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Fractional Brownian Motion and its Application in Financial Mathematics

Diplomarbeit
zur Erlangung des ersten akademischen Grades

Diplommathematiker

(Wirtschaftsmathematik)

vorgelegt von

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Tag der Einreichung: 01.03.2015

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Abstract

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2 Gaussian Process and Brownian Motion

In this section we start off the general concept of probability spaces and stochastic processes. Of this, a most important case we then discribe, is Gaussian process. It bring us to introduce the Brownian Motion as a fine example.

2.1 Probability Space and Stochastic Process

DEFINITION 2.1. Let $\mathscr{A}$ be a collection of subsets of a set Ω. $\mathscr{A}$ is then a σ- *Algebra* on Ω if it satisfies the following conditions:

(i) $\Omega \in \mathscr{A}$.

(ii) For any set $F \in \mathscr{A}$, its complement $F^c \in \mathscr{A}$.

(iii) If a serie $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathscr{A}$, then $\cup_{n\in\mathbb{N}} F_n \in \mathscr{A}$.

DEFINITION 2.2. A mapping $\mathcal{P}$ is said to be a *probability measure* from $\mathscr{A}$ to $\mathscr{B}(\mathbb{R}^n)$, if $\mathcal{P}\left[\sum_{n=1}^{\infty} F_n\right] = \sum_{n=1}^{\infty} \mathcal{P}\left[F_n\right]$ for any $\{F_n\}_{n\in\mathbb{N}}$ disjoint in $\mathscr{A}$ satisfying $\sum_{n=1}^{\infty} F_n \in \mathscr{A}$.

DEFINITION 2.3. A *probability space* is defined as a triple $(\Omega, \mathscr{A}, \mathcal{P})$ of a set Ω, a σ-Algebra $\mathscr{A}$ of Ω and a measure $\mathcal{P}$ from $\mathscr{A}$ to $\mathscr{B}(\mathbb{R}^n)$.

The σ- Algebra generated of all open sets on $\mathbb{R}^n$ is called the *Borel σ- Algebra* which we denote as usual by $\mathscr{B}(\mathbb{R}^n)$. Let μ be a probability measure on $\mathbb{R}^n$. Indeed, $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mu)$ is a special case that probability space on $\mathbb{R}^n$. A function f mapping from $(\mathcal{D}, \mathscr{D}, \mu)$ into $(\mathcal{E}, \mathscr{E}, \nu)$ is *measurable* if its collection of the inverse image of $\mathscr{E}$ is a subset of $\mathscr{D}$. A *random variable* is a $\mathbb{R}^n$-valued measurable function on some probability space. Let $\mathcal{P}$ represent a probability measure, recall that in probability theory, for $B \in \mathscr{B}(\mathbb{R}^n)$ we call $\mathcal{P}\left[\{X \in B\}\right]$ the *distribution* of X. We write also $\mathcal{P}_X[\cdot]$ or $\mathcal{P}[X]$ for convenience of the notation above.

DEFINITION 2.4. Let $(\Omega, \mathscr{A}, \mathcal{P})$ be a probability space. A n-dimensional *stochastic process* (X_t) is a family of random variable such that $X_t(\omega) : \Omega \longrightarrow \mathbb{R}^n, \forall t \in T$, where T denotes the set of Index of Time.

DEFINITION 2.5. A stochastic process $(X_t)_{t\in T}$ is said to be *stationary*, if the joint distribution

$$\mathcal{P}\left[X_{t_1}, \ldots, X_{t_n}\right] = \mathcal{P}\left[X_{t_1+\tau}, \ldots, X_{t_n+\tau}\right]$$

for $t_1, \ldots, t_n$ and $t_1 + \tau, \ldots, t_n + \tau \in T$.

Remark that, definition 2.5 means the distribution of a stationary process is independent of a shift of time.

2.2 Normal Distribution and Gaussian Process

DEFINITION 2.6 (1-dimensional normal distribution)**.** A $\mathbb{R}$-valued random variable X is said to be *standard normal distributed*, if its distribution can be discribed as

$$\mathcal{P}\left[X \leq x\right] = \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-\frac{u^2}{2}} \, du$$

for $x \in \mathbb{R}$.

DEFINITION 2.7. A $\mathbb{R}$-valued random variable X is said to be *normal distributed* with a *mean* μ and a *variance* σ^2, if

$$(X - \mu)/\sigma$$

is standard normal distributed.

