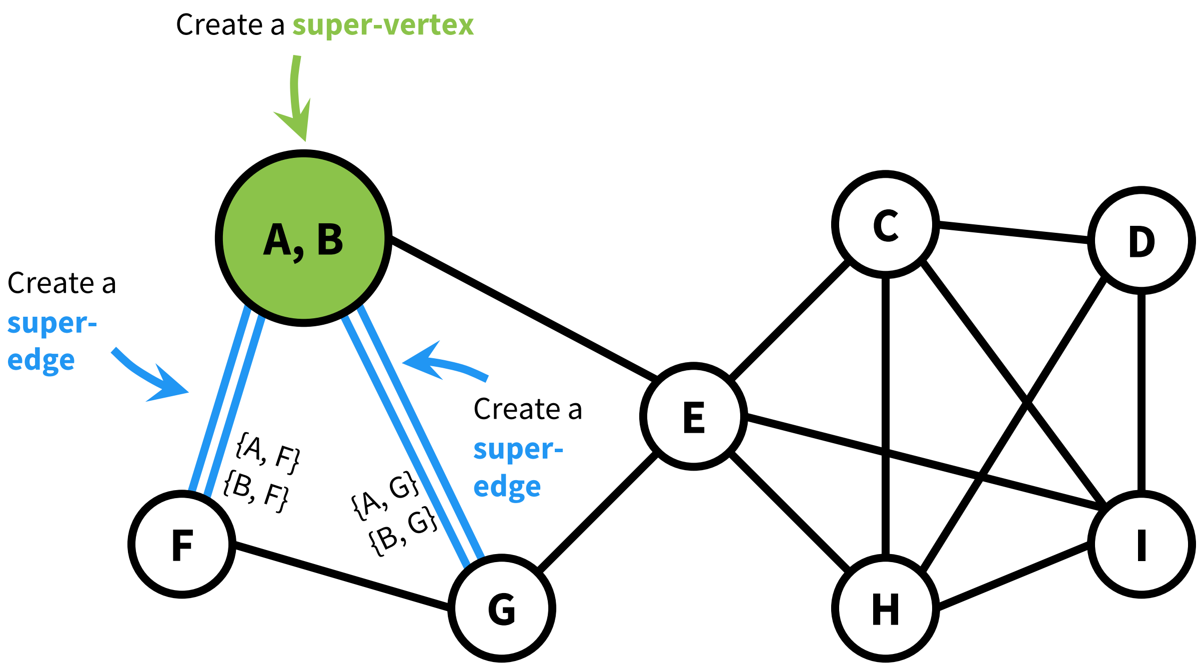

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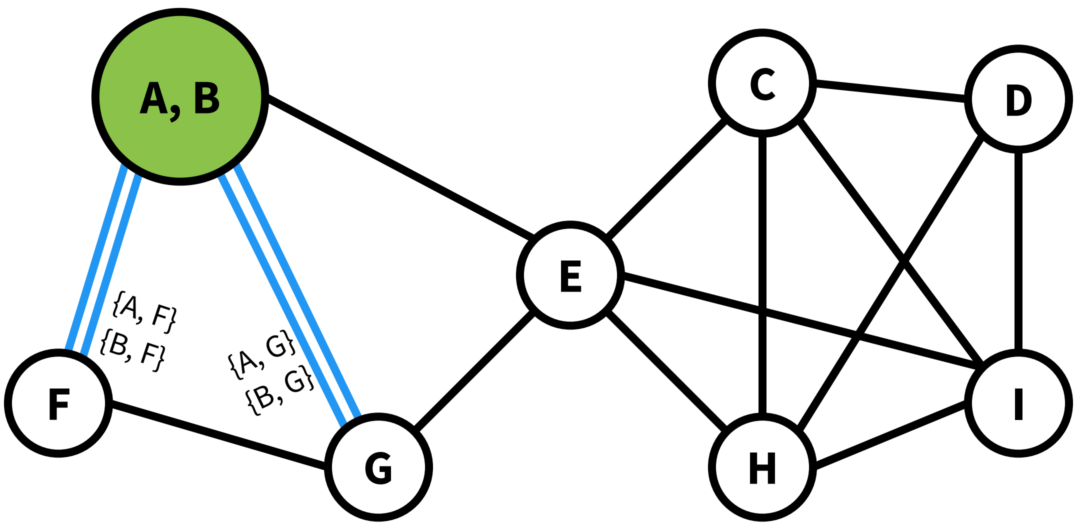

