

# Linearly Stabilized Schemes for the Time Integration of Stiff Nonlinear PDEs

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# Introduction

- Goal: time stepping for stiff nonlinear PDEs.
  - Stability
  - Accuracy
  - Simplicity
  - Efficiency

# Example

Consider the heat equation,

$$u_t = u_{xx}, \quad x \in \Omega, \quad t > 0.$$

Discretize in space:

$$U' = LU, \quad U \in \mathbb{R}^N, \quad t > 0.$$

Explicit:  $U^{n+1} = G(U^n, U^{n-1}, \dots, LU^n, LU^{n-1}, \dots)$ , but  $\Delta t \leq Ch^2$ .

Implicit:  $AU^{n+1} = b$ ; unconditionally stable, but must solve a linear system.

# Example

Now compare with

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u}, \quad x \in \Omega, \quad t > 0.$$

and

$$U' = F(U), \quad U \in \mathbb{R}^N, \quad t > 0.$$

Explicit:  $U^{n+1} = G(U^n, U^{n-1}, \dots, F(U^n), F(U^{n-1}), \dots)$ , but  $\Delta t \leq Ch^2$ .

Implicit:  $AU^{n+1} = b(U^{n+1})$ ; unconditionally stable, but must solve a nonlinear system because nonlinearity is in the stiff term.

## Example

Comparing side-by-side:

$$u_t = u_{xx}, \quad x \in \Omega, t > 0,$$

Explicit:  $\Delta t \leq Ch^2$

Implicit: unconditionally stable;  
solution to linear system

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u}, \quad x \in \Omega, t > 0,$$

Explicit:  $\Delta t \leq Ch^2$

Implicit: unconditionally stable;  
solution to nonlinear system

### Summary: What We Like

Explicit: simple; handles nonlinear terms with no added difficulty.

Implicit: large time steps

# Example

Modify the equation,

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u} - u_{xx} + u_{xx}, \quad x \in \Omega, \quad t > 0,$$

and discretize in space,

$$U' = F(U) - LU + LU, \quad U \in \mathbb{R}^N, \quad t > 0.$$

Use implicit-explicit time stepping, e.g.

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^n) - LU^n + LU^{n+1}.$$

# Linear Stability

More generally, from  $U' = F(U)$ , we can modify as

$$U' = \underbrace{F(U) - pLU}_{(\star)} + pLU, \quad p > 0,$$

and apply a time stepping scheme that treats  $(\star)$  explicitly.

**Is this unconditionally stable?**



# Scalar test equation

Standard case:

$$U' = F(U)$$

Linearize  $\rightarrow$  Diagonalize  $\rightarrow$  Test equation:

$$u' = \lambda u$$

Apply time stepping method:

$$u^{n+1} = \xi(\lambda \Delta t) u^n.$$

Unconditional stability:

$$|\xi(\lambda \Delta t)| \leq 1 \quad \text{for all} \quad \lambda \Delta t < 0.$$

With linear modification:

$$U' = F(U) - pLU + pLU$$

Linearize  $\rightarrow$  Diagonalize  $\rightarrow$  Test equation:

$$\begin{aligned} u' &= \lambda u - p\lambda u + p\lambda u \\ &= (1 - p)\lambda u + p\lambda u \end{aligned}$$

# Implicit-explicit Euler

Applied to the test equation,  $u' = (1 - p)\lambda u + p\lambda u$ , yields

$$\frac{u^{n+1} - u^n}{\Delta t} = (1 - p)\lambda u^n + p\lambda u^{n+1}.$$

The amplification factor is

$$\xi_1(\lambda\Delta t) = \frac{1 + (1 - p)\lambda\Delta t}{1 - p\lambda\Delta t}.$$

Impose unconditional stability:

$$|\xi_1(\lambda\Delta t)| \leq 1 \text{ for all } \lambda\Delta t < 0 \iff p \geq 1/2.$$

# Explicit-implicit-null (EIN)

Duchemin and Eggers (2014) use Richardson extrapolation to get second order. The amplification factor is

$$\xi_{EIN}(\lambda\Delta t) = 2\xi_1^2(\lambda\Delta t/2) - \xi_1(\lambda\Delta t).$$

and

$$|\xi_{EIN}(\lambda\Delta t)| \leq 1 \text{ for all } \lambda\Delta t < 0 \iff p \geq 2/3.$$

# Implicit-explicit multistep methods

An alternative for second and higher order methods: IMEX multistep methods.

