

Randomized Algorithms II

Hashtables

Outline for Today

More randomized algorithms!

Hashing Basics

Universal Hash Functions

What's the Source of the Randomness?

Hashing Basics

Randomized Algorithms

A randomized algorithm is an algorithm that incorporates **randomness** as part of its operation.

Often aim for properties like ...

- Good average-case behavior

- Getting exact answers with high probability

- Getting answers that are close to the right answer

Data Structures

	Sorted linked lists	Sorted arrays	Balanced BSTs
Search	$O(n)$ expected & worst-case	$O(\log n)$ expected & worst-case	$O(\log n)$ expected & worst-case <small>$O(n)$ worst-case for generic BSTs</small>
Insert/Delete	$O(n)$ expected & worst-case <small>without a pointer to the element</small>	$O(n)$ expected & worst-case	$O(\log n)$ expected & worst case

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Insert/Delete	$O(n)$ expected & worst-case <small>without a pointer to the element</small>	$O(n)$ expected & worst-case	$O(\log n)$ expected & worst case	$O(1)$ expected $O(n)$ worst-case <small>without a pointer to the element</small>

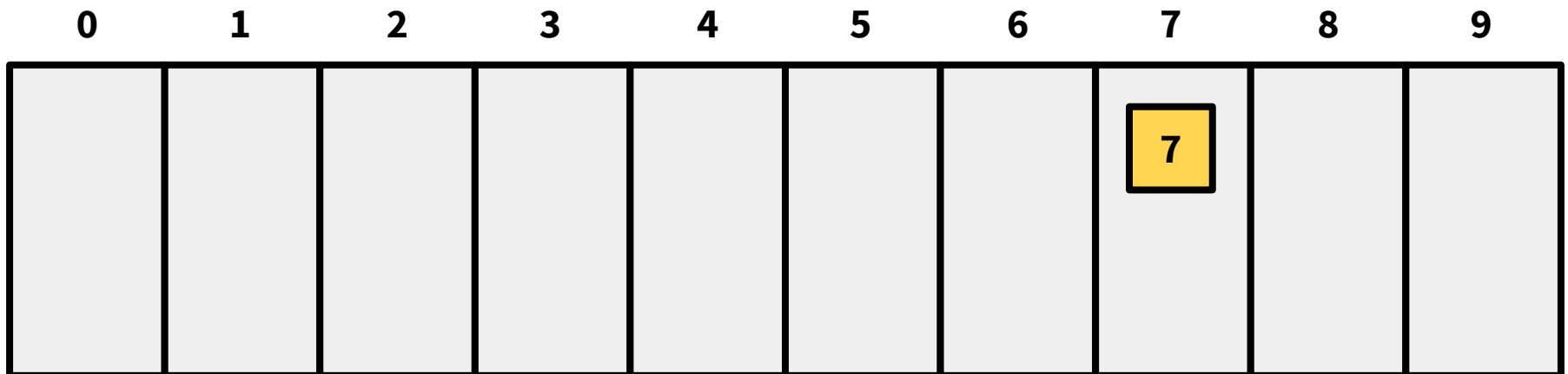
Direct Addressing

Direct addressing means we can know the memory position of an element based on the element directly.

How might we get $O(1)$ -time? Try [direct addressing](#)!

A simple case: #Buckets > Maximum possible number, so one item per address.

```
insert(7)
```



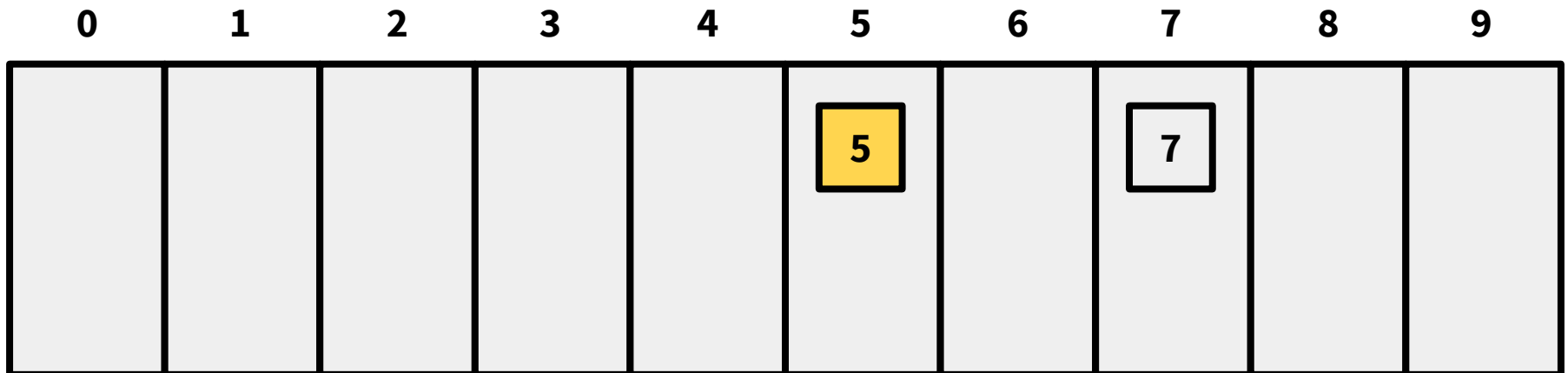
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One item per address.

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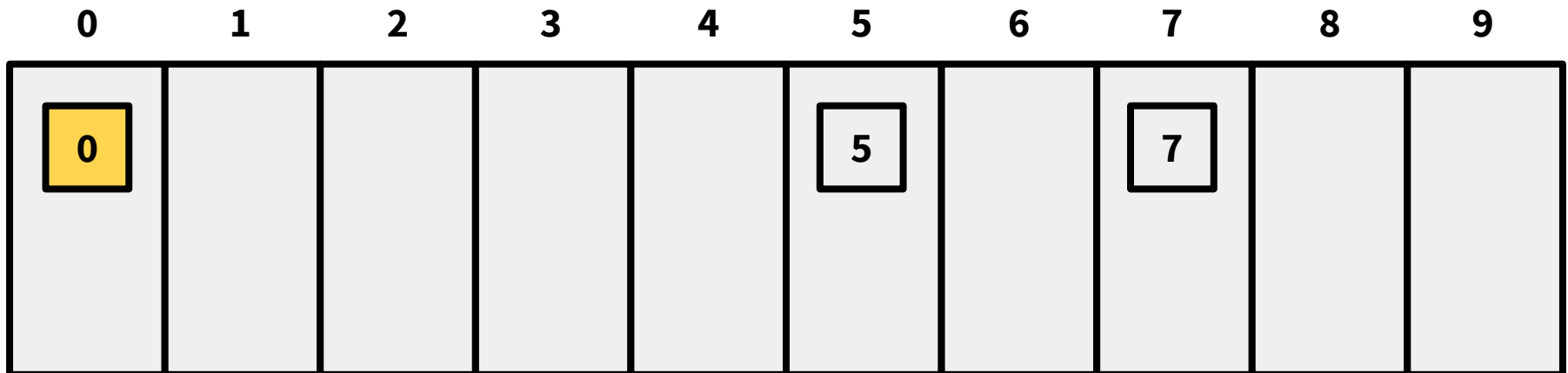
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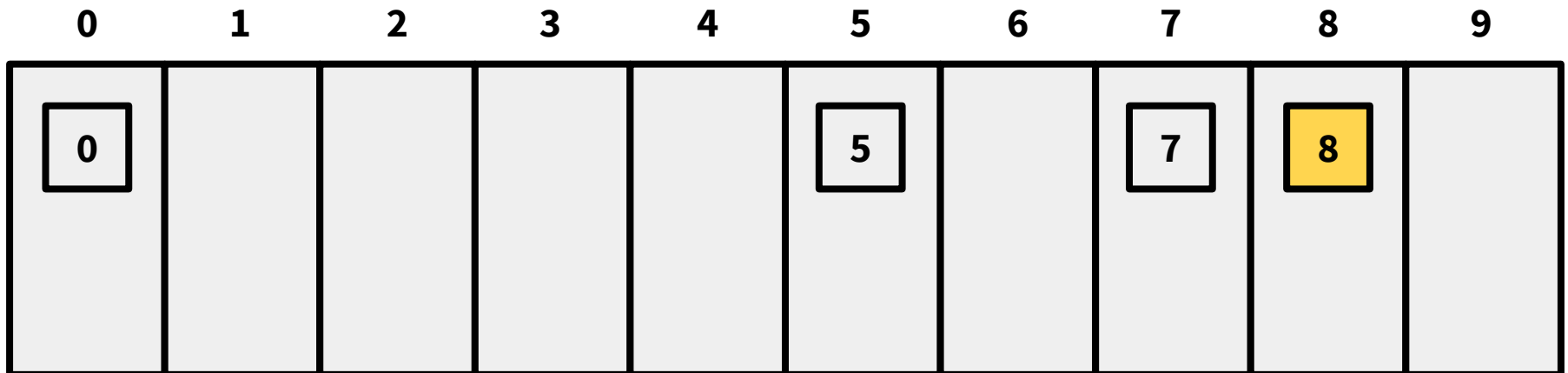
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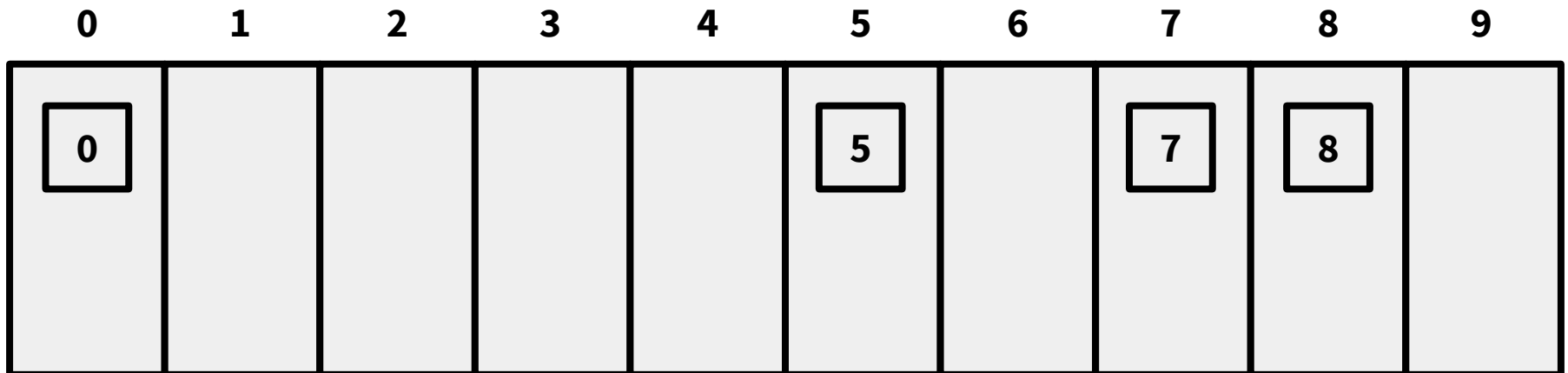
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insert(7) search(7)

insert(5) search(2)

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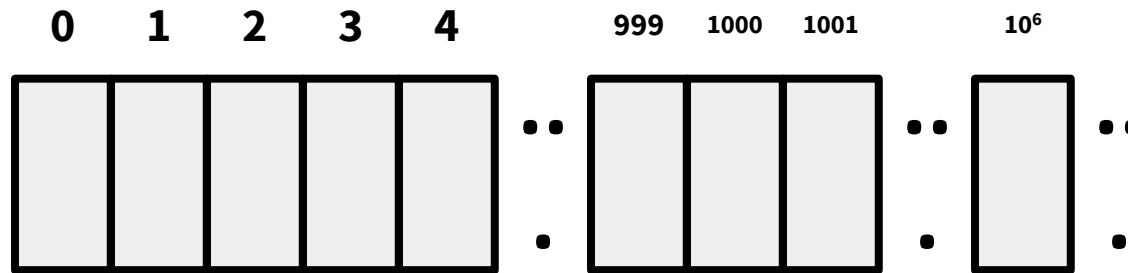
What's the issue with this approach? 🤔

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Similar to **counting_sort**, if the set of items being inserted/deleted is large (e.g. $\{0, 1, 2, \dots, 999, 1000, \dots, 10^6, \dots\}$), then the **sheer space** required to maintain this data structure becomes an issue.

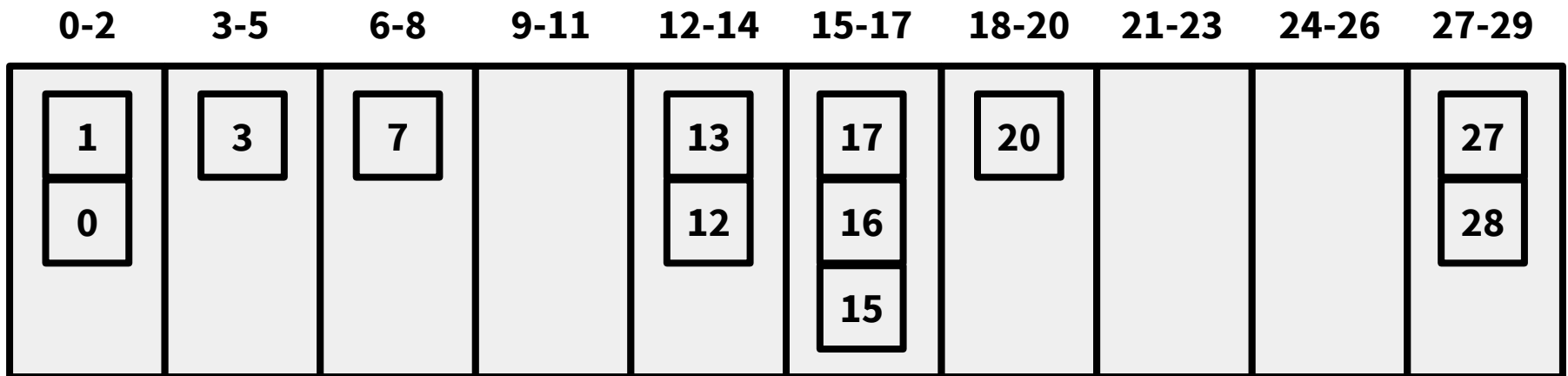


Direct Addressing

How might we get **$O(1)$** -time? Try **direct addressing**!

Can we fix this issue by **assigning multiple items per address**, like case (2) of **bucket_sort**?

Sometimes, this binning approach is useful. `search(12)` still runs pretty fast.

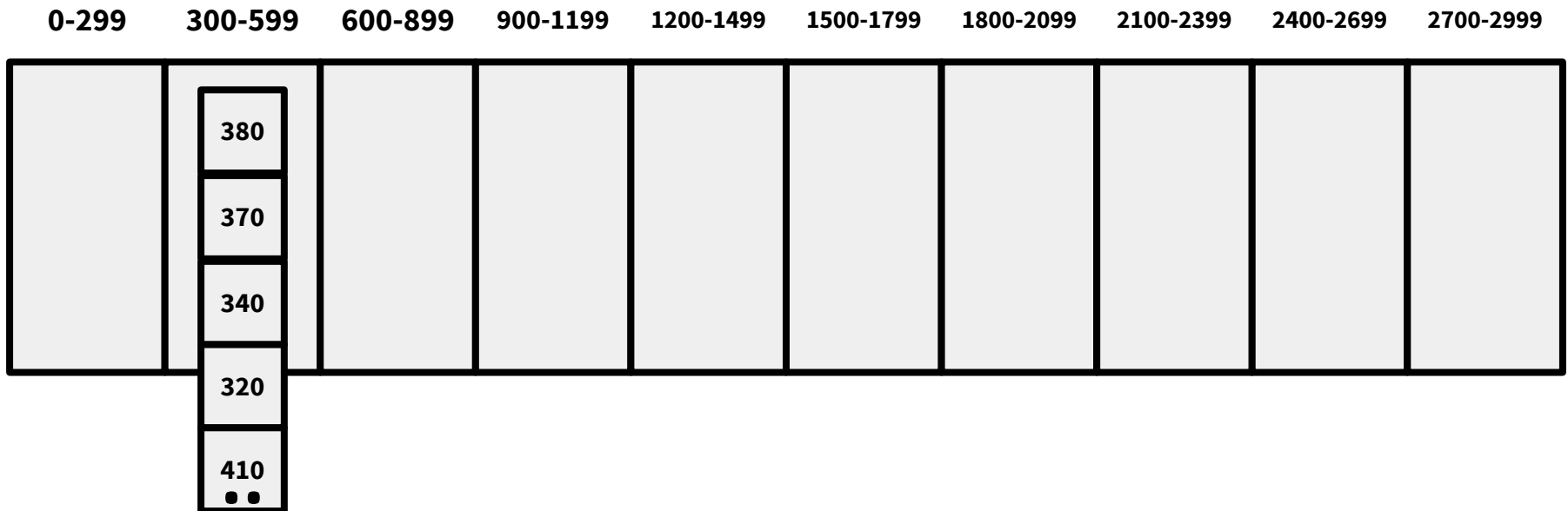


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Other times, it causes an issue. `search(432)` is slow.

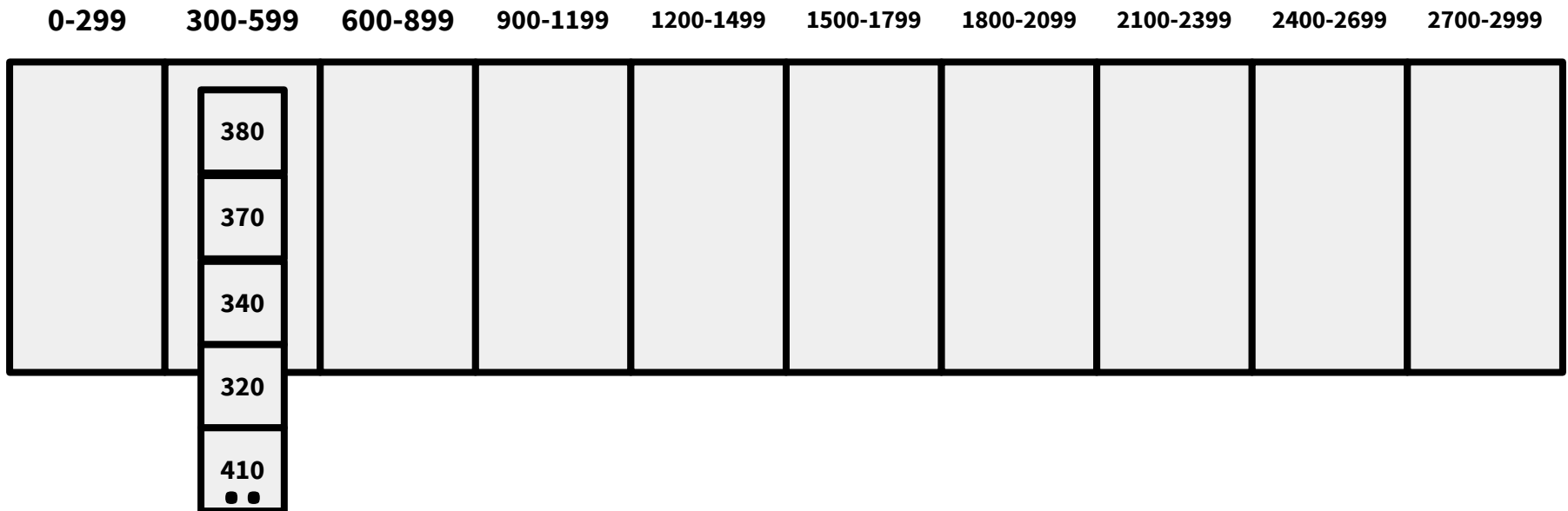


Direct Addressing

This is an example of a hash table.

Although one with a very basic bucketing strategy.

Can we do better?



Terminology

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$|U|$ is really big.

What is $|U|$ if U is the set of ASCII strings of length 16? 🤔

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We hash the keys to n buckets.

$|U| \gg n$; i.e. $|U|$ is a lot bigger than n .

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e.g. all the valid twitter sentences

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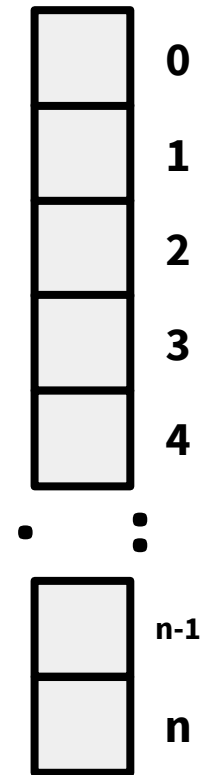
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There's a hash function $h: U \rightarrow \{1, \dots, n\}$ that maps keys to buckets.

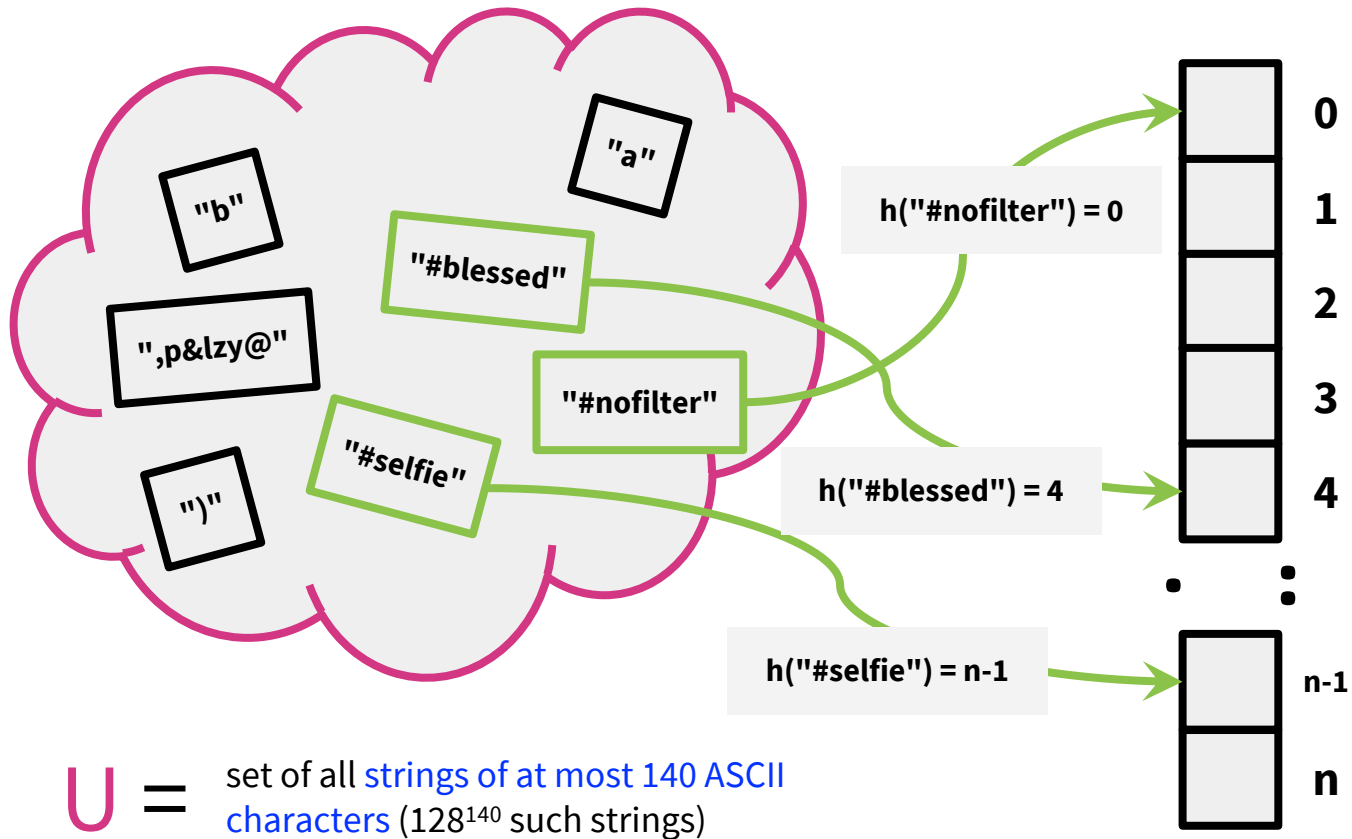
An Example



U = set of all strings of at most 140 ASCII characters (128^{140} such strings)

And we'll need to store a small subset of U (say, the ones that might be trending [hashtags on Twitter](#)); we're assuming the number of hashtags $\leq n$, the number of buckets.

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Hash Tables (with chaining)

List of n buckets.

Each bucket stores an unsorted linked list.

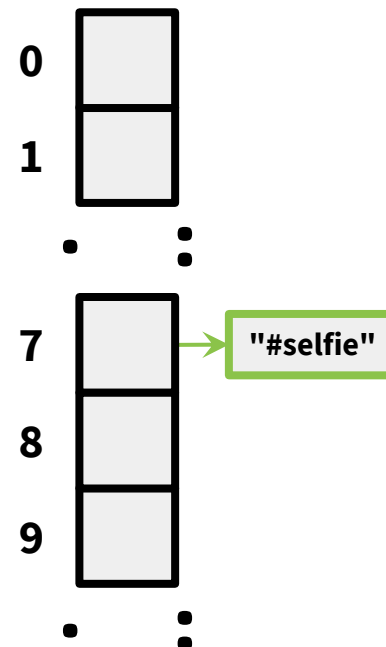
`insert` in $O(1)$ since it's unsorted; `search` in $O(n)$.

$h: U \rightarrow \{1, \dots, n\}$ can be any function

For example, suppose it's length.

Suppose we insert a bunch of keys and then search.

`insert("#selfie")`



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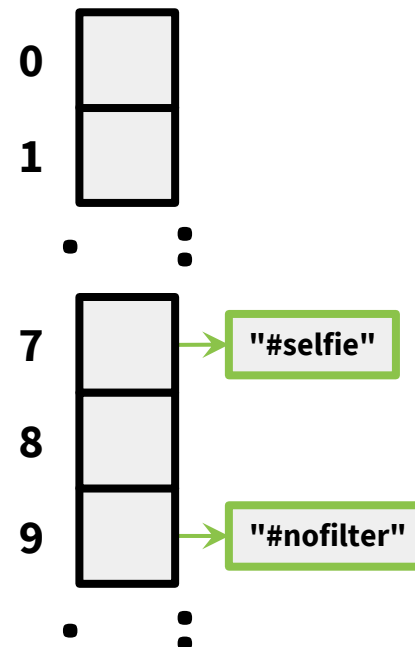
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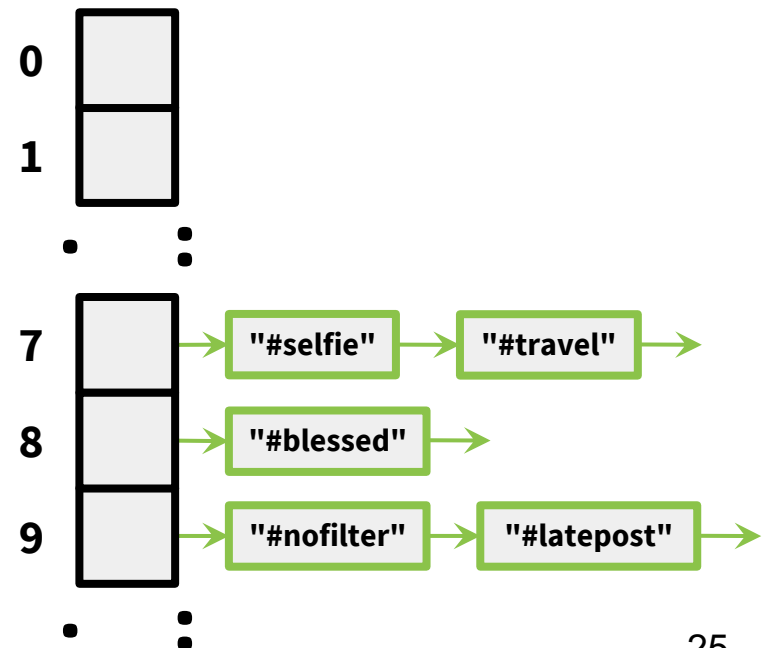
```
insert("#blessed")
```

```
insert("#travel")
```

```
insert("#latepost")
```

```
search("#travel")
```

Scans through
all elements in
bucket
 $h(\text{"#travel"})$



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Not really. In fact, it's quite terrible since there are a lot of hashtags that share the same lengths.

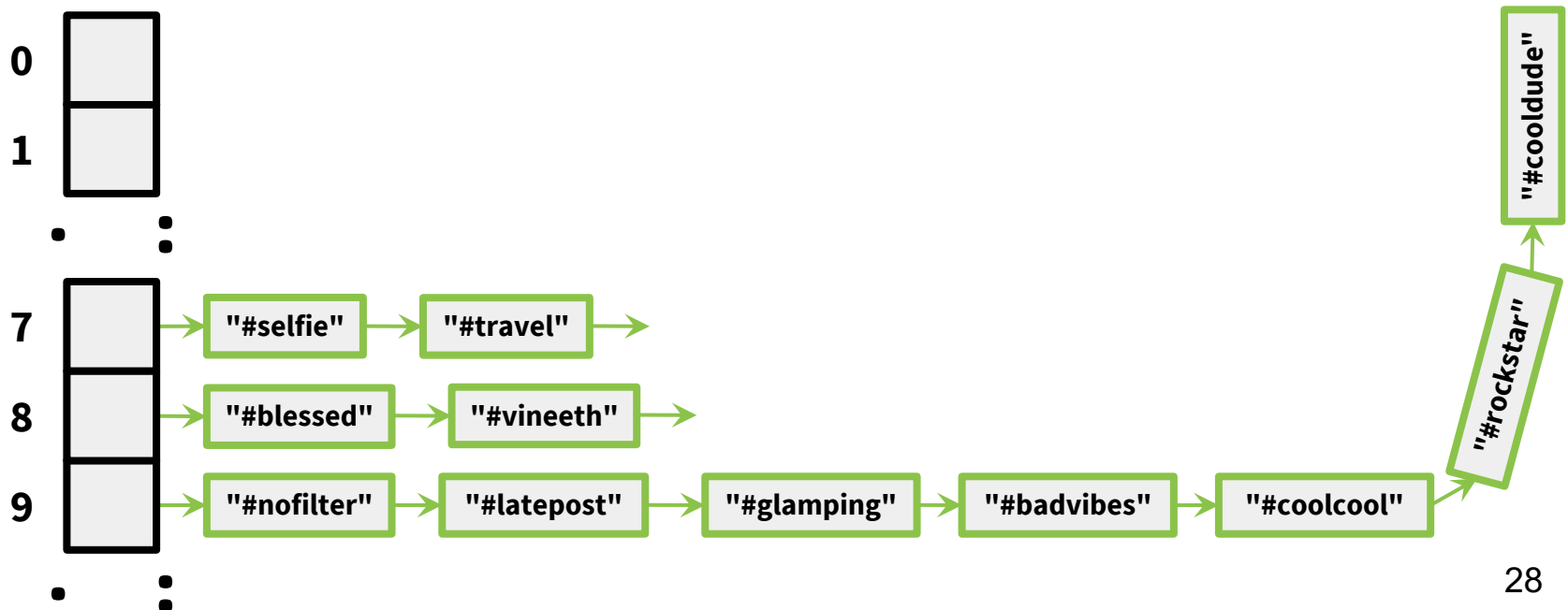
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So how do we choose a better h ?

The items need to be spread out in the buckets.



Designing Hash Functions

One h to Rule Them All?

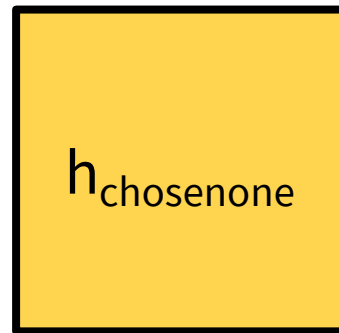
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1. You choose your hash function $h_{\text{chosenone}}$.



2. An adversary gives your hash function n items to hash.

Is it possible to construct $h_{\text{chosenone}}$ such that you're guaranteed that all buckets will have size $O(1)$? This would be ideal. 🤔

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You probably couldn't think of how. Why not? *It's impossible!*

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$h_{\text{chosenone}}$ is defined from a domain of $|U|$ items to a range of n buckets. By the pigeonhole principle, at least one of the buckets receives at least $|U|/n$ items. Recall that $|U| \gg n$, so $|U|/n > n$; therefore at least one of the buckets receives at least n items.

Notation indicating $U_{\text{bigbucket}}$ is a function of $h_{\text{chosenone}}$



Let's call the set of items that get hashed to this bucket $U_{\text{bigbucket}}(h_{\text{chosenone}})$ where $U_{\text{bigbucket}} \subset U$. The adversary could choose to hash n items from $U_{\text{bigbucket}}$. This is a valid set of n items, and results in one bucket with all n items, by construction. Therefore, (1) is impossible.