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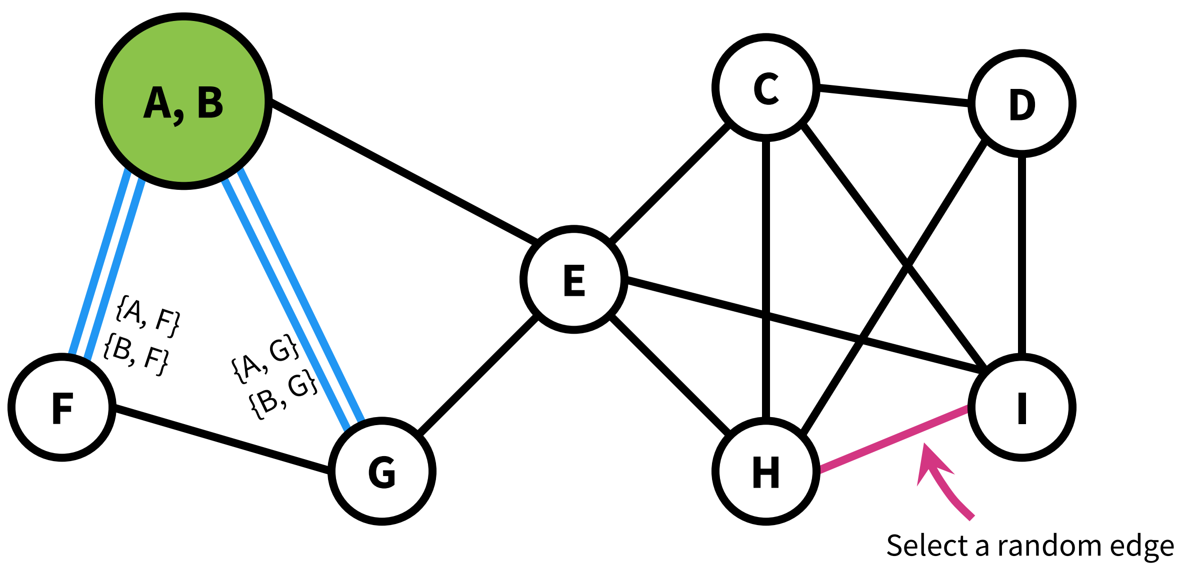

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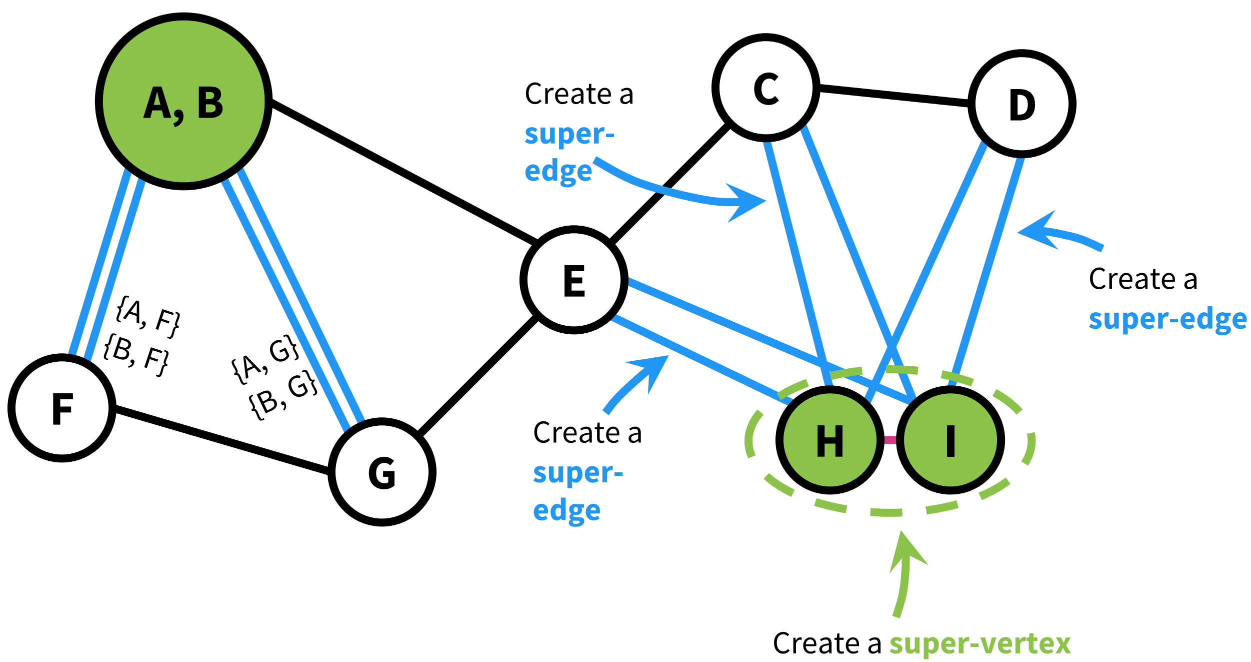

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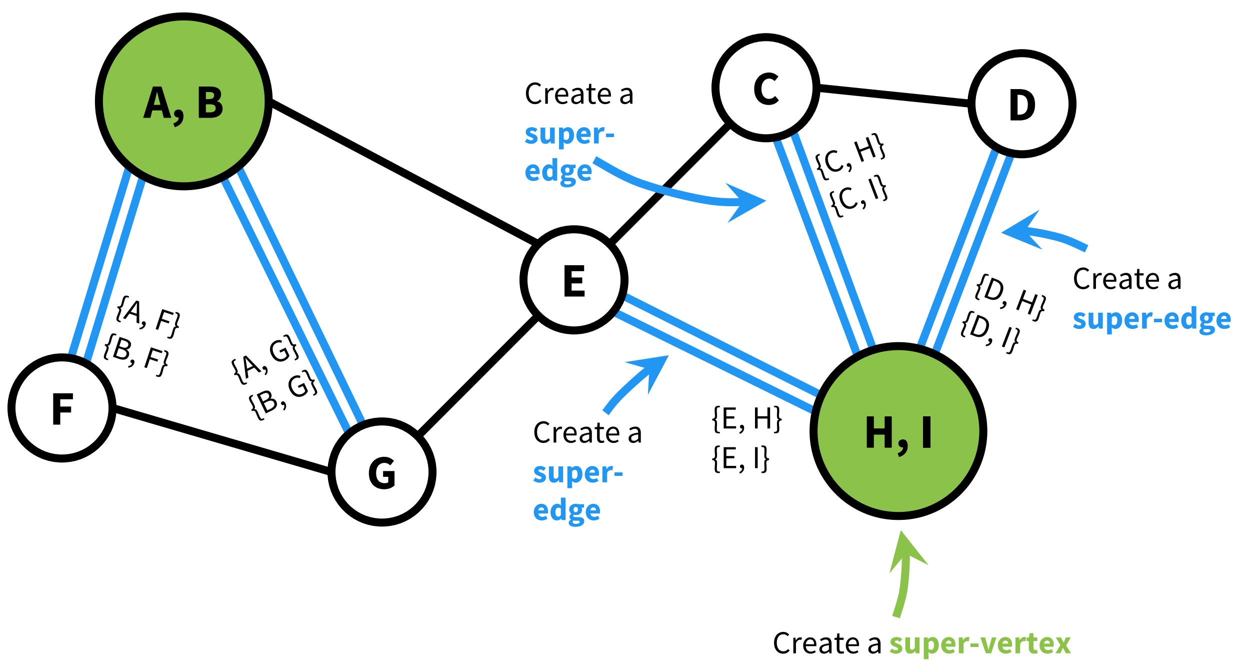