**Table :** Parameter restriction for select IMEX methods.

Order	Method	$p \in$
1	IMEX-Euler	$[1/2, \infty)$
2	SBDF2	$[3/4, \infty)$
	CNAB	$[1, \infty)$
	mCNAB	$[8/9, \infty)$
	CNLF	$[1/2, \infty)$
3	SBDF3	$[7/8, 2]$
4	SBDF4	$[11/12, 5/4]$

# Understanding the methods: 3 key properties

Do the methods work as advertised?  
Examine this with two test problems,

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u},$$

and

$$u_t = \Delta(u^5).$$

# Test Problem 1

First test problem:

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u}, \quad 0 < x < 10, \quad t > 0,$$

with initial condition

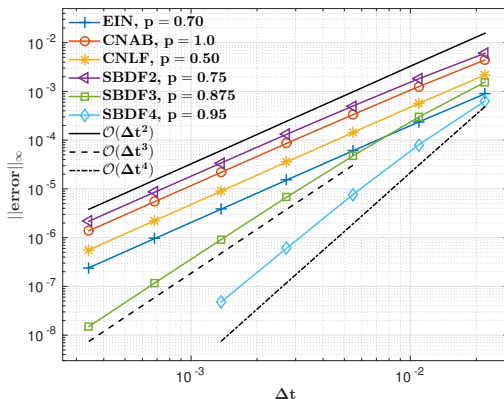
$$u(x, 0) = 1 + 0.10 \sin\left(\frac{\pi}{5}x\right),$$

and boundary conditions  $u(0, t) = 1 = u(10, t)$ .

Stabilized as

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u} - pu_{xx} + pu_{xx}.$$

# Numerical convergence test



**Figure :** Numerical convergence of linearly stabilized schemes at time  $T = 0.35$ . Compared against a reference solution generated using an explicit 3rd order RK with  $\Delta t = 1.46 \times 10^{-5}$ . Stabilized by adding and subtracting  $pu_{xx}$ .

## Selecting $p$ for more general problems

How did we choose  $p$ ? Consider

$$u' = \lambda u - p\lambda u + p\lambda u$$

and

$$U' = F(U) - pLU + pLU.$$

With the test equation, we derived a restriction on  $p$ . More generally, the restriction applies to  $p\lambda_L/\lambda_F$ . For test problem 1 with centred differences, we find

$$\frac{p\lambda_L}{\lambda_F} \approx p(1 + (D_1 \bar{u}_j^n)^2).$$



# Rejecting methods with bounded $p$ -parameter restrictions

The selection of  $p$  for SBDF3 is dictated by

$$\max_{1 \leq j \leq N} \frac{7}{8} \frac{1}{1 + (D_1 \bar{u}_j^n)^2} \leq p \leq \min_{1 \leq j \leq N} \frac{2}{1 + (D_1 \bar{u}_j^n)^2}.$$

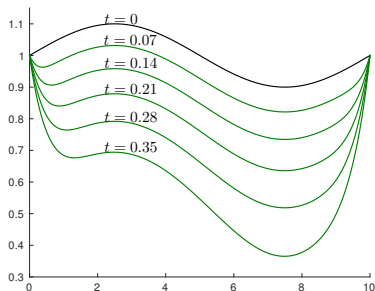


Figure : Numerical solution to test problem 1 .

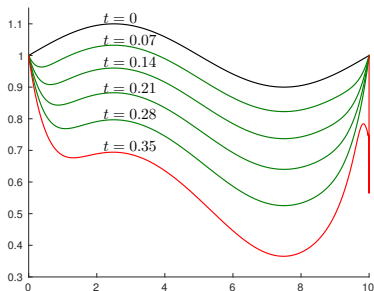


Figure : Development of instabilities using SBDF3,  $p = 1.625$ .

