Linearly Stabilized Schemes for Nonlinear Parabolic PDEs

Kevin Chow and Steve Ruuth
Department of Mathematics, Simon Fraser University

Introduction

We seek an effective strategy to handling the severe time step restrictions resulting from the spatial discretization of **nonlinear parabolic PDEs**.

The Approach

Typical implicit methods require solution to a nonlinear system. We modify the equation with a well-chosen linear system and solve in an implicit-explicit (IMEX) manner:

$$u' = \underbrace{pL_h u}_{\text{implicit}} + \underbrace{F(u,t) - pL_h u}_{\text{explicit}}, \quad p > 0. \qquad (1)$$

A Modified Test Equation

We apply IMEX linear multistep methods (LMM) [1] to the **modified test equation**

$$u' = p\lambda u + (1-p)\lambda u. \qquad (2)$$

One views λ as an eigenvalue of the Jacobian of the linearized F.
Our analysis is predicated on determining the range of p for which the scheme is absolutely stable for all $\Delta t > 0$.

Table 1: Parameter ranges for select IMEX LMM.

Order	Method	$p \in$
1	IMEX-Euler	$(1/2, \infty)$
2	SBDF2	$(3/4, \infty)$
	CNAB	$(1, \infty)$
	mCNAB	$(8/9, \infty)$
	CNLF	$(1/2, \infty)$
3	SBDF3	$(7/8, 2)$
4	SBDF4	$(11/12, 5/4)$

Examples and Applications

Nonlinear parabolic PDEs are of significant interest with applications in numerous areas. We present here examples from simulating interface motion, image processing, and the modelling of phase separation in binary alloys.