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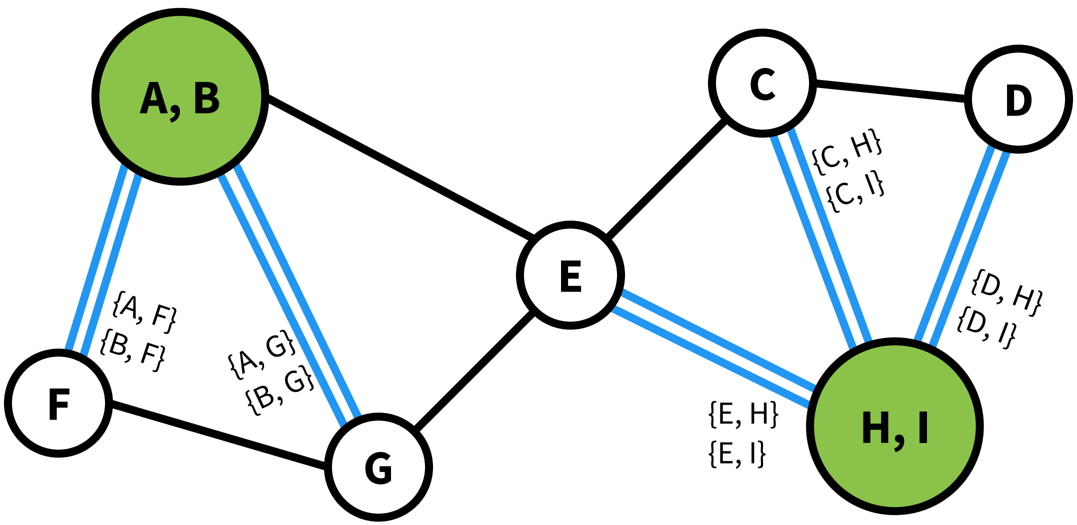

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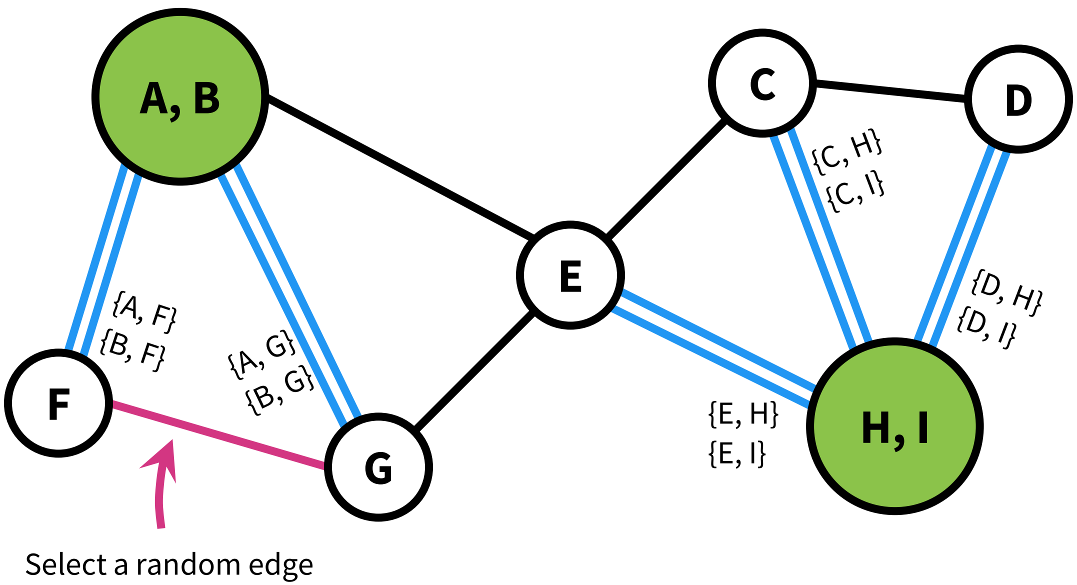

