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1)

a) $P(x) = 1 + x + x^2 + x^3 + x^4 + \dots = \boxed{\dfrac{1}{1-x}}$

b) $N(x) = 1 + x^5 + x^{10} + x^{15} + \dots = 1 + u + u^2 + \dots$ where $u = x^5$

$N(x) = \dfrac{1}{1-u} = \boxed{\dfrac{1}{1-x^5}}$

c) $N(x) \cdot P(x) = NP(x) = \dfrac{1}{1-x^5} \cdot \dfrac{1}{1-x} = \boxed{\dfrac{1}{(1-x^5)(1-x)}}$

d) $P(x) \cdot N(x) \cdot D(x) \cdot Q(x) \cdot H(x) =$

$\left(\dfrac{1}{1-x}\right) \cdot \left(\dfrac{1}{1-x^5}\right) \cdot \left(1 + x^{10} + x^{20} + \dots\right) \cdot \left(1 + x^{25} + x^{50} + \dots\right) \cdot \left(1 + x^{50} + x^{100} + \dots\right)$

$= \dfrac{1}{1-x} \cdot \dfrac{1}{1-x^5} \cdot \dfrac{1}{1-x^{10}} \cdot \dfrac{1}{1-x^{25}} \cdot \dfrac{1}{1-x^{50}} =$

$= \boxed{\dfrac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}}$

e) By converting the closed form (boxed above) into an infinite series (a generating function of the form $1+?x + ?x^2 + \dots$) we can check the coefficient of the x^{50} term to get the number of ways to make 50 cents using pennies, nickels, dimes, quarters, and half-dollars.

2.

a) $C(x) = x^3 + x^4 + x^5 + \ldots = x^3(1 + x + x^2 + \ldots) = \boxed{\dfrac{x^3}{1-x}}$

b) $g(x) = 1 + x + x^2$

c) $Co(x) = 1 + x^2$

d) $P(x) = 1 + x^4 + x^8 + x^{12} + x^{16} + \ldots =$ let $u = x^4$: $1 + u + u^2 + u^3 + \ldots$

$= \dfrac{1}{1-u} = \boxed{\dfrac{1}{1-x^4}}$

e) $C(x)g(x)Co(x)P(x) \Rightarrow$ {Convolution Rule}

$= \dfrac{x^3}{1-x} \cdot (1+x+x^2) \cdot (1+x^2) \cdot \dfrac{1}{1-x^4}$ ← diff of squares

$= \dfrac{x^3(1+x+x^2)}{(1-x)(1-x^2)} = \dfrac{x^3(1+x+x^2)}{(1-x)(1-x)(1+x)} = \boxed{\dfrac{x^3(1+x+x^2)}{(1-x)^2(1+x)}}$

f) $E(x) = \dfrac{\cancel{x^3}(1+x+x^2)}{(1-x)^2(1+x)}$; $x^3 E(x) = \dfrac{1+x+x^2}{(1-x)^2(1+x)}$;

$\dfrac{1+x+x^2}{(1-x)^2(1+x)} = \dfrac{A}{1+x} + \dfrac{B}{1-x} + \dfrac{C}{(1-x)^2}$;

$1 + x + x^2 = A(1-x)^2 + B(1-x)(1+x) + C(1+x)$

$$x^3E(x) = \frac{1+x+x^2}{(1-x)^2(1+x)} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

$$x^3E(x) \Rightarrow 1+x+x^2 = A(1-x)^2 + B(1+x)(1-x) + C(1+x)$$

$x=1;\quad 2C=3;\ \boxed{C=3/2}$

$x=0;\quad 1=A+B+C$

$x=-1;\quad 1=4A;\ \boxed{A=1/4}$

$$\boxed{x^3E(x) = \frac{1}{4(1+x)} + \frac{3}{2(1-x)^2} - \frac{3}{4(1-x)}}$$

$$B = 1 - \frac{1}{4} - \frac{3}{2} = \boxed{\frac{-3}{4}}$$

F continued: If n <3, then the coefficient is 0 and there are no ways to pick a negative number of donuts. If n =3, there is only 1 way to pick 0 donuts (the null set). Let us restrict, then, n to be greater than 3, because our sequence has no $x^{-1, -2, -3}$ terms $\{[x^n]*x^3 = [x^{n-3}]\}$.

