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Fractional Brownian Motion and Applications in Financial Mathematics

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Abstract

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2 Gaussian Process and Brownian Motion

In this section we start off by looking at some general concepts of probability spaces and stochastic processes. Of this, a most important case we then discribe is Gaussian process. Within the framework of Gaussian processes, one could specific a stationary and independent behaviour of increments of it. This lead us to introduce the Brownian motion as a fine example.

2.1 Probability Space and Stochastic Process

Definition 2.1. Let $\mathscr{A}$ be a collection of subsets of a set Ω. $\mathscr{A}$ is said to be a σ- *Algebra* on Ω, if it satisfies the following conditions:

(i) $\Omega \in \mathscr{A}$.

(ii) For any set $F \in \mathscr{A}$, its complement $F^c \in \mathscr{A}$.

(iii) If there is a sequence such that $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathscr{A}$, then $\cup_{n\in\mathbb{N}} F_n \in \mathscr{A}$.

Definition 2.2. A mapping $\mathcal{P}$ is said to be a *probability measure* from $\mathscr{A}$ to $\mathscr{B}(\mathbb{R}^n)$, if $\mathcal{P}\left[\sum_{n=1}^{\infty} F_n\right] = \sum_{n=1}^{\infty} \mathcal{P}\left[F_n\right]$ for any $\{F_n\}_{n\in\mathbb{N}}$ disjoint in $\mathscr{A}$ satisfying $\sum_{n=1}^{\infty} F_n \in \mathscr{A}$.

Definition 2.3. A *probability space* is defined as a triple $(\Omega, \mathscr{A}, \mathcal{P})$ of a set Ω, a σ-Algebra $\mathscr{A}$ of Ω and a measure $\mathcal{P}$ from $\mathscr{A}$ to $\mathscr{B}(\mathbb{R}^n)$.

The σ- Algebra generated of all open sets on $\mathbb{R}^n$ is called the *Borel σ- Algebra* which we denote as usual by $\mathscr{B}(\mathbb{R}^n)$. Let μ be a probability measure on $\mathbb{R}^n$. Indeed, $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mu)$ is a special case that probability space on $\mathbb{R}^n$. A function f mapping from $(\mathcal{D}, \mathscr{D}, \mu)$ into $(\mathcal{E}, \mathscr{E}, \nu)$ is *measurable*, if its collection of the inverse image of $\mathscr{E}$ is a subset of $\mathscr{D}$. A *random variable* is a $\mathbb{R}^n$-valued measurable function on some probability space. Let $\mathcal{P}$ represent a probability measure, recall that in probability theory, for $B \in \mathscr{B}(\mathbb{R}^n)$ we call $\mathcal{P}\left[\{X \in B\}\right]$ the *distribution* of X. We write also $\mathcal{P}_X[\cdot]$ or $\mathcal{P}[X]$ for convenience for those notations .

Definition 2.4. Let $(\Omega, \mathscr{A}, \mathcal{P})$ be a probability space. A n-dimensional *stochastic process* $(X_t)_{t\in T}$ is a family of random variable such that $X_t(\omega) : \Omega \longrightarrow \mathbb{R}^n, \forall t \in T$, where T denotes the set of Index of Time.

Some basically definitions, which are needed in following sections, are given.

Definition 2.5. A stochastic process $(X_t)_{t\in T}$ is said to be *stationary*, if the joint distribution

$$\mathcal{P}\left[X_{t_1}, \ldots, X_{t_n}\right] = \mathcal{P}\left[X_{t_1+\tau}, \ldots, X_{t_n+\tau}\right]$$

for $t_1, \ldots, t_n$ and $t_1 + \tau, \ldots, t_n + \tau \in T$.

Remark that, Definition 2.5 means the distribution of a stationary process is independent of a shift of time.

Definition 2.6. Let $(X_t)_t$ be a stochastic process.

$$\varsigma_X(t,s) := \mathrm{Cov}(X_t, X_s)$$

is called *autocovariance* between s, t and

$$\eta_X(t,s) := \frac{\mathrm{Cov[X_t, X_s]}}{\sqrt{\mathrm{Var}[X_t]\mathrm{Var}[X_s]}}$$

is called *autocorrelation* betwenn s, t.

If $(X_t)_t$ is stationary process, we write $\varsigma_X(\tau)$ for $\varsigma_X(t, t+\tau)$ for any t. $\eta_X(\tau)$ is used in the same way.

We use a notation $X \sim Y$ represents X equals Y *in distribution*.

Definition 2.7. A stochastic process $(X_t)_{t\in T}$ is said to be *α-self similar* if $(X_{ct_1}, \ldots, X_{ct_k}) \sim (c^\alpha X_{t_1}, \ldots, c^\alpha X_{t_k})$ for any $t_1, \ldots, t_k, ct_1, \ldots, ct_k \in T$ and $c > 0$.

2.2 Normal Distribution and Gaussian Process

Definition 2.8 (1-dimensional normal distribution)**.** A $\mathbb{R}$-valued random variable X is said to be *standard normal distributed* or *standard Gaussian*, if its distribution can be discribed as

$$\mathcal{P}\left[X \leq x\right] = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\ du \tag{2.1}$$

for $x \in \mathbb{R}$.

The integrand of (2.1) is also called *density function* of a Gaussian random variable.

Definition 2.9. A $\mathbb{R}$-valued random variable X is said to be *normal distributed* or *Gaussian* with a *expected value* μ and a *variance* σ^2, if

$$(X-\mu)/\sigma$$

is standard Gaussian.

Proposition 2.10. Let X be a $\mathbb{R} - \mathit{valued}$ Gaussian random variable with expected value μ and variance σ^2, then it is distributed as

$$\mathcal{P}[X \leq x] = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{(u-\mu)^2}{2\sigma^2}}\ du$$

Proof. Suppose $X = \sigma Y + \mu$ with Y standard Gaussian. We denote this mapping by $g(y) : y \to \sigma y + \mu$ and give the inverse $g^{-1}(x) : x \to \frac{(x-\mu)}{\sigma}$. The distribution function of X is

$$\begin{aligned}\int_\Omega \mathcal{P}[X \in dx] &= \int_\Omega \mathcal{P}[Y \circ g \in dy] \\ &= \int_{\mathbb{R}\circ g} f_X \circ g^{-1}(y)\, dy \\ &= \int_{\mathbb{R}} \sigma \frac{1}{\sqrt{2\pi}} \exp\{\frac{(\frac{(y-\mu)}{\sigma})^2}{2}\}\, dy \\ &= (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\{-\frac{(y-\mu)^2}{2\sigma^2}\}\, dy,\end{aligned}$$

where f_X is density function of X. $\square$

It is denoted by $X \sim (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}\, dx$, if X is standard Gaussian. In order to verifying the behaviour of a normal distributed random variable we use the characteristic function in probability theory, Cf.[1].

Theorem 2.11. Let X be a $\mathbb{R}$-valued Gaussian random variable. The characteristic function of X

$$\Psi_X(\xi) := \int_{\mathbb{R}} e^{ix\xi} \mathcal{P}\left[X \in dx\right] = e^{i\mu\xi - \frac{1}{2}(\sigma\xi)^2} \tag{2.2}$$

for $\xi \in \mathbb{R}$.

Proof. Cf.[12]. We assume firstly X is standard Gaussian. In terms of the Definion of characteristic function of a standard Gaussian X, integrating its density function over $\mathbb{R}$ we get

$$\Psi_X(\xi) = \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi}\, dx,$$

take differentiating both sides of the equation by ξ, then

$$\begin{aligned}\Psi'_X(\xi) &= \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} ix\, dx \\ &= (-i) \cdot \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} (\frac{d}{dx} e^{-\frac{x^2}{2}}) e^{ix\xi}\, dx \\ &\overset{part.int.}{=} -\int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} \xi\, dx \\ &= -\xi \Psi_X(\xi).\end{aligned}$$

Obviously, $\Psi(\xi) = \Psi(0) e^{-\frac{\xi^2}{2}}$ is the solution of the partial differential equation, and $\Psi(0)$ equals 1, hence $\Psi(\xi) = e^{-\frac{\xi^2}{2}}$. In particular, the characteristic function of a Gaussian random variable with a expected value μ and a variance σ^2, which denoted by $\Psi_{X_{\mu,\sigma^2}}(\xi)$, is $e^{i\mu\xi - \frac{1}{2}(\sigma\xi)^2}$. To achieve this result, we just need to substitute x by $(x-\mu)/\sigma$ in the previous calculation. $\square$

Definition 2.12. Let X be a $\mathbb{R}^n$-valued random vector. X is said to be *normal distributed* or *Gaussian*, if for any $d \in \mathbb{R}^n$ such that $d^T X$ is Gaussian in $\mathbb{R}$.

Definition 2.13. A stochastic process $(X_t)_{t\in T}$ is said to be *Gaussian process* if the joint distribution of any finite instance is Gaussian, that means $(X_{t_1}, \ldots, X_{t_n})$ has joint Gaussian distribution in $\mathbb{R}^n$ for $t_1, \ldots, t_n \in T$.

The definition immediately shows every instance X_t in Gaussian process is Gaussian.

Corollary 2.14. Let $(X_t)_{t\in T}$ be a stochastic process. The following condition is equivalent to Definition 2.13.

$$\sum_j^n c_{t_j} X_{t_j} \tag{2.3}$$

is Gaussian for any $t_1, \ldots, t_n \in T$.

Proof. It is clear due to Definition 2.12. $\square$

Lemma 2.15. Let X be a $\mathbb{R}^n$-valued normal distributed ramdon vector. Then its characteristic function is

$$\mathrm{E}\, e^{i\xi^T X} = e^{i\xi^T m - \frac{1}{2}\xi^T \Sigma \xi}. \tag{2.4}$$

For $\xi \in \mathbb{R}^n$. Where $m \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n\times n}$ are *mean vector* , *covariance matrix* of X respectively. Furthermore, the density function of X is

$$(2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}}\, e^{-\frac{1}{2}(x-m)^T \Sigma^{-1}(x-m)}. \tag{2.5}$$

Remark, the equation (2.4) can also be as definition of characteristic function of a n-dimensional normal distributed random variable. I.e., any normal distributed random variable can be characterized by form of the equation (2.4).

Proof. Since X normal distributed on $\mathbb{R}^n$, then $\xi^T X$ is normal distributed on $\mathbb{R}$. Due to the Theorem 2.11, there is

$$\begin{aligned}
\mathrm{E}e^{i\xi^T X} &= \mathrm{E}e^{i\cdot 1\cdot \xi^T X} \\
&= e^{i\mathrm{E}[\xi^T X] - \frac{1}{2}\mathrm{Var}[\xi^T X]} \\
&= e^{i\xi^T \mathrm{E}[X] - \frac{1}{2}\xi^T \mathrm{Var}[X]\xi} \\
&= e^{i\xi^T m - \frac{1}{2}\xi^T \Sigma \xi}.
\end{aligned}$$

Moreover, since Σ symmetirc and positive definit, there exist $\Sigma^{-1}, \Sigma^{\frac{1}{2}}$ and $\Sigma^{-\frac{1}{2}}$.

$$
\begin{aligned}
& (2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{ix^T\xi} e^{-\frac{1}{2}(x-m)^T\Sigma^{-1}(x-m)}\ dx \\
= \quad & (2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{ix^T\xi} e^{i(x-m)^T\xi} e^{-\frac{1}{2}(x-m)^T\Sigma^{-1}(x-m)}\ dx \\
= \quad & (2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} e^{im^T\xi} \int_{\mathbb{R}^n} e^{i(x-m)^T\xi} e^{i(x-m)^T\xi} e^{-\frac{1}{2}(x-m)^T\Sigma^{-1}(x-m)}\ dx \\
\overset{y=\Sigma^{-\frac{1}{2}}x}{=} \quad & (2\pi)^{-\frac{n}{2}} e^{im^T\xi} \int_{\mathbb{R}^n} e^{i(\Sigma^{\frac{1}{2}}y)\xi}\ e^{-\frac{1}{2}|y|^2}\ dy \\
= \quad & (2\pi)^{-\frac{n}{2}} e^{im^T\xi} \int_{\mathbb{R}^n} e^{iy^T(\Sigma^{\frac{1}{2}}\xi)}\ e^{-\frac{1}{2}|y|^2}\ dy \\
\overset{\text{Fourier transformation}}{=} \quad & e^{im^T\xi} e^{-\frac{1}{2}|\Sigma^{\frac{1}{2}}\xi|^2} \\
= \quad & e^{im^T\xi} e^{-\frac{1}{2}\xi^T\Sigma\xi}
\end{aligned}
$$

In terms of the uniqueness theorem of characteristic function (in [1], p.199, Satz 23.4), then we can deduce (2.5) is density function of X. □

Theorem 2.16. A linear combination of independent normal distributed random variable (or vector) is Gaussian.

Proof. We suppose $X_1, \cdots, X_m$ are independent random vectors on $\mathbb{R}^n$ and $c_1, \cdots, c_m \in \mathbb{R}$. Let have a look at the chracteristic function of it,

$$
\begin{aligned}
\mathrm{E}e^{i\xi^T\sum_{j=1}^m (c_jX_j)} \quad & \overset{independent}{=} \quad \prod_{j=1}^m \mathrm{E}e^{i\xi^T(c_jX_j)} \\
& = \quad \prod_{j=1}^m \exp\left(i\xi^T \mathrm{E}[c_jX_j] - \frac{1}{2}\xi^T \mathrm{Var}[c_jX_j]\xi \right) \\
& = \quad \exp\left(i\xi^T \mathrm{E}[\sum_{j=1}^m c_jX_j] - \frac{1}{2}\xi^T \textstyle\sum_{j=1}^m Var[c_jX_j]\xi \right) \\
& \overset{independent}{=} \quad \exp\left(i\xi^T \mathrm{E}[\sum_{j=1}^m c_jX_j] - \frac{1}{2}\xi^T \mathrm{Var}[\sum_{j=1}^m c_jX_j]\xi \right),
\end{aligned}
$$

which is a form of characteristc function of normal distribution. That means $\sum_{j=1}^m c_jX_j$ is Gaussian. □

Example 2.17 (Bivariate Normal Distribution)**.** Cf.[9], p.241, Example 8.6. Suppose S_1, S_2 are independent random variables and have standard normal distributions. $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ has standard normal joint distribution since they are independent. We define

$$
\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \sigma_1, & 0 \\ \sigma_2\rho, \sigma_2(1-\rho^2)^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \tag{2.6}
$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 \in \mathbb{R}, -1 \leq \rho \leq 1$. Again, Y_1, Y_2 are Gaussian and the joint distribution $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is also Gaussian. We set $\mathrm{E}[Y_1] = \mu_1, \mathrm{E}[Y_2] = \mu_2$ for short. Since S_1, S_2 are independent,

$$\begin{aligned}
\mathrm{Var}[Y_1] &= \mathrm{Var}[\sigma_1 S_1] \\
&= \sigma_1^2, \\
\mathrm{Var}[Y_2] &= \mathrm{Var}[\sigma_2 \rho S_1] + \mathrm{Var}[\sigma_2 (1-\rho^2)^{\frac{1}{2}} S_2] \\
&= \sigma_2^2 \rho^2 + \sigma_2^2 (1-\rho^2) \\
&= \sigma_2^2, \\
\mathrm{Cov}[Y_1, Y_2] &= \mathrm{E}[(Y_1 - \mathrm{E}[Y_1])(Y_2 - \mathrm{E}[Y_2])] \\
&= \mathrm{E}[Y_1 Y_2 - \mu_1 Y_2 - \mu_2 Y_1 + \mu_1 \mu_2] \\
&= \mathrm{E}[(\sigma_1 S_1 + \mu_1)(\sigma_2 \rho S_1 + \sigma_2 (1-\rho^2)^{\frac{1}{2}} S_2 + \mu_2)] - \mu_1 \mu_2 \\
&= \sigma_1 \sigma_2 \underbrace{\mathrm{E}[S_1^2]}_{=1} \rho + \mu_1 \sigma_2 \rho \underbrace{\mathrm{E}[S_1]}_{=0} + \sigma_1 \sigma_2 (1-\rho^2)^{\frac{1}{2}} \underbrace{\mathrm{E}[S_1 S_2]}_{=\mathrm{E}[S_1]\mathrm{E}[S_2]=0} \\
&+ \mu_1 \sigma_2 (1-\rho^2)^{\frac{1}{2}} \underbrace{\mathrm{E}[S_2]}_{=0} + \sigma_1 \underbrace{\mathrm{E}[S_1]}_{=0} \mu_2 + \mu_1 \mu_2 - \mu_1 \mu_2 \\
&= \rho \sigma_1 \sigma_2,
\end{aligned}$$

that means the corrlation of Y_1, Y_2 is ρ. Because of the equation (2.5), the joint density function

$$f_{Y_1, Y_2}(y_1, y_2) = (2\pi)^{-1} (\det(\Sigma))^{-\frac{1}{2}} \exp\left((y_1 - \mu_1) \Sigma^{-1} (y_2 - \mu_2) \right),$$

where $\Sigma = \begin{pmatrix} \sigma_1^2, & 0 \\ \sigma_2^2 \rho^2, & \sigma_2^2 (1-\rho^2) \end{pmatrix}$

Indeed,

$$\det(\Sigma) = (1-\rho^2) \sigma_1^2 \sigma_2^2$$

and

$$\Sigma^{-1} = \frac{\begin{pmatrix} \sigma_2^2 (1-\rho^2), & 0 \\ -\sigma_2^2 \rho, & \sigma_1^2 \end{pmatrix}}{(1-\rho^2) \sigma_1^2 \sigma_2^2}.$$

Namely,

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2\pi (1-\rho^2)^{\frac{1}{2}} \sigma_1 \sigma_2} \exp\left(-\frac{1}{2(1-\rho^2)} (z_1^2 - 2\rho z_1 z_2 + z_2^2) \right) \tag{2.7}$$

where $z_1 = \frac{y_1 - \mu_1}{\sigma_1}, z_2 = \frac{y_2 - \mu_2}{\sigma_2}$.

Corollary 2.18. Let Y_1, Y_2 be $\mathbb{R}$-valued random variables and $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ has a joint normal distribution, then the conditional expected value of Y_2 given Y_1

$$\mathrm{E}[Y_2|Y_1 = y_1] = \mathrm{E}[Y_2] + \rho(y_1 - \mathrm{E}[Y_1])\frac{\sigma_2}{\sigma 1},$$

and the conditional variance of Y_2 given Y_2

$$\mathrm{Var}[Y_2|Y_1 = y_1] = \sigma_1^2(1-\rho^2).$$

Where σ_1, σ_2 are standard deviations of Y_1, Y_2 and ρ is the correlation of Y_1, Y_2.

Proof. Recall the equation (2.7), we can specify the joint density function if σ_1, σ_2, ρ are known. As result of this, $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ has a form of the equation (2.6). Suppose S_1, S_2 are independent standard normal distributed random variables. Now we have

$$\begin{aligned} S_1 &\sim \frac{(Y_1 - \mathrm{E}[Y_1])}{\sigma_1} \\ Y_2 &\sim \sigma_2\rho S_1 + \sigma_2(1-\rho^2)^{\frac{1}{2}}S_2 + \mathrm{E}[Y_2], \end{aligned}$$

more precisely,

$$Y_2 \sim \sigma_2\rho\frac{(Y_1 - \mathrm{E}[Y_1])}{\sigma_1} + \sigma_2(1-\rho^2)^{\frac{1}{2}}S_2 + \mathrm{E}[Y_2].$$

Take expectation of both sides,

$$\mathrm{E}[Y_2|Y_1 = y_1] = \sigma_2\rho\frac{(y_1 - \mathrm{E}[Y_1])}{\sigma_1} + \mathrm{E}[Y_2].$$

Now consider

$$\begin{aligned} \mathrm{Var}[Y_2|Y_1 = y_1] &= \mathrm{E}[(Y_2 - \mu_{Y_2|Y_1})^2|Y_1 = y_1] \\ &= \int_{-\infty}^{\infty} (y_2 - \mu_{Y_2|Y_1})^2 f_{Y_2|Y_1}(y_2, y_1)\ dy_2 \\ &= \int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1)\right]^2 f_{Y_2|Y_1}(y_2, y_1)\ dy_2, \end{aligned}$$