$$u_t = |\nabla u| \operatorname{div}(\nabla \phi(\nabla u))$$

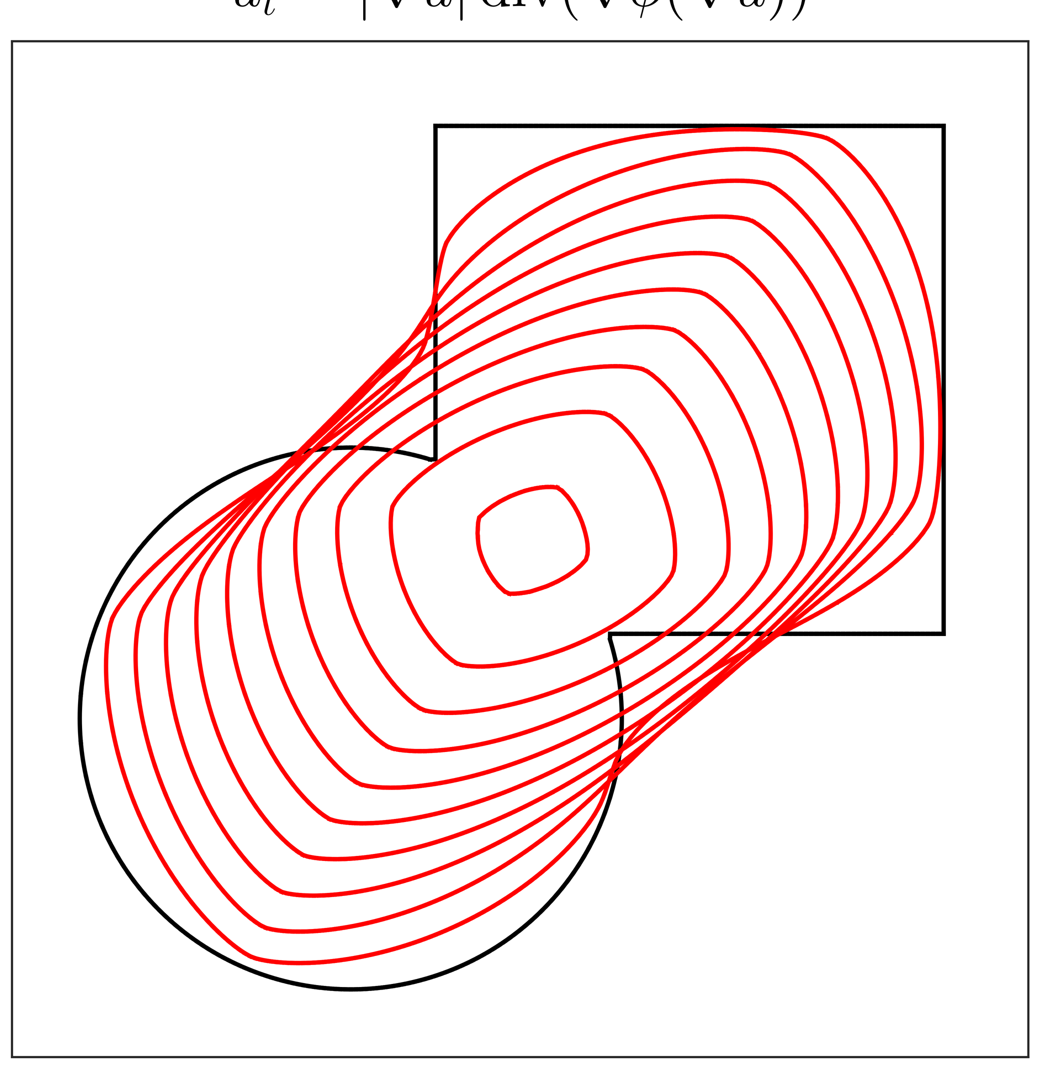

Figure 1: Black curve marks the initial state.

$$u_t = -\Delta\nabla \cdot \left(\nabla u / \sqrt{|\nabla u|^2 + \epsilon^2}\right) + \lambda_D(f - u)$$

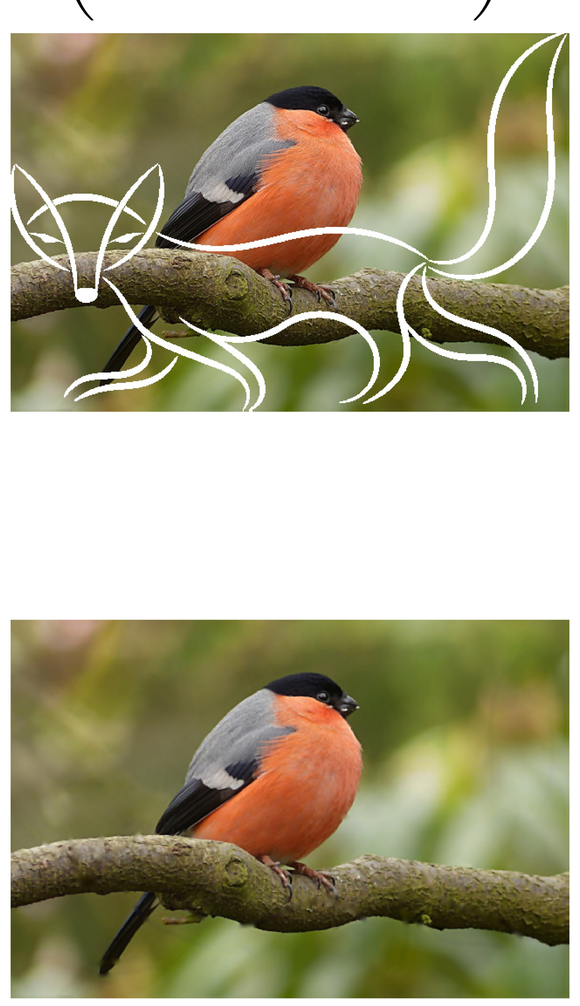

Figure 2: Image inpainting [2]: (above) damaged photograph, (below) restored image.

