Linearly Stabilized Schemes for the Time Integration of Stiff Nonlinear PDEs

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Introduction

- Focus on time stepping for stiff nonlinear PDEs.
 - Stability
 - Accuracy
 - Efficiency
 - Simplicity



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Consider the heat equation,

$$u_t = u_{xx}, \quad x \in \Omega, \quad t > 0.$$

Discretize in space:

$$U' = LU, \quad U \in \mathbb{R}^N, \quad t > 0.$$

Explicit: $U^{n+1} = G(U^n, U^{n-1}, \dots, LU^n, LU^{n-1}, \dots)$, but $\Delta t \leq Ch^2$. Implicit: $AU^{n+1} = b$; unconditionally stable, but must solve a linear system.

Now compare with

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u}, \quad x \in \Omega, \quad t > 0.$$

and

$$U' = F(U), \quad U \in \mathbb{R}^N, \quad t > 0.$$

Explicit: $U^{n+1} = G(U^n, U^{n-1}, \dots, F(U^n), F(U^{n-1}), \dots)$, but $\Delta t \leq Ch^2$. Implicit: $AU^{n+1} = b(U^{n+1})$; unconditionally stable, but must solve a nonlinear system because nonlinearity is in the stiff term.

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Comparing side-by-side:

$$u_t = u_{xx}, \quad x \in \Omega, t > 0,$$

Explicit: $\Delta t \leq Ch^2$

Implicit: unconditionally stable;

solution to linear system

 $u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u}, \quad x \in \Omega, t > 0,$

Explicit: $\Delta t \leq Ch^2$

Implicit: unconditionally stable; solution to nonlinear system

Summary: What We Like

Explicit: simple; handles nonlinear terms with no added difficulty.

Implicit: large time steps

Modify the equation,

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u} - u_{xx} + u_{xx}, \quad x \in \Omega, \quad t > 0,$$

and discretize in space,

$$U' = F(U) - LU + LU, \quad U \in \mathbb{R}^N, \quad t > 0.$$

Use implicit-explicit time stepping, e.g.

$$\frac{U^{n+1}-U^n}{\Delta t}=F(U^n)-LU^n+LU^{n+1}.$$



Linear Stability

More generally, from U' = F(U), we can modify as

$$U' = \underbrace{F(U) - pLU}_{(\star)} + pLU, \quad p > 0,$$

and apply a time stepping scheme that treats (\star) explicitly.

Is this unconditionally stable?



Scalar test equation

Standard case:

$$U' = F(U)$$

 $\mbox{Linearize} \rightarrow \mbox{Diagonalize} \rightarrow \mbox{Test} \\ \mbox{equation:} \\$

$$u' = \lambda u$$

Apply time stepping method:

$$u^{n+1} = \xi(\lambda \Delta t) u^n.$$

Unconditional stability:

$$|\xi(\lambda \Delta t)| \le 1$$
 for all $\lambda \Delta t < 0$.

With linear modification:

$$U' = F(U) - pLU + pLU$$

Linearize \rightarrow Diagonalize \rightarrow Test equation:

$$u' = \lambda u - p\lambda u + p\lambda u$$
$$= (1 - p)\lambda u + p\lambda u$$

Implicit-explicit Euler

Applied to the test equation, $u' = (1 - p)\lambda u + p\lambda u$, yields

$$\frac{u^{n+1}-u^n}{\Delta t}=(1-p)\lambda u^n+p\lambda u^{n+1}.$$

The amplification factor is

$$\xi_1(\lambda \Delta t) = \frac{1 + (1 - p)\lambda \Delta t}{1 - p\lambda \Delta t}.$$

Impose unconditional stability:

$$|\xi_1(\lambda \Delta t)| \le 1$$
 for all $\lambda \Delta t < 0 \iff p \ge 1/2$.



Explicit-implicit-null (EIN)

Duchemin and Eggers (2014) use Richardson extrapolation to get second order. The amplification factor is

$$\xi_{EIN}(\lambda \Delta t) = 2\xi_1^2(\lambda \Delta t/2) - \xi_1(\lambda \Delta t).$$

and

$$|\xi_{EIN}(\lambda \Delta t)| \le 1$$
 for all $\lambda \Delta t < 0 \iff p \ge 2/3$.



Implicit-explicit multistep methods

An alternative for second and higher order methods: IMEX multistep methods.

Table: Parameter restriction for select IMEX methods.

Order	Method	$p \in$
1	IMEX-Euler	$[1/2,\infty)$
2	SBDF2 CNAB mCNAB CNLF	$[3/4,\infty)$ $[1,\infty)$ $[8/9,\infty)$ $[1/2,\infty)$
3	SBDF3	[7/8, 2]
4	SBDF4	[11/12, 5/4]

Comparing the methods

Do the methods work as advertised? Examine this with two test problems,

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u},$$

and

$$u_t = \Delta(u^5).$$

Test Problem 1

First test problem:

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u}, \quad 0 < x < 10, \quad t > 0,$$

with initial condition

$$u(x,0) = 1 + 0.10\sin\left(\frac{\pi}{5}x\right),\,$$

and boundary conditions u(0, t) = 1 = u(10, t). Stabilized as

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u} - pu_{xx} + pu_{xx}.$$

Numerical convergence test

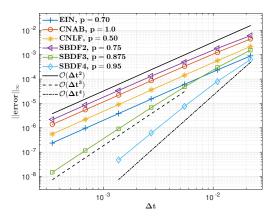


Figure : Numerical convergence of linearly stabilized schemes at time T=0.35. Compared against a reference solution generated using an explicit 3rd order RK with $\Delta t=1.46\times 10^{-5}$. Stabilized by adding and subtracting pu_{xx} .