After multiplying both sides by the density function of Y_1 and integrating it by y_1, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathrm{Var}[Y_2|Y_1 = y_1] f_{Y_1}(y_1)\, dy_1 \\ = \quad & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1)\right]^2 \underbrace{f_{Y_2|Y_1}(y_2, y_1)\, f_{Y_1}(y_1)}_{f_{Y_1,Y_2}(y_1,y_2)}\, dy_2\ dy_1 \\ \iff \quad & \\ & \mathrm{Var}[Y_2|Y_1 = y_1] \underbrace{\int_{-\infty}^{\infty} f_{Y_1}(y_1)\, dy_1}_{1} \\ = \quad & \mathrm{E}\left[(Y_2 - \mu_2) - (\frac{\rho\sigma_2}{\sigma_1})(Y_1 - \mu_1)\right]^2 \end{aligned}$$

multiplying right side out, we see

$$\begin{aligned}
\mathrm{Var}[Y_2|Y_1 = y_1] &= \underbrace{\mathrm{E}[(Y_2-\mu_2)^2]}_{\sigma_2^2} - 2\frac{\rho\sigma_2}{\sigma_1}\underbrace{\mathrm{E}[(Y_1-\mu_1)(Y_2-\mu_2)]}_{\rho\sigma_1\sigma_2} \\
&+ \frac{\rho^2\sigma_2^2}{\sigma_1^2}\underbrace{\mathrm{E}[(Y_1-\mu_1)^2]}_{\sigma_1^2} \\
&= \sigma_2^2 - 2\rho^2\sigma^2 + \rho^2\sigma_2^2 \\
&= \sigma_2^2 - \rho^2\sigma_2^2.
\end{aligned}$$

□

Theorem 2.19. Let X be a Gaussian random variable, then

$$\mathrm{E}[\exp(\beta X)] = \exp(\beta\mu + \frac{1}{2}\beta^2\sigma^2). \tag{2.8}$$

Where μ and σ are $\mathrm{E}[X]$ and $\mathrm{Var}[X]$ respectively.

Proof.

$$\begin{aligned}
&\mathrm{E}[\exp(\beta X)] \\
= \; &(2\pi\sigma^2)^{-\frac{1}{2}}\int_{\mathbb{R}} \exp(\beta x)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \\
= \; &(2\pi\sigma^2)^{-\frac{1}{2}}\int_{\mathbb{R}} \exp(\beta x)\exp\left(-\frac{(x^2-2x\mu+\mu^2)}{2\sigma^2}\right)dx \\
= \; &(2\pi\sigma^2)^{-\frac{1}{2}}\int_{\mathbb{R}} \exp\left(-\frac{x^2-2(\beta\sigma^2+\mu)x+\mu^2}{2\sigma^2}\right)dx \\
= \; &(2\pi\sigma^2)^{-\frac{1}{2}}\int_{\mathbb{R}} \exp\left(-\frac{x^2-2(\beta\sigma^2+\mu)x+(\beta\sigma^2+\mu)^2-(\beta\sigma^2+\mu)^2+\mu^2}{2\sigma^2}\right)dx \\
= \; &(2\pi\sigma^2)^{-\frac{1}{2}}\int_{\mathbb{R}} \exp\left(-\frac{(x-(\beta\sigma^2+\mu))^2+\mu^2-(\beta\sigma^2+\mu)^2}{2\sigma^2}\right)dx \\
= \; &\exp\left(\frac{(\beta\sigma^2+\mu)^2-\mu^2}{2\sigma^2}\right)\underbrace{(2\pi\sigma^2)^{-\frac{1}{2}}\int_{\mathbb{R}} \exp\left(-\frac{(x-(\beta\sigma^2+\mu))^2}{2\sigma^2}\right)dx}_{1} \\
= \; &\exp\left(\frac{\beta^2\sigma^4+2\mu\beta\sigma^2}{2\sigma^2}\right) \\
= \; &\exp(\mu\beta + \frac{1}{2}\beta^2\sigma^2)
\end{aligned}$$

□

2.3 Brownian Motion

The Brownian motion was first introduced by Bachelier in 1900 in his PhD thesis. We now give the common definition of it.

Definition 2.20. Let $(B_t)_{t\geq 0}$ be a $\mathbb{R}^n$-valued stochastic process. (B_t) is called *Brownian motion* if it satisfies the following conditions:

(i) $B_0 = 0$ a.s. .

(ii) $(B_{t_1} - B_{t_0}), \dots, (B_{t_n} - B_{t_{n_1}})$ are independet for $0 = t_0 < t_1 < \dots < t_n$ and $n \in \mathbb{N}$.

(iii) $B_t - B_s \sim B_{t-s}$, for $0 \leq s \leq t < \infty$.

(iv) $B_t - B_s \sim \mathcal{N}(0, t-s)^{\otimes n}$.

(v) B_t is continuous in t a.s. .

A usual saying for *(ii)* and *(iii)* is the Brownian motion has independent, stationary increments. In (iv), N represent a random variable which has a normal distribution. B_t is normal distributed due to (ii). It is clear that the increments of Brownian motion is stationary.

Proposition 2.21. Let (B_t) be $\mathbb{R}$-valued Brownian motion. Then the covariance of B_m, B_n for $m, n \geq 0$ is $m \wedge n$.

Proof. Without loss of generality, we assume that $m \geq n$, then

$$\begin{aligned} \mathrm{E}[B_m B_n] &= \mathrm{E}[(B_m - B_n)B_n] + \mathrm{E}[B_n^2] \\ &= \mathrm{E}[B_m - B_n]\mathrm{E}[B_n] + n \\ &= n. \end{aligned}$$

$\square$

Proposition 2.22. Let (B_t) be $\mathbb{R}$-valued Brownian motion. Then $B_{cm} \sim c^{\frac{1}{2}} B_m$.

Proof. Because B_m is normal distributed for any $m > 0$, we then get

$$\begin{aligned} \mathrm{E}[e^{i\xi B_{cm}}] &= e^{-\frac{1}{2}cm\xi^2} \\ &= e^{-\frac{1}{2}(c(m)^{\frac{1}{2}}\xi)^2} \\ &= \mathrm{E}[e^{i\xi c^{\frac{1}{2}} B_m}]. \end{aligned}$$

$\square$

Theorem 2.23. A $\mathbb{R}$-valued Brownian motion is a Gaussian process.

Proof. The following idea using the independence of increments to prove the claim come from [12]. We choose $0 = t_0 < t_1 < \dots < t_n$, for $n \in \mathbb{N}$. Define $V = (B_{t_1}, \dots, B_{t_n})^T$,

$K = (B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}})^T$ and $A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$. Let us see the characteristic function of V,

$$
\begin{aligned}
& \mathrm{E}[e^{i\xi^T V}] \\
= \quad & \mathrm{E}[e^{i\xi^T AK}] \\
= \quad & \mathrm{E}[e^{iA^T \xi K}] \\
= \quad & \mathrm{E}[\exp(i(\xi^{(1)} + \cdots + \xi^{(n)}, \xi^{(2)} + \cdots + \xi^{(n)}, \cdots, \xi^{(n)})] \\
\cdot \quad & (B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})^T) \\
\overset{ind.increments}{=} \quad & \prod_{j=1}^{n} \mathrm{E}[\exp(i(\xi^{(j)+\cdots+\xi^{(n)}})(B_{t_j} - B_{t_{t-1}}))] \\
\overset{stat.increments}{=} \quad & \prod_{j=1}^{n} \exp(-\frac{1}{2}(t_j - t_{j-1})(\xi^{(j)} + \cdots + \xi^{(n)})^2) \\
= \quad & \exp\left(-\frac{1}{2}\sum_{j=1}^{n}(t_j - t_{j-1})(\xi^{(j)} + \cdots + \xi^{(n)})^2\right) \\
= \quad & \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n} t_j(\xi^{(j)} + \cdots + \xi^{(n)})^2 - \sum_{j=1}^{n} t_{j-1}(\xi^{(j)} + \cdots + \xi^{(n)})^2\right)\right) \\
= \quad & \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n-1} t_j((\xi^{(j)} + \cdots + \xi^{(n)})^2 - (\xi^{(j+1)} + \cdots + \xi^{(n)})^2) + t_n(\xi^{(n)})^2\right)\right) \\
= \quad & \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n-1} t_j\xi^{(j)}(\xi^{(j)} + 2\xi^{(j+1)} + \cdots + 2\xi^{(n)}) + t_n(\xi^{(n)})^2\right)\right) \\
= \quad & \exp\left(-\frac{1}{2}\left(\sum_{j,h=1}^{n} (t_j \wedge t_h)\xi^{(j)}\xi^{(h)}\right)\right).
\end{aligned}
$$

Recall with Proposition 2.3, $(t_j \wedge t_h)_{t,h=1,\ldots,n}$ is the covariance matrix of V and therefore it is symmetric and positive definit. The mean vector of it is zero, then we have been proved that the characteristic function is a form of some normal distributed random vector, i.e., V is Gaussian. □

Schilling gave in his lecture [12] the relationship between a one-dimensional Brownian motion and a n-dimensional Brownian motion. In fact, $(B_t^{(l)})_{l=1,\ldots,n}$ is Brownian motion if and only if $B_t^{(l)}$ is Brownian motion and all of the component are independent. Using this independence and the theorem of fubini in the characteristic function for high dimensional Brownian motion we can say a n-dimensional Brownian motion is also a Gaussian process.

Definition 2.24. Let $(X_t)_{t \in T}$ be a stochastic process. $(Y_t)_{t \in T}$ is defined on the same probability space as $(X_t)_{t \in T}$ and said to be *modification* of $(X_t)_{t \in T}$, if

$$\mathcal{P}[X_t = Y_t] = 1 \quad \forall \quad t \in T.$$

Theorem 2.25 (Kolmogorov Chentsov)**.** Let $(X_t)_{t \geq 0}$ be a stochastic process on $\mathbb{R}^n$ such that

$$[|X_j - X_k|^\alpha] \leq c|j-k|^{1+\beta} \quad \forall \quad j,k \geq 0 \quad \text{and} \quad j \neq k,$$

for $\alpha, \beta > 0, c < \infty$. Then $(X_t)_t$ has a modification $(Y_t)_t$ with continuous sample path such that

$$\mathrm{E}[(\frac{|Y_j - Y_k|}{|j-k|^\gamma})^\alpha] < \infty$$

for all $\gamma \in (0, \frac{\beta}{\alpha})$.