$$u_t = -\epsilon^2 \Delta^2 u + \Delta(u^3 - u)$$

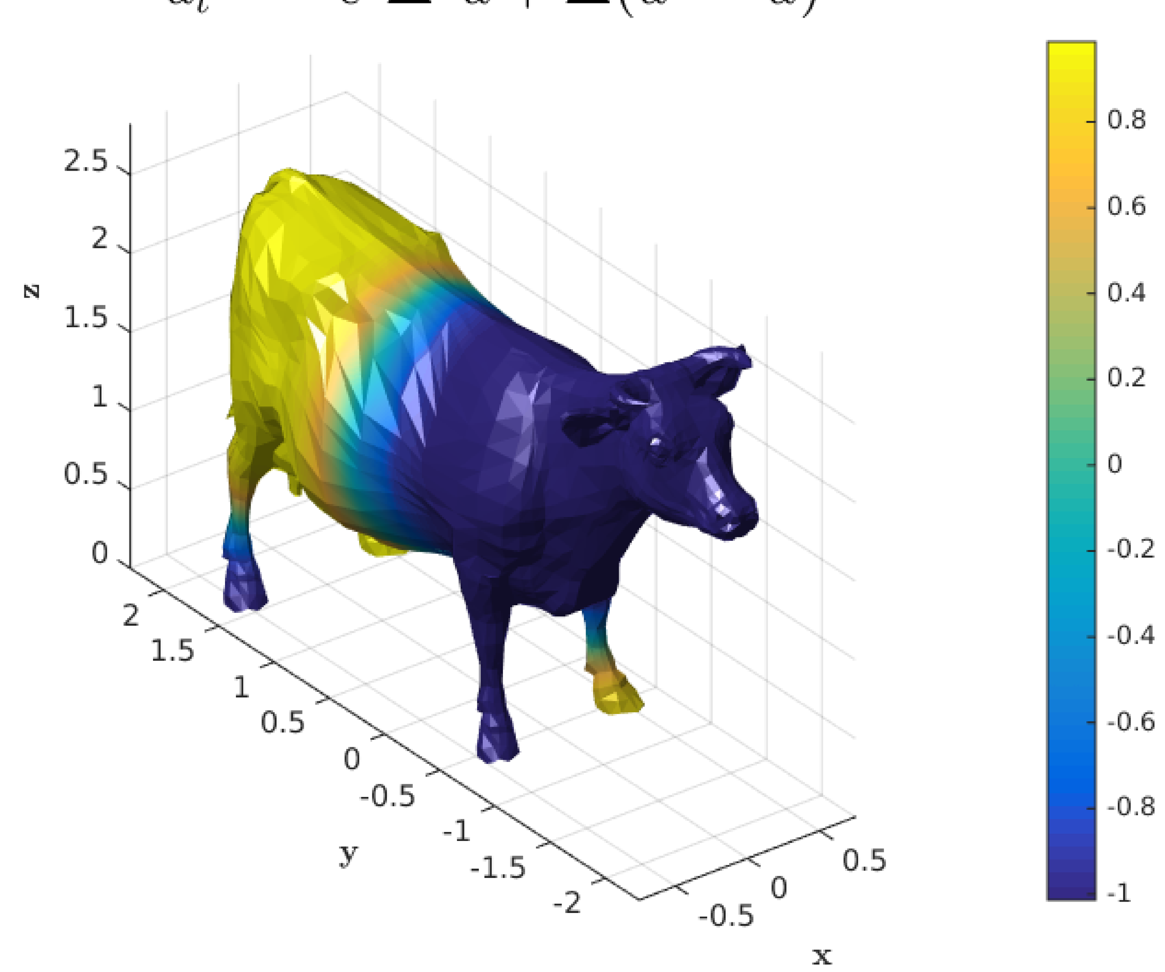

Figure 3: Spinodal decomposition starting from random initial conditions on a cow-shaped surface.

Related Works

A similar approach in [3] recommends Richardson extrapolation (EIN) for the push to second order, as was suggested in [4] but not implemented. Our experiments found this to be less efficient and showed a reduction in the order of accuracy when large values of p were needed.