Failure of SBDF3 and SBDF4

How did we choose p? Consider

$$u' = \lambda u - p\lambda u + p\lambda u$$

and

$$U' = F(U) - pLU + pLU.$$

With the test equation, we derived a restriction on p. More generally, the restriction applies to $p\lambda_L/\lambda_F$. For test problem 1 with centred differences, we find

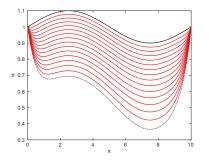
$$\frac{\rho\lambda_L}{\lambda_F}pprox
ho(1+(D_1ar{u}_j^n)^2),$$



Failure of SBDF3 and SBDF4

The selection of p for SBDF3 is dictated by

$$\max_{1 \leq j \leq N} \frac{7}{8} \frac{1}{1 + (D_1 \bar{u}_j^n)^2} \leq p \leq \min_{1 \leq j \leq N} \frac{2}{1 + (D_1 \bar{u}_j^n)^2}.$$



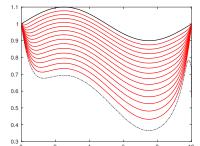


Figure: Numerical solution to test problem 1.

Figure : Development of instabilities using SBDF3, p = 1.625.

Test Problem 2

Second test problem:

$$u_t = \Delta(u^5), \quad (x, y) \in [0, 1]^2, \quad t > 0,$$

with initial and boundary conditions set such that the exact solution is

$$u(x, y, t) = \left(\frac{4}{5}(2t + x + y)\right)^{1/4}.$$

Stabilize with $p\Delta u$; $p\lambda_L/\lambda_F \approx p/(8(1+t))$.



Numerical convergence test

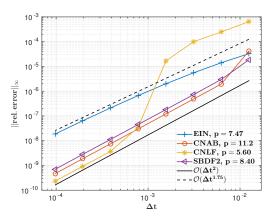


Figure : Numerical convergence of linearly stabilized schemes at time T=0.40. Compared against a reference solution generated using an explicit 3rd order RK with $\Delta t = 6.25 \times 10^{-6}$. Stabilized using $p\Delta u$; $p\lambda_L/\lambda_F \approx p/(8(1+t))$.

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Error constant

Discretizing $u'=(1-p)\lambda u+p\lambda u$, we observe that the discretization error grows with p.

How does the error behave as we increase p?

Examine the coefficient of the leading order error term.

Table: Coefficient of leading order error term as applied to the test equation.

Method	Coefficient
EIN	$\frac{1}{2}(p-p^2)$
SBDF2	$\frac{2}{3}p - \frac{5}{18}$
CNAB	$\frac{1}{2}p - \frac{1}{4}$
CNLF	$p - \frac{1}{6}$

Amplification factor as $\Delta t \rightarrow \infty$

Methods with small amplication factor as $\Delta t \to \infty$ perform well as large step-sizes.

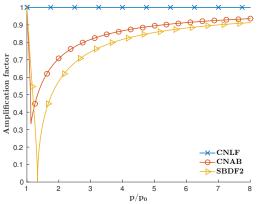


Figure : Amplification factor as $\Delta t \to \infty$.



Numerical Experiments

We consider two classes of problems to demonstrate the effectiveness of our new methods:

- Image inpainting.
- Mean curvature motion.

Image Inpainting

In Schönlieb and Bertozzi (2011), the authors proposed the fourth order inpainting model

$$u_t = -\Delta \nabla \cdot \left(\frac{\nabla u}{\sqrt{\left|\nabla u\right|^2 + \epsilon^2}} \right) + \lambda_0 (u_0 - u),$$

and numerical solution by the first order accurate method

$$\frac{u^{n+1} - u^n}{\Delta t} = -\Delta \nabla \cdot \left(\frac{\nabla u^n}{\sqrt{|\nabla u^n|^2 + \epsilon^2}} \right) + \lambda_0 (u_0 - u^n) + p_1 \Delta^2 u^n - p_1 \Delta^2 u^{n+1} + p_0 \lambda u^n - p_0 \lambda u^{n+1}.$$

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Image inpainting





Figure : $TV-H^{-1}$ image restoration.

Table : Iteration counts for $TV-H^{-1}$ image restoration.

	Δt	Iterations
SBDF1	0.30	1002
SBDF2	0.54	401
CNAB	0.64	347

Motion by mean curvature

We evolve the level set equation for motion by mean curvature,

$$u_t = \kappa |\nabla u| = |\nabla u| \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right)$$

on an initial dumbbell-shaped curve in 3D. We solve to time T=0.75 on a $256\times128\times128$ periodic grid.

Motion by mean curvature

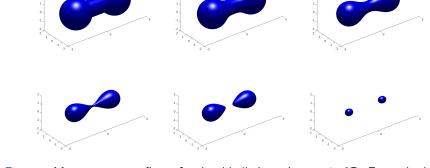


Figure : Mean curvature flow of a dumbbell-shaped curve in 3D. From the left to right, top to bottom, the plots show the evolution at times t = 0, 0.10, 0.30, 0.525, 0.55, 0.75.

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Motion by mean curvature

On a machine with an Intel[®] Core TM i5-4570 CPU@3.20GHz running MATLAB 2014b:

- Forward Euler: 3000 time steps \rightarrow over 28 minutes.
- SBDF2: 75 time steps \rightarrow under 100 seconds.

Conclusion

Contributions of this thesis mainly fall into two categories:

- Further developed linearly stabilized schemes.
 - Outlined a framework for developing new methods of this type.
 - Identified properties necessary for effective schemes.
- Proposed new methods that outperform existing ones.
 - IMEX multistep methods and exponential time differencing methods.

Future work:

- Oevelopment higher order methods without the deficiencies exhibited by ETD methods.
- ② A comparison with popular algorithms for nonlinear stiff PDEs, particularly for image inpainting.