# Test Problem 2

Second test problem:

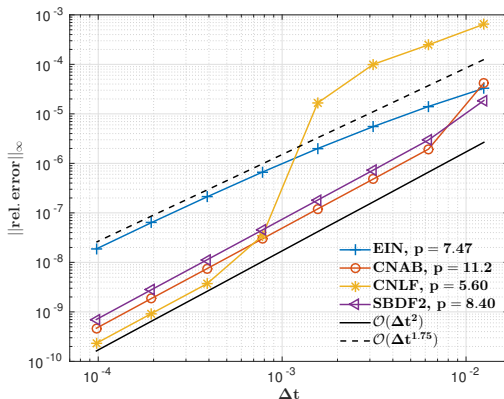
$$u_t = \Delta(u^5), \quad (x, y) \in [0, 1]^2, \quad t > 0,$$

with initial and boundary conditions set such that the exact solution is

$$u(x, y, t) = \left( \frac{4}{5}(2t + x + y) \right)^{1/4}.$$

Stabilize with  $p\Delta u$ ;  $p\lambda_L/\lambda_F \approx p/(8(1+t))$ .

# Numerical convergence test



**Figure :** Numerical convergence of linearly stabilized schemes at time  $T = 0.40$ . Compared against a reference solution generated using an explicit 3rd order RK with  $\Delta t = 6.25 \times 10^{-6}$ . Stabilized using  $p\Delta u$ ;  $p\lambda_L/\lambda_F \approx p/(8(1+t))$ .

## Error coefficient and dependence on $p$

Discretizing  $u' = (1 - p)\lambda u + p\lambda u$ , we observe that the discretization error grows with  $p$ .

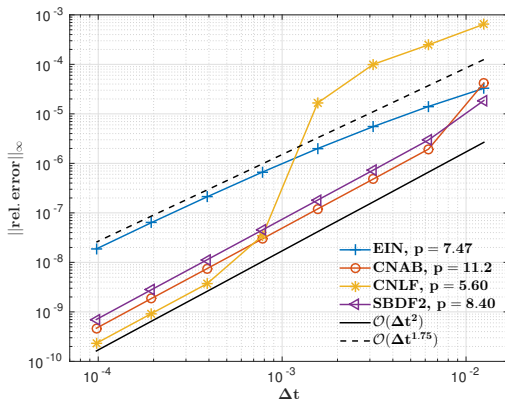
**How does the error behave as we increase  $p$ ?**

Examine the coefficient of the leading order error term.

**Table :** Coefficient of leading order error term as applied to the test equation.

Method	Coefficient
EIN	$\frac{1}{2}(p - p^2)$
SBDF2	$\frac{2}{3}p - \frac{5}{18}$
CNAB	$\frac{1}{2}p - \frac{1}{4}$
CNLF	$p - \frac{1}{6}$

# Large values of $p$ needed to stabilize



**Figure :** Numerical convergence of linearly stabilized schemes at time  $T = 0.40$ . Compared against a reference solution generated using an explicit 3rd order RK with  $\Delta t = 6.25 \times 10^{-6}$ . Stabilized using  $p\Delta u$ ;  $p\lambda_L/\lambda_F \approx p/(8(1+t))$ .

## Amplification factor as $\Delta t \rightarrow \infty$

Methods with small amplification factor as  $\Delta t \rightarrow \infty$  perform well as large step-sizes.

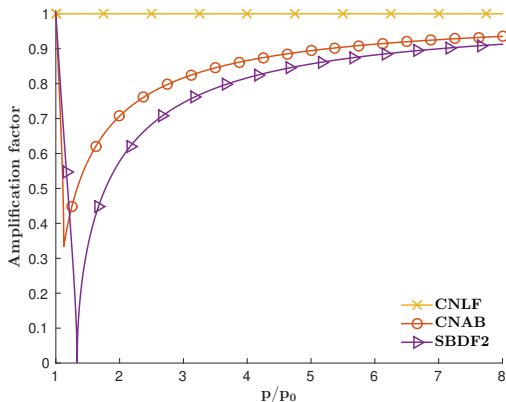


Figure : Amplification factor as  $\Delta t \rightarrow \infty$ .

