Simple graphs

CS 206: Discrete Structures II

Simple graphs

A simple graph is a graph $G = (V, E)$ such that

- edges are undirected
- there are no self-loops
- there is at most one edge between any two vertices

Degree of a vertex

Two vertices are adjacent if there's an edge between them.

The set of vertices adjacent to X is the neighborhood of X, denoted $N(X)$.

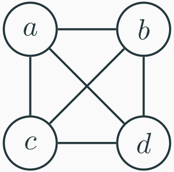
Each edge is incident to its endpoints.

The degree of a vertex is the number of edges incident to it.

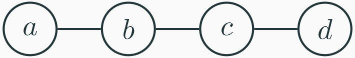
$$\sum_v \deg(v) = 2|E|$$

Common graphs

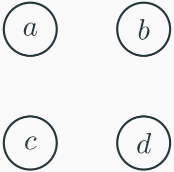
Complete graph K_n



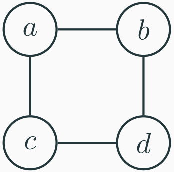
Path graph P_n



Empty graph



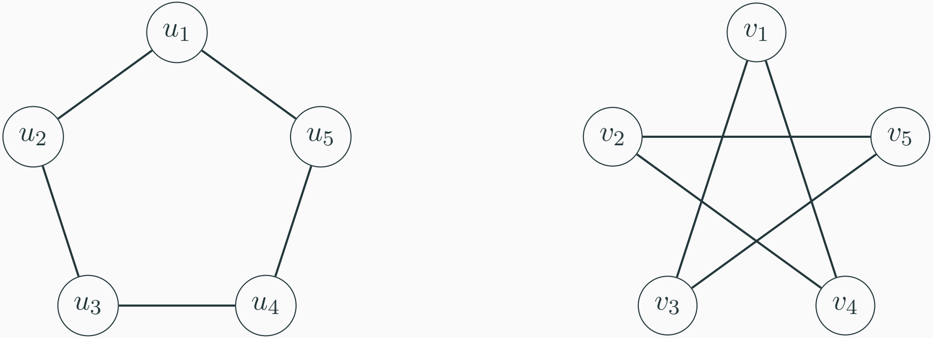
Cycle C_n



Isomorphism

An isomorphism is a bijection of the vertices of two graphs

$$(u, v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H)$$



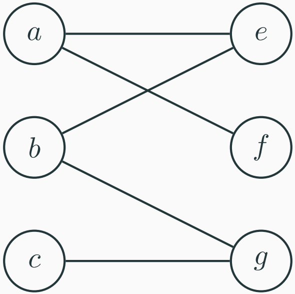
Isomorphism

Graph isomorphism is an equivalence relation:

- reflexive: $G \simeq G$
- symmetric: $G \simeq H \Rightarrow H \simeq G$
- transitive: $G \simeq H \wedge H \simeq I \Rightarrow G \simeq I$

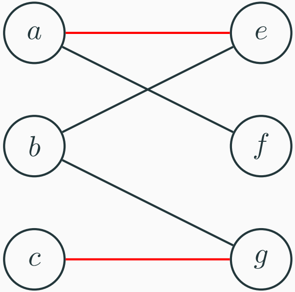
Bipartite graphs

A bipartite graph has vertices that can be partitioned into two sets, such that all edges are between these two sets.



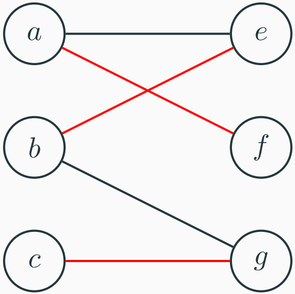
Matching

A matching is a set of non-adjacent edges.



Matching

A perfect matching is a matching where every vertex is incident to exactly one edge in the set.



Hall's theorem

Theorem (Hall's theorem)

We can match every vertex in A with one in B iff for all subsets S of A,

$$|S| \leq |N(S)|$$

Let's prove the $\Rightarrow$ direction.

Suppose we have a matching M that matches every vertex in A.

For any $S \subseteq A$, we need $|S| \leq |N(S)|$.

Let $M(S)$ be the vertices matched to every $x \in S$:

- $|S| = |M(S)|$
- $M(S) \subseteq N(S)$, so $|M(S)| \leq |N(S)|$.

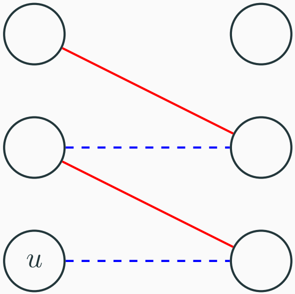
Hence $|S| \leq |N(S)|$. $\square$

Hall's theorem

Now let's prove the $\Leftarrow$ direction.

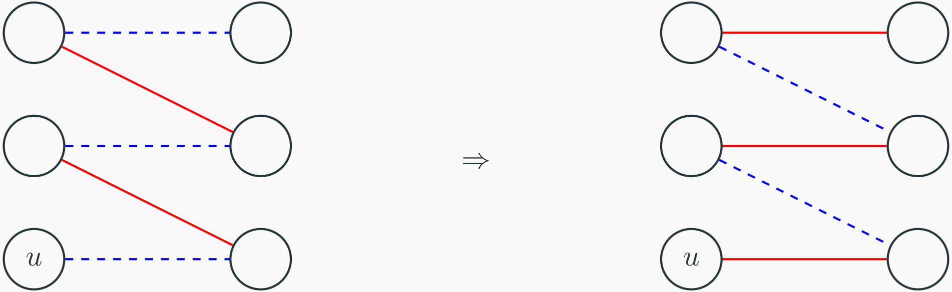
We'll do this by contradiction, so suppose the subset condition holds, but a maximal matching M doesn't match some vertex $u \in A$.

Let's build a path from u by alternating edges not in M and edges in M.



Hall's theorem

Note that the path can't end in B. If it did, we could swap edges in/not in M and increase the size of our matching:

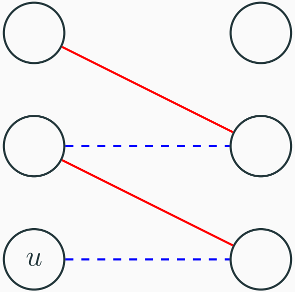


Hall's theorem

Of the vertices u can reach via this path:

- let A_u be those in A (and $u \in A_u$)
- let B_u be those in B

The matching M provides a bijection from $A_u - \{u\}$ to B_u, so



$$|A_u - \{u\}| = |B_u|$$

or

$$|A_u| - 1 = |B_u|$$

Hall's theorem

We also have that

$$N(A_u) \subseteq B_u$$

Suppose $b \in B$ is connected to $a \in A_u$.

If this edge is in M, then b is in B_u.

If not, for a path ending in a, we could add (a, b), flip the edges in/not in M, and get a larger matching.

So

$$|N(A_u)| \leq |B_u|$$

Hall's theorem

So we have:

- $|B_u| = |A_u| - 1 < |A_u|$
- $|N(A_u)| \leq |B_u|$

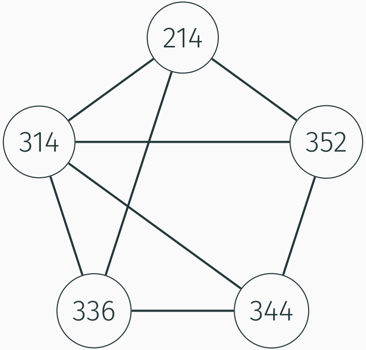
Thus $|N(A_u)| < |A_u|$

But this violates the subset condition assumption! $\square$

Scheduling final exams

If any student is taking both course A and course B,

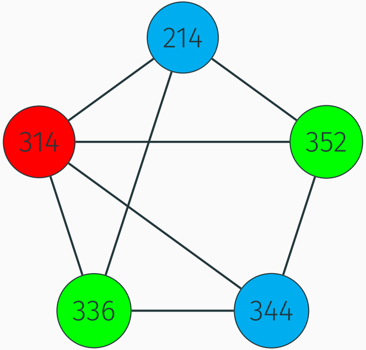
- they must be scheduled at different times
- represent this as an edge between those classes



Graph coloring

Then if final exam time slots are represented by colors,

- each vertex needs a color
- adjacent vertices can't be the same color



Graph coloring

A coloring of a graph is an assignment of colors to vertices such that no adjacent vertices have the same color.

A graph G is k-colorable if there's a coloring that uses at most k colors.

The minimum number of colors needed is called the graph's chromatic number, denoted $\chi(G)$.

Coloring common graphs

How many colors would we need for these?

- even cycles
- odd cycles
- complete graphs
- empty (no edge) graphs
- bipartite graphs

Coloring common graphs

- $\chi(C_{\text{even}}) = 2$
- $\chi(C_{\text{odd}}) = 3$
- $\chi(K_n) = n$
- $\chi(\text{Empty}_n) = 1$
- $\chi(\text{Bipartite}_n) = 2$

Theorem

Let Δ be the max degree of G. Then $\chi(G) \leq \Delta + 1$.

Proof.

Induction on the number of vertices k.

For $k \leq 1$, we have $\Delta = 0$, and we can use 1 color.

Assume it holds for k, and G has $k + 1$ vertices.

Remove any vertex v. Then the inductive hypothesis gives us a coloring of $G - \{v\}$.

Then add v back.