Proof. See [7], p.519. □

Lemma 2.26. Let $(B_t)_{t \geq 0}$ be Brownian motion. Then

$$\mathrm{E}[B_t^{2k}] = (2k-1)!!t^{2k}$$

for $k \in \mathbb{N}_0$.

Proof. Cf.[12]. Taking expectation of B_t^{2k}, we get

$$\begin{aligned}
\mathrm{E}[B_t^{2k}] &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2t}} \, dx \\
&\overset{x=\sqrt{2ty}}{=} \frac{2^k t^k}{\sqrt{\pi}} \int_0^\infty y^{k-\frac{1}{2}} e^{-y} \, dy \\
&= \frac{2^k t^k}{\sqrt{\pi}} \int_0^\infty y^{k+\frac{1}{2}-1} e^{-y} \, dy \\
&= \frac{2^k t^k}{\sqrt{\pi}} \Gamma(k + \frac{1}{2}) \\
&= \frac{2^k t^k}{\sqrt{\pi}} \Gamma(\frac{1}{2}) \prod_{j=1}^{k} (j - \frac{1}{2}) \\
&= 2^k t^k \prod_{j=1}^{k} (\frac{2j-1}{2}) \\
&= (2k-1)!! \cdot t^k
\end{aligned}$$

□

Corollary 2.27. Let $(B_t)_{t \geq 0}$ be Brownian motion. Then B_t is γ-Hölder continuous on a compact set of time almost surely for all $\gamma < \frac{1}{2}$.

Proof. Because of Lemma 2.26, we have

$$\begin{aligned} \mathrm{E}[(B_t - B_s)^{2k}] &= \mathrm{E}[B_{t-s}^{2k}] \\ &= (2k-1)!! \cdot |t-s|^k. \end{aligned}$$

In terms of the Theorem of Kolmogorov Chenstov, B_t is γ-Hölder continuous a.s. for $\gamma \in (0, \frac{k}{2k})$. □

3 Stable Measures and Stable Integrals

In order to represent a integration form of fractional Brownian motion, we deal with the stable integral in this section. In fact, fractional Brownian motion is a Gaussian process with zero mean. To show Gaussian properties of it, we define it by a stable integral which can imaged as stochastic process of stable variables on time.

3.1 Stable Variables

Definition 3.1. Let X be a random variable. X is said to have a stable distribution, if there exist $0 < \gamma \leq 2, \delta \geq 0, -1 \geq \kappa \geq 1, \theta \in \mathbb{R}$ such that its characteristic function can be described as following

$$\mathrm{E}[\exp i\xi X] = \begin{cases} \exp\{i\xi\theta - |\delta\xi|^{\gamma}(1 - i\kappa \cdot sgn(\xi) \tan \frac{\gamma\pi}{2})\}, & \text{if} \quad \gamma \in (0,1) \cup (1,2], \\ \exp\{i\xi\theta - |\delta\xi|(1 + i\frac{2}{\pi}\kappa \cdot sgn(\xi) \ln |\xi|)\}, & \text{if} \quad \gamma = 1. \end{cases} \tag{3.9}$$

Where

$$sgn(x) = \begin{cases} 1, & \text{if} \quad x > 1 \\ 0, & \text{if} \quad x = 0 \\ -1, & \text{if} \quad x < 0. \end{cases}$$

Notice, we write $\Lambda(\gamma, \kappa, \theta, \delta)$ as one random variable whose characteristic function equals (3.9).

Theorem 3.2. X is Gaussian if and only $X \sim \Lambda(\gamma, \kappa, \theta, \delta)$ with $\gamma = 2$.

Proof. In one hand, if X is Gaussian, indeed, γ must equal 2. On the other hand, if $\gamma = 2$, then $i\kappa \cdot sgn(\xi) \tan \frac{\gamma\pi}{2}$ vanishes since $\tan(pi) = 0$. Therefore, X is Gaussian because $\mathrm{E}[\exp i\xi X] = \exp\{i\xi\theta - |\delta\xi|^2\}$. $\square$

Remark, if $\gamma = 2$, then κ is irrelevant in Definition. We specific $\kappa = 0$ without loss of generality. For instance, $B_t \sim \Lambda(2, 0, 0, \frac{\sqrt{t}}{2})$ when $(B_t)_t$ is Brownian motion.

Definition 3.3. A random variable X is said to be *symmetric* if X and $-X$ have the same distribution.

Proposition 3.4. Let X be have a stable distribution. X is *symmetric* if and only if $X \sim \Lambda(\gamma, 0, 0, \delta)$. I.e. its characteristic function has the form

$$\mathrm{E}[\exp\{i\xi X\}] = \exp\{-|\delta\xi|^{\gamma}\} \tag{3.10}$$

Proof. The Definition of symmetricity implies

$$\begin{aligned}
& \exp\{i\xi\theta - |\delta\xi|^\gamma(1 - i\kappa \cdot sgn(\xi)\tan\frac{\gamma\pi}{2})\} \\
= \; & \mathrm{E}[i\xi X] \\
= \; & \mathrm{E}[i\xi(-X)] \\
= \; & \mathrm{E}[i(-\xi)X] \\
= \; & \exp\{i(-\xi)\theta - |\delta\xi|^\gamma(1 - i\kappa \cdot sgn(-\xi)\tan\frac{\gamma\pi}{2})\},
\end{aligned}$$

this requires $\theta = \kappa = 0$. □

Corollary 3.5. Let $(B_t)_t$ be Brownian motion, then B_t has a symmetric stable distribution.

Proof. It is clear due to the previous Proposition. □

3.2 Stable Random Measures

In this subsection we suppose $(\mathrm{D}, \mathscr{D})$ and $(\mathrm{E}, \mathscr{E})$ are probability spaces, $\kappa(\cdot) : \Omega \to [-1, 1]$ is a measurable function.

Definition 3.6. Let ν be a measure such that

$$\nu : \mathscr{D} \to \mathscr{E}.$$

ν is said to be *independently scattered*, if $\nu[D_1], \ldots, \nu[D_n]$ are independent for any $D_1, \ldots, D_n$ *disjoint* $\in \mathscr{D}$.

For the next definition we need a notation

$$\mathscr{G} = \{D \in \mathscr{D} : \mu[D] < \infty, \, \mu : \mathscr{D} \to \mathscr{E}\}. \tag{3.11}$$

Definition 3.7. Let ν be an independent cattered and σ-additive set function such that

$$\nu : \mathscr{G} \to \mathrm{L}^\infty(\Omega, \mathscr{A}, \mathcal{P}).$$

ν is said to be *stable random measure* on $(D, \mathscr{D})$ with control measure μ, degree γ and skewness intensity $\kappa(\cdot)$ if

$$\nu[F] \sim \Lambda\left(\gamma, \frac{\int_F \kappa(x)\,\mu[dx]}{\mu[F]}, 0, (\mu[F])^{\frac{1}{\gamma}}\right) \tag{3.12}$$

for $F \in \mathscr{D}$.

Samorodnitsky and Taqqu show the existence of stable measures, see [8], pp.119∼120.

Example 3.8. Suppose $[0, T]$ is a index set and $0 = t_0, t_1, \ldots, t_k \in [0, T]$ for $k \in \mathbb{N}$. We show the mapping $\nu : \mathscr{B}([0, T]) \to \mathscr{B}(\mathbb{R})$, where $\nu[A_j](\omega) := B_{t_{j+1}}(\omega) - B_{t_j}(\omega), A_j = [t_j, t_{j+1})$.
Firstly, we show ν is independently scattered and σ-additive. We take $\{A_j\}$ such that $\cup_{j=1}^{\infty} A_j = [0, T]$. $\{\nu[A_k]\}_{k=1}^{\infty}$ has independent elements since $B_{t_1} - B_{t_0}, \ldots, B_{t_{j+1}} - B_{t_j}$ are independent.
Secondly,

$$\begin{aligned} \nu[(\cup_{j=1}^{\infty} A_j)] &= B_T - B_1 \\ &= \sum_{j=1}^{\infty} (B_{t_{j+1}} - B_{t_j}) \\ &= \sum_{j=1}^{\infty} \nu[A_j]. \end{aligned}$$

Finally,

$$\begin{aligned} \mathrm{E}[\exp(i\xi\nu[A_j])] &= \mathrm{E}[\exp(i\xi(B_{t_{j+1}} - B_{t_j})] \\ &= \exp(-\frac{(t_{j+1} - t_j)\xi^2}{2}) \end{aligned}$$

Comparing with (3.9), we deduce the control measure must be $\frac{|\cdot|}{2}$. In fact, $\nu[A_j] \sim \Lambda(2, 0, 0, \frac{|t_{j+1} - t_j|}{2})$.

3.3 Stable Integrals

Samorodnitsky and Taqqu defined an Integral with respect to stable measure as stochastic process in [8]. The stable Integral is given as

$$\int_F f(x)\, \nu(dx). \tag{3.13}$$

Where $f : F \to \mathbb{R}$ is a measurable function such that

$$\begin{cases} \int_F |f(x)|^{\gamma} \mu(dx) < \infty, & \text{if} \quad \gamma \in (0, 1) \cup (1, 2], \\ \int_F |\kappa(x) f(x) \ln |f(x)|| \mu(dx) < \infty, & \text{if} \quad \gamma = 1, \end{cases} \tag{3.14}$$

, γ, μ, κ are, respectively, degree, control measure and skewness intensity of the stable measure ν.

Some properties of the stable function are given by Samorodnitsky and Taqqu.

Proposition 3.9. Let $J(f)$ be a stable integral as form of (3.13). Then

$$J(f) \sim \Lambda(\gamma, \kappa, \theta, \delta)$$

for the degree, control measure, skewness intensity, respectively,

$$\begin{aligned}
\gamma &\in (0, 2], \\
\kappa &= \frac{\int_F \kappa(x)|f(x)|^\gamma \cdot sgn(f(x))\mu(dx)}{\int_F |f(x)|^\gamma \mu(dx)}, \\
\theta &= \begin{cases} 0, \quad \text{if} \quad \gamma \in (0,1) \cup (1,2], \\ -\frac{2}{\pi} \int_F \kappa(x) f(x) \ln |f(x)| \mu(dx), \quad \text{if} \quad \gamma = 1, \end{cases} \\
\delta &= \left(\int_F |f(x)^\gamma| \mu(dx) \right)^{\frac{1}{\gamma}},
\end{aligned}$$

of the stable measure ν.