$$[x^n]\,x^3E(x) \Rightarrow \frac{1}{4}(1-x+x^2+\dots) + \frac{3}{2}(1+2x+3x^2+\dots) + \left(-\frac{3}{4}\right)$$

$$[x^n]\,x^3E(x) = \frac{1}{4}(-1)^n + \frac{3}{2}(n+1) - \frac{3}{4}$$

$$[x^{n-3}]\,E(x) = \frac{1}{4}(-1)^{n-3} + \frac{3}{2}(n-2) - \frac{3}{4}$$

$$= 0.25(-1)^{n-3} + 1.5(n-2) - 0.75$$

$$[x^{n-3}]\,E(x) = 1.5n - 3.75 + 0.25(-1)^{n-3}$$

n has to be greater than 3.

Closed form for $[x^n]$

3.

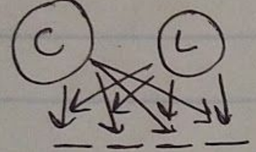
a) Songbirds: $x^2+x^4+x^6+\dots = x^2(1+x^2+x^4+x^6+\dots)$

let $u=x^2$: $u(1+u+u^2+u^3+\dots) = \frac{u}{1-u} = \boxed{\frac{x^2}{1-x^2}}$

Freddy: $\boxed{1+x}$ Cats: $x^2+x^3+x^4+\dots = \boxed{\frac{x^2}{1-x}}$

Dogs: Two brands (sets) to pick from. $= \boxed{2^n}$

Say $n=4$



$= 2^n$ } the set of all bit strings where 0 = chihuahua & 1 = labrador.

Dogs: $2^2x^2+2^3x^3+2^4x^4+\dots$

$= 4x^2+8x^3+16x^4+\dots = 4x^2(1+2x+4x^2+\dots)$

$= 4x^2(1+u+u^2+\dots)$ where $u=2x \to \frac{4x^2}{1-u} = \boxed{\frac{4x^2}{1-2x}}$

Pets(x) = Songbirds(x) Freddy(x) Cats(x) Dogs(x)

$= \frac{x^2}{1-x^2}\cdot\frac{x^2}{1-x}\cdot 1+x \cdot \frac{4x^2}{1-2x}$

$= \frac{x^4\cdot 4x^2}{(1-x)(1-2x)}\cdot\frac{1+x}{1-x^2} = \frac{4x^6}{(1-2x)}\cdot\frac{1}{(1-x)}\cdot\frac{\cancel{1+x}}{(1-x)\cancel{(1+x)}}$

$= \frac{4x^6}{(1-x)^2(1-2x)}$ This matches $P(x)$.

b) $P(x) = \frac{4x^6}{(1-x)^2(1-2x)}$: $4x^6 g(x) = \frac{1}{(1-x)^2(1-2x)}$

$\frac{1}{(1-x)^2(1-2x)} = \frac{A}{(1-x)^2} + \frac{B}{(1-x)} + \frac{C}{(1-2x)}$; $1 = A(1-2x) + B(1-x)(1-2x) + C(1-x)^2$

let $x=1$; $1 = A(1-2) \Rightarrow \boxed{A=-1}$ let $x=0$: $A+B+C=1$

let $x=1/2$; $1 = C(1/2)^2 \Rightarrow \boxed{C=4}$ $B = 1-A-C = 2-4 = -2$; $\boxed{B=-2}$

$4x^6 g(x) = \frac{-1}{(1-x)^2} - \frac{2}{(1-x)} + \frac{4}{(1-2x)}$; $1+2x+3x^2+\ldots \to n+1$; $1+x+x^2+\ldots \to 1$; $1+2x+4x^2+8x^3+\ldots \to 2^n$

$[x^n][4x^6 g(x)] = -1(n+1) - 2(1) + 4(2^n)$

$4\cdot[x^{n-6}](g(x)) = -n-1-2+4\cdot 2^n = 2^{n+2} - n - 3$

$4[x^{n-6}]g(x) = 2^{(n-6)+2} - (n-6) - 3 = 2^{n-4} - n + 3$

$4[x^{n-6}][2^{n-4} - n + 3]$

{Const factor} $[x^{n-6}][4[2^{n-4} - n + 3]] = [x^n][2^{n-2} - 4n + 12]$

$P(n) = 2^{n-2} - 4n + 12$ BUT only for $n \geq 6$

Lets just say that Miss McGillicuddy shouldn’t take out less than 6 animals. That's not funky enough.

