CS 206 Homework 6

Fall 2020

1. We conduct an experiment of counting the number of tosses of a fair coin until getting three consecutive heads. We record the flips until we get three heads (for example, $HTTTTHHH$). Let's define a geometric random variable X that represents the number of tosses of a fair coin until we get three consecutive heads. Possible values for X are positive integers, such as $x \in \{1, 2, 3, 4...\}$.

   (a) Enumerate all possible outcomes with length 3, 4, 5, or 6.

   (b) These records give us a sample space Ω, and we can decompose Ω into four sets:

   - $A_1 = \{(H, H, H)\}$
   - $A_2 = \{(T, *, *, ..., *)\}$ (e.g., all records starting with T)
   - $A_3 = \{(H, T, *, *, ..., *)\}$ (e.g., all records starting with H, T)
   - $A_4 = \{(H, H, T, *, *, ..., *)\}$ (e.g., all records starting with H, H, T)

   Please compute $\mathbb{P}(A_1)$, $\mathbb{P}(A_2)$, $\mathbb{P}(A_3)$, and $\mathbb{P}(A_4)$.
   (E.g., $\mathbb{P}(A_2)$ is the probability that a randomly chosen record is in the set A_2.)

   (c) Please find $\mathbb{E}[X]$ using these four decompositions.
   Hint: You could write an equation using $\mathbb{E}[X], \mathbb{E}[X + 1], \mathbb{E}[X + 2]$, and $\mathbb{E}[X + 3]$.

2. Suppose you design an algorithm to multiply two n-bit integers x and y. The general multiplication technique takes $T(n) = \mathcal{O}(n^2)$ time. For a more efficient algorithm, you first split each of x and y into their left and right halves, which are $n/2$ bits long. For example, if $x = 10001101_2$, then $x_L = 1000_2$ and $x_R = 1101_2$, and $x = 2^4 \cdot x_L + x_R$. Then the product of x and y can be re-written as the following:

$$x \cdot y = 2^n \cdot (x_L \cdot y_L) + 2^{n/2} \cdot (x_L \cdot y_R + x_R \cdot y_L) + (x_R \cdot y_R)$$

   (a) Based on the rewritten multiplication formula, write a recurrence relation for $T(n)$, where all running time of summations/subtractions can be approximated by $\mathcal{O}(n)$. By using the Master theorem, find the $\mathcal{O}(\cdot)$ running time.

   (b) A clever person remarks that

$$(x_L \cdot y_R + x_R \cdot y_L) = (x_L + x_R) \cdot (y_L + y_R) - (x_L \cdot y_L) - (x_R \cdot y_R)$$

   and this can be used to improve your recursion process. How does it change your recursion equation? Please write a new recurrence and use the Master theorem to find the corresponding $\mathcal{O}(\cdot)$.

3. Suppose a plant puts out a new shoot, and that shoot has to grow two months before it is strong enough to support branching. If it branches every month after that at the growing point, write a recursion equation to describe number of branches after n months, $B(n)$, and find its closed form, where $B(1) = 1$ and $B(2) = 1$.

   (Note: your answer should be more than just "this looks like X, which has closed form Y". If you think it's X, you should prove/derive that.)

   See the following diagram of the plant at months $n = 0$ through $n = 5$: