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1.

1. Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

a) x^8y^9 in $(3x+2y)^{17}$.

$? \, x^8 y^9 \to 8 = k,$

$9 = n-k = 17-8.$

$\therefore$ Coeff: $\binom{17}{8} \times 3^8 \times 2^9 \quad \{(3x)^8 \,\&\, (2y)^9\}$

$$\Rightarrow \boxed{\frac{17! \times 3^8 \times 2^9}{8!\,9!}}$$

b) a^6b^6 in $(a^2+b^3)^5$. $\quad n=5$

$(a^2)^3(b^3)^2 = a^6b^6$.

$k = 3$;

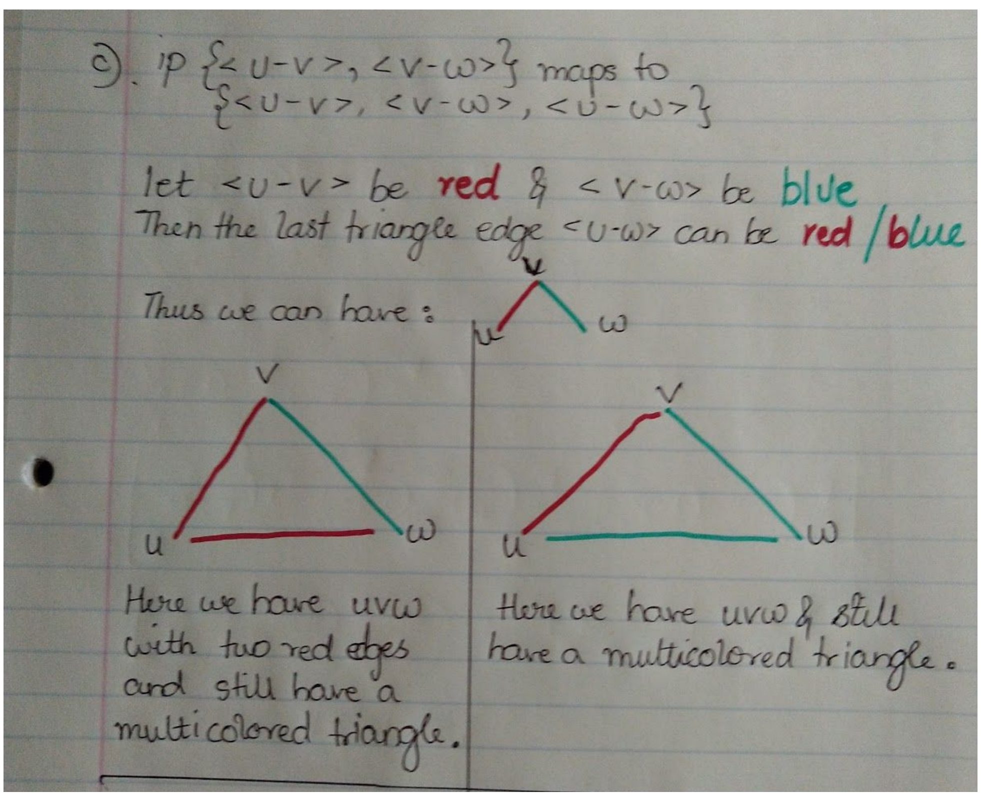
$n-k = 2$; $\therefore$ Coeff $= \binom{5}{3}(a^2)^{5-2}(b^3)^{5-3}$

$$= \frac{5!}{3!\,2!} = \boxed{10}$$

2. ->

a. Triangles are of the form $V_1V_2V_3$ where V_x represents a vertex. If I have 6 vertices and I am asked how many triangles there are in the shape, then I am essentially asked how many ways I can choose 3 vertices from a set of 6. Ordering does not matter, i.e. ABC is the same triangle as ACB and CAB and BAC and BCA, and we cannot have repetition either. Therefore this is a combination without repetition, and the answer is 6 choose 3 = 20 triangles.

b. Note that an incident pair {<u-v> and <v-w>} can be written as uv^2w. Now in a triangle uvw, we can pick one of the vertices to square and place it in the center.
   i. uv^2w
   ii. vu^2w
   iii. uw^2v
   iv. Each of the three above are a bijection to incident pairs where the squared vertex is the center of ip. If there are 20 triangles, then choosing one out of each 3 of the vertices to square is 20 * (3 choose 1) = 60 incident pairs. Note that ordering does not matter as we can order them ourselves, we are only concerned with choosing one of the three.

c. 
c). ip {<u-v>, <v-w>} maps to {<u-v>, <v-w>, <u-w>}

let <u-v> be red & <v-w> be blue
Then the last triangle edge <u-w> can be red / blue

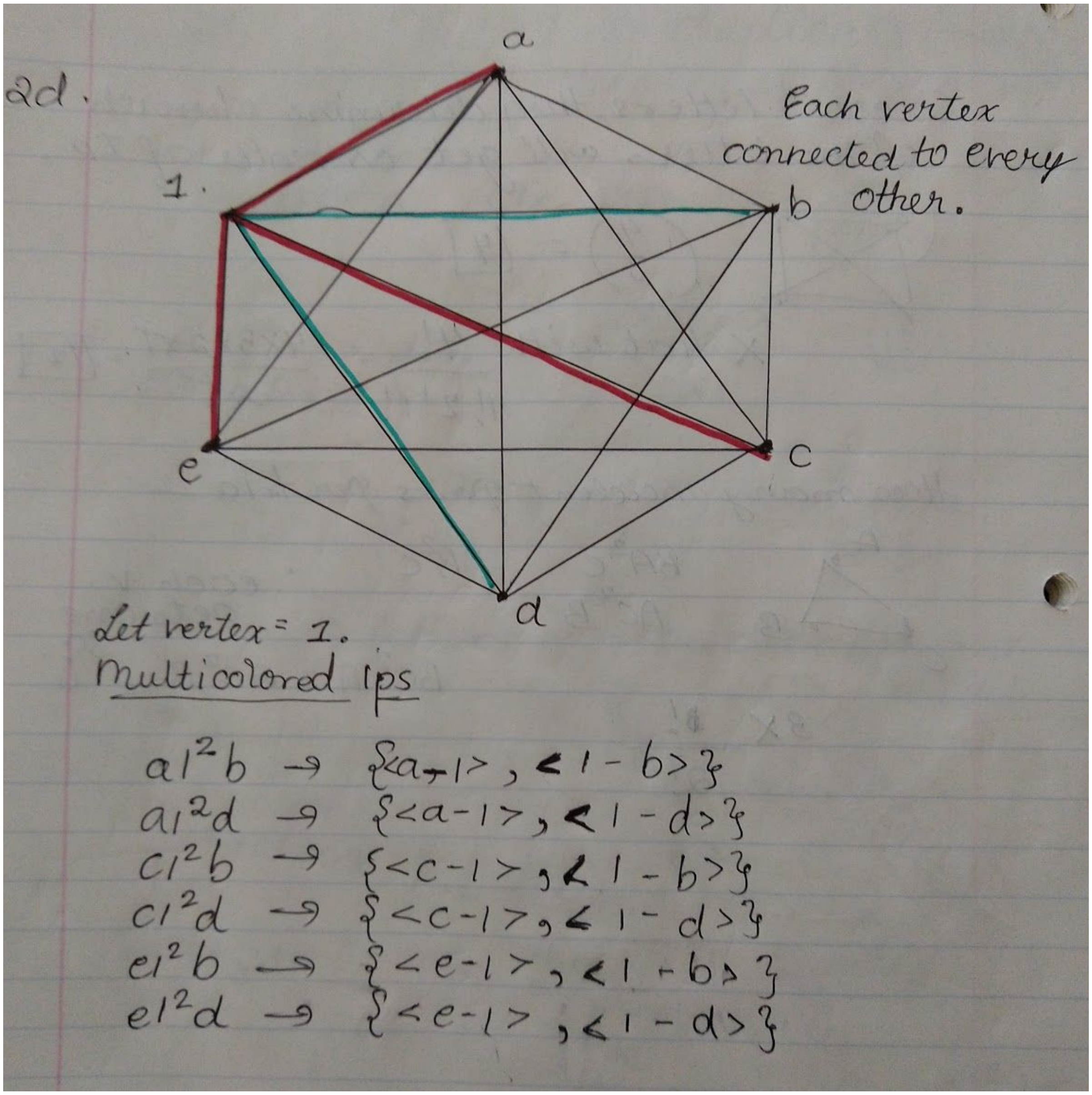
Thus we can have :

Here we have uvw with two red edges and still have a multicolored triangle.

Here we have uvw & still have a multicolored triangle.

To start with we had a multicolored incident pair u-v and v-w, one red and the other blue. If the third edge we add is red (left) then we still have a multicolored triangle (edge v-w is blue). If the third edge we add is blue (right) then we still have a multicolored triangle (edge u-v is red). Thus this mapping is 2 to 1 as starting with the same two edges and adding a third edge of a color from (2 choose 1) = 2 still results in the same 1 multicolored triangle.

d.



i. These are the only possible multicolored ips from one center. There are no other multicolored ips. In other words the pairs are red/blue only. There are no red/red or blue/blue pairs as these would not be multicolored ips.

ii. An arrangement of maximizing blues and reds such that one is only 1 greater than the other maximizes the number of multicolored ips that we can have such at a given center. If we do not try to maximize the number of blues/reds or reds/blues then we will have lesser multicolored ips.

iii. Since we have exhaustively listed all multicolored ips above and can add no other multicolored ip, we know that each center of ip in the graph K_6, in this case it is the vertex 1, can only have 6 multicolored ips (all listed above).

iv. This arrangement of maximizing blues and reds such that one is only 1 greater than the other maximizes the number of multicolored ips that we can have such at a given center.If we do not try to maximize the number of blues/reds or reds/blues then we will have lesser multicolored ips.

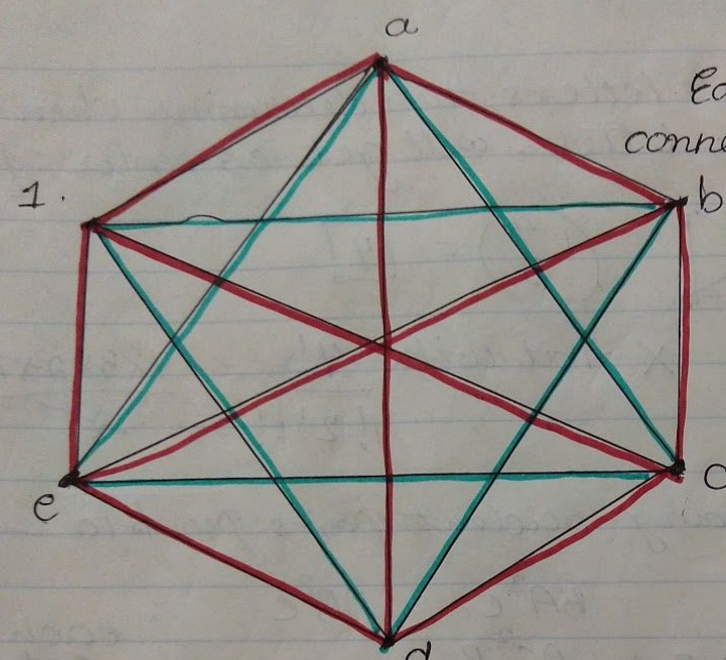
e. Since K_6 has only 6 vertices and each vertex has 6 multicolored ips, we can show that the graph K_6 can contain at most 36 multicolored ips.

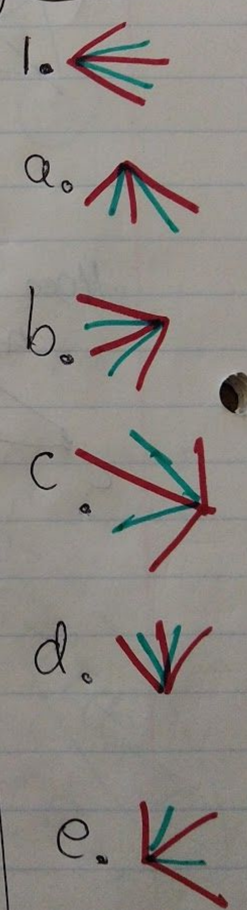
   i. A vertex with 5 edges coming out of it (vertex 1 in picture 2d) must have x red edges and y blue edges or x blue edges and y red edges (since either arrangement still gives us the same # of multicolored ips). In this case there are 3 red edges and 2 blue edges but it could have been 3 blue/2 red and in both cases we have a maximum of 6 multicolored ips.

   ii. Therefore consider 6 standalone vertices with 5 edges coming out of them with 3red/2blue or 3blue/2red, then there are exactly 6 vertices * 6 multicolored ips = 36 multicolored ips all together. This is reflected in the fact that each vertex has edges converging to it that reflect vertex 1. In the image below, we see that the remaining 5 vertices a,b,c,d, and e are all rotations of 1. Hence we have 6*6 = at most 36 multicolored ips.

   iii. Image of rotation is shown on the next page:

Each vertex connected to every other.





t vertex = 1.

ulticolored ips

$a1^2b \rightarrow \{\langle a-1\rangle, \langle 1-b\rangle\}$

$a1^2d \rightarrow \{\langle a-1\rangle, \langle 1-d\rangle\}$

$c1^2b \rightarrow \{\langle c-1\rangle, \langle 1-b\rangle\}$

$c1^2d \rightarrow \{\langle c-1\rangle, \langle 1-d\rangle\}$

$e1^2b \rightarrow \{\langle e-1\rangle, \langle 1-b\rangle\}$

$e1^2d \rightarrow \{\langle e-1\rangle, \langle 1-d\rangle\}$

iv.

f. A three person group is simply a triangle. Note that in our diagram above we have two blue triangles cae and 1bd. These are two uniform three-vertex groups. If the red edges were blue and the blue red, then we would have two uniform triangles in the center of the graph 'cae' and '1bd' be red.
   i. But how does 36 multicolored ips imply this result?
   ii. There are 6 choose 3 = 20 triangles in K_6. We know from c, that multicolored ips to multicolored triangles is a 2 to 1 function. Therefore if there are exactly 36 multicolored ips, then there must be 36/2 = 18 multicolored triangles.
   iii. If we counted exactly 20 triangles in K_6 and there are only 18 multicolored triangles then we know by inclusion/exclusion that there are 20-18 = 2 non-multi colored triangles in K_6.
   iv. Let a multi colored triangle signify a three person group of people who do not know each other (since multicolored ips signifies a pair of people who do not know each other for example's sake). Then non-multi colored triangles signify a three person group of people who do know each other.
      1. If there are 2 non multicolored triangles then this directly means that there are 2 three person groups who know each other in this graph.

3. 5-Card hands (13 ranks) (4 suits):
   a. We can pick any suit for each of the 5 cards, however we can only choose the first card out of 10 choices (A - 10). That is 10 * 4^5 = 10,240.

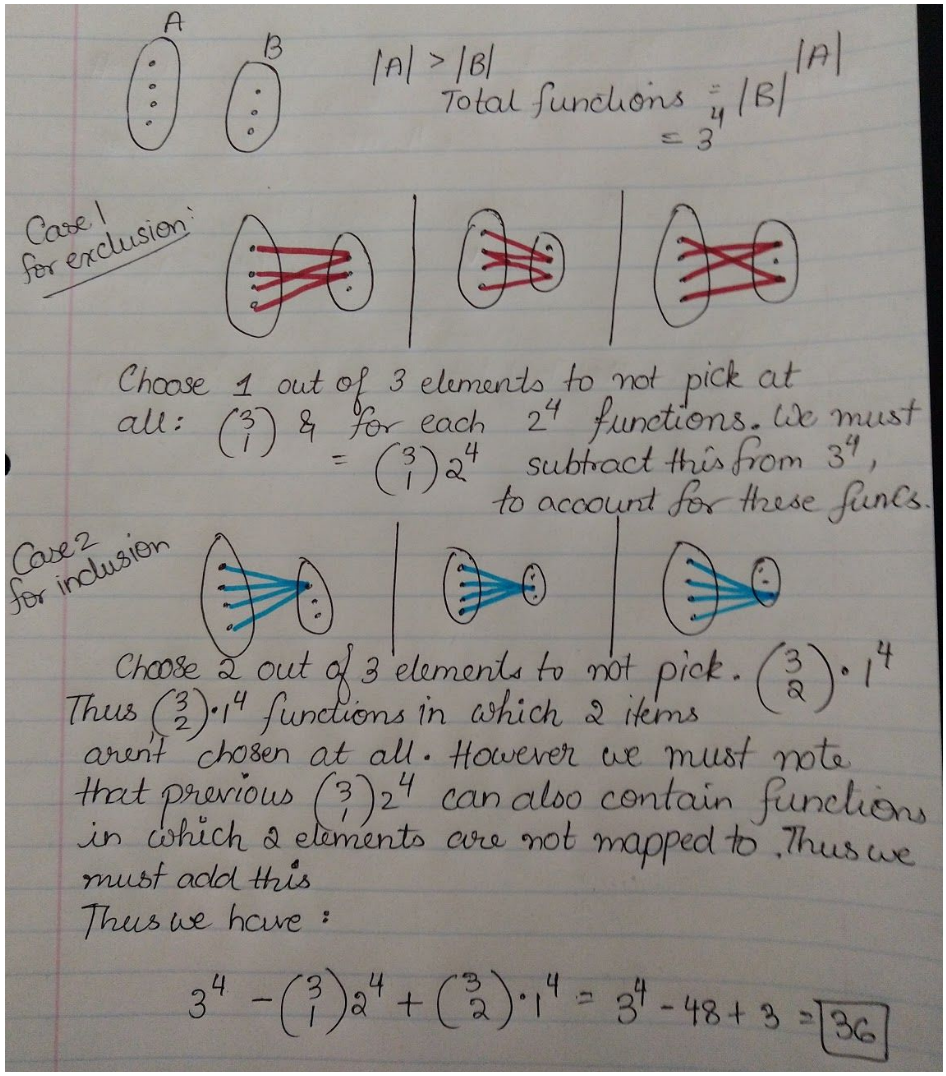
   b. We can pick one suit, then select 5 cards from the 13 cards in that suit. That is: (4 choose 1) * (13 choose 5) = 5148.

   c. We can pick one suit, then choose from A-10 for the first card to determine the sequence: (4 choose 1) * (10 choose 1) = 40.

   d. We know that all cards cannot be of the same suit. However we must have a sequence.
      i. (a) gave us the number of sequences where each card could be any suit.
      ii. (c) gave us the number of straight flushes where we had a straight but all cards were of the same suit.
      iii. | a | - | c | gives us the number of straights, i.e sequences which are not straight flushes.
      iv. Answer: 10,240 - 40 = 10,200.

4. ->
    a. For a function to be a bijection from A to B, the function must be injective (|A| <= |B|) and surjective (|A| >= |B|). In a bijective function, the cardinality of both the sets are equal. For each pigeon in A, there is a pigeonhole in B. Suppose the cardinality of both sets is n, then the A_1, the first pigeon, can choose B_x pigeonhole in B (n choose 1). The second pigeon in A has (n-1 choose 1) choices for pigeonholes. … The last pigeon in A has to choose the remaining pigeonhole in B (1 choose 1). Thus the first element in A could be mapped to n items. The second to n-1 items…. The second to last to 2 items. The last to 1 item. Therefore there are n! ways to map the items of A to B. Hence there are |B|! bijections from A to B.
    b. ->
        i. If |A| > (1) |B|, then the function cannot be injective (pigeonhole principle -> at least k+1 = 1+1 = 2 elements in A map to some element in B; but this means the function cannot be injective).
        ii. If |A| <= |B|, then we can have an injective function as each element of A can be mapped to at least 1 distinct element of B (1 pigeonhole for each pigeon). Thus given that |A| <= |B|, we have: |B|! /(|B| - |A|)! different ways to map elements of A to distinct elements of B. Note that if |A| = |B|, then this becomes |B|! / (1)! = |B|!. If |A| < |B| then the last element of A, has |B| - |A-1| remaining elements of B that it can map to. After all elements of A are mapped, we have |B| -|A| elements that have not been mapped. This is a falling factorial basically.