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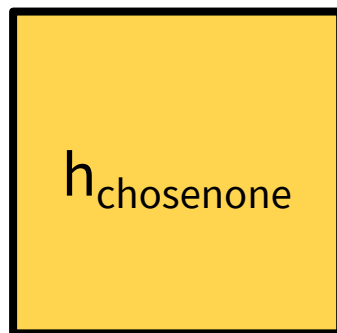
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Can you think of such an $h_{\text{chosenone}}$? 🤔

Probably not. This is the same question as **(1)**! Since the adversary is choosing the n items, **there's no randomness anywhere in the process.**


As a result, for at least one bucket, the **expected** size of the bucket will be trivially just the size.

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In order for all buckets to have **expected** size $O(1)$ after hashing any n items, we need to **introduce randomness**.

Where? Well there's only one option ... **in our choice of hash function**.

We will randomly choose h from a large set of hash functions!

(There won't be an h to rule them all).



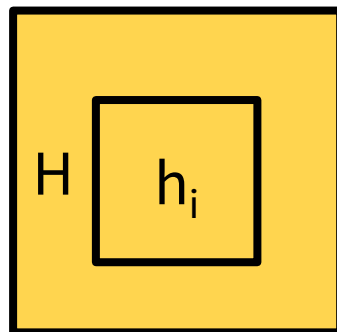
Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

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Yes! But it's not very useful.

Let H be the set of n hash functions where h_i hashes all keys in the entire universe to bucket i . With probability $1/n$, h_b will be chosen, then bucket b will have all the n keys hashed to it. Otherwise, bucket b will be empty.

$$\begin{aligned} E[\text{size_of}(b)] &= P(\text{all keys hashed to it}) \cdot n + P(0 \text{ keys hashed to it}) \cdot 0 \\ &= (1/n) \cdot n \\ &= \mathbf{1} \end{aligned}$$

But $P(\text{lots of keys get hashed to one bucket}) = 1$.

This is not good. Requiring all buckets to have expected $O(1)$ size is not enough! Maybe we should be using a different metric.

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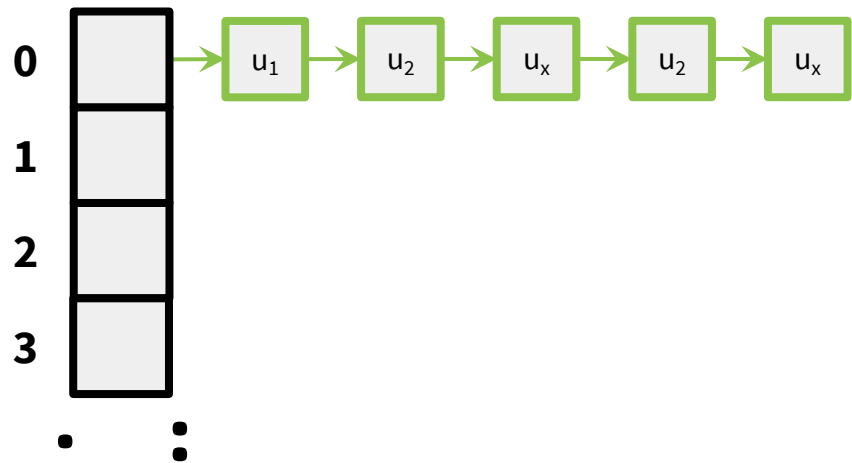
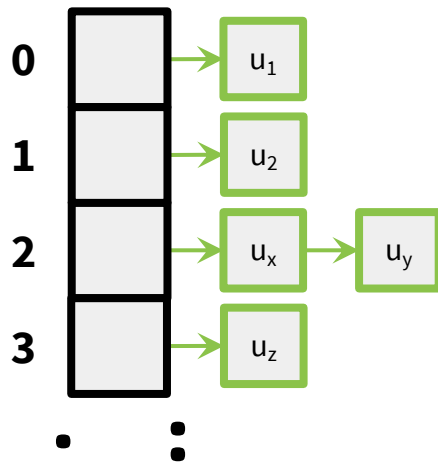
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As an analogy for the difference between (3) and (4), consider the “small classes illusion.” Suppose a university offers 10 classes, 9 of which have 1 person in them and the last of which has 500 persons in them. Using reasoning from (3), the university might tout average class sizes of ~ 50 , when in reality, it should report much higher class sizes experienced by the average student, as in (4).

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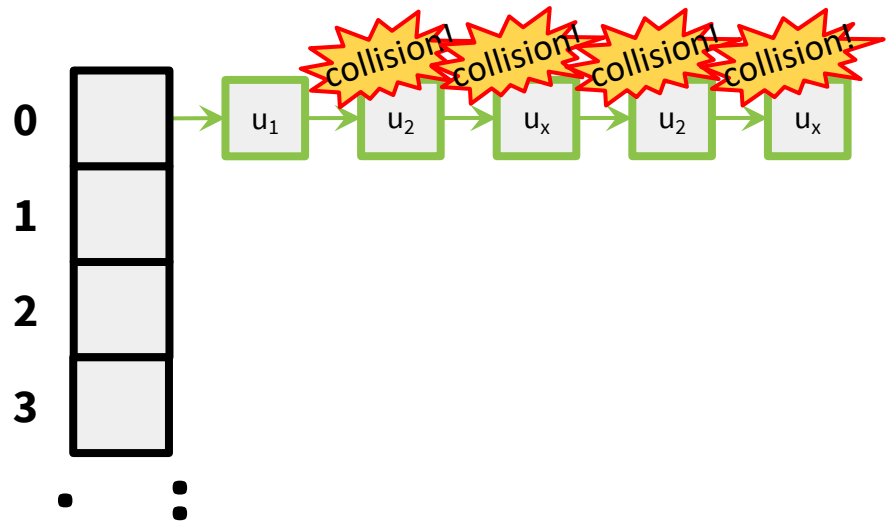
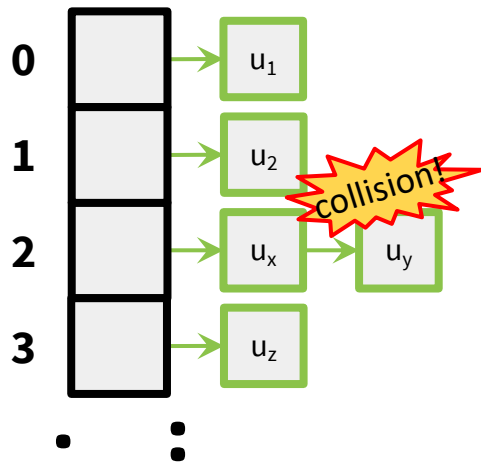
We can think of this statement in terms of minimizing the expected number of collisions.



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Yes! This time it's possible.

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
"a"	0	0	0	0	1	1	1	1
"b"	0	0	1	1	0	0	1	1
"c"	0	1	0	1	0	1	0	1

The 0's and 1's represent the buckets i.e. h_8 hashes "b" to bucket 1.