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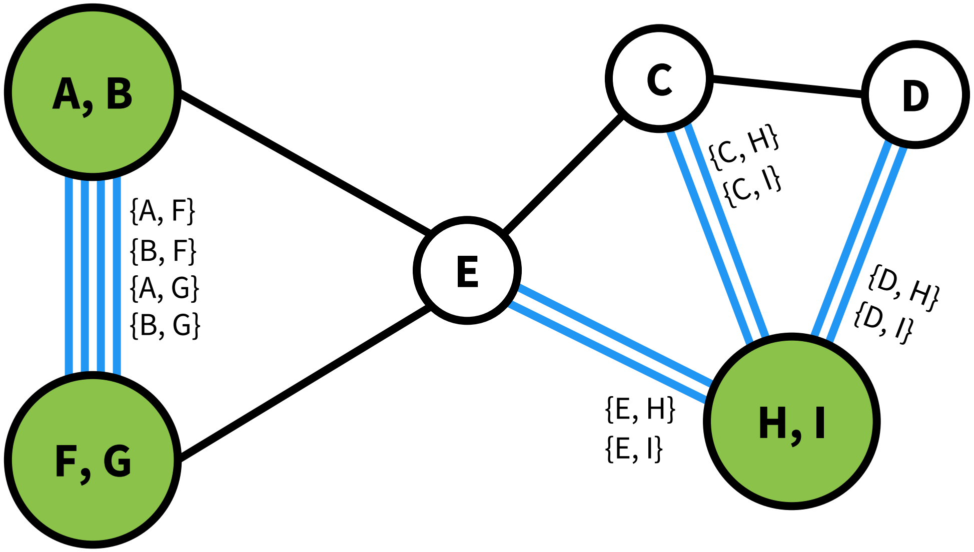

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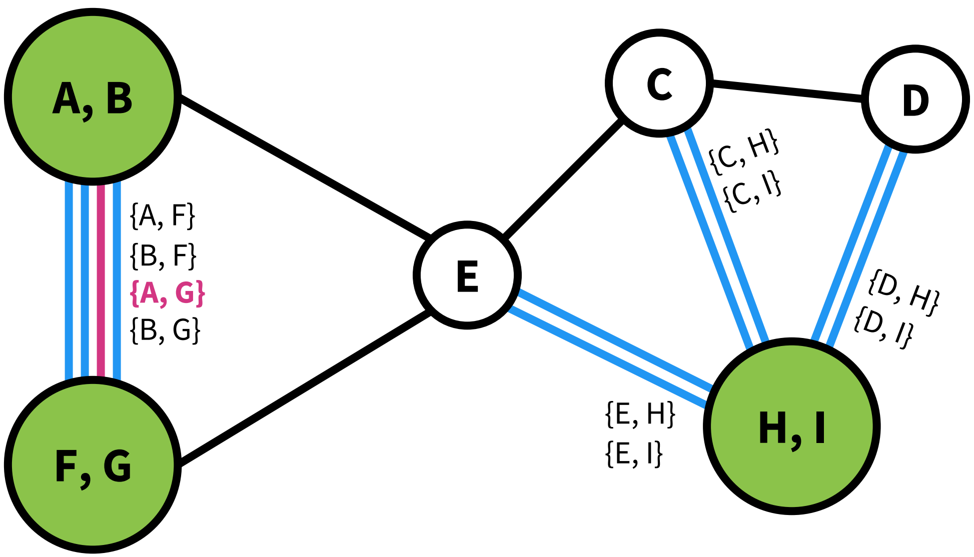

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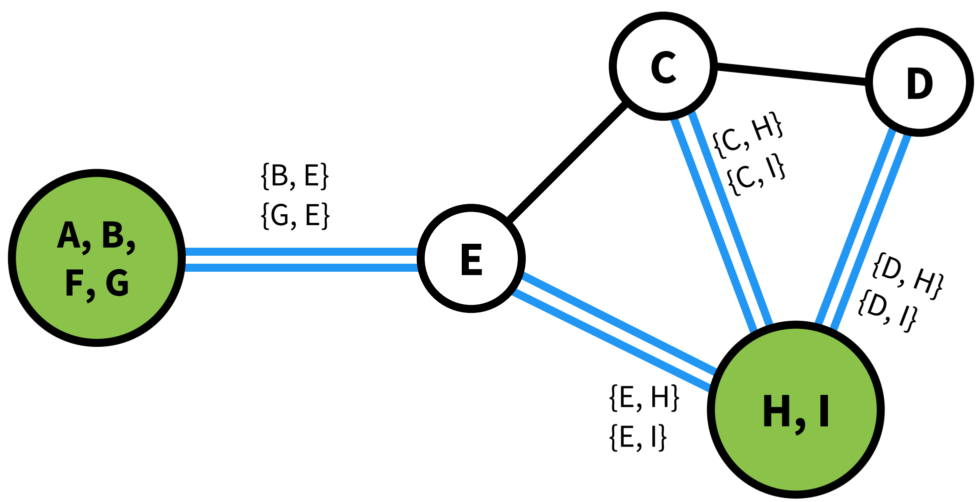