4.

a. GoodCount having a set of strings that have good counts implies that no string starts with a right bracket because then the running total would go to -1. Furthermore in each element of GoodCount, the number of right brackets equals the number of left brackets (start and end sum is 0). For every left bracket, there is a right bracket. However since the count never goes below -1, we also know for a string with n left brackets and n right brackets, the last (nth) right bracket MUST come at the very end of the string. If it did not then we would have a situation where running total would go to -1.

   i. For example: [[ ]]] [. Here sum goes to -1 in the second to last bracket.

   ii. Thus we know that each element of GoodCount has a left starting bracket and a right ending bracket (if it didn't it would violate good count property).

   iii. Let us consider the x^0 term. Here we have 0 left brackets. Thus we have 0 right brackets. However this does not classify as a wrap since there are no left and right brackets in a good count property fulfilling way.

   iv. Thus we must remove this x^0 term from consideration. Then algebraically we then get:

   v.
$$C(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$
$$C(x) - c_0 = c_1 x + c_2 x^2 + c_3 x^3 + \dots$$
$$\underbrace{C(x) - c_0} = x\underbrace{(c_1 + c_2 x + c_3 x^2 + \dots)}$$

Wraps of GoodCount Set $\Rightarrow$ $x \cdot C(x)$. $\therefore$ Wraps of Strings from goodcount set $= \boxed{xC(x)}$

b. This question asks why each after appending strings from the set of GoodCount we get a string which is also an element of the set GoodCount.

   i. Unless a string has ALL its RIGHT brackets lined up together after its left brackets, then that string can be rewritten as some other string appended to by another string.

ii. [[[ ]]] cannot be written as a wrap of any other string but itself. Because, if it could, then we would violate the good count property (thus the strings which can be written as wraps of some strings must not have all of its n right brackets grouped up towards the end).

iii. If they do not, then we are guaranteed that at least 2 strings exist which can be appended to each other to form the string s. Because then the first ½ length(S) brackets of string S are not n left brackets. For some strings like this:

1. [[ ] [ ]] [ ] [ ] can be [[ ] [ ]] + [ ] [ ]. This is guaranteed as the first and last half the string are not left brackets and right brackets respectively. Note that [[ ] [ ]] [ ] [ ] = [[ ] [ ]] + [ ] + [ ] is not possible as [ ] is a duplicate and thus we have s = $s_1s_2s_2$ where s_2 repeats and thus the set of strings appended to form s are not unique.
2. Given the property of goodcount then, each string starts its first letter with a running count of +1. We can form a new string **as soon as** this running count hits 0. Then start again and stop as soon as the running count hits 0. And repeat until we run out of runway in s. Those mini strings can be appended to form the string S.
3. But why are they unique? Because if we stop as soon as the count hits 0 after starting from 1, then we know that that mini string we have can be rewritten as NO other string. By picking strings $s_1...s_k$, we are assured that those are the smallest possible strings in GoodCount that when appended in order of 1,2,...k, form s.
4. How are we assured of this? Because if they did not then we would have a running total go to 0 after the initial +1 for some string s_i and we would have ourselves another mini string s_i. However since we selected the string which gets to running a total of 0 earliest in the first place, s_i would be that string we selected in the first place. Hence our selection would be unique.

c. I do not have the answer for how to conclude C = 1 + xC + (xC)^2 + …

   i. But I do have the rest

(1) $C = 1 + xC + (xC)^2 + (xC)^3 + \dots$

(2) $C = 1 + u + u^2 + u^3 = \frac{1}{1-u} = \frac{1}{1-xC}$

(3) $C = \frac{1}{1-xC}$: $C^2x - C + 1 = 0$; {Quadratic f.} $= C = \frac{1 \pm \sqrt{1-4(x)(1)}}{2(x)}$