Proof. See.[8], p.124, Proposition 3.4.1 . □

Proposition 3.10. The stable integral is linear, in fact,

$$J(c_1 f_1 + c_2 f_2) \overset{a.s.}{=} c_1 J(f_1) + c_2 J(f_2) \tag{3.15}$$

for any f_1, f_2 integrable with respect to some stable measure and real numbers c_1, c_2.

Proof. See [8], p.117, Property 3.2.3 . □

4 Fractional Brownian Motion

The fractional Brownian motion(FBM) was defined by Kolmogorov primitively. After that Mandelbrot and Van Ness has present the work in detail. This section is concerned with the definition and some properties of it.

4.1 Definition of Fractional Brownian Motion

Mandelbrot and Van Ness [10] gave a integration presentation of the definion of FBM.

Definition 4.1. Let $(U_H(t))_{t\in\mathbb{R}}$ be a $\mathbb{R}$-valued stochatstic process and H be such that $0 < H < 1$. $(U_H(t))$ is said to be *fractional Brownian motion* if

$$U_H(t) - U_H(s) \quad = \quad \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{\mathbb{R}} \mathbb{1}_{\{t\geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u\leq s\}}(-u)^{H-\frac{1}{2}}\, dB_u\right) \tag{4.1}$$

for $t \geq s, t, s \in \mathbb{R}$. Where (B_u) is defined as two-sides Brownian motion, the integral is in sense of stable integral as in previous section. H is called Hurst exponent or Hurst index of FBM.

As usual, we set $U_H(0) = 0$, then equation (4.1) is equivalent to

$$U_H(t) \quad = \quad \frac{1}{(\Gamma(H+\frac{1}{2}))^2}\left(\int_{\mathbb{R}} \mathbb{1}_{\{t\geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u\leq 0\}}(-u)^{H-\frac{1}{2}}\, dB_u\right). \tag{4.2}$$

Lemma 4.2. The equation (4.2) is well-defined, $U_H(t)$ has stable distribution and

$$U_H(t) \sim \Lambda(2, 0, 0, \frac{1}{\Gamma(H+\frac{1}{2})}(\int_{\mathbb{R}} |f(u)^2|\ \frac{du}{2})^{\frac{1}{2}}),$$

Where $f(x)$ is the integrand of integral in (4.2).

Proof. Firstly, B_t is Gaussian and symmetric stable measure with zero mean and $\frac{|\cdot|}{2}$ as the control measure of it shown in Example 3.8.
Secondly, the well-definition was refered to by Samorodnitksy and Taqqu in [8], p.321, Proposition 7.2.6 for not only $H = \frac{1}{2}$, but also $H \neq \frac{1}{2}$. Which satisfies, in other words, the condition $\int_{-\infty}^{\infty} f^2(u)\ \frac{du}{2} < \infty$.
Finally, in terms of Proposition 3.9, we get the claim. □

Remark that, if we take $H = \frac{1}{2}$ choosing a restriction of the integrand on $\mathbb{R}_+$, $(U_2(t))_{t\geq 0}$ is a Brownian motion.

Theorem 4.3. Let $(U_H(t))_t$ be a FBM. Then $U_H(t) \sim \mathcal{N}(0, \frac{1}{\Gamma(H+\frac{1}{2})^2}(\int_{\mathbb{R}} |f(u)^2|\ du))$.

Proof. In terms of Lemma 4.2, $\mathrm{E}[i\xi U_H(t)] = \exp\{-\xi^2 \frac{1}{2\Gamma(H+\frac{1}{2})^2}(\int_{\mathbb{R}} |f(u)^2|\, du)\}$. The rest is clear thanks to the form of characteristic function of a Gaussian random variable. □

Lemma 4.4. Let $(U_H(t))_t$ be a FBM. Then $U_H(t)$ has an expected value 0 and variance $t^{2H}\,\mathrm{E}U_H^2(1)$ for any $t \in \mathbb{R}$.

Proof. It is clear that U_H is Gaussian with zero mean due to Lemma 4.2. We suppose that $t \geq s \geq 0, c(H) = \frac{1}{(\Gamma(H+\frac{1}{2}))^2}$.

$$\begin{aligned}
& \mathrm{E}[(U_H(t) - U_H(s))^2] \\
= \quad & c(H)\mathrm{E}[\Big(\int_{\mathbb{R}} [\mathbb{1}_{\{t\geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{s\geq u\}} \cdot (s-u)^{H-\frac{1}{2}}\ U_H(u)\Big)^2] \\
\overset{\text{Theorem 4.3}}{=} \quad & c(H)\mathrm{E}[\int_{\mathbb{R}} \Big([\mathbb{1}_{\{t\geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{s\geq u\}} \cdot (s-u)^{H-\frac{1}{2}}\Big)^2\ du] \\
= \quad & c(H)\mathrm{E}[\int_{\mathbb{R}} \Big(\mathbb{1}_{\{t-s\geq u\}} \cdot (t-s-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{0\geq u\}} \cdot (-u)^{H-\frac{1}{2}}\Big)^2\ du] \\
\overset{m=t-s}{=} \quad & c(H)\mathrm{E}[\int_{\mathbb{R}} \Big(\mathbb{1}_{\{m\geq u\}} \cdot (m-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{0\geq u\}} \cdot (-u)^{H-\frac{1}{2}}\Big)^2\ du] \\
\overset{u=ml}{=} \quad & c(H)\mathrm{E}[\int_{\mathbb{R}} \Big(\mathbb{1}_{\{m\geq ml\}} \cdot (m-ml)^{H-\frac{1}{2}} - \mathbb{1}_{\{0\geq ml\}} \cdot (-ml)^{H-\frac{1}{2}}\Big)^2\ m \cdot dl] \\
= \quad & c(H)\mathrm{E}[\int_{\mathbb{R}} \Big(\mathbb{1}_{\{1\geq l\}} \cdot (1-l)^{H-\frac{1}{2}} - \mathbb{1}_{\{0\geq l\}} \cdot (-l)^{H-\frac{1}{2}}\Big)^2 \cdot m^{2H-1} \cdot m\ dl] \\
= \quad & c(H)m^{2H}\mathrm{E}[U_H(1)^2] \\
= \quad & c(H)(t-s)^{2H}\mathrm{E}[U_H(1)^2]
\end{aligned} \tag{4.3}$$

Using the same calculation, we get

$$\mathrm{E}[(U_H(t)^2] = c(H)t^{2H}\mathrm{E}[U_H(1)^2]. \tag{4.4}$$

(4.4) is variance of $U_H(t)$ due to $\mathrm{E}[(U_H(t)] = 0$. □

To normalize the variance, a definition of standard FBM is given.

Definition 4.5. A stochastic process $(U_H(t))_t$ is said to be a *standrad fractional Brownian motion*(sFBM) if

$$U_H(t) = \hat{c}(H) \int_{\mathbb{R}} \mathbb{1}_{\{t\geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u\leq 0\}}(-u)^{H-\frac{1}{2}}\ dB_u\,. \tag{4.5}$$

Where $\hat{c}(H) = \frac{1}{\mathrm{E}[U_H(1)^2]}$.

We consider from now on sFBM as FBM.

Theorem 4.6. Let $(U_H(t))_t$ be a FBM. The Covariance function of $U_H(t), U_H(s)$ is $\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ for $t, s \in \mathbb{R}$.

Proof. Cf.[10], Theorem 5.3 .

$$\begin{aligned}
\mathrm{Cov}[U_H(t), U_H(s)] &= \mathrm{E}[U_H(t)U_H(s)] \\
&= \frac{1}{2}\left(\mathrm{E}[U_H(t)^2] + \mathrm{E}[U_H(s)^2] - \mathrm{E}[(U_H(t) - U_H(s))^2]\right) \\
&\overset{(4.4)}{=} \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \qquad (4.6)
\end{aligned}$$

□

THEOREM 4.7. $(U_H(t))_t$ is Gaussian process.

Proof. We just need to prove that for an arbitrary finite linear combination of values of time is Gaussian. We take $t_1, \ldots, t_k \in T, c_1, \ldots, c_k \in \mathbb{R}$, and the stable integral $J(f)$ is a linear functional with $\gamma = 2, \kappa = 0, \theta = 0, \delta = (\frac{1}{2}\int_{-\infty}^{\infty} f^2(u)\ du)^{\frac{1}{2}}$ due to Corollary 3.15. Suppose $f_1, \ldots, f_k$ are integrands of the form of stable integral of $U_H(t_1), \ldots, U_H(t_k)$. Consider now, according to the Minkowski inequality,

$$\begin{aligned}
\int_{-\infty}^{\infty} (\sum_{j=1}^{k} c_j f_j)^2\ du &\leq \sum_{j=1}^{k} \underbrace{\int_{-\infty}^{\infty} (c_j f_j)^2\ du}_{<\infty} \\
&< \infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{j=1}^{k} c_j U_H(t_j) &= \sum_{j=1}^{k} c_j J(f_j) \\
&= J(\sum_{j=1}^{k} c_j f_j) \\
&\sim \Lambda(2, 0, 0, (\frac{1}{2}\int_{-\infty}^{\infty} (\sum_{j=1}^{k} c_j f_j)^2\ du)^{\frac{1}{2}})
\end{aligned}$$

is Gaussian and the rest is clear. □

COROLLARY 4.8. Let $(U_H(t))_t$ be a FBM, then $(U_H(t))_t$ has stationary and H-self similar increments .

Proof. Assume that $s \geq u$. Because the joint distribution of $(U_H(s), U_H(u))^T$ is Gaussian, $(1, -1) \cdot (U_H(s), U_H(u))^T$ is Gaussian. In other words, $U_H(s) - U_H(u) \sim \mathcal{N}(0, (s-u)^{2H})$ which is only dependent on $(s-u)$ and $(U_H(t))$ has therefore stationary increments.
$(U_H(t))$ has zero mean and $\mathrm{Var[U_H(s)]} = s^{2H}\mathrm{Var}[U_H(1)]$ we get $U_H(s) \sim s^H U_H(1)$ due to it is Gaussian. To show FBM has H-self similar increments, we have to prove $(U_H(zt_1), U_H(zt_2), \ldots, U_H(zt_n))) \sim (z^H U_H(t_1), z^H U_H(t_2), \ldots, z^H U_H(t_n))$ for any $z > 0$.

