Linearly Stabilized Schemes for the Time Integration of Stiff Nonlinear PDEs

Kevin Chow

December 9, 2016

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Table of contents

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1. Introduction
   - Example
2. Linear Stability
   - Test equation
   - IMEX Euler
   - EIN
   - IMEX Multistep
3. Comparing the Methods
   - Test Problem 1
   - Test Problem 2
4. Numerical Experiments
   - Image Inpainting
   - Motion by Mean Curvature
5. Conclusion

Introduction

1. Focus on time stepping for stiff nonlinear PDEs.
    - Stability
    - Accuracy
    - Efficiency
    - Simplicity

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    - Accuracy
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Example

Consider the heat equation,

$$u_t = u_{xx}, \quad x \in \Omega, \quad t > 0.$$

Discretize in space:

$$U' = LU, \quad U \in \mathbb{R}^N, \quad t > 0.$$

Explicit: $U^{n+1} = G(U^n, U^{n-1}, \ldots, LU^n, LU^{n-1}, \ldots)$, but $\Delta t \leq Ch^2$.
Implicit: $AU^{n+1} = b$; unconditionally stable, but must solve a linear system.

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Example

Now compare with

$$u_t = \frac{u_{xx}}{1+u_x^2} - \frac{1}{u}, \quad x \in \Omega, \quad t > 0.$$

and

$$U' = F(U), \quad U \in \mathbb{R}^N, \quad t > 0.$$

Explicit: $U^{n+1} = G(U^n, U^{n-1}, \ldots, F(U^n), F(U^{n-1}), \ldots)$, but $\Delta t \leq Ch^2$.
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2016-12-04

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Example

Comparing side-by-side:

$$u_t = u_{xx}, \quad x \in \Omega, t > 0,$$

Explicit: $\Delta t \leq Ch^2$
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Explicit: simple; handles nonlinear terms with no added difficulty.
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Example

Modify the equation,

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u} - u_{xx} + u_{xx}, \quad x \in \Omega, \quad t > 0,$$

and discretize in space,

$$U' = F(U) - LU + LU, \quad U \in \mathbb{R}^N, \quad t > 0.$$

Use implicit-explicit time stepping, e.g.

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^n) - LU^n + LU^{n+1}.$$

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Linear Stability

More generally, from $U' = F(U)$, we can modify as

$$U' = \underbrace{F(U) - pLU}_{(\star)} + pLU, \quad p > 0,$$

and apply a time stepping scheme that treats $(\star)$ explicitly.

Is this unconditionally stable?

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Scalar test equation

Standard case:

$$U' = F(U)$$

Linearize $\to$ Diagonalize $\to$ Test equation:

$$u' = \lambda u$$

With linear modification:

$$U' = F(U) - pLU + pLU$$

Linearize $\to$ Diagonalize $\to$ Test equation:

$$\begin{aligned} u' &= \lambda u - p\lambda u + p\lambda u \\ &= (1-p)\lambda u + p\lambda u \end{aligned}$$

Apply time stepping method:

$$u^{n+1} = \xi(\lambda \Delta t) u^n.$$

Unconditional stability:

$$|\xi(\lambda \Delta t)| \leq 1 \quad \text{for all} \quad \lambda \Delta t < 0.$$

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└Linear Stability
  └Test equation
    └Scalar test equation

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With linear modification: $U' = F(U) - pLU + pLU$ Linearize $\to$ Diagonalize $\to$ Test equation: $u' = \lambda u - p\lambda u + p\lambda u = (1-p)\lambda u + p\lambda u$

Apply time stepping method: $u^{n+1} = \xi(\lambda \Delta t) u^n.$

Unconditional stability: $|\xi(\lambda \Delta t)| \leq 1$ for all $\lambda \Delta t < 0.$

Implicit-explicit Euler

Applied to the test equation, $u' = (1-p)\lambda u + p\lambda u$, yields

$$\frac{u^{n+1} - u^n}{\Delta t} = (1-p)\lambda u^n + p\lambda u^{n+1}.$$

The amplification factor is

$$\xi_1(\lambda \Delta t) = \frac{1 + (1-p)\lambda \Delta t}{1 - p\lambda \Delta t}.$$

Impose unconditional stability:

$$|\xi_1(\lambda \Delta t)| \leq 1 \text{ for all } \lambda \Delta t < 0 \iff p \geq 1/2.$$

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Explicit-implicit-null (EIN)

Duchemin and Eggers (2014) use Richardson extrapolation to get second order. The amplification factor is

$$\xi_{EIN}(\lambda \Delta t) = 2\xi_1^2(\lambda \Delta t/2) - \xi_1(\lambda \Delta t).$$

and

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Implicit-explicit multistep methods

An alternative for second and higher order methods: IMEX multistep methods.

Table : Parameter restriction for select IMEX methods.

Order	Method	$p \in$
1	IMEX-Euler	$[1/2, \infty)$
2	SBDF2	$[3/4, \infty)$
	CNAB	$[1, \infty)$
	mCNAB	$[8/9, \infty)$
	CNLF	$[1/2, \infty)$
3	SBDF3	$[7/8, 2]$
4	SBDF4	$[11/12, 5/4]$

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Comparing the methods

Do the methods work as advertised?
Examine this with two test problems,

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u},$$

and

$$u_t = \Delta(u^5).$$

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Test Problem 1

First test problem:

$$u_t = \frac{u_{xx}}{1+u_x^2} - \frac{1}{u}, \quad 0 < x < 10, \quad t > 0,$$

with initial condition

$$u(x,0) = 1 + 0.10 \sin\left(\frac{\pi}{5}x\right),$$

and boundary conditions $u(0,t) = 1 = u(10,t)$.
Stabilized as

$$u_t = \frac{u_{xx}}{1+u_x^2} - \frac{1}{u} - pu_{xx} + pu_{xx}.$$

2016-12-04
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└Comparing the Methods
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Numerical convergence test

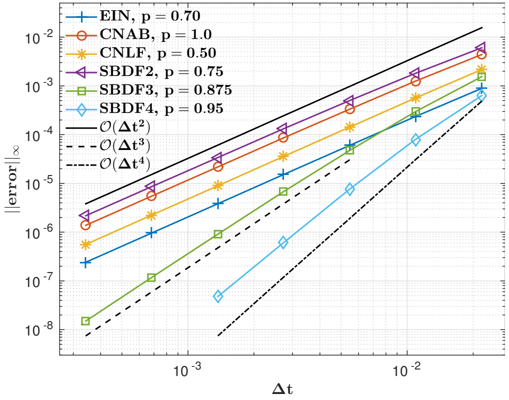

Figure : Numerical convergence of linearly stabilized schemes at time $T = 0.35$. Compared against a reference solution generated using an explicit 3rd order RK with $\Delta t = 1.46 \times 10^{-5}$. Stabilized by adding and subtracting pu_{xx}.

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Linearly Stabilized Schemes
└Comparing the Methods
  └Test Problem 1
    └Numerical convergence test

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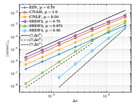

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Failure of SBDF3 and SBDF4

How did we choose p? Consider

$$u' = \lambda u - p\lambda u + p\lambda u$$

and

$$U' = F(U) - pLU + pLU.$$

With the test equation, we derived a restriction on p. More generally, the restriction applies to $p\lambda_L/\lambda_F$. For test problem 1 with centred differences, we find

$$\frac{p\lambda_L}{\lambda_F} \approx p(1 + (D_1 \bar{u}_j^n)^2),$$

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Failure of SBDF3 and SBDF4

The selection of p for SBDF3 is dictated by

$$\max_{1\le j\le N} \frac{7}{8}\frac{1}{1+(D_1\bar{u}_j^n)^2} \le p \le \min_{1\le j\le N} \frac{2}{1+(D_1\bar{u}_j^n)^2}.$$

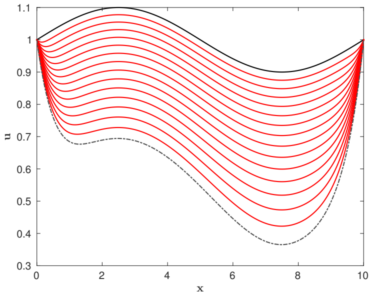

Figure : Numerical solution to test problem 1 .

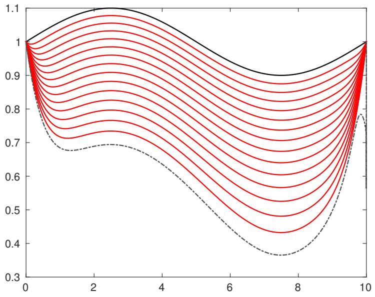

Figure : Development of instabilities using SBDF3, $p = 1.625$.

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Figure : Numerical solution to test problem 1 .

Figure : Development of instabilities using SBDF3, $p = 1.625$.

Test Problem 2

Second test problem:

$$u_t = \Delta(u^5), \quad (x, y) \in [0,1]^2, \quad t > 0,$$

with initial and boundary conditions set such that the exact solution is

$$u(x, y, t) = \left(\frac{4}{5}(2t + x + y)\right)^{1/4}.$$

Stabilize with $p\Delta u$; $p\lambda_L/\lambda_F \approx p/(8(1+t))$.

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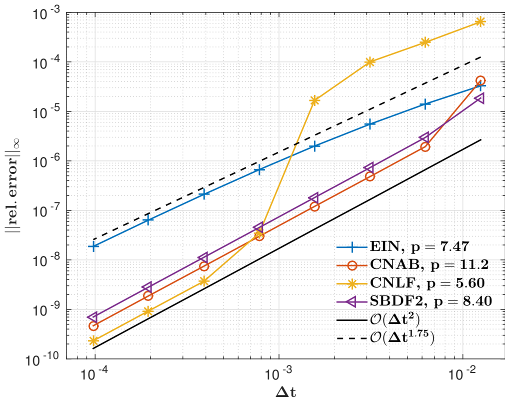

Figure : Numerical convergence of linearly stabilized schemes at time $T = 0.40$. Compared against a reference solution generated using an explicit 3rd order RK with $\Delta t = 6.25 \times 10^{-6}$. Stabilized using $p\Delta u$; $p\lambda_L/\lambda_F \approx p/(8(1+t))$.

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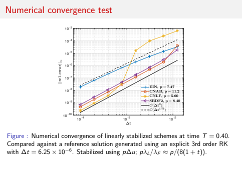
Linearly Stabilized Schemes
└Comparing the Methods
└Test Problem 2
└Numerical convergence test



Error constant

Discretizing $u' = (1-p)\lambda u + p\lambda u$, we observe that the discretization error grows with p.

How does the error behave as we increase p?

Examine the coefficient of the leading order error term.

Table : Coefficient of leading order error term as applied to the test equation.

Method	Coefficient
EIN	$\frac{1}{2}(p-p^2)$
SBDF2	$\frac{2}{3}p - \frac{5}{18}$
CNAB	$\frac{1}{2}p - \frac{1}{4}$
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Amplification factor as $\Delta t \to \infty$

Methods with small amplication factor as $\Delta t \to \infty$ perform well as large step-sizes.

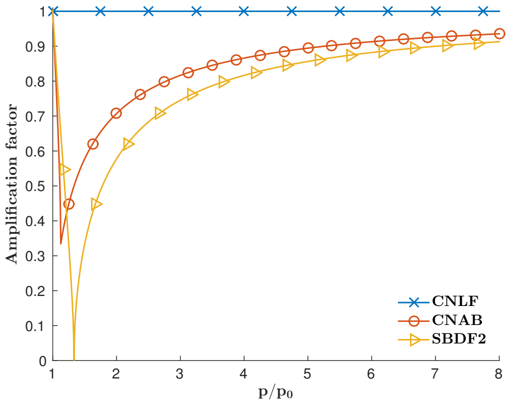

Figure : Amplification factor as $\Delta t \to \infty$.