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Let H be the exhaustive set of all hash functions that map elements in the universe U to buckets 1 to n , which has size $|H| = n^{|U|}$.

e.g. Suppose $U = \{“a”, “b”, “c”\}$ and $n = 2$ (there are 2 buckets). H would be a set of 8 hash functions. One h would map “a”, “b”, and “c” all to bucket 0. Another h would map “a” and “b” to bucket 0 and “c” to bucket 1. etc. etc.

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
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$$E[\text{number of items in } u_x\text{'s bucket}] = \sum_{y=1}^n P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} 1/n$$



Percentage of hash functions that hash u_x and u_y to the same bucket:
 $P[h(u_x)=h(u_y)]=n/n^2=1/n$

This is also the **probability of a collision!**

$$= 1 + (n-1)/n$$

$$\leq 2$$

The Good News


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Yes! This is great news! It means that we can choose H to be the exhaustive set of all hash functions, and the insert, delete, search operations on any n elements will have an expected runtime of $O(1)$ per operation.

The Bad News


The exhaustive set of all hash functions is HUUUGE!!!

How many bits would it take to represent
one of the $n^{|U|}$ hash functions in this H ? 🤔

 Like really huge.

The Bad News

The exhaustive set of all hash functions is HUUUGE!!!

How many bits would it take to represent one of the $n^{|U|}$ hash functions in this H ? 🤔 $\log n^{|U|} = |U| \log n$.  Like really huge.

To see why, consider how much memory it would take to write down the name of one of the 8 hash functions from earlier. You could assign h_1 the id 000, h_2 the id 001, etc. So 8 hash functions requires $\log 8 = 3$ bits to write down.

$|U| \log n$ bits is even enough to do direct addressing! So it's pointless to spend efforts for hashing.

H Is Too Big

How can we fix this issue of the size of H?

Universal Hash Functions

H Is Too Big

How can we fix this issue of the size of H?

Pick from a smaller set H, that still guarantees (4).

Recall the bound that allowed us to achieve this guarantee:

$$E[\text{number of items in } u_x\text{'s bucket}] = \sum_{y=1}^n P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} 1/n$$

$$= 1 + (n-1)/n$$

$$\leq 2$$

This step is the key!

Percentage of hash functions that hash u_x and u_y to the same bucket:
 $P[h(u_x)=h(u_y)]=n/n^2=1/n$

This is also the **probability of a collision!**

Universal Hash Family

This bound is so important, there's a special name for sets H that satisfy it.

A **hash family** is a fancy name for a set of hash functions.

A **universal hash family** describes a set of hash functions that satisfy the bound: $P_{h \in H}[h(u_x) = h(u_y)] \leq 1/n$, i.e., the probability of collision is bounded by $1/n$.

The exhaustive set of hash functions is an example of a universal hash family but, as discussed previously, it's too big to be practical.

A Smaller Universal Hash Family

Identifying new smaller universal hash families is an active field of research in Computer Science, especially in Cryptology.

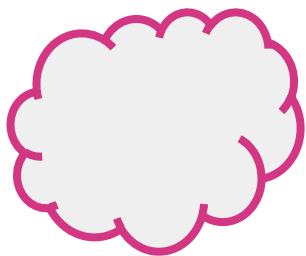
One of the well-studied universal hash families:

To hash **an integer** x in $\{0, \dots, |U|-1\}$ to a bucket $\{1, \dots, n\}$:

$$h_{a,b}(x) = ((ax + b) \bmod p) \bmod n$$

for some prime $p \geq |U|$ and $a \in \{1, \dots, p-1\}$ and $b \in \{0, \dots, p-1\}$

To select an $h_{a,b}$ from this family:



p

a

b

1. Determine $|U|$.
e.g. 100, x in 0~99

2. Find the smallest
prime $p \geq |U|$.
e.g., 101

3. Let **a** be a
random number in
 $\{1, \dots, p-1\}$.
e.g., 10

4. Let **b** be a
random number in
 $\{0, \dots, p-1\}$.
e.g. 5

Show an example on board: $h_{10,5}(x) = ((10x+5) \bmod 101) \bmod 10$

How Small Is This H?

There are $p-1$ choices for **a** and p choices for **b**,
so $|H| = p(p-1) = O(p^2) = O(|U|^2)$.

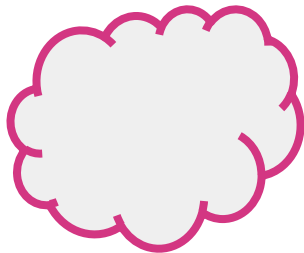
(the last step is based on the prime gap theorem)

That's much better than $n^{|U|}$.

The number of bits need to store an h is $\log |U|^2 = O(\log |U|) \ll O(|U| \log n)$.

Why is it a universal hash family?

Briefly explain $P(h(x)-h(y)=0) = P((a(x-y) \bmod p) \bmod n = 0) = 1/n$



p

a

b

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e.g. 100, x in $0 \sim 99$

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Another Universal Hash Family

Another of the well-studied universal hash families (using matrix multiplication!):

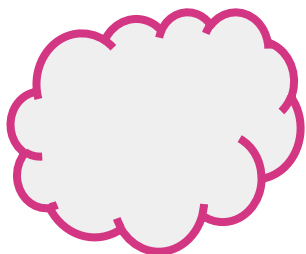
To hash a u -bit string x (i.e. bit string of length u) to a bucket $\{1, \dots, n\}$ (i.e. bit string of length $b = \log(n)$)

E.g., hash 8-bit strings like 10100110 to 4 buckets, each bucket is represented by a 2-bit string like 01.

$$h_A(x) = (Ax) \bmod 2$$

for some $b \times u$ (e.g. 2×8) matrix A of 0's and 1's, using binary (modulo 2) arithmetic.

To select an h_A from this family:



1. Determine $|U|$.

u

2. $u = \log(|U|)$.

b

3. $b = \log(n)$.

A

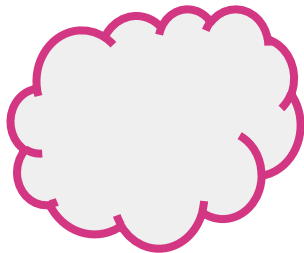
4. Let A be a $b \times u$ random matrix of 0's and 1's.

How Small Is This H?

How many possible binary matrices of size $b \times u$ for **A**?

$$2^{ub} = O(|U|^{\log(n)}).$$

That's much better than $n^{|U|}$, but larger than the other universal hash family $O(|U|^2)$.



1. Determine $|U|$.

u

2. $u = \log(|U|)$.

b

3. $b = \log(n)$.

