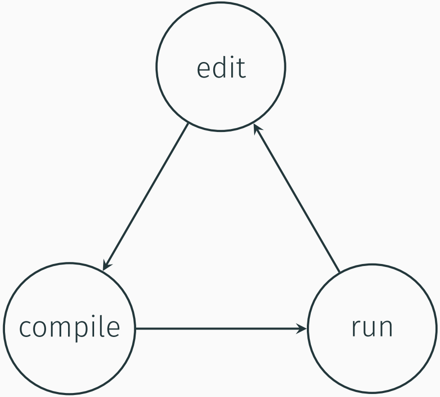
Directed graphs and partial orders

CS 206: Discrete Structures II

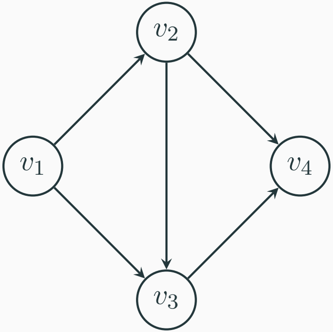
Directed graphs



Directed graphs (or digraphs) are a model of a set of things where you can go from one thing to another.

- state transitions
- web links
- twitter followers

Directed graphs



A directed graph G contains:

- vertices $V(G)$
- edges $E(G)$

(often written $G = (V, E)$)

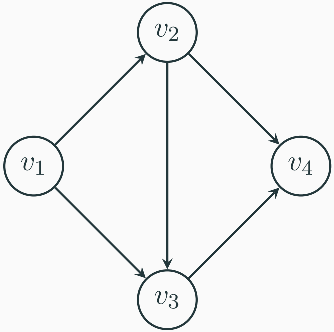
Directed edges

A directed edge (u, v)
(or $\langle u \to v \rangle$)

- u is the tail
- v is the head

$E \subseteq V \times V$

Degree of a vertex

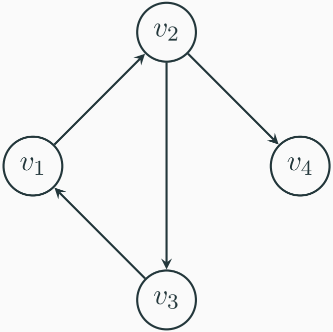


- indegree: # incoming edges
- outdegree: # outgoing edges

For example,

- v_2 has indegree 1
- v_2 has outdegree 2

Walks



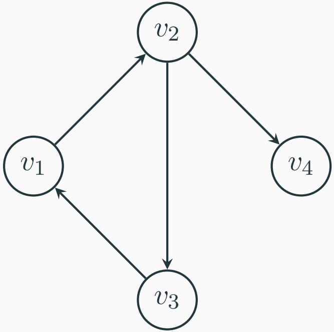
A walk is a sequence of vertices connected by edges.
For example,

$$v_1 \to v_2 \to v_3 \to v_1 \to v_2 \to v_4$$

A closed walk ends where it began:

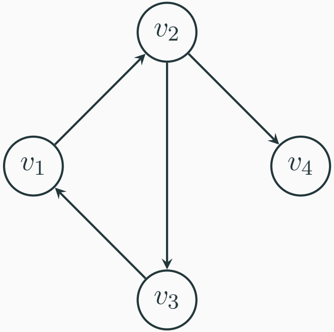
$$v_1 \to v_2 \to v_3 \to v_1$$

Paths



A path is a walk that doesn't repeat vertices.
For example,

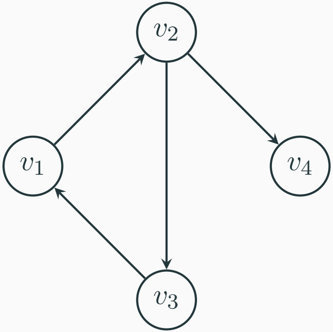
$$v_3 \to v_1 \to v_2 \to v_4$$



A cycle is a closed walk (with length > 0) where all vertices are unique (except the first and last): For example,

$$v_1 \to v_2 \to v_3 \to v_1$$

Merging paths



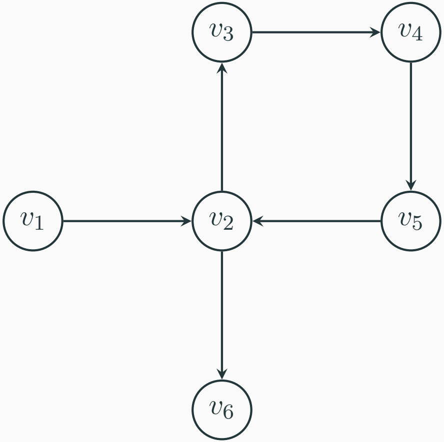
Given a walk that ends at v and another that starts there, we can merge these into a single new walk:

- $f = v_1 \to v_2 \to v_3$
- $g = v_3 \to v_1 \to v_2 \to v_4$

Then

- $f\hat{\ }g = v_1 \to v_2 \to v_3 \to v_1 \to v_2 \to v_4$

Finding paths



Theorem:
A shortest walk is a path.