Obviously, the former and the latter of the term are Gaussian and $\mathrm{Var}[U_H(zt_i), U_H(zt_j)] = \mathrm{Var}[z^H U_H(t_i), z^H U_H(t_j)] = \frac{1}{2} z^{2H}(t_i^{2H} + t_j^{2H} - |t_i - t_j|^{2H})$. Thus they have the same covariance matrix and zero mean in their characteristic function. Using uniqueness theorem we can prove the claim. $\square$

4.2 Regularity

Theorem 4.9 (Kolmogorov Chentsov)**.** FBM has almost surely continuous sample path.

Proof. Cf.[10] Proposition 4.1 . Let $(U_H(t))_t$ be FBM with Hurst index H. Fix α such that $1 < \alpha H$. Let look at the expectation of $(U_H(t) - U_H(s))^\alpha$ using the same calculation in (4.3)

$$\begin{aligned} \mathrm{E}[(U_H(t) - U_H(s))^\alpha] &= |t-s|^{\alpha H} \cdot \underbrace{\mathrm{E}\left(\int_{\mathbb{R}} \mathbb{1}_{\{1 \geq u\}} \cdot (1-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u \leq 0\}} (-u)^{H-\frac{1}{2}} \, dB_u\right)^\alpha}_{c(\alpha,H)} \\ &= c(\alpha, H) \cdot |t-s|^{\alpha H}. \end{aligned} \tag{4.7}$$

We choose $\beta = \alpha H - 1$ and $\gamma \in (0, H - \frac{1}{\alpha})$ then the rest follows from Theorem 2.25 . $\square$

Remark, $U_H(t)$ is, in fact, γ-Hölder continuous with $\gamma < H$ almost surely.

Theorem 4.10. The sample path of FBM is almost surely not differentiable.

Proof. Cf. [10] Proposition 4.2 . Fix $\omega \in \Omega$, we assume $c > 0, t_j \to s$.

$$\begin{aligned} & \mathcal{P}[\limsup_{t \to s} |\frac{U_H(t) - U_H(s)}{t - s}| > c] \\ = \; & \mathcal{P}[\lim_{j \to \infty} \sup_{t_j \neq s} |\frac{U_H(t_j) - U_H(s)}{t_j - s}| > c] \end{aligned} \tag{4.8}$$

Since continuity of measures from above, then

$$\begin{aligned} (4.8) \quad &= \lim_{j \to \infty} \mathcal{P}[\sup_{t_j \neq s} |\frac{U_H(t_j) - U_H(s)}{t_j - s}| > c] \\ &\geq \lim_{j \to \infty} \mathcal{P}[|\frac{U_H(t_j) - U_H(s)}{t_j - s}| > c] \\ &= \lim_{j \to \infty} \mathcal{P}[\frac{|(t_j - s)^H U_H(1)}{t_j - s}| > c] \\ &= \lim_{j \to \infty} \mathcal{P}[|(t_j - s)^{H-1} U_H(1)| > c] \\ &= \lim_{j \to \infty} \mathcal{P}[|U_H(1)| > \underbrace{|t_j - s|^{1-H}}_{\overset{j\to\infty}{\longrightarrow} 0} c] \\ &\overset{j \to \infty}{\longrightarrow} 1 \end{aligned}$$

□

THEOREM 4.11. Let $(U_H(k))_k$ be FBM. The conditional expectation of $U_H(s)$ given $U_H(t) = x$ is

$$\frac{|\frac{s}{t}|^{2H} + 1 - |\frac{s}{t} - 1|^{2H}}{2} \cdot x$$

for all $s < t$.

Proof. Cf. [10] Theorem 5.3. Taking conditional expectation of $U_H(s)$ given $U_H(t)$,

$$\begin{aligned}
& \mathrm{E}[U_H(s)|U_H(t)] \\
\overset{\text{Corollary 2.18}}{=} \quad & \mu_s + \rho_{s,t}(\frac{\sigma_s}{\sigma_t}U_H(t) - \mu_t) \\
= \quad & \rho\frac{\sigma_s}{\sigma_t}U_H(t) \\
= \quad & \frac{\rho \cdot \sigma_s\sigma_t \cdot U_H(t)}{\sigma_t^2} \\
= \quad & \frac{\mathrm{E}[U_H(s)U_H(t)]}{\mathrm{E}[U_H^2(t)]} \cdot U_H(t) \\
\overset{(4.6)}{=} \quad & \frac{s^{2H} + t^{2H} - |s-t|^{2H}}{2\mathrm{E}[U_H^2(t)]} \cdot U_H(t) \\
= \quad & \frac{s^{2H} + t^{2H} - |s-t|^{2H}}{2t^{2H}} \cdot U_H(t) \\
= \quad & \frac{|\frac{s}{t}|^{2H} + 1 - |\frac{s}{t} - 1|^{2H}}{2} \cdot U_H(t)
\end{aligned}$$

□

4.3 Fractional Brownian Noise

DEFINITION 4.12. Let $(U_H(t))_t$ be a FBM. The *fractional Brownian noise* is a sequence $(S_k)_k$ forms

$$S_H(k) \quad = \quad U_H(k+1) - U_H(k)$$

for $k \in \mathbb{R}$.

PROPOSITION 4.13. Fractional Brownian noise is stationary and its autocovariance is

$$\varsigma_{S_H}(\tau) \quad = \quad \frac{1}{2}(|\tau+1|^{2H} - 2|\tau|^{2H} + |\tau-1|^{2H}) \tag{4.9}$$

for $\tau \in \mathbb{R}$.

Proof. Cf. [11], p.333, Proposition 7.2.9 . The first part of the claim is clear due to FBM has stationary increments.
In terms of definition of fractional Brownian noise we have

$$\begin{aligned}
\varsigma_{S_H}(\tau) &= \mathrm{E}[S_H(0)S_H(\tau)] \\
&= \mathrm{E}[(U_H(1)-U_H(0))(U_H(\tau+1)-U_H(\tau))] \\
&= \mathrm{E}[(U_H(1)-U_H(0))(U_H(\tau+1)-U_H(\tau))] \\
&= \mathrm{E}[U_H(1)(U_H(\tau+1)-U_H(\tau))] \\
&= \mathrm{E}[U_H(1)U_H(\tau+1)]-\mathrm{E}[U_H(1)U_H(\tau)] \\
&= \varsigma_{U_H}(1,\tau+1)-\varsigma_{U_H}(1,\tau) \\
&\overset{(4.6)}{=} \frac{1}{2}(1+|1+\tau|^{2H}-\tau^{2H})-\frac{1}{2}(1+\tau^{2H}-|1-\tau|^{2H}) \\
&= \frac{1}{2}(|1+\tau|^{2H}-2|\tau|^{2H}+|1-\tau|^{2H})
\end{aligned}$$

for $\tau \in \mathbb{R}$. □

Definition 4.14. A stationary stochastic process $(X_t)_t$ is said to have *long memory* if its autocovariance $\varsigma_X(\tau)$ tend to 0 with so slowly such that $\sum_{\tau=-\infty}^{\infty}\varsigma_X(\tau)$ diverges for $\tau \in \mathbb{Z}$.

Theorem 4.15. The fractional Brownian noise has long memory.

Proof. Cf. [11], p.335, Proposition 7.2.10 .
Without loss of generality, we suppose $\tau \in \mathbb{Z}_0^+$. For $\tau \in \mathbb{Z}^-$, we deal with it in a similar way and $\tau = 0$ is clear. Following (4.9), we have

$$\begin{aligned}
&\varsigma(\tau) \\
&= \frac{1}{2}\tau^{2H-2}\{\tau^2[(1+\frac{1}{\tau})^{2H}-2+(1-\frac{1}{\tau})^{2H}]\} \\
&= \frac{1}{2}\tau^{2H-2}\{\frac{(1+\frac{1}{\tau})^{2H}-1}{\frac{1}{\tau^2}}-\frac{1-(1-\frac{1}{\tau})^{2H}}{\frac{1}{\tau^2}}\}
\end{aligned}$$

We deal with the former of the content in { } with L'Hôpital's rule as τ tend to infinity.

$$\begin{aligned}
&\lim_{\tau\to\infty}\frac{(1+\frac{1}{\tau})^{2H}-1}{\frac{1}{\tau^2}} \\
&= \lim_{\tau\to\infty}\frac{2H(1+\frac{1}{\tau})^{2H-1}(-\frac{1}{\tau^2})}{-\frac{4}{\tau^3}} \\
&= \lim_{\tau\to\infty}\frac{H(1+\frac{1}{\tau})^{2H-1}}{\frac{1}{\tau}}.
\end{aligned}$$

We calculate the Latter of the content in { } in a similar way. Then

$$\begin{aligned}
\lim_{\tau\to\infty} \quad & \varsigma(\tau) \\
= \quad & \lim_{\tau\to\infty} \tau^{2H-2}\frac{1}{2}\{\frac{H(1+\frac{1}{\tau})^{2H-1}}{\frac{1}{\tau}} - \frac{H(1-\frac{1}{\tau})^{2H-1}}{\frac{1}{\tau}}\} \\
\overset{\text{L'Hôpital}}{=} \quad & \lim_{\tau\to\infty} \frac{1}{2}\tau^{2H-2}\{\frac{H(2H-1)(1+\frac{1}{\tau})^{2H-2}(-\frac{1}{\tau^2})}{-\frac{1}{\tau^2}} - \frac{H(2H-1)(1-\frac{1}{\tau})^{2H-2}\frac{1}{\tau^2}}{-\frac{1}{\tau^2}}\} \\
= \quad & \lim_{\tau\to\infty} \frac{1}{2}\tau^{2H-2}\{H(2H-1)(1+\frac{1}{\tau})^{2H-2} + H(2H-1)(1-\frac{1}{\tau})^{2H-2}\} \\
= \quad & \lim_{\tau\to\infty} \frac{1}{2}\tau^{2H-2}2H(2H-1) \\
= \quad & \lim_{\tau\to\infty} H(2H-1)\tau^{2H-2}
\end{aligned}$$

Convergence of $\sum\limits_{\tau=-n}^{n} \varsigma(\tau)$ as $n \to \infty$ requires $2H-2 < -1$, namely, $H < \frac{1}{2}$. Otherwise, if $\frac{1}{2} < H < 1$, $\sum\limits_{\tau=-\infty}^{\infty} \varsigma(\tau)$ diverges. With $\lim\limits_{\tau\to\infty} \varsigma(\tau) = 0$, we achieve the claim. □

Corollary 4.16. Let S_H be fractional Brownian noise , $\tau \in \mathbb{Z}$ and $\varsigma_{S_H}(\cdot)$ its autocovariance. Then $\sum_{\tau\in\mathbb{Z}} \varsigma^2_{S_H}(\tau) < \infty$ if and only if $H < \frac{3}{4}$.

