Probability: Random variables

CS 206: Discrete Structures II

Independence

Intuitively, rolling two dice should be “independent”.

The result of one roll doesn’t affect the other.

Formally, events A and B being independent means

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

Independence

Since we have

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

another way to define independence is

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

Independence

For example, flip two coins. We assume they're independent, so the probability of two heads is

$$\mathbb{P}(HH) = \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

where the events are:

- A: heads on the first coin
- B: heads on the second coin

Independence

We can extend this to > 2 events. For events A, B, and C to be mutually independent, we need

- $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$
- $\mathbb{P}(B \cap C) = \mathbb{P}(B)\,\mathbb{P}(C)$
- $\mathbb{P}(A \cap C) = \mathbb{P}(A)\,\mathbb{P}(C)$
- $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$

Independence

For example, flip 3 coins (call their resulting heads/tails values c_1, c_2, and c_3), and define these events:

- $A_1 : c_1 = c_2$
- $A_2 : c_2 = c_3$
- $A_3 : c_3 = c_1$

Are these events mutually independent?

Independence

For example, flip 3 coins (call their resulting heads/tails values c_1, c_2, and c_3), and define these events:

- $A_1 : c_1 = c_2$
- $A_2 : c_2 = c_3$
- $A_3 : c_3 = c_1$

$A_1 = \{HHH, HHT, TTH, TTT\}$

$\mathbb{P}(A_1) = \frac{1}{2}$

Similarly for A_2 and A_3.

Independence

For example, flip 3 coins (call their resulting heads/tails values c_1, c_2, and c_3), and define these events:

- $A_1 : c_1 = c_2$
- $A_2 : c_2 = c_3$
- $A_3 : c_3 = c_1$

$A_1 \cap A_2 = \{HHH, TTT\}$

$\mathbb{P}\left(A_1 \cap A_2\right) = \frac{1}{4} = \mathbb{P}\left(A_1\right) \mathbb{P}\left(A_2\right)$

Similarly for $A_2 \cap A_3$ and $A_1 \cap A_3$.

Independence

For example, flip 3 coins (call their resulting heads/tails values c_1, c_2, and c_3), and define these events:

- $A_1 : c_1 = c_2$
- $A_2 : c_2 = c_3$
- $A_3 : c_3 = c_1$

$A_1 \cap A_2 \cap A_3 = \{HHH, TTT\}$

$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{4} \neq \mathbb{P}(A_1)\,\mathbb{P}(A_2)\,\mathbb{P}(A_3)$

So the events are pairwise independent, but not mutually independent.

Independence

Note that two events being independent does not mean they're disjoint.

If they're disjoint, then one occurring implies the other doesn't, so

$$\mathbb{P}(A \cap B) = 0 \neq \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Independence

Suppose we roll two dice, and define the following events:

- E: the sum is 7
- F: the first die rolls a 3
- G: the second die rolls a 4

For these to be independent, we'd need $\mathbb{P}(E|FG) = \mathbb{P}(E)$.

But if we know both F and G occur, then E must occur as well:

$$\mathbb{P}(E|FG) = 1 \neq \frac{1}{6} = \mathbb{P}(E)$$

Independence

Suppose we want to send a message from A to B, and there are four parallel channels connecting the two. For the message to fail, all four would have to fail. Let S_i denote the event where channel i fails, and these are mutually independent.

What's the probability of successfully sending the message?

Independence

Suppose we want to send a message from A to B, and there are four parallel channels connecting the two. For the message to fail, all four would have to fail. Let S_i denote the event where channel i fails, and these are mutually independent.

Then the probability of sending the message is

$$\begin{aligned}
\mathbb{P}\left(\text{success}\right) &= 1 - \mathbb{P}\left(\text{failure}\right) \\
&= 1 - \mathbb{P}\left(\text{all fail}\right) \\
&= 1 - \mathbb{P}\left(S_1 \cap S_2 \cap S_3 \cap S_4\right) \\
&= 1 - \mathbb{P}\left(S_1\right)\mathbb{P}\left(S_2\right)\mathbb{P}\left(S_3\right)\mathbb{P}\left(S_4\right)
\end{aligned}$$

Random variables

A random variable R maps a sample space S to some codomain V:

$$R : S \to V$$

Examples, if we roll a die:

- R_1 could be the number rolled
- R_2 could be the square of the number rolled

Random variables

For example, we could toss 3 coins and define these random variables:

- C: the number of heads
- M: 1 if all coins are the same, 0 otherwise

Then C maps to a codomain of $[0, 3]$, while M maps to $\{0, 1\}$.