We use a notation $X \sim Y$, which means X and Y have the same distribution. In similar way it is denoted by $X \sim (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \, dx$, if it is standard normal distributed. In order to identifing the behaviour of a normal distributed random variable we recall the characteristic function in probability theory, see[1].

PROPOSITION 2.8. Let X be a $\mathbb{R}$-valued standard normal distributed random variable. The characteristic function of X

$$\Psi_X(\xi) := \int_{\mathbb{R}} e^{ix\xi} \mathcal{P}\left[X \in dx\right] = e^{-\frac{\xi^2}{2}} \tag{2.1}$$

for $\xi \in \mathbb{R}$.

Proof. According to the definion of characteristic function

$$\Psi_X(\xi) = \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} \, dx,$$

take differentiating both sides of the equation by ξ, then

$$\begin{aligned} \Psi'_X(\xi) &= \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} ix \, dx \\ &= (-i) \cdot \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} \left(\frac{d}{dx} e^{-\frac{x^2}{2}}\right) e^{ix\xi} \, dx \\ &\overset{part.int.}{=} -\int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} \xi \, dx \\ &= -\xi \Psi_X(\xi). \end{aligned}$$

Obviously, $\Psi(\xi) = \Psi(0) e^{-\frac{\xi^2}{2}}$ is the solution of the partial differential equation above, and $\Psi(0)$ is equal to 1. $\square$

In particular, the characteristic function of a normal distributed random variable with a mean μ and a variance σ^2, which denoted by $\Psi_{X_{\mu,\sigma^2}}(\xi)$, is $e^{i\mu\xi-\frac{1}{2}(\sigma\xi)^2}$. To achieve this result, we just need to substitute x by $(x-\mu)/\sigma$ in the calculation before.

DEFINITION 2.9. Let X be a $\mathbb{R}^n$-valued random variable. X is said to be *normal distributed*, if for any $d \in \mathbb{R}^n$ such that $d^T X$ is normal distributed on $\mathbb{R}$.

PROPOSITION 2.10. Let X be a $\mathbb{R}^n$-valued normal distributed. Then there exist $m \in \mathbb{R}^n$ and a positive definite symmetric matrix $\Sigma \in \mathbb{R}^{n\times n}$ such that,

$$\mathrm{E}\, e^{i\xi^T X} = e^{i\xi^T m - \frac{1}{2}\xi^T \Sigma \xi} \tag{2.2}$$

For $\xi \in \mathbb{R}^n$. Furthermore, the density function of X is

$$(2\pi)^{-\frac{d}{2}} \, (\det \Sigma)^{-\frac{1}{2}} \, e^{-\frac{1}{2}(x-m)^T \Sigma^{-1}(x-m)} \; dx\,. \tag{2.3}$$

Remark, the equation (2.2) can also be as definition of characteristic function of a n-dimensional normal distributed random variable. I.e., any normal distributed random variable can be characterized by form of the equation (2.2).

Proof. Since X normal distributed on $\mathbb{R}^n$, then $\xi^T X$ is normal distributed on $\mathbb{R}$. Due to the proposition 2.8 there is

$$\begin{aligned} \mathrm{E}e^{i\xi^T X} &= \mathrm{E}e^{i\cdot 1\cdot \xi^T X} \\ &= e^{i\mathrm{E}[\xi^T X]-\frac{1}{2}\mathrm{Var}[\xi^T X]} \\ &= e^{i\xi^T \mathrm{E}[X]-\frac{1}{2}\xi^T \mathrm{Var}[X]\xi}. \end{aligned}$$

According to the uniqueness theorem of characteristic function (Satz 23.4 in [1]), then we can deduce the density function of the equation (2.3). □

A normal distributed normal random variable can be characterized by its mean and variance respectively mean vector and covariance vector because of the characteristic function.

COROLLARY 2.11. A linear combination of independent normal distributed random variables has normal distribution.

Proof. In general case, we suppose $Y_1, \cdots, Y_m$ are independent random variables on $\mathbb{R}^n$, for $c_1, \cdots, c_m \in \mathbb{R}$. Let have a look at the chracteristic function of it,

$$\begin{aligned}
\mathrm{E}e^{i\xi^T \sum_{j=1}^m (c_j X_j)} &\overset{independent}{=} \prod_{j=1}^m \mathrm{E}e^{i\xi^T (c_j X_j)} \\
&= \prod_{j=1}^m \exp\left(i\xi^T \mathrm{E}[c_j X_j] - \frac{1}{2}\xi^T \mathrm{Var}[c_j X_j]\xi \right) \\
&= \exp\left(i\xi^T \mathrm{E}[\sum_{j=1}^m c_j X_j] - \frac{1}{2}\xi^T \sum\nolimits_{j=1}^m Var[c_j X_j]\xi \right) \\
&\overset{independent}{=} \exp\left(i\xi^T \mathrm{E}[\sum_{j=1}^m c_j X_j] - \frac{1}{2}\xi^T \mathrm{Var}[\sum_{j=1}^m c_j X_j]\xi \right),
\end{aligned}$$

which is a form of characteristc function of normal distribution. That means $\sum_{j=1}^m c_j X_j$ is normal distributed. $\square$

DEFINITION 2.12. Let $(X_t)_{t\in T}$ be a $\mathbb{R}^n$-valued stochastic process. (X_t) is said to be a *gaussian process* if

$$c_1^T X_{t_1} + \cdots + c_n^T X_{t_n}$$

has a normal distribution for any $c_1 \cdots c_n \in \mathbb{R}^n$, $t_1 \ldots t_n \in T$ and $n \in \mathbb{N}$.

The definition immediately shows for every X_t in gaussian process has a normal distribution.

EXAMPLE 2.13 (Bivariate Normal Distribution)**.** Suppose S_1, S_2 are independent random variables on $\mathbb{R}$ and have standard normal distributions. $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ is standard normal distributed since any lineare combination of independent normal distributed random variables has normal distribution. We define

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \sigma_1, & 0 \\ \sigma_2\rho, \sigma_2(1-\rho^2)^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \tag{2.4}$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 \in \mathbb{R}, -1 \le \rho \le 1$. Again, Y_1, Y_2 are normal distributed and the joint distribution $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is normal. Note $\mathrm{E}[Y_1] = \mu_1, \mathrm{E}[Y_2] = \mu_2$. Since S_1, S_2 are independent,

$$\begin{aligned}
\mathrm{Var}[Y_1] &= \mathrm{Var}[\sigma_1 S_1] \\
&= \sigma^2, \\
\mathrm{Var}[Y_2] &= \mathrm{Var}[\sigma_2 \rho S_1] + \mathrm{Var}[\sigma_2(1-\rho^2)^{\frac{1}{2}} S_2] \\
&= \sigma_2^2\rho^2 + \sigma_2^2(1-\rho^2) \\
&= \sigma_2^2, \\
\mathrm{Cov}[Y_1, Y_2] &= \mathrm{E}[(Y_1 - \mathrm{E}[Y_1])(Y_2 - \mathrm{E}[Y_2])] \\
&= \mathrm{E}[Y_1Y_2 - \mu_1 Y_2 - \mu_2 Y_1 + \mu_1\mu_2] \\
&= \mathrm{E}[(\sigma_1 S_1 + \mu_1)(\sigma_2\rho S_1 + \sigma_2(1-\rho^2)^{\frac{1}{2}} S_2 + \mu_2)] - \mu_1\mu_2 \\
&= \sigma_1\sigma_2 \underbrace{\mathrm{E}[S_1^2]}_{=1} \rho + \mu_1\sigma_2\rho \underbrace{\mathrm{E}[S_1]}_{=0} + \sigma_1\sigma_2(1-\rho^2)^{\frac{1}{2}} \underbrace{\mathrm{E}[S_1S_2]}_{=\mathrm{E}[S_1]\mathrm{E}[S_2]=0} \\
&+ \mu_1\sigma_2(1-\rho^2)^{\frac{1}{2}} \underbrace{\mathrm{E}[S_2]}_{=0} + \sigma_1 \underbrace{\mathrm{E}[S_1]}_{=0} \mu_2 + \mu_1\mu_2 - \mu_1\mu_2 \\
&= \rho\sigma_1\sigma_2,
\end{aligned}$$

that means the corrlation of Y_1, Y_2 is ρ. Because of the equation (2.3), joint the density function

$$f_{Y_1,Y_2}(y_1, y_2) = (2\pi)^{-1}(\det(\Sigma))^{-\frac{1}{2}} \exp\left((y_1 - \mu_1)\Sigma^{-1}(y_2 - \mu_2)\right),$$

where $\Sigma = \begin{pmatrix} \sigma_1^2, & 0 \\ \sigma_2^2\rho^2, & \sigma_2^2(1-\rho^2) \end{pmatrix}$

Indeed,

$$\det(\Sigma) = (1-\rho^2)\sigma_1^2\sigma_2^2$$

and

$$\Sigma^{-1} = \frac{\begin{pmatrix} \sigma_2^2(1-\rho^2), & 0 \\ -\sigma_2^2\rho, & \sigma_1^2 \end{pmatrix}}{(1-\rho^2)\sigma_1^2\sigma_2^2}.$$

Namely,

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}\sigma_1\sigma_2} \exp\left(-\frac{1}{2(1-\rho^2)}(z_1^2 - 2\rho z_1 z_2 + z_2^2)\right) \tag{2.5}$$

where $z_1 = \frac{y_1 - \mu_1}{\sigma_1}, z_2 = \frac{y_2 - \mu_2}{\sigma_2}$.