    c. We will apply the principle of inclusion exclusion to determine the number of onto functions from A to B. Note that if |A| < |B| then we cannot have surjectivity as there is at least 1 element of B not being mapped to. Thus we are working an A and B such that |A| >= |B|.
        i. We know that there are $|B|^{|A|}$ total functions.
        ii. However some of these functions result in 1 or more elements of B not mapped to by anything from element A. We must therefore ensure that we subtract these types of functions using inclusion exclusion.


A B

$|A| > |B|$

Total functions $= |B|^{|A|}$

$= 3^4$

Case 1 for exclusion:

Choose 1 out of 3 elements to not pick at all: $\binom{3}{1}$ & for each 2^4 functions. We must $= \binom{3}{1}2^4$ subtract this from 3^4, to account for these funcs.

Case 2 for inclusion

Choose 2 out of 3 elements to not pick. $\binom{3}{2}\cdot 1^4$

Thus $\binom{3}{2}\cdot 1^4$ functions in which 2 items aren't chosen at all. However we must note that previous $\binom{3}{1}2^4$ can also contain functions in which 2 elements are not mapped to. Thus we must add this

Thus we have:

$$3^4 - \binom{3}{1}2^4 + \binom{3}{2}\cdot 1^4 = 3^4 - 48 + 3 = \boxed{36}$$

iii.

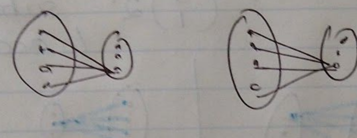
iv. Here we have taken the total number of functions $3^4 = 81$ and subtracted those functions (excluded) those in which 1 element of B doesn't get mapped to and then added those in which 2 elements of B don't get mapped to because case 1 and case 2 have some common elements. For example the image shows which functions are being deleted twice. And why they must be added once, so we do not delete the same function twice from the total:

v. Why do we add? Because we are subtracting some functions twice/more than once.

For example

Case 1

→ lets exclude 1, & exclude 2,



all map to 3 but we subtract this twice

Thus we must add the case where we exclude 1 & 2 from case 2 so that $-2+1 = -1$ & we only subtract one function once from the total.

This is inclusion/exclusion.

vi. Thus we can tweak the formula for inclusion/exclusion of sets to create an onto function formula. The derivation is shown below and formula is highlighted in brown (I've generalized using a smaller example):

vii.

$|B| = n = 3 \quad |A| = m = 4$

$$3^4 - \binom{3}{1}2^4 + \binom{3}{2}1^4$$
$$= n^m - \binom{n}{1}2^m + \binom{n}{2}1^m$$

$$= (-1)^0 n^m + (-1)^1\binom{n}{1}2^m + (-1)^2\binom{n}{2}1^m$$
$$= (-1)^0 n^m + (-1)^1\binom{n}{1}(n-1)^m + (-1)^2\binom{n}{2}(n-2)^m$$

$$= (-1)^0\binom{n}{0}n^m + (-1)^1\binom{n}{1}(n-1)^m + (-1)^2\binom{n}{2}(n-2)^m$$

Sum goes from 0 to $n-1$.

$$\sum_{k=0}^{n-1} \rightarrow \underbrace{\left[(-1)^k\binom{n}{k}n^m\right]}_{k=0}, \underbrace{\left[(-1)^k\binom{n}{k}(n-k)^m\right]}_{k=1}, \underbrace{\left[(-1)^k\binom{n}{k}(n-k)^m\right]}_{k=2}$$

$$\Rightarrow \boxed{\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}n^m} \leftarrow \text{\# Onto functions } f: A \rightarrow B \text{ provided } |A| \geq |B|$$

5. ->:

   a.