Random variables

Consider all possible outcomes and the values that C and M would take on in each case:

Outcome	C	M
TTT	0	1
TTH	1	0
THT	1	0
THH	2	0
HTT	1	0
HTH	2	0
HHT	2	0
HHH	3	1

We can consider events defined on random variables, such as the event where $C = 2$.

This corresponds to the outcomes $\{THH, HTH, HHT\}$.

So we can say $\mathbb{P}\left(C = 2\right) = 3/8$, since all outcomes have an equal probability of $1/8$.

Random variables

For another example, consider the event where $C \leq 1$.

This corresponds to the outcomes $\{TTT, TTH, THT, HTT\}$.

So we can say $\mathbb{P}\left(C \leq 1\right) = 4/8 = 1/2$.

We can also define events such as "$C \cdot M$ is odd".

For each outcome, evaluate C and M individually, and then look at their product.

Outcome	C	M	$C \cdot M$
TTT	0	1	0
TTH	1	0	0
THT	1	0	0
THH	2	0	0
HTT	1	0	0
HTH	2	0	0
HHT	2	0	0
HHH	3	1	3

So $\mathbb{P}(C \cdot M \text{ is odd}) = 1/8$.

Random variables

A random variable that takes on only the values 0 and 1 is called an indicator random variable (e.g., M in the previous example).

It's also sometimes called a Bernoulli variable.

An event naturally gives rise to an indicator random variable. For event E, define the random variable I_E that has value 1 if event E occurs, and 0 otherwise.

Example: pick 4 balls

Suppose we have 20 balls, labelled $1 - 20$, and we randomly pick 4 of them.

Let X denote the largest number picked. For example, if we picked balls 2, 7, 5, and 13, then $X = 13$.

So X is a random variable mapping the sample space to the range $[4, 20]$.

What is $\mathbb{P}(X = 1)$?

What is $\mathbb{P}(X = 4)$?

Example: pick 4 balls

What is $\mathbb{P}(X = i)$?

This means the largest ball is i and the other three must be less than i.

From the $i - 1$ values that are less than i, there are $\binom{i-1}{3}$ ways to choose three.

Then the probability is $\binom{i-1}{3} / \binom{20}{4}$.

Independence of random variables

As with events, we can define indpenedence of random variables.

Two random variables R_1 and R_2 are independent if

- the events $[R_1 = x_1]$ and $[R_2 = x_2]$ are independent

for all values of x_1 and x_2.

Independence of random variables

For the three coin flips example, we had these random varibles:

- C: number of heads
- M: whether all flips give the same result

Are these independent?

Independence of random variables

For these to be independent, we need (for all x_1, x_2)

$$\mathbb{P}\left(C = x_1 \cap M = x_2\right) = \mathbb{P}\left(C = x_1\right) \cdot \mathbb{P}\left(M = x_2\right)$$

Consider $x_1 = 2$ and $x_2 = 1$. Then

$$\mathbb{P}\left(C = 2 \cap M = 1\right) = 0 \neq \mathbb{P}\left(C = 2\right) \cdot \mathbb{P}\left(M = 1\right)$$

So they're not independent.

Expected value

The expected value (or mean) of a random variable is a weighted sum of the values it can take on:

$$E[R] = \sum_{x \in \text{range}(R)} x \cdot \mathbb{P}(x)$$

For example, if we roll one die and let R be the number rolled:

$$E[R] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{7}{2}$$

Expected value

Let's flip a coin and let the random variable $R = 0$ if it's heads, 1 if it's tails.

Then

$$E[R] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

For a biased coin, that has probability $1/4$ of heads and $3/4$ of tails:

$$E[R] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{3}{4}$$

Expected value

Suppose you're given the option of flipping one of two coins, both biased, with these prizes. Which should you choose?