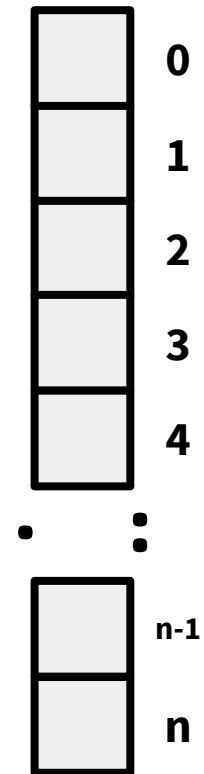
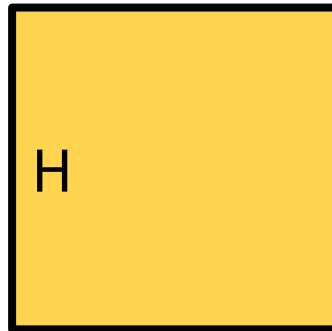
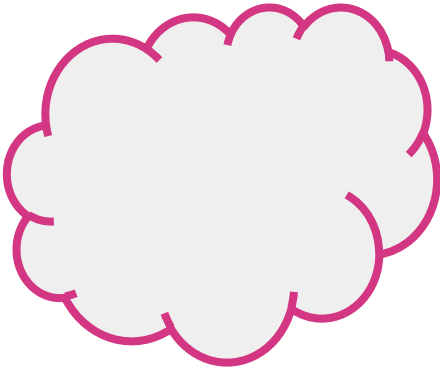
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Hash Tables

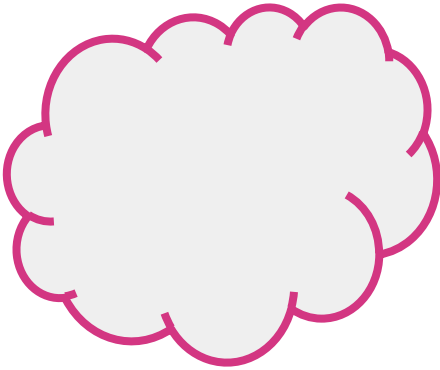
Let's say you wanted to implement a hash table ...

1. You choose your set of hash functions H , likely a universal hash family like $H = \text{mod } p \text{ mod } n$.



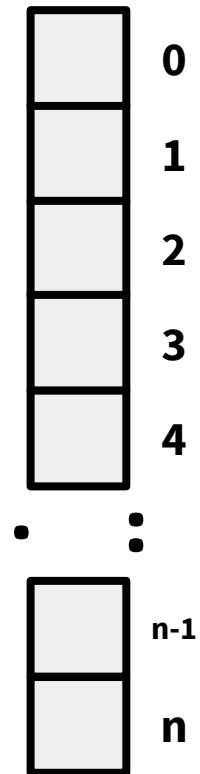
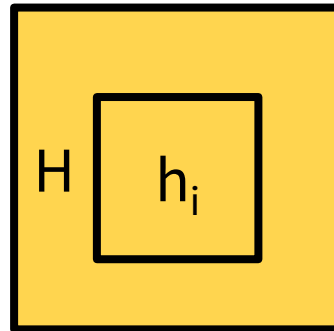
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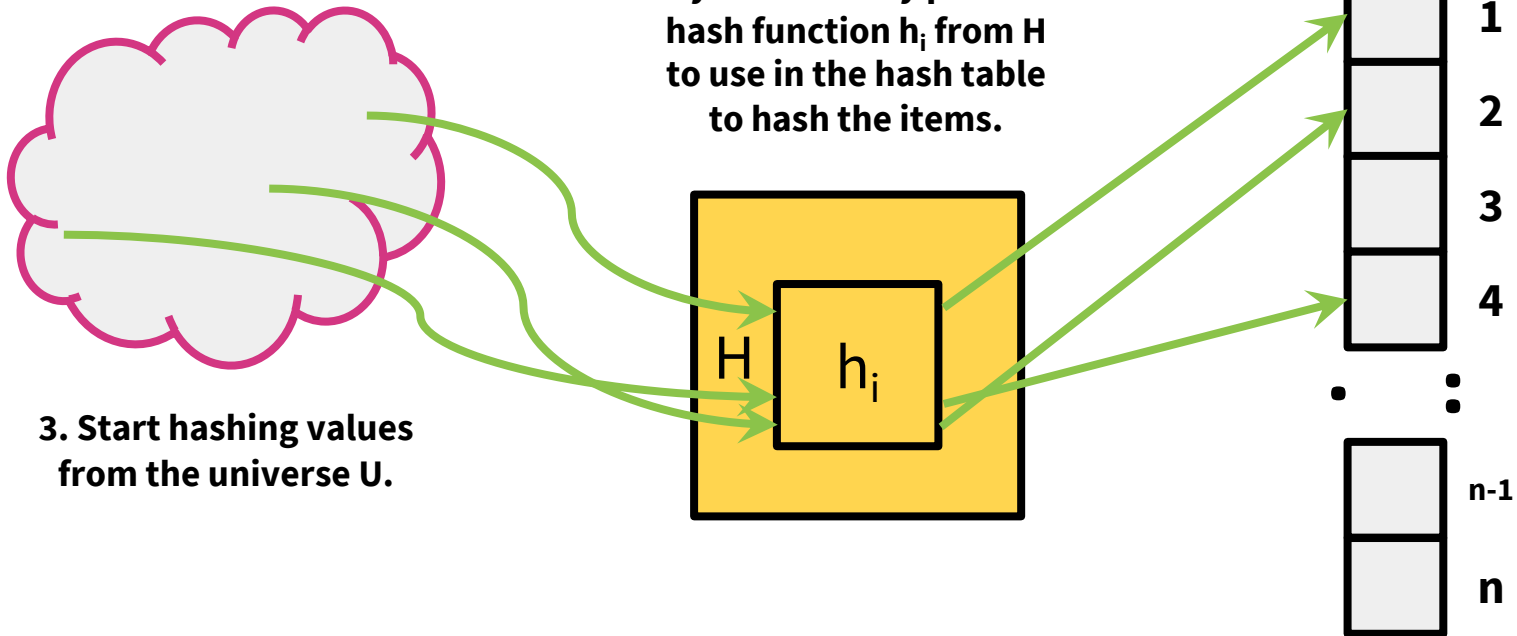
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2. When the client initializes a hash table, you randomly pick a hash function h_i from H to use in the hash table to hash the items.

3. Start hashing values from the universe U .



What's the Source of the Randomness?

As was the case in quicksort, we want the **average-case runtime** for **a specific input** to be low.

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Same thing here with hash tables.

Summary

Randomized Algorithms

Hashing Basics and Terminology
Designing Hash Functions
Universal Hash Functions

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Acknowledgement: Part of the materials are adapted from Virginia Williams and David Eng's lectures on algorithms. We appreciate their contributions.