Proof. Cf. [11], p.72, Lemma 6.3. In Theorem 4.15, we have $\varsigma^2_{S_H}(\tau) \propto H^2(2H-1)^2\tau^{4H-4}$. The sum of it is finite if and only if, according to the same reason as in Theorem 4.15, $\tau^{4H-4} < -1$. That means $H < \frac{3}{4}$. □

4.4 FBM is not Semimartingale for $H \neq \frac{1}{2}$

Let look at our Integration representation for FBM, in the case of FBM with Hurst index $\frac{1}{2}$, it must be an ordinary Brownian motion. Except for this, we will show FBM is not a seimimartingal.

Definition 4.17. The *Hermite polynomials* form as following

$$\eta_n(u) = (-1)^n e^{\frac{u^2}{2}}(\frac{\partial^n}{\partial u^n}e^{-\frac{u^2}{2}}), \tag{4.10}$$

for $u \in \mathbb{R}, n \in \mathbb{N}_0$.

Proposition 4.18. Let $(\eta_n)_{n\in\mathbb{N}}$ be a family of Hermite polynomials, W a standard Gaussian variable, $f, g : \mathbb{R} \to \mathbb{R}$ continuously differentiable. It has then following properties

(i) $\eta_{n+1}(u) = (n+1)\eta_n(u)$ and $\eta_{n+2}(u) = u\cdot\eta_{n+1}(u) - (n+1)\eta_n(u)$ for all $n \in \mathbb{N}_0, u \in \mathbb{R}$.

(ii) Let W, V be standard Gaussian distributed such that (W, V) have a disjoint Gaussian distribution. Then

$$\int_\Omega \eta_j(W)\cdot\eta_k(V)\,\mathcal{P} = \begin{cases} j!\,(\mathrm{E}[WV])^j & \text{if } j=k, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Let W be standard Gaussian distributed, then

$$\frac{1}{j!}\int_\Omega \eta_j(W)\eta_k(W)\,\mathcal{P} = \begin{cases} 1 & \text{if } j=k, \\ 0 & \text{otherwise.} \end{cases}$$

Remark that, (iii) means, in fact, $\{\frac{1}{\sqrt{j!}}\cdot\eta_j\}_{j=0}^\infty$ is an orthonormal basis in $\mathcal{L}^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), e^{-\frac{x^2}{2}}\,dx)$.

Proof. See [11] p.3, Propostion 1.3. □

Lemma 4.19. Let $(U_H(t))_t$ be a FBM, W standrad Gaussian variable, and $f:\mathbb{R}\to\mathbb{R}$, Borel-measurable function such that $\mathrm{E}[f^2(G)] < \infty$. Then,

$$\frac{1}{n}\sum_{j=1}^n f(U_H(j) - U_H(j-1)) \overset{\text{in } \mathcal{L}^2(\mathcal{P})}{\rightarrow} \mathrm{E}[f(W)],$$

as n tend to ∞. In particular,

$$\sum_{j=1}^n |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\beta \overset{\text{in } \mathcal{L}^2(\mathcal{P})}{\rightarrow} \begin{cases} 0 & \text{if } \beta > \frac{1}{H} \\ \infty & \text{if } \beta < \frac{1}{H} \end{cases} \tag{4.11}$$

as n tend to ∞.

Proof. C.f. [11], p.17, Theorem 2.1 .
Firstly, because $\mathrm{E}[f^2(W)] < \infty$ one has $f \in \mathcal{L}^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), e^{-\frac{x^2}{2}}\,dx)$. In terms of Proposition 4.18(iii), taking expectation

$$\mathrm{E}[f(x)] = \mathrm{E}[\sum_{j=0}^\infty \frac{a_j H_j(x)}{\sqrt{j!}}],$$

for $x \in \mathbb{R}$. Notice $H_0 = 1$ due to (4.10). Setting $x = W$, Equating coefficients leads to

$a_0 = \mathrm{E}[f(W)]$. Moreover,

$$\begin{aligned}
& \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n} f(U_H(j)-U_H(j-1)) - \mathrm{E}[f(W)]\}^2] \\
= \;& \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n} (f(U_H(j)-U_H(j-1)) - \mathrm{E}[f(W)])\}^2] \\
= \;& \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n} (\sum_{k=0}^{\infty} \frac{a_k}{\sqrt{k!}} H_k(U_H(j)-U_H(j-1))) - \mathrm{E}[f(W)]\}^2] \\
= \;& \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n} (\sum_{k=1}^{\infty} \frac{a_k}{\sqrt{k!}} H_k(U_H(j)-U_H(j-1)))\}^2]
\end{aligned}$$

Consider now

$$\mathrm{E}[f^2(W)] < \infty,$$

which requires $\sum\limits_{k=1}^{\infty}(a_k)^2 < \infty$. Then

$$\begin{aligned}
& \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n} f(U_H(j)-U_H(j-1)) - \mathrm{E}[f(W)]\}^2] \\
= \;& \frac{1}{n^2}\mathrm{E}[\sum_{k=1}^{\infty}\frac{a_k^2}{k!}(\sum_{j=1}^{n} H_k(U_H(j)-U_H(j-1)))^2] \\
= \;& \frac{1}{n^2}\sum_{k=1}^{\infty}\frac{a_k^2}{k!}\sum_{j=1,m=1}^{n} \mathrm{E}[H_k(U_H(j)-U_H(j-1))H_k(U_H(m)-U_H(m-1)] \\
\overset{\text{Proposition 4.18(ii)}}{=} \;& \frac{1}{n^2}\sum_{k=1}^{\infty} a_k^2 \sum_{j=1,m=1}^{n} (\mathrm{E}[(U_H(j)-U_H(j-1))(U_H(m)-U_H(m-1))])^k \\
= \;& \frac{1}{n^2}\sum_{k=1}^{\infty} a_k^2 \sum_{j=1,m=1}^{n} (\mathrm{E}[S_H(j-1)S_H(m-1)])^k \\
= \;& \frac{1}{n^2}\sum_{k=1}^{\infty} a_k^2 \sum_{j=1,m=1}^{n} (\varsigma_{S_H}(j-m))^k
\end{aligned}$$

And

$$\begin{aligned}
|\varsigma_{S_H}(k)| \;&=\; |\varsigma_{S_H}(|k|)| \\
&=\; \mathrm{E}[(U_H(1)-U_H(0))(U_H(|x|+1)-U_H(|x|))] \\
&\overset{\text{Cauchy Schwartz}}{\leq}\; \underbrace{\sqrt{\mathrm{E}[U_H(1)^2]}}_{=1} \cdot \underbrace{\sqrt{\mathrm{E}[U_H(|x|+1)-U_H(|x|)]^2}}_{=1} \\
&=\; 1
\end{aligned}$$

Consequently, $(\varsigma_{S_H}(j-m))^k \leq |\varsigma_{S_H}(j-m)|$. In fact,

$$
\begin{aligned}
& \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n} f(U_H(j)-U_H(j-1)) - \mathrm{E}[f(W)]\}^2] \\
\leq\ & \frac{1}{n^2}\underbrace{\sum_{k=1}^{\infty} a_k^2}_{=:\alpha<\infty} \sum_{j=1,m=1}^{n} |\varsigma_{S_H}(j-m)| \\
=\ & \frac{\alpha}{n^2}\sum_{j=1,m=1}^{n} |\varsigma_{S_H}(j-m)| \\
=\ & \frac{\alpha}{n^2} 2\cdot \sum_{j=1}^{n}\sum_{m<j} |\varsigma_{S_H}(j-m)| \\
\leq\ & \frac{\alpha}{n^2} 2n \sum_{k=1}^{n-1} |\varsigma_{S_H}(k)| \\
=\ & \frac{2\alpha}{n}\sum_{k=0}^{n-1} |\varsigma_{S_H}(k)|,
\end{aligned}
$$

As in the proof in Theorem 4.15, $\sum\limits_{k=0}^{n-1} |\varsigma_{S_H}(k)| \propto H(2H-1)\sum\limits_{j=1}^{n-1} j^{2H-2} \propto H(2H-1)n \cdot n^{2H-2} \propto n^{2H-1}$ and $\frac{2\alpha}{n}\sum\limits_{k=0}^{n-1}|\varsigma_{S_H}(k)| \propto n^{2H-2}$ as k goes to infinity. This leads $\mathrm{E}[\{\frac{1}{n}\sum\limits_{j=1}^{n} f(U_H(j)-U_H(j-1)) - \mathrm{E}[f(W)]\}^2] = 0$ due to $n^{2H-2} \to 0$ for $0 < H < 1$ as $n \to \infty$.
Secondly, we apply previous result for (4.11), In fact,

$$
\begin{aligned}
& \sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\beta \\
=\ & \frac{1}{n^{\beta H}}\sum_{j}^{n} |U_H(j)-U_H(j-1)|^\beta \\
=\ & \frac{1}{n^{\beta H-1}}\frac{1}{n}\sum_{j}^{n} |U_H(j)-U_H(j-1)|^\beta \\
\longrightarrow\ & n^{1-\beta H}\mathrm{E}[|W|^\beta]
\end{aligned}
$$

Due to $\mathrm{E}[|W|^\beta] < \infty$, (4.11) holds as well as $n \to \infty$. $\square$

Theorem 4.20. FBM is not a semimartingale for $H \neq \frac{1}{2}$.