Coin	$\mathbb{P}$(heads)	Prize for heads	$\mathbb{P}$(tails)	Prize for tails
C_1	9/10	\$10	1/10	\$20
C_2	4/5	\$5	1/5	\$50

Expected value

Coin	$\mathbb{P}$(heads)	Prize for heads	$\mathbb{P}$(tails)	Prize for tails
C_1	9/10	$10	1/10	$20
C_2	4/5	$5	1/5	$50

$$E[C_1] = \frac{9}{10} \cdot 10 + \frac{1}{10} \cdot 20 = 11$$
$$E[C_2] = \frac{4}{5} \cdot 5 + \frac{1}{5} \cdot 50 = 14$$

So the second coin, on average, pays more.

Expected value

Suppose we have events R and S, where $S = 1/R$.

Is it true that $E[S] = 1/E[R]$?

Consider R being the roll of a die, and S its reciprocal. Then

$$E[R] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{7}{2}$$
$$E[S] = \frac{1}{1} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{6} + \cdots + \frac{1}{6} \cdot \frac{1}{6} = \frac{49}{120}$$

So in general, these are not equivalent.

Linearity of expectation

Given two random variables R_1 and R_2:

$$E(R_1 + R_2) = E(R_1) + E(R_2)$$

For example, if we roll two dice, what is the expected value of their sum?

We could calculate this two ways:

- weighted average of all possible sums
- using linearity of expectation

Linearity of expectation

If we roll two dice, what are the possible sums?

		R_2					
		1	2	3	4	5	6
R_1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Linearity of expectation

Sum	Probability
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

So the expected value is

$$2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} +$$

$$8 \cdot \frac{5}{36} + 9 \cdot \frac{4}{36} + 10 \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36}$$

which equals

$$\frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36}$$

$$= \frac{252}{36}$$

$$= 7$$

Linearity of expectation

Here's another approach:

Let R_1 be the roll of the first die, and R_2 the second.

Then we want $E(R_1 + R_2)$, but this is just $E(R_1) + E(R_2)$, and we know the expected value for a single die is $7/2$. So

$$\begin{aligned} E(R_1 + R_2) &= E(R_1) + E(R_2) \\ &= 7/2 + 7/2 \\ &= 7 \end{aligned}$$

Conditional expectation

We can also condition this on some event A:

$$E[R|A] = \sum_{x \in \text{range}(R)} x \cdot \mathbb{P}\left(x|A\right)$$

Conditional expectation

For example, let's roll a single die, and let R be the value rolled. If we assume we roll at least a 4, what's the expected value?

$$\begin{aligned} E[R|R \geq 4] &= \sum i \cdot \mathbb{P}\left(R = i | R \geq 4\right) \\ &= 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot \frac{1}{3} + 5 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} \\ &= 5 \end{aligned}$$

Law of total expectation

Recall the law of total probability:

$$\mathbb{P}(A) = \mathbb{P}(A|E)\,\mathbb{P}(E) + \mathbb{P}\left(A|\bar{E}\right)\mathbb{P}\left(\bar{E}\right)$$

If we generalize this slightly to a set of events E_i that partition the sample space:

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A|E_i)\,\mathbb{P}(E_i)$$

Law of total expectation

Law of total probability:

$$\mathbb{P}\left(A\right) = \sum_{i} \mathbb{P}\left(A|E_i\right) \mathbb{P}\left(E_i\right)$$

We can similarly define a law of total expectation:

$$E(R) = \sum_{i} E(R|A_i)\mathbb{P}\left(A_i\right)$$

Law of total expectation

Example: Suppose the population is 49% male and 51% female.

And suppose that the expected height of a random guy is 180 cm, and a random girl 165 cm.

What is the expected height of a random person?

Law of total expectation

Example: Suppose the population is 49% male and 51% female.

And suppose that the expected height of a random guy is 180 cm, and a random girl 165 cm.

$$\begin{aligned} E(H) &= E(H|M)\mathbb{P}(M) + E(H|F)\mathbb{P}(F) \\ &= 180 \cdot 0.49 + 165 \cdot 0.51 \\ &\approx 172.5 \text{ cm} \end{aligned}$$