$C = \left(1 \pm \sqrt{1-4x}\right)/2x$

(4) $D(x) = 2x(C(x))$. if $D(x) = d_0 + d_1x^1 + d_2x^2 + \dots$

$\frac{D(x)}{2(x)} = C(x) = \frac{d_0x^{-1} + d_1x^0 + d_2x + d_3x^2}{2}$

$[x^n]C(x) = [x^n]\left[\cancel{d_0} + \frac{d_1 + d_2x + d_3x^2 + \dots}{2}\right]$

$[x^n]C(x) = \frac{d_{n+1}}{2}$ {ensure no x^i where $i<0$}

$C_n = \frac{d_{n+1}}{2}$

   ii.

d.

$d(x) = 1 - \sqrt{1-4x}$

$C_n = \dfrac{d_n+1}{2}$; $C_0 = \dfrac{d_0+1}{2}$

$c_0 = \dfrac{d_1}{2}$

$2(c_0) = d_1 = 2$

$\mathcal{G} \subseteq$ goodcount s.t $|\mathcal{G}| = 0 \rightarrow \mathcal{G} = \emptyset$

Only 1 $\emptyset$ in all subsets of goodcount

if $d_1 = 2$ then $\dfrac{d'(0)}{1!} = 2$

Verify: $\dfrac{d}{dx}\left(1 - \sqrt{1-4x}\right)$

$= \frac{1}{2} 4 (1-4x)^{-1/2}$

$= \dfrac{4}{2\sqrt{1-4x}}$; $d'(0) = \dfrac{4}{1 \cdot 2} = 2$; $\dfrac{d'(0)}{1!} = d'(0)$

d)

$\boxed{d_1 = d'(0) = 2}$

This is actually part e.

d)

$D(x) = 1 - (1-4x)^{1/2}$

$D'(x) = 2(1-4x)^{-1/2}$

$D''(x) = 4(1-4x)^{-3/2}$

$D'''(x) = 24(1-4x)^{-5/2}$

$D^{iv}(x) = 240(1-4x)^{-7/2}$

$D^{5}(x) = 3360(1-4x)^{-9/2}$

$D^{6}(x) = 60{,}480$

$d_1 = \boxed{2}$; $d_2 = \frac{4}{2!} = \boxed{2}$; $d_3 = \frac{24}{3!} = \boxed{4}$; $d_4 = \frac{240}{4!} = \boxed{10}$;

$d_5 = \frac{3360}{5!} = \boxed{28}$; $d_6 = \frac{60480}{6!} = \boxed{84}$;

$\langle \underset{1}{2}, \underset{2}{2}, \underset{3}{4}, \underset{4}{10}, \underset{5}{28}, \underset{6}{84}, \ldots \rangle$

~~Vefr~~ Verify: Let us choose d_6 for least chance of accidental correct ans.

$$d_6 = \frac{(2(6)-3)(2(6)-5)(2(6)-7)(2(6)-9)(2(6)-12)2^6}{6!}$$

$$= \frac{9 \times 7 \times 5 \times 3 \times 1 \times 2^6}{6!} = \boxed{84}$$

Our results match $d_6 = 84$ ✓.

8)

$$C_n = \frac{1}{n+1}\binom{2n}{n} \Rightarrow$$

$$C_n = \frac{1}{n+1} \cdot \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = \frac{0!}{1!0!} = 1 \qquad C_3 = \frac{6!}{4!3!} = 5$$

$$C_1 = \frac{2!}{2!1!} = 1 \qquad C_4 = \frac{8!}{5!4!} = 14.$$

$$C_2 = \frac{4!}{3!2!} = 2 \qquad C_5 = \frac{10!}{6!5!} = 42$$

$C_n = \langle 1, 1, 2, 5, 14, 42, \ldots \rangle$

Recall $d_{n+1} = \langle 2, 2, 4, 10, 28, 84, \ldots \rangle$

$\frac{d_{n+1}}{2}$ exactly matches C_n.

Therefore we conclude that the two share the relationship $C_n = \frac{d_{n+1}}{2}$ & that $\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$.