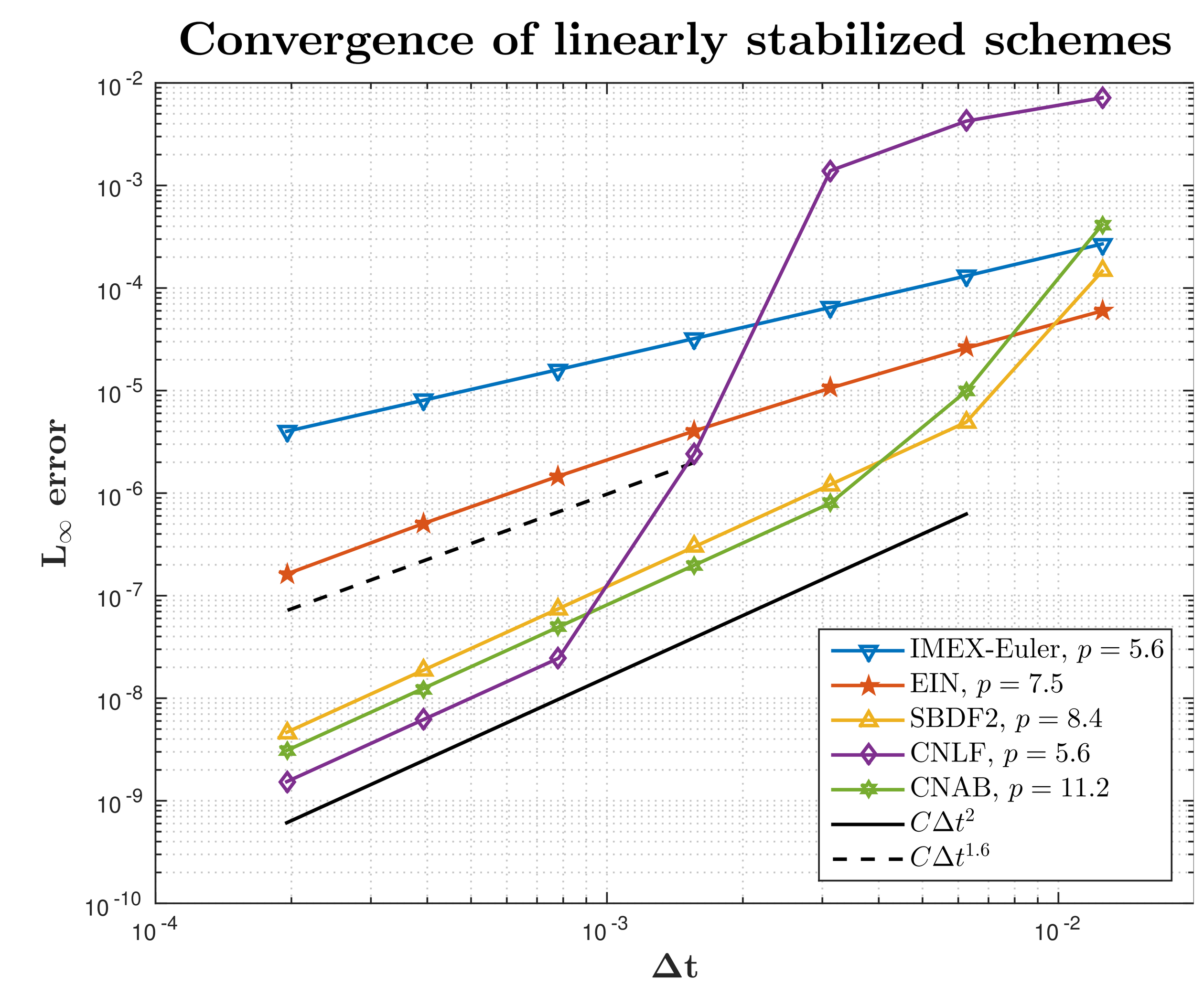

Figure 4: Rates of convergence to an exact solution to $u_t = \Delta(u^5)$.

Summary

The proposed methods deliver a significant improvement in efficiency over commonly used numerical methods and are remarkably simple to implement. The second order methods (SBDF2, CNAB) are recommended as an unbounded parameter range is crucial for practical applications.

References

[1] U. M. Ascher, S. J. Ruuth, and B. T. R. Wetton. Implicit-Explicit Methods for Time-Dependent Partial Differential Equations. *SIAM J. Numer. Anal.*, 32(3):797–823, 1995.

[2] C. B. Schönlieb and A. Bertozzi. Unconditionally stable schemes for higher order inpainting. *Commun. Math. Sci.*, 9(2):413–457, 2011.

[3] L. Duchemin and J. Eggers. The Explicit-Implicit-Null method: Removing the numerical instability of PDEs. *J. Comput. Phys.*, 263:37–52, 2014.

[4] P. Smereka. Semi-implicit level set methods for curvature and surface diffusion motion. *J. Sci. Comput.*, 19(December):439–456, 2003.