It's connected to at most Δ vertices, but we have $\Delta + 1$ colors, so we can pick any unused color for v. $\square$

Graph coloring

Uses of graph coloring:

- scheduling server updates
- assigning radio station frequencies
- allocating registers for variables
- coloring maps

Walks, paths, cycles

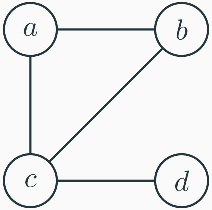
Similar to directed graphs:

- a walk is a sequence of vertices connected by edges
  - the length of the walk is the number of edges
- a path is a walk where all vertices are distinct
- a closed walk starts and ends at the same vertex
- a cycle is a closed walk of length > 2
  - (vertices are distinct except the start and end)

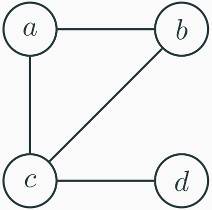
Subgraphs

H is a subgraph of G if:

- $V(H) \subseteq V(G)$
- $E(H) \subseteq E(G)$



For example, we can define a cycle of a graph as a subgraph that's isomorphic to some C_n

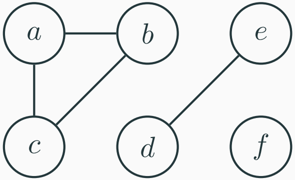


A subgraph isomorphic to C_3:

- $V = \{a, b, c\}$
- $E = \{(a, b), (b, c), (a, c)\}$

Connected components

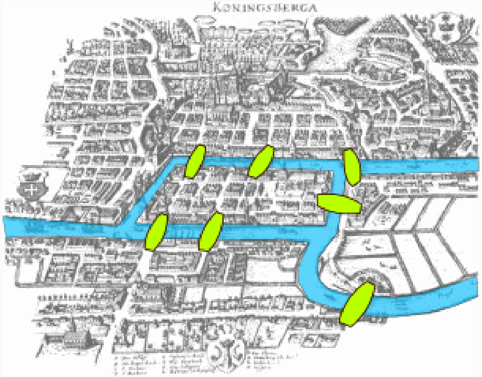
A connected component of a graph is a set of vertices that can all reach each other.



- $\{a, b, c\}$
- $\{d, e\}$
- $\{f\}$

Seven bridges of Königsberg

Can you devise a walk through the city of Königsberg that crosses each bridge exactly once?



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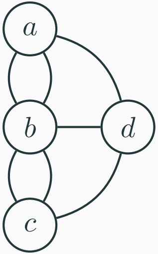
Euler tours

- An Euler walk uses each edge exactly once
- An Euler tour also ends where it started



- $a \to b \to c \to b \to a \to d \to c$ (misses $b \to d$)
- $a \to b \to c \to b \to d \to a \to b$ (misses $b \to c$)
- ...

Euler tours

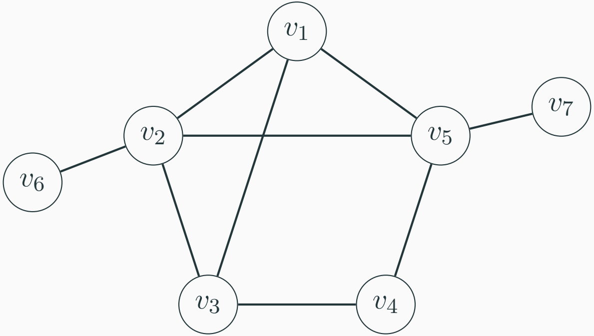
What do these have to do with vertex degrees?

- An Euler tour exists if all vertices have even degree.
- An Euler walk exists if exactly two vertices have odd degree.

Hamiltonian cycles

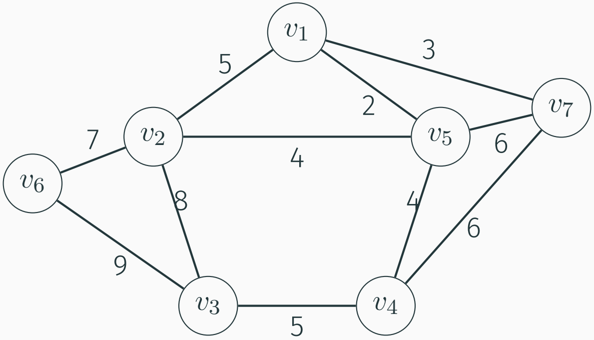
Instead of edges, we can try to visit every vertex exactly once:

- Such a path is a Hamiltonian path
- Such a cycle is a Hamiltonian cycle



Traveling salesman problem

The traveling salesman problem (TSP) is to find, given a graph with edge weights, a Hamiltonian cycle of minimum cost.



Traveling salesman problem

One possible route:

- $v_1 \to v_2 \to v_6 \to v_3 \to v_4 \to v_5 \to v_7 \to v_1$
- cost is $5 + 7 + 9 + 5 + 4 + 6 + 3 = 39$