How to get from v_1 to v_6?

Finding paths

Theorem

A shortest walk in a digraph is a path.

Proof.

Let w be the minimum length walk from u to v. Assume w is not a path, that is, that some vertex x is repeated:

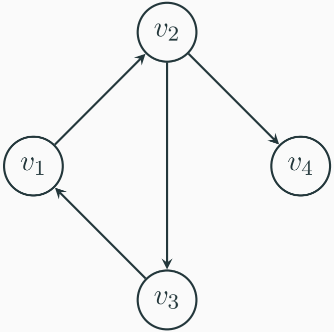
$$u \to \cdots \to x \to \cdots \to x \to \cdots \to v$$

Then cut out the $x \to \cdots \to x$ part to get a shorter walk:

$$u \to \cdots \to x \to \cdots \to v$$

$\square$

Distances in graphs



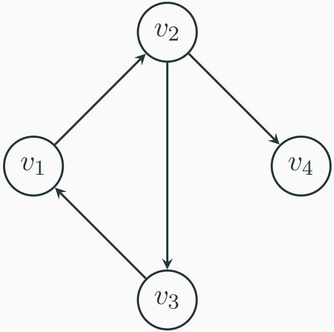
The distance from u to v is the length of the shortest path.

$$\text{dist}(v_1, v_4) = 2$$

Note: the triangle inequality holds:

$$\text{dist}(u, v) \leq \text{dist}(u, x) + \text{dist}(x, v)$$

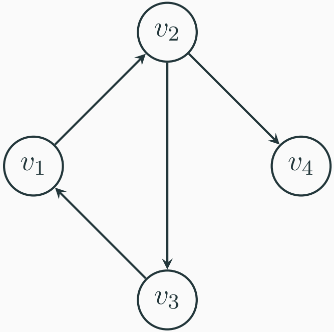
Adjancency matrices



If G has n vertices, we can represent it by an $n \times n$ adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

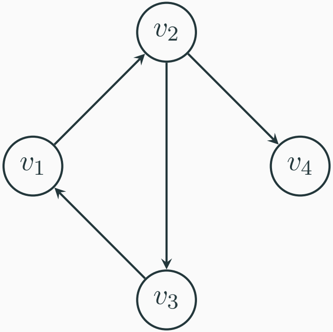
Adjacency matrices



$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

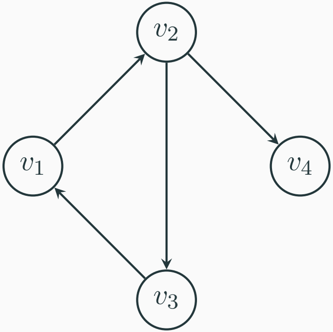
Adjacency matrices



$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A^k counts the number of length k walks!

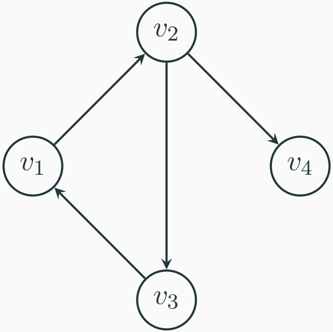
Walk relations



We can define a walk relation G^* where $u\ G^*\ v$ means that there is a walk from u to v in graph G:

- $v_1\ G^*\ v_4$
- $v_3\ G^*\ v_2$

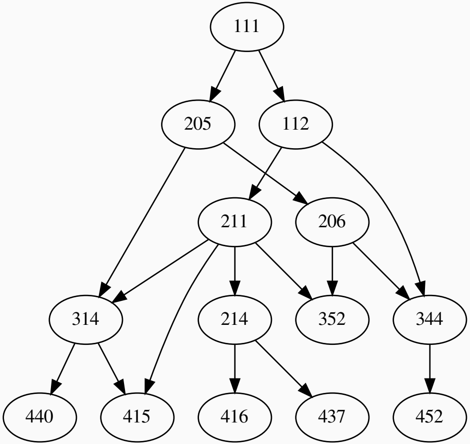
Walk relations



If G^n means there is a length n walk from u to v, then

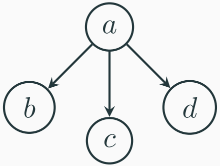
$$G^* = \bigcup_{i=0}^{n-1} G^i$$

DAGs



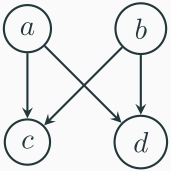
A directed acyclic graph (DAG) is a directed graph with no cycles.