5a) Let $n=3$, $m=4$, & $r=2$.

Then we have:

$$= \binom{3}{0}\binom{4}{2} + \binom{3}{1}\binom{4}{1} + \binom{3}{2}\binom{4}{0}$$

$$= 1\times 6 + 3\times 4 + 1\times 3$$

$$= 6 + 12 + 3 = 9 + 12 = \boxed{21}$$

That is same as $\binom{3+4}{2} = \binom{7}{2}$

$$= \frac{7!}{2!\,5!} = \frac{7\times 6}{2\times 1} = 7\times 3 = \boxed{21}$$

Therefore, this binomial coefficient is same as $\binom{n+m}{r}$.

   b. Suppose I have n males and m females. How many ways can I select r people from this group of males and females?

      1. I can choose 0 males and r females, 1 male and r-1 females, 2 males and r-2 females, … r males and 0 females.
      2. OR
      3. I can view these groups of people as the same set and choose r people from the set.
      4. In case 1 I would have (n choose 0)(m choose r) + (n choose 1)(m choose r-1) + … + (n choose r)(m choose 0). In case 2 I would have (n+m choose r).
      5. Both would give me the same number of choices.

c. Combinatorial Proof: A dog breeder has n breeds of dogs with 1 male dog and 1 female dog for each breed. In total thus there are 2n dogs. Let's suppose you wish to start your own dog breeding business so you decide to buy half the dogs.

   i. If there are 2n dogs and you wish to take home n dogs from the breeder (half his stock), then the # of ways to select dogs to take home is (2n choose n).

   ii. To ensure that the dogs don't start breeding unless the breeder wants them to, the dog breeder puts a fence between all the male dogs and the female dogs. Thus on the left side of the fence there are n male dogs (for the n breeds) and similarly on the right side of the fence there are n female dogs.

   iii. Your business plan varies and on this particular day you wish to know how many ways you can select x male dogs and y female dogs. This set Z is the union of the disjoint sets of x/y where each x/y set corresponds to your choice of picking x males and y females. The xs come from the left side of the fence and the ys from the right side of the fence. Note that (n choose k) is equal to (n choose n-k) {factorials are the same}.

      1. If you're not a good businessman and you want to pick 0 males and all females, then you have (n males choose 0)(n females choose n). Here x = 0, y = n. Given that n choose 0 equals n choose n, we really have (n choose 0)^2.
      2. If you want to pick 1 male and the rest females, then you have (n males choose 1)(n females choose n-1). Here x=1, y=n-1. Given that n choose 0 equals n choose n, we really have (n choose 1)^2.
      3. If you want to pick 2 males and the rest females, then you have (n males choose 2)(n females choose n-2). Here x=2, y=n-2. Given that n choose 0 equals n choose n, we really have (n choose 2)^2.
      4. …
      5. If you want to pick n-1 males and the last dog as female, then you have (n males choose n-1)(n females choose 1). Here x=n-1, y=1. Given that n choose 0 equals n choose n, we really have (n choose n-1)^2.
      6. If you want to pick all males, then you have (n males choose n)(n females choose 0). Here x=n, y=0. Given that n choose 0 equals n choose n, we really have (n choose n)^2.

iv. These sets are all disjoint (i.e the number of males and females in each set differ). The size of this set is thus $= \sum_{k=0}^{n} \binom{n}{k}^2$. Because we are summing up the (n choose x)^2 where x goes from k=0 to k=n (# males chosen).

v. The set Z now contains all the sets where you have x males and y females but the total number of dogs add up to n. Thus you have a set Z describing the number of ways to select n dogs from the original 2n number of dogs that the breeder had.

vi. Thus this equality holds because you have counted the same thing but in different ways: $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.