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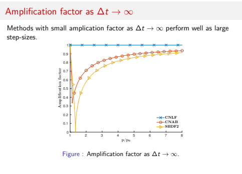
└Comparing the Methods
└Test Problem 2
└Amplification factor as $\Delta t \to \infty$



Numerical Experiments

We consider two classes of problems to demonstrate the effectiveness of our new methods:

1. Image inpainting.
2. Mean curvature motion.

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Image Inpainting

In Schönlieb and Bertozzi (2011), the authors proposed the fourth order inpainting model

$$u_t = -\Delta\nabla \cdot \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \epsilon^2}} \right) + \lambda_0(u_0 - u),$$

and numerical solution by the first order accurate method

$$\frac{u^{n+1} - u^n}{\Delta t} = -\Delta\nabla \cdot \left(\frac{\nabla u^n}{\sqrt{|\nabla u^n|^2 + \epsilon^2}} \right) + \lambda_0(u_0 - u^n)$$
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Image inpainting

Figure : TV-H^{-1} image restoration.

Table : Iteration counts for TV-H^{-1} image restoration.

	Δt	Iterations
SBDF1	0.30	1002
SBDF2	0.54	401
CNAB	0.64	347

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Motion by mean curvature

We evolve the level set equation for motion by mean curvature,

$$u_t = \kappa \left|\nabla u\right| = \left|\nabla u\right| \nabla \cdot \left(\frac{\nabla u}{\left|\nabla u\right|} \right)$$

on an initial dumbbell-shaped curve in 3D. We solve to time $T = 0.75$ on a $256 \times 128 \times 128$ periodic grid.

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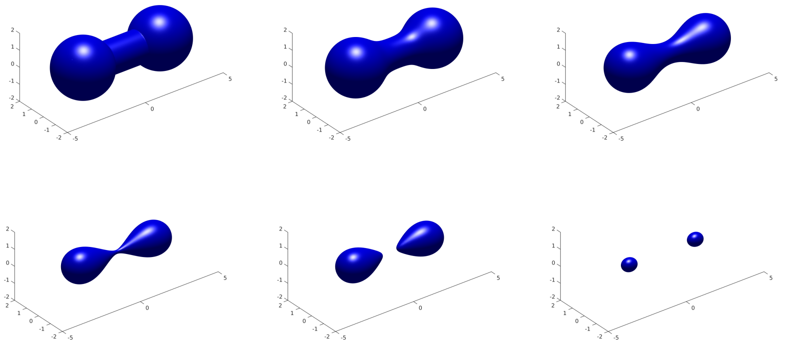

Figure : Mean curvature flow of a dumbbell-shaped curve in 3D. From the left to right, top to bottom, the plots show the evolution at times $t = 0$, 0.10, 0.30, 0.525, 0.55, 0.75.

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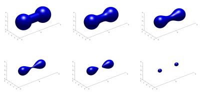

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Motion by mean curvature

On a machine with an Intel® Core™ i5-4570 CPU@3.20GHz running MATLAB 2014b:

- Forward Euler: 3000 time steps $\rightarrow$ over 28 minutes.
- SBDF2: 75 time steps $\rightarrow$ under 100 seconds.

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Conclusion

Contributions of this thesis mainly fall into two categories:

1. Further developed linearly stabilized schemes.
    - Outlined a framework for developing new methods of this type.
    - Identified properties necessary for effective schemes.
2. Proposed new methods that outperform existing ones.
    - IMEX multistep methods and exponential time differencing methods.

Future work:

1. Development higher order methods without the deficiencies exhibited by ETD methods.
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