Final Project for PHYSICS 67

For the final project, I learned more about model fitting using a very simple constant model. In particular, I practiced taking the first and second derivatives of a summation expression (i.e., the χ^2 value) in order to derive closed-form expressions for the minimum value and uncertainty.

First Derivative

Recall that $\chi^2 = \sum_i \left(\frac{n_i - p_0}{\sigma_i}\right)^2$. Thus, $\frac{d}{dp_0}\chi^2 = \sum_i 2\left(\frac{n_i - p_0}{\sigma_i}\right)\left(-\frac{1}{\sigma_i}\right) = -2\sum_i \frac{n_i - p_0}{\sigma_i^2}$.

Second Derivative

From above, $\frac{d}{dp_0}\chi^2 = -2\sum_i \frac{n_i - p_0}{\sigma^2}$. Thus, $\frac{d^2}{dp_0^2} = 2\sum_i \frac{1}{\sigma_i^2}$.

Discussion I

By visual inspection, the plots I came up with by deriving the closed-form expressions for the first- and second-derivatives match the ones produced using numerical methods for derivative computation.

Optimizing χ^2

$$\begin{aligned}
\frac{d}{dp_0}\chi^2 &= 0 \\
-2\sum_i \frac{n_i - p_0}{\sigma_i^2} &= 0 \\
\sum_i \frac{n_i - p_0}{\sigma_i^2} &= 0 \\
\sum_i \frac{n_i}{\sigma_i^2} - \sum_i \frac{p_0}{\sigma_i^2} &= 0 \\
\sum_i \frac{n_i}{\sigma_i^2} - p_0 \sum_i \frac{1}{\sigma_i^2} &= 0 \\
\sum_i \frac{n_i}{\sigma_i^2} &= p_0 \sum_i \frac{1}{\sigma_i^2} \\
p_0 &= \boxed{\frac{\sum_i \frac{n_i}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2}}}
\end{aligned}$$

Solving for the Uncertainty

Note that $\Delta\chi^2 = a(\delta p_0)^2$ where $a = \frac{1}{2}\frac{d^2}{dp_0^2}\chi^2 \approx 0.23$. Furthermore, note that $a\sigma_{p_0}^2 = 1$. Solving for σ_{p_0} gives us $\sigma_{p_0} = 0.877$.

Discussion II

The expressions obtained for p_0 and σ_{p_0} are exactly those obtained from inverse variance weighting!