# Numerical Experiments

We consider two classes of problems to demonstrate the effectiveness of our new methods:

- 1 Image inpainting.
- 2 Mean curvature motion.

# Image Inpainting

In Schönlieb and Bertozzi (2011), the authors proposed the fourth order inpainting model

$$u_t = -\Delta \nabla \cdot \left( \frac{\nabla u}{\sqrt{|\nabla u|^2 + \epsilon^2}} \right) + \lambda_0(u_0 - u),$$

and numerical solution by the first order accurate method

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} = & -\Delta \nabla \cdot \left( \frac{\nabla u^n}{\sqrt{|\nabla u^n|^2 + \epsilon^2}} \right) + \lambda_0(u_0 - u^n) \\ & + p_1 \Delta^2 u^n - p_1 \Delta^2 u^{n+1} + p_0 \lambda u^n - p_0 \lambda u^{n+1}. \end{aligned}$$



# Image inpainting



Figure :  $\text{TV-H}^{-1}$  image restoration.

Table : Iteration counts for  $\text{TV-H}^{-1}$  image restoration.

	$\Delta t$	Iterations
IMEX Euler	0.30	1002
SBDF2	0.54	401
CNAB	0.64	347

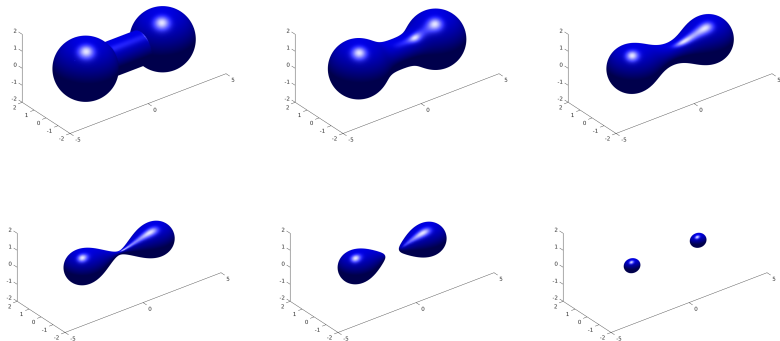
# Motion by mean curvature

We evolve the level set equation for motion by mean curvature,

$$u_t = \kappa |\nabla u| = |\nabla u| \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right)$$

on an initial dumbbell-shaped curve in 3D. We solve to time  $T = 0.75$  on a  $256 \times 128 \times 128$  periodic grid over  $(-5, 5) \times (-2, 2) \times (-2, 2)$ .

# Motion by mean curvature



**Figure :** Mean curvature flow of a dumbbell-shaped curve in 3D. From the left to right, top to bottom, the plots show the evolution at times  $t = 0, 0.10, 0.30, 0.525, 0.55, 0.75$ .

# Motion by mean curvature

On a machine with an Intel<sup>®</sup>Core<sup>™</sup> i5-4570 CPU@3.20GHz running MATLAB 2014b:

- Forward Euler: 3000 time steps  $\rightarrow$  over 28 minutes.
- SBDF2: 175 time steps  $\rightarrow$  under 2 minutes.

# Conclusion

Contributions of this thesis mainly fall into two categories:

- ① Further developed linearly stabilized schemes.
  - Outlined a framework for developing new methods of this type.
  - Identified three properties necessary for effective schemes.
- ② Proposed new methods that outperform existing ones.
  - IMEX multistep methods and exponential time differencing methods.

Future work:

- ① Development higher order methods without the deficiencies exhibited by ETD methods.
- ② A comparison with popular algorithms for nonlinear stiff PDEs, particularly for image inpainting.