Extract an unknown function in a non-parametric way

November 12, 2020

1. The standard Bayes approach already contains the "sequential" calibration of parameters.

2. To avoid data sensitive to low-T constrains parameter at high-T requires better functional parametrization

3. A non-parametric way to parametrize the unknown functional of $\hat{q}(T)$.

The standard Bayes approach

Suppose the following dependence of two sets of observables y on parameters x.

- $\mathbf{y}_1 = \mathbf{y}_1(x_1)$ only depends on x_1.
- $\mathbf{y}_2 = \mathbf{y}_2(x_1, x_2)$ depends on both x_1, x_2.

Suppose, people have measured $\mathbf{y}_1$ long ago in the past: $\mathbf{y}_1^{\mathrm{exp}} \pm \delta\mathbf{y}_1^{\mathrm{exp}}$, and then someone used it to extract information of x_1,

$$P_1(x_1) = \mathrm{Prior}(x_1)\mathrm{Likelihood}\left(\frac{\mathbf{y}_1(x_1) - \mathbf{y}_1^{\mathrm{exp}}}{\delta\mathbf{y}_1^{\mathrm{exp}}}\right)$$

Few years later,$\mathbf{y}_2^{\mathrm{exp}} \pm \delta\mathbf{y}_2^{\mathrm{exp}}$ are measured, but we don't want to vary x_1 arbitrarily when extracting x_2, so we use $P_1(x_1)$ as informative prior:

$$\begin{aligned} P_{12}(x_1, x_2) &= P_1(x_1)\mathrm{Prior}(x_2)\mathrm{Likelihood}\left(\frac{\mathbf{y}_2(x_1, x_2) - \mathbf{y}_2^{\mathrm{exp}}}{\delta\mathbf{y}_2^{\mathrm{exp}}}\right) \\ &= \mathrm{Prior}(x_1)\mathrm{Prior}(x_2)\mathrm{Likelihood}\left(\frac{\mathbf{y}_2(x_1, x_2) - \mathbf{y}_2^{\mathrm{exp}}}{\delta\mathbf{y}_2^{\mathrm{exp}}}\right)\mathrm{Likelihood}\left(\frac{\mathbf{y}_1(x_1) - \mathbf{y}_1^{\mathrm{exp}}}{\delta\mathbf{y}_1^{\mathrm{exp}}}\right) \end{aligned}$$

The same as calibrated to both dataset simultaneously.

The subtly is in the parametrization of unknown functionals

For example, usually, we try temperature dependence of $\hat{q}$

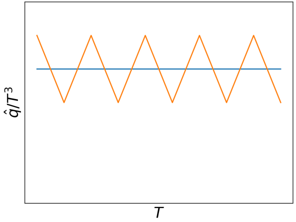
$$\frac{\hat{q}}{T^3} = A + B\left(\frac{T}{T_c}\right)^C$$

The problem: A, B, and C controls the temperature dependence in a highly correlated manner $\rightarrow$ May leads to correlated change of all parameters when we include new dataset.

Step-function like parameterization

$$\frac{\hat{q}}{T^3} = a_0(1 \sim 1.5T_c) + a_1(1.5 \sim 2T_c) + a_2(2 \sim 2.5T_c) + a_3(2.5 \sim 3T_c) + a_4(3 \sim 4T_c)$$

Decorrelate from one temperature to another. Potential problem: lack of adjacent correlation:

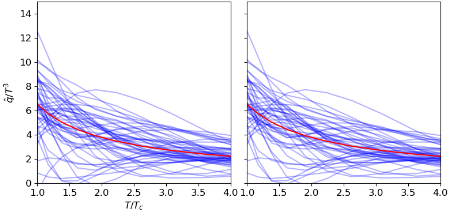


To avoid the yellow case, we can manipulate the choice of prior in a correlated way! Not long range correlation, but finite-range correlation that $P_0(x \rightarrow x_1) = P_1(x_1)$. $\rightarrow$ no need to change the model computation procedure, just need a carefully designed prior.

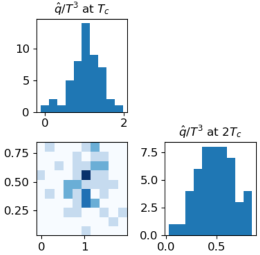
A proposed way to parametrize unknown functional form of $\hat{q}/T^3$

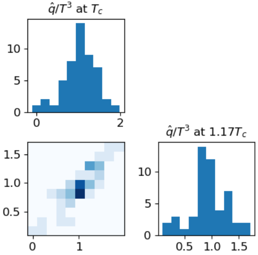
- Treat $\hat{q}/T^3$ as some unknown function in a functional space that:
  - If I know $\hat{q}/T^3$ at low temperature, I am still agnostic to its high-T behavior.
  - If I know $\hat{q}/T^3$ at T_1, its value at adjacent temperature must be close to the value at T_1.
- Easily modeled by a random function sampled by the Gaussian process.

$$P[\hat{q}(T)] \sim \mathcal{GP}\left(\mathrm{mean} = \hat{q}_0, \mathrm{cov} = Ce^{-\frac{(\ln T - \ln T')^2}{2\sigma^2}}\right)$$



"Infinitely" many points, but effectively only $\frac{\ln(T_{\max}/T_{\min})}{\sigma}$ d.o.f.. A smooth version of "independently vary" $\hat{q}$ at temperature far apart with short-range correlation.





Workflow

Run physical model:

- Generate, e.g. 100, realization of $P[\hat{q}(T)]$.
- Model predictions: centrality, energy, collision system, p_T dependent.

The following jobs is purely statistical.

- The fact that the parameterization $\hat{q}(T)$ is uncorrelated from $\hat{q}(T')$ if T and T' are sufficently different prevents low-T-sensitive data from constraining high-T parameters.
- Use emulators to learn the mapping from $\hat{q}(T_i, i = 1, 2, 3 \cdots)$ to observables.
- Using Bayes theorem to get the posterior distribution of $\hat{q}$:

$$P(\hat{q}[T]) = \int D[\hat{q}(T)] P_{\mathcal{GP}}[\hat{q}(T)] \mathrm{Likelihood}\left(\frac{\mathbf{y}_{\mathrm{emulator}}[\hat{q}(T)] - \mathbf{y}^{\mathrm{exp}}}{\delta \mathbf{y}^{\mathrm{exp}}}\right)$$

A non-parametric way to extract $\hat{q}(T)$
I do not know any application like this before in literature. We may need to use a "toy" model to evaluate its performance before using real model.