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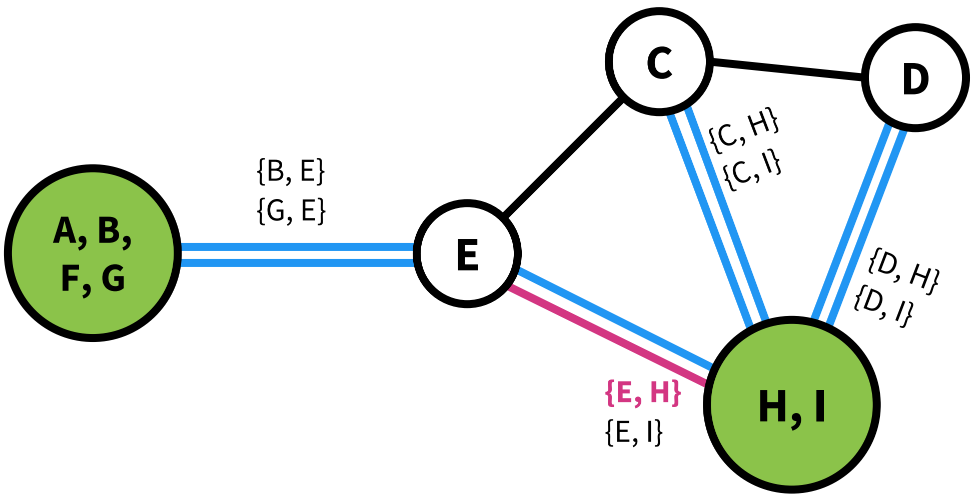
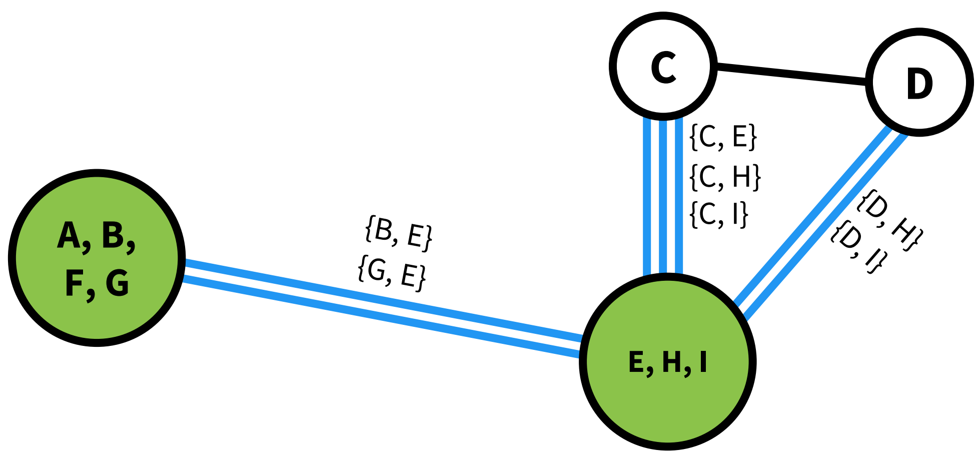


Karger’s Algorithm



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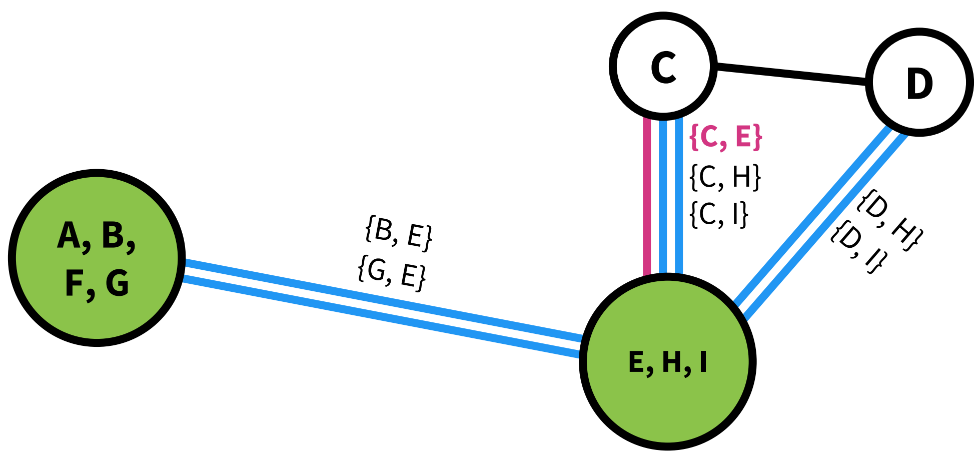

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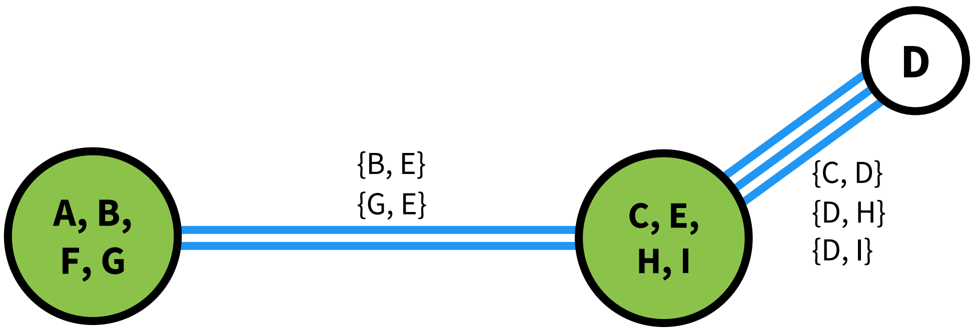
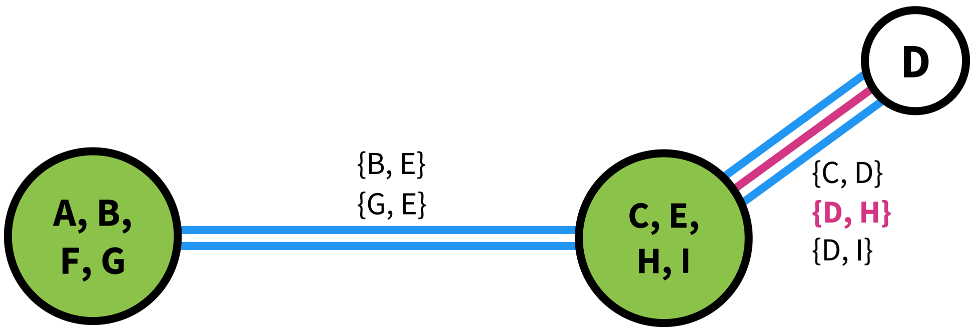