DAGs



A minimum in a DAG is a node that can reach every other node.

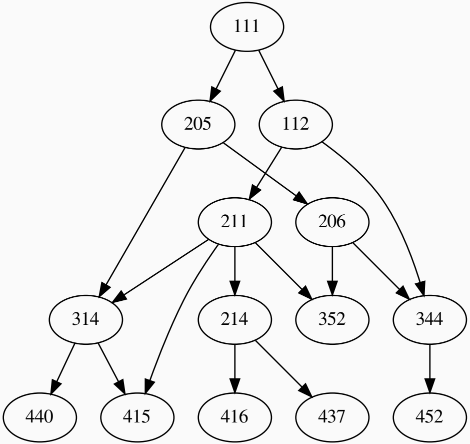
- a



A minimal node is one not reachable by any other node.

- a
- b

Scheduling

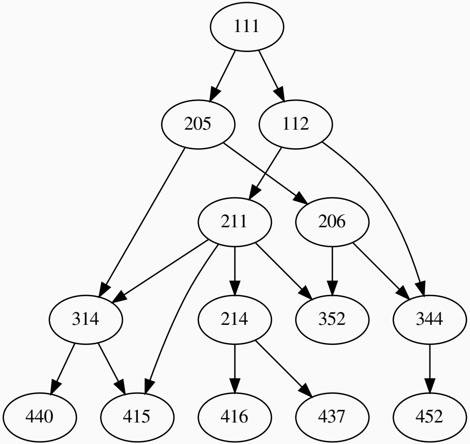


To build a schedule:

- pick a minimal node
- remove it
- repeat

This is a topological sort.

Chains



A chain is a sequence of connected nodes.

- $111 \to 205 \to 206$

An antichain is a set of nodes with no connections between them.

- $\{314, 214, 352\}$

Partial orders

Two ways of looking at this:

- a digraph $G = (V, E)$
- a binary relation from V to V

Relation properties

- transitivity: $a \; R \; b$ and $b \; R \; c \Rightarrow a \; R \; c$
- reflexivity: $a \; R \; a$
- irreflexivity: $\neg(a \; R \; a)$
- symmetry: $a \; R \; b \Rightarrow b \; R \; a$
- asymmetry: $a \; R \; b \Rightarrow \neg(b \; R \; a)$
- antisymmetry: $a \; R \; b$ and $b \; R \; a \Rightarrow a = b$

Partial order

A partial order is a binary relation that is

- transitive
- reflexive
- antisymmetric

The relation is often written $\preceq$

Strict partial orders

A strict partial order is a binary relation that is

- transitive
- irreflexive
- asymmetric

The relation is often written $\prec$

Note: a strict partial order is a dag.

Natural numbers

For example, consider $\mathbb{N}$ under the usual $\leq$ relation.

- transitive: $a \leq b, b \leq c \Rightarrow a \leq c$
- reflexive: $a \leq a$
- antisymmetric: $a \leq b, b \leq a \Rightarrow a = b$

So this is a partial order!

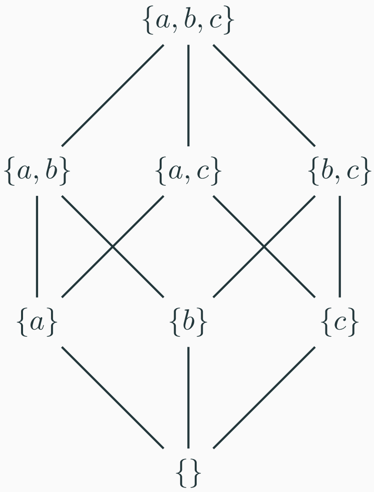
Natural numbers

But natural numbers under $\leq$ have another property:

- for any $x, y \in \mathbb{N}$, either $x \leq y$ or $y \leq x$ (or both)

So this is a total order!

Subsets



Consider the subsets of $\{a, b, c\}$ and how they are ordered under the $\subseteq$ relation.

- $\{\} \subseteq \{b, c\}$
- $\{a\} \subseteq \{a, c\}$
- ...

$\subseteq$ is transitive, reflexive, and antisymmetric, so it's a partial order.

(This is called a Hasse diagram)

Equivalence relations

An equivalence relation is

- transitive
- reflexive
- symmetric

For example, $=$ is an equivalence relation.

Equivalence classes

Given an equivalence relation $R : A \to A$, and element a

the equivalence class $[a]_R$ is

$$\{x \in A | a \; R \; x\}$$

Equivalence classes

For example, consider $\mathbb{N}$ and equality mod 5.

- $[3] = \{3, 8, 13, 18, 23, \ldots\}$
- $[5] = \{0, 5, 10, 15, 20, \ldots\}$
- ...

Note: equivalence relations yield a partition of their domain!