Conditional probability

CS 206: Discrete Structures II

Fall 2020

Conditional probability

If we roll two dice, what is the probability that their sum is at least 8?

Conditional probability

If we roll two dice, and the first is a 5, what is the probability that their sum is at least 8?

Conditional probability

Conditional probability is the probability of A happening given that B happens:

$$\mathbb{P}\left(A|B\right) = \frac{\mathbb{P}\left(A \cap B\right)}{\mathbb{P}\left(B\right)}$$

Conditional probability example

Consider a random 4-bit binary string.

What's the probability it has 2 consecutive 0s, if the first bit is 0?

Let E represent having 2 consecutive 0s.

Let F represent the first bit being 0.

Example: coin flips

Say we flip two coins. What's the probability of getting two heads if:

- the first flip is heads?
- at least one flip is heads?

Hockey game

A hockey team plays in a best-of-3 tournament.

- They have a 50/50 chance of winning their first game.
- If they win a game, then 2/3 chance to win the next game.
- If they lose a game, then 1/3 chance to win the next game.

Define two events:

- A: they win the tournament
- B: they win their first game

Hockey game

$A =$ win the tournament, $B =$ win the first game

What is $\mathbb{P}(A|B)$?

Hockey game

$A =$ win the tournament, $B =$ win the first game

What is $\mathbb{P}\left(B|A\right)$?

Bayes' theorem

Bayes' theorem provides a way to relate these two conditional probabilities:

$$\mathbb{P}\left(B|A\right) = \frac{\mathbb{P}\left(A|B\right)\mathbb{P}\left(B\right)}{\mathbb{P}\left(A\right)}$$

Bayes' theorem derivation

By definition of conditional probability:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \qquad \mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

Then multiplying by the denominators:

$$\mathbb{P}(A|B)\,\mathbb{P}(B) = \mathbb{P}(A \cap B) \qquad \mathbb{P}(B|A)\,\mathbb{P}(A) = \mathbb{P}(A \cap B)$$

Then we can equate the left-hand sides and divide by $\mathbb{P}(A)$:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\,\mathbb{P}(B)}{\mathbb{P}(A)}$$

Example: a medical clinic

Event		Frequency
A	have liver disease	10%
B	are alcoholic	5%
$B\|A$	alcoholics among those with liver disease	7%

What's the probability of liver disease if you're an alcoholic?

Example: medical tests

Suppose we have a test for a particular disease, but the test isn't perfect:

- 10% false negative rate
- 5% false positive rate

And we know 1% of the population has this disease.

If you test positive, what's the probability you actually have the disease?

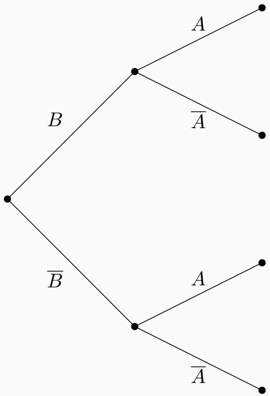
Example: medical tests

$H =$ you have the disease, $P =$ you have a positive test

What is $\mathbb{P}(H|P)$?

Law of Total Probability

$$\mathbb{P}(A) = \mathbb{P}(A|B)\,\mathbb{P}(B) + \mathbb{P}\left(A|\overline{B}\right)\mathbb{P}\left(\overline{B}\right)$$



Law of Total Probability

For example, let's flip a coin.

If it's heads, we'll roll a die.

If it's tails, we'll roll two dice and sum them.

What's the probability we get a (sum of) 2?

- A: sum is 2
- B: coin flip is heads

Bayes' theorem + Law of Total Probability

Bayes' theorem:

$$\mathbb{P}\left(B|A\right) = \frac{\mathbb{P}\left(A|B\right)\mathbb{P}\left(B\right)}{\mathbb{P}\left(A\right)}$$

We can rewrite the denominator using the law of total probability:

$$\mathbb{P}\left(B|A\right) = \frac{\mathbb{P}\left(A|B\right)\mathbb{P}\left(B\right)}{\mathbb{P}\left(A|B\right)\mathbb{P}\left(B\right) + \mathbb{P}\left(A|\overline{B}\right)\mathbb{P}\left(\overline{B}\right)}$$

Example: Bayesian spam filtering

S = message is spam, W = message contains "watch"

$$\mathbb{P}(S|W) = \frac{\mathbb{P}(W|S)\,\mathbb{P}(S)}{\mathbb{P}(W)}$$

$$\mathbb{P}(W) = \mathbb{P}(W|S)\,\mathbb{P}(S) + \mathbb{P}\left(W|\overline{S}\right)\mathbb{P}\left(\overline{S}\right)$$

$$\mathbb{P}(S|W) = \frac{\mathbb{P}(W|S)\,\mathbb{P}(S)}{\mathbb{P}(W|S)\,\mathbb{P}(S) + \mathbb{P}\left(W|\overline{S}\right)\mathbb{P}\left(\overline{S}\right)}$$

Law of Total Probability

The Law of Total Probability can be extended to > 2 events.

If we have:

- E_1, E_2, E_3 disjoint
- $\mathbb{P}(E_1 \cup E_2 \cup E_3) = 1$

Then

$$\mathbb{P}(A) = \mathbb{P}(A|E_1)\,\mathbb{P}(E_1) + \mathbb{P}(A|E_2)\,\mathbb{P}(E_2) + \mathbb{P}(A|E_3)\,\mathbb{P}(E_3)$$

Inclusion-exclusion

We can apply an inclusion-exclusion-like rule in conditional probabilities:

$$\mathbb{P}\left(A \cup B|C\right) = \mathbb{P}\left(A|C\right) + \mathbb{P}\left(B|C\right) - \mathbb{P}\left(A \cap B|C\right)$$

Inclusion-exclusion

Suppose we roll two dice. What's the probability that

- they sum to at least 8
- or are both prime

if the first die comes up 5?