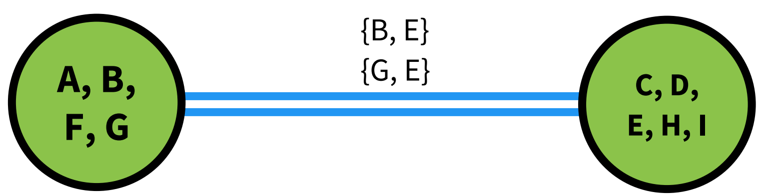
Karger’s Algorithm



Karger's Algorithm

The minimum cut is given by the remaining super-vertices.

e.g. The cut is "{A, B, F, G} and {C, D, E, H, I}"; the edges that cross this cut are {B, E} and {G, E}.



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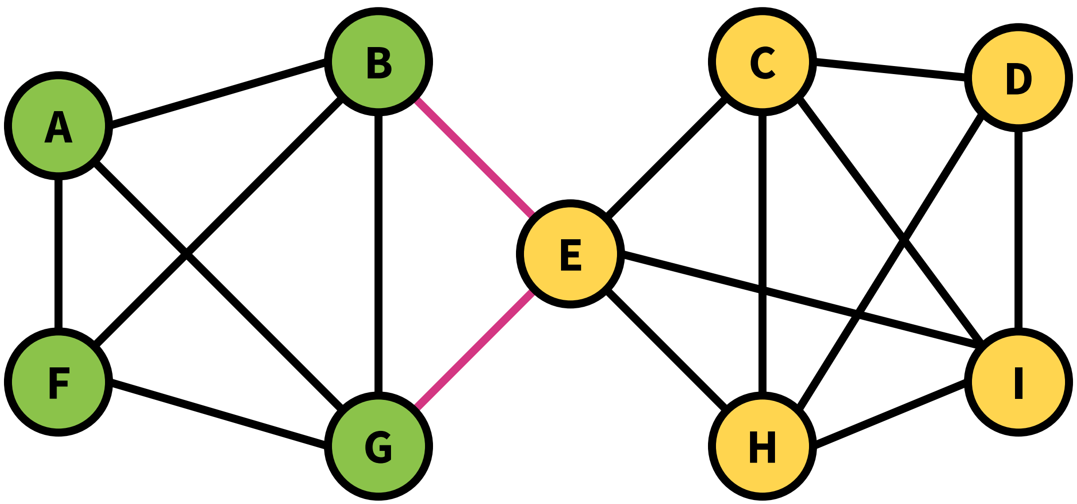


Karger's Algorithm

```
algorithm karger(G=(V,E)):
  G' = {supervertex(v) for v in V}
  Eu'v' = {(u,v)} for (u,v) in E
  Eu'v' = {} for (u,v) not in E
  F = {{(u,v)} for (u,v) in E}
  while |G'| >= 2:
    {(u,v)} = uniform random edge in F
    merge_supervertices(u, v)
    F = F \ Eu'v'
  return cut of the remaining super-vertices

algorithm merge_supervertices(u, v):
  x' = supervertex(u' ∪ v')
  for w' in G' \ {u',v'}:
    Ex'w' = Eu'w' ∪ Ev'w'
    Remove u' and v' from G' and add x'
```

These are the super-edges.

The while loop executes |V| - 2 times.

Removes all edges in the super-edge between super-vertices u' and v'.

This takes O(|V|).

u' is the super-vertex containing u; v' is the super-vertex containing v.

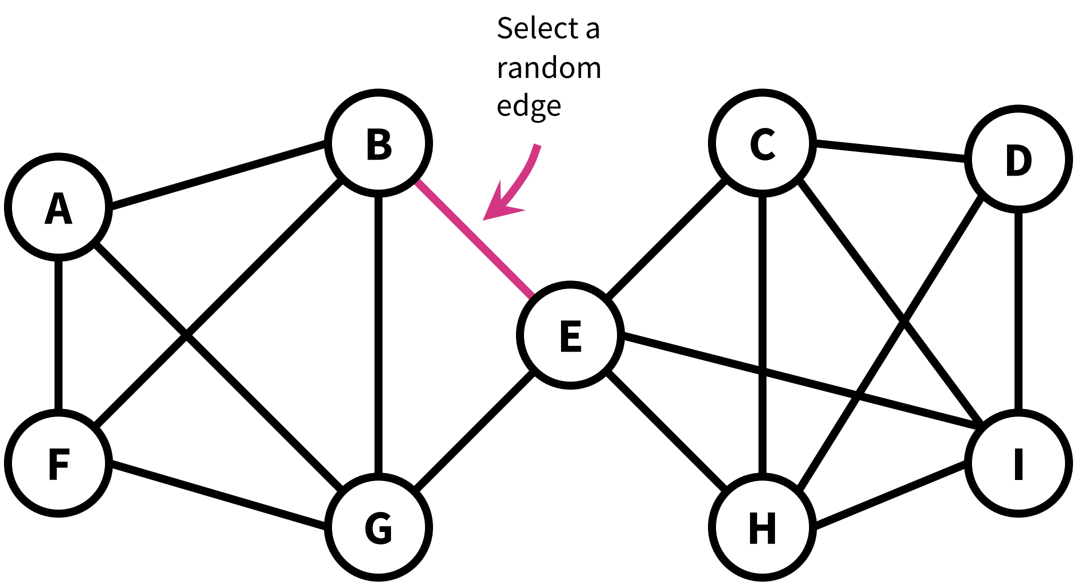
Runtime: $O(|V|^2)$

We can do better with fancy data structures, but this is fine for now.