COROLLARY 2.14. Let Y_1, Y_2 be normal distributed random variables the conditional mean of Y_2 given Y_1

$$\mathrm{E}[Y_2|Y_1 = y_1] = \mathrm{E}[Y_2] + \rho(y_1 - \mathrm{E}[Y_1])\frac{\sigma_2}{\sigma 1},$$

and the conditional variance of Y_2 given Y_2

$$\mathrm{Var}[Y_2|Y_1 = y_1] = \sigma_1^2(1-\rho^2).$$

Where σ_1, σ_2 are standard deviations of Y_1, Y_2 and ρ is the correlation of Y_1, Y_2.

Proof. Let S_1, S_2 be independent standard normal distributed random variables. It is clear from the equation 2.4 that

$$\begin{aligned} S_1 &\sim \frac{(Y_1 - \mathrm{E}[Y_1])}{\sigma_1} \\ Y_2 &\sim \sigma_2 \rho S_1 + \sigma_2(1-\rho^2)^{\frac{1}{2}S_2} + \mathrm{E}[Y_2], \end{aligned}$$

more precisely,

$$Y_2 \sim \sigma_2 \rho \frac{(Y_1 - \mathrm{E}[Y_1])}{\sigma_1} + \sigma_2(1-\rho^2)^{\frac{1}{2}S_2} + \mathrm{E}[Y_2].$$

And

$$\mathrm{E}[Y_2|Y_1 = y_1] = \sigma_2 \rho \frac{(y_1 - \mathrm{E}[Y_1])}{\sigma_1} + \mathrm{E}[Y_2]$$

Now consider

$$\begin{aligned} \mathrm{Var}[Y_2|Y_1 = y_1] &= \mathrm{E}[(Y_2 - \mu_{Y_2|Y_1})^2 | Y_1 = y_1] \\ &= \int_{-\infty}^{\infty} (y_2 - \mu_{Y_2|Y_1})^2 f_{Y_2|Y_1}(y_2, y_1)\ dy_2 \\ &= \int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1) \right]^2 f_{Y_2|Y_1}(y_2, y_1)\ dy_2 \end{aligned}$$

Multiply both sides by the density function of Y_1 and integral it over by y_1, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathrm{Var}[Y_2|Y_1 = y_1] f_{Y_1}(y_1)\, dy_1 \\ = \quad & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1) \right]^2 \underbrace{f_{Y_2|Y_1}(y_2, y_1)\, f_{Y_1}(y_1)}_{f_{Y_1,Y_2}(y_1,y_2)} dy_2\ dy_1 \\ \Longleftrightarrow \quad & \\ & \mathrm{Var}[Y_2|Y_1 = y_1] \underbrace{\int_{-\infty}^{\infty} f_{Y_1}(y_1)\, dy_1}_{1} \\ = \quad & \mathrm{E}\left[(Y_2 - \mu_2) - (\frac{\rho\sigma_2}{\sigma_1})(Y_1 - \mu_1) \right]^2 \end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}[Y_2|Y_1 = y_1] &= \underbrace{\mathrm{E}[(Y_2-\mu_2)^2]}_{\sigma_2^2} - 2\frac{\rho\sigma_2}{\sigma_1}\underbrace{\mathrm{E}[(Y_1-\mu_1)(Y_2-\mu_2)]}_{\rho\sigma_1\sigma_2} \\
&+ \frac{\rho^2\sigma_2^2}{\sigma_1^2}\underbrace{\mathrm{E}[(Y_1-\mu_1)^2]}_{\sigma_1^2} \\
&= \sigma_2^2 - 2\rho^2\sigma^2 + \rho^2\sigma_2^2 \\
&= \sigma_2^2 - \rho^2\sigma_2^2
\end{aligned}
$$

□

3 Fractional Brownian Motion

4 Fractional Ornstein Uhlenbeck Process Model

5 Application in Financial Mathematics

6 Conclusion

References

[1] BAUER, H. (2002). Wahrscheinlichkeitstheorie(5th. durchges. und verb. Aufl.). Berlin: W. de Gruyter