Proof. Without loss of generality, we set the Time index $T = [0,1]$. Choosing $\beta = 2$ in (4.11), we suppose $U_H(t)$ were a semimartingale.
Case $H < \frac{1}{2}$. Then $\sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^2 = \infty$ contradicts that semimartingale has finite quadratic variation.
Case $H > \frac{1}{2}$. $\sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^2 = 0$. On the one hand, according to Doob-Meyer

decomposition, $U_H(t) = M(t) + A(t)$, where $M(t)$ is a local martingal and $A(t)$ is local finite variation process. Due to $A(t)$ is local finite, we have $0 =< U_H, U_H >=< M, M >$, where $< \cdot, \cdot >$ is denoted for quadratic variation. Consequently, $M(t)$ is zero process due to Cauchy Schwarz inequality. In other words, $U_H(t) = A(t)$ which has finite variation. On the ohter Hand, choosing $1 < \gamma < \frac{1}{H}$, then $\sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\gamma \to \infty$. Precisily,

$$\begin{aligned}
&\infty \leftarrow \sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\gamma \\
&\leq \underbrace{\sup_{1 \leq j \leq n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^{\gamma-1}}_{\overset{(\gamma-1)-\text{Hölder}}{\rightarrow} 0} \cdot \sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|,
\end{aligned}$$

this leads to $\sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})| \to \infty$ contradicts mentioned above that $U_H(t)$ has finite variation.

Given all that, FBM is not a semimartingal for $H \neq \frac{1}{2}$. $\square$

5 Fractional Ornstein-Uhlenbeck Process

In this section we turn our attention on the fractional Ornstein-Uhlenbeck process (for short, we denote FOU). Consider the following stochastic dynamics

$$dX_t = -aX_t\,dt + \gamma\,dU_H(t), \tag{5.1}$$

where $(X_t)_{t\geq 0}$ is a stochastic process, $a \in \mathbb{R}, \gamma \in \mathbb{R}_+$ and $(U_H(t))_{t\geq 0}$ a FBM with Hurst exponent H. In fact, given a initial solution $X_0(\omega) = b(\omega)$, then it is

$$X_t(\omega) = b(\omega) - a\int_0^t X_u(\omega)\,du + \gamma U_H(t)(\omega) \tag{5.2}$$

for $t \geq 0$.

Due to $U_H(t)$ is α-Hölder-continuous with $\alpha < H$, we have following

Lemma 5.1. Let $U_H(t)$ be a FBM and $u, d \in \mathbb{R}$ such that $d \leq s$. Then there exist a Riemann-Stieljes integral such that

$$\int_d^s e^{au}\,dU_H(u) = e^{as}U_H(s) - e^{ad}U_H(d) - a\int_d^s U_H(u)e^{au}\,du\,. \tag{5.3}$$

Proof. See. [4], p.11, Proposition A.1 . □

Theorem 5.2. $\hat{X}_t^{b,H} := e^{-at}\left(b + \gamma\int_0^t e^{au}\,dU_H(u)\right)$ is the solution that solves (5.2) for $u \geq 0$.

Proof. Cf. [4], p.11, Proposition A.1 . We define

$$Y(t) := \int_0^t X_u\,du,$$

for $t \geq 0$. Rewirte (5.2) with Y_t and $Y(0) = 0$,

$$Y'(t) = b - aY(t) + \gamma U_H(t)$$

And the solution of that linear differential equation with $Y(0) = 0$ is

$$Y(t) = e^{-at}\int_0^t e^{au}(b + \gamma U_H(u))\,du,$$

in terms of Definition above, using (5.3)

$$
\begin{aligned}
& X(t) = Y'(t) \\
= \; & -ae^{-at}\int_0^t e^{au}(b+\gamma U_H(u))\, du + e^{-at}e^{at}(b+\gamma U_H(t)) \\
= \; & -ae^{-at}\int_0^t e^{au}(b+\gamma U_H(u))\, du + b + \gamma U_H(t) \\
= \; & e^{-at}\left(\underbrace{-a\int_0^t e^{at}\gamma U_H(u)\, du + e^{at}\gamma U_H(t)}_{\gamma\int_0^t e^{au}\, dU_H(u)} - a\int_0^t e^{au} du \cdot b\right) + b \\
= \; & e^{-at}\left(\gamma\int_0^t e^{au}\, dU_H(u) - e^{au}|_{u=0}^{u=t}\cdot b\right) + b \\
= \; & e^{-at}\left(\gamma\int_0^t e^{au}\, dU_H(u) + b\right)
\end{aligned}
$$

□

Let $\hat{X}_{H,t} := \hat{X}_t^{\gamma\int_{-\infty}^0 e^{au}\, dU_H(u),H} := e^{-at}\left(\gamma\int_{-\infty}^t e^{au}\, dU_H(u)\right)$.

THEOREM 5.3. $(\hat{X}_{H,t})_{t\geq 0}$ is Gaussian and stationary process.

Proof. Fix $\epsilon^j > 0 H \in (0,1)$, then there exists $\{u_0 < u_1^j < \cdots < u_{k_j}^j \leq t_j\}$ such that

$$
|\int_{-\infty}^{t_j} e^{au}\, dU_H(u) - \sum_{l=0}^{k_j-1} e^{au_l^j}(U_H(u_{l+1}^j) - U_H(u_l^j))| < \epsilon^j.
$$

for $0 \leq j \leq d$.

In one hand, we calculate the characterstic function of $(\hat{X}_{t_1},\ldots,\hat{X}_{t_d})$ approximately with respect with $\epsilon^1,\ldots,\epsilon^d$.

$$
\mathrm{E}[\exp\{i\sum_{j=1}^d \xi_j \hat{x}_{t_j}\}] \approx \mathrm{E}[\exp\{i\sum_{j=1}^d \xi_j\left(\sum_{l=0}^{k_j-1} e^{au_l^j}(U_H(u_{l+1}^j) - U_H(u_l^j))\right)\}]
$$

Notice that since $(U_H(t))$ is Gaussian process, any linear transform of its instance is Gaussian again. In other words,

$$
\sum_{j=1}^d \xi_j\left(\sum_{l=0}^{k_j-1} e^{au_l^j}(U_H(u_{l+1}^j) - U_H(u_l^j))\right)
$$

is Gaussian. Passing $\epsilon^1,\ldots,\epsilon^d$ to zero, it implies immediately $(\hat{X}_{t_1},\ldots,\hat{X}_{t_d})$ is Gaussian due to the continuaty theorem of characteristic function.

On the other hand,

$$\mathrm{E}[\exp\{i\sum_{j=1}^{d}\xi_j\hat{x}_{t_j}\}] \approx \mathrm{E}[\exp\{i\sum_{j=1}^{d}\xi_j\left(\sum_{l=0}^{k_j-1}e^{au_l^j}(U_H(u_{l+1}^j)-U_H(u_l^j))\right)\}]$$

Since $\{U_H(t)\}$ has stationary increments, then

$$\begin{aligned} & \mathrm{E}[\exp\{i\sum_{j=1}^{d}\xi_j\left(\sum_{l=0}^{k_j-1}e^{au_l^j}(U_H(u_{l+1}^j)-U_H(u_l^j))\right)\}] \\ = \ & \mathrm{E}[\exp\{i\sum_{j=1}^{d}\xi_j\left(\sum_{l=0}^{k_j-1}e^{au_l^j}(U_H(u_{l+1}^j+\tau)-U_H(u_l^j+\tau))\right)\}] \end{aligned}$$

which is convergent if ϵ's tend to zero and must be equal $\mathrm{E}[\exp\{i\sum_{j=1}^{d}\xi_j\hat{X}_{t_j+\tau}\}]$, i.e., $(\hat{X}_t)$ is stationary process. □

Lemma 5.4. Let H be that $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then

$$\varsigma_{X_H}(\tau) = \frac{1}{2}\gamma^2\sum_{k=1}^{N}a^{-2k}\left(\prod_{j=0}^{2k-1}(2H-j)\right)\tau^{2(H-k)} + \mathcal{O}(\tau^{2(H-N-1)})$$

for $N \in \mathbb{N}_0, \tau \in \mathbb{R}$

Proof. See [4], p.7, Theorem 2.3 . □

6 Applications in Financial Mathematics

6.1 Fractional Black-Scholes Model

blah...blah First we give some basic concepts in following. Through this section we denote $\mathcal{F}_t^X$ the filtration of a stochatic process (X_t). Our finance market consist of $\hat{X} := (X_t^0, \ldots, X_t^n)$.

Definition 6.1. A $\mathbb{R}^{n+1}$-valued stochastic process $(\hat{\xi}_t)_t := (\xi_t^0, \xi_t^1, \ldots, \xi_t^n)_{t\in[0,T]}$ is said to be a *strategy* , if $\hat{\xi}_t \in \mathcal{F}_j^X$ for $0 \le j \le t$.

Definition 6.2. A stochastic process $(V_t)_{t\in[0,T]}$ is said to be *value process* respect to strategy $\hat{\xi}_{t\in[0,T]}$, if

$$V_t = \sum_{k=0}^{n} \xi_t^k X_t^k =: \hat{\xi}_t \cdot \hat{X}_t.$$

for $t \in [0, T]$.

Definition 6.3. A stochastic process $(\tilde{V}_{t\in[0,T]})$ is said to be *discounted value process* of a value process $(V_t)_t t \in [0, T]$ respect to $\hat{\xi}_{t\in[0,T]}$, if

$$\tilde{V}_t = \frac{V_t}{X_t^0}$$

for $t \in [0, T]$.

Definition 6.4. A strategy $(\hat{\xi}_t)$ is said to be *self-financing*, if

$$V_T = V_0 + \sum_{j=1}^{m} \hat{\xi}_{s_j} \cdot (\hat{X}_{s_j} - \hat{X}_{s_{j-1}}) \tag{6.1}$$

for $0 = s_0 \le s_1 \le \cdots \le s_m = T$.

Definition 6.5. A self-financing strategy $\hat{\xi}$ is said to be *arbitrage*, if its discounted process satisfies following conditions

(i) $\mathcal{P}[\tilde{V}_T - \tilde{V}_0] = 1$.
(ii) $\mathcal{P}[\tilde{V}_T > 0] > 0$.

To be specific, our finance market is modeled with two stochastic processes that a process of a riskless asset $(A_t)_t$ and a process of price of a stock $(S_t)_t$. The stock is assumed that it pays no dividends. Setting initial values $A_0 = 1, S_0 = 1$, we give our fractional Black-Scholes model as follows

$$\begin{aligned} A_t &= \exp(rt) \\ S_t &= \exp((rt + \mu(t) + \sigma U_H(t)), t \in [0,T], \end{aligned} \tag{6.2}$$

where $r \in \mathbb{R}, \sigma \in \mathbb{R}_+, \sup_{t\in[0,T]} \mu(t) < \infty$.

Lemma 6.6. Let $(X_t)_{t