Karger's Algorithm

We got really lucky!

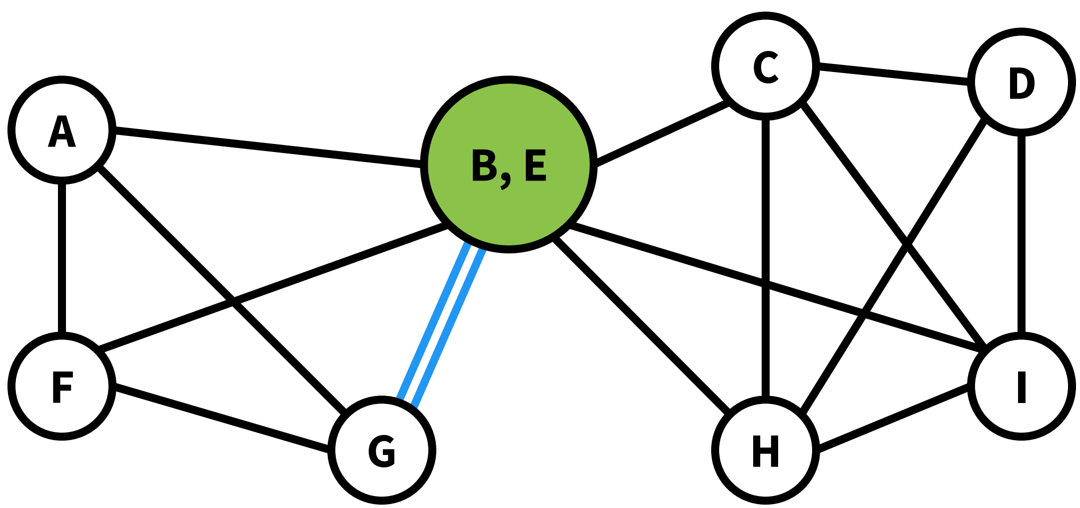
e.g. Suppose we had chosen this edge.



Karger’s Algorithm

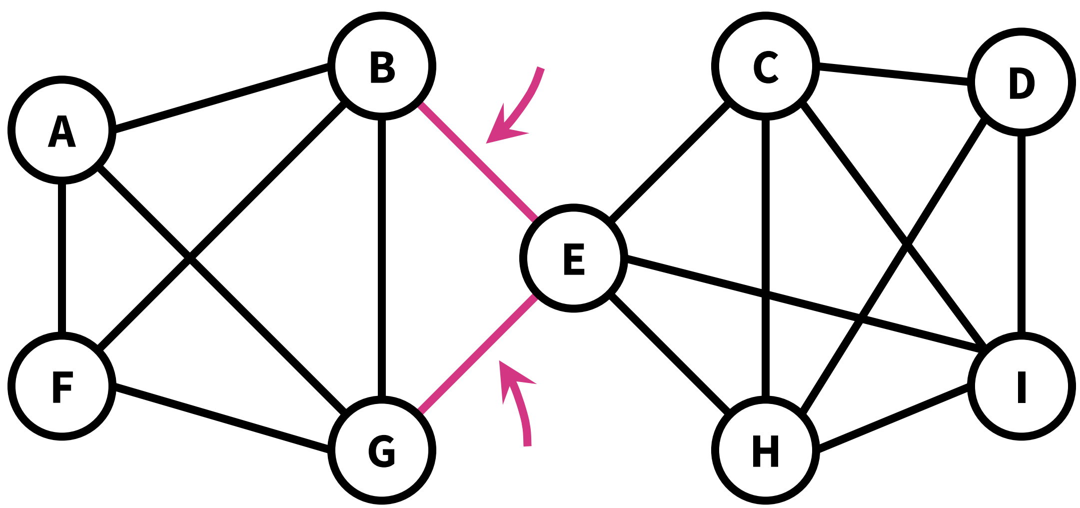
We got really lucky!

e.g. Suppose we had chosen this edge. Now there’s no way to return a cut that separates B and E.



Karger's Algorithm

If fact, if Karger's algorithm ever randomly selects **edges in the min-cut**, then it will be incorrect.



Karger's Algorithm

The probability that Karger's algorithm returns a minimum cut is …

$$\geq 1 / \binom{n}{2}$$

where n is the number of nodes.

Proof:

Suppose S^* is a min-cut and suppose we select edges $e_1, e_2, \ldots, e_{n-2}$.

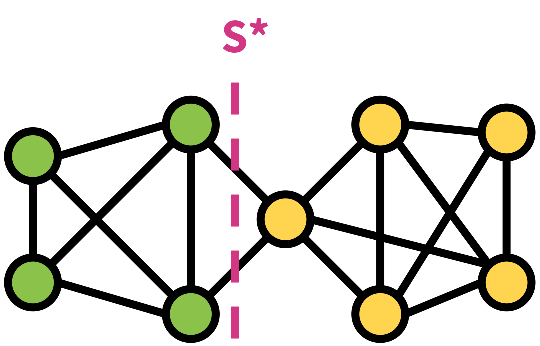
Then P(karger returns S^*) = P(no e_i crosses S^*)

= P(e_1 doesn't cross S^*)

$\times$ P(e_2 doesn't cross S^* | e_1 doesn't cross S^*)

…

$\times$ P(e_{n-2} doesn't cross S^* | $e_1, \ldots, e_{n-3}$ doesn't cross S^*)



Karger's Algorithm

The probability that Karger's algorithm returns a minimum cut is …

$$\geq 1 / \binom{n}{2}$$

Proof, cont.:

Suppose, after j-1 iterations, `karger` hasn't messed up yet! What's the probability of messing up now?

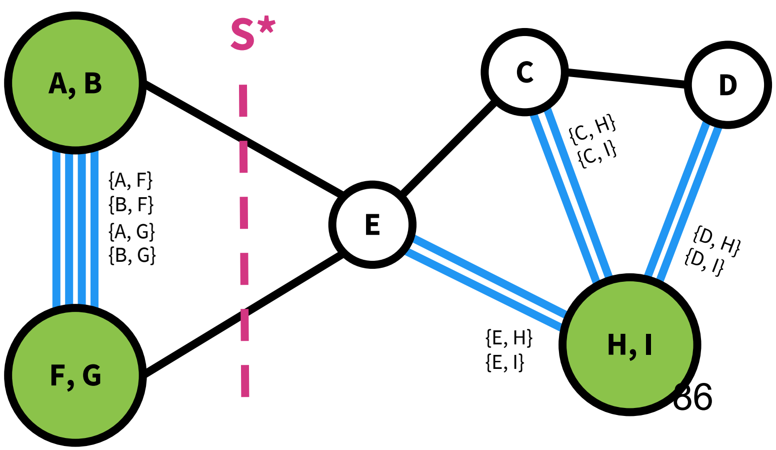
Suppose there are k edges that cross **S***.

All remaining vertices must have degree at least k (otherwise there would be a smaller cut).

So there are at least **(n-j+1)k/2** total edges.

So the probability that `karger` chooses one of the k edges crossing **S*** at step j is at most

$$\frac{k}{\frac{(n-j+1)k}{2}} = \frac{2}{n - j + 1}$$



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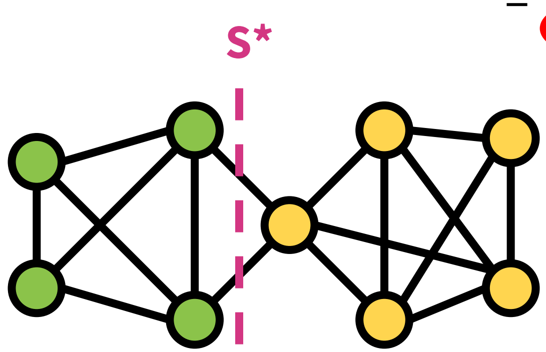
…

$\times$ P(e_{n-2} doesn't cross S^* | $e_1, \ldots, e_{n-3}$ doesn't cross S^*)

$$\geq \frac{(n-2)}{n} \frac{(n-3)}{(n-1)} \frac{(n-4)}{(n-2)} \frac{(n-5)}{(n-3)} \frac{(n-6)}{(n-4)} \cdots \frac{4}{6} \frac{3}{5} \frac{2}{4} \frac{1}{3}$$



= 2/(n(n-1))

= 1/(nC2)

Karger's Algorithm

1/(nC2) isn't all that great …

For our example of n = 9, 1/(9C2) = 0.028.

Suppose we want to find the min-cut with probability 0.9.

What can we do? 🤔

Karger's Algorithm

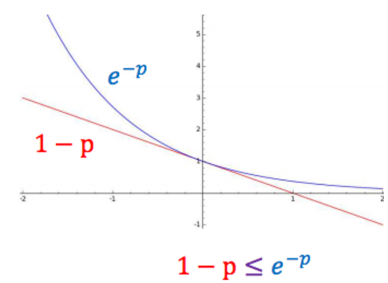
1/(nC2) isn't all that great …

For our example of n = 9, 1/(9C2) = 0.028.

Suppose we want to find the min-cut with probability 0.9.
What can we do? 🤔

How many times T do we need to repeat `karger` to obtain this probability?

Note that if P(find the min-cut after 1 time) ≥ 1/(nC2), then
P(don't find the min-cut after 1 time ≤ 1 - 1/(nC2)



P(find the min-cut after T times) ≥ 0.9
⇔ P(don't find the min-cut after T times) ≤ 0.1.
P(don't find the min-cut after T times) = $(1 - 1/(nC2))^T$
$\leq (e^{-1/(nC2)})^T = 0.1$

T = (nC2) ln (1/0.1) times

Suppose we want to find the min-cut with probability p.
Then we must repeat Karger **T = (nC2) ln (1/(1-p)) times**.

Karger's Algorithm

$T = (nC2) \ln (1/(1-p))$ times = $O(|V|^2)$ times, so the overall runtime is $O(|V|^4)$.

- Treating 1-p as a constant.
- If we use union-find data structures, then we can do better.
- This might seem lousy, but then consider that enumerating over all possible cuts to find the min-cut requires $O(2^{|V|})$.
- This is a huge improvement!

Karger’s Algorithm

```
algorithm karger_loop(G=(V,E), threshold):
  cur_min_cut = None
  n = V.length, p = threshold
  for t = 1 to (nC2)ln(1/(1-p)) :
    candidate_cut = karger(G)
    if candidate_cut.size < cur_min_cut.size:
      cur_min_cut = candidate_cut
  return cur_min_cut
```

Runtime: $O(|V|^4)$

Karger's Algorithm

Key point: Whenever we have a Monte-Carlo algorithm with a small probability of success, we can boost the probability of success by repeating it a bunch of times and taking the best solution!

Many statistical machine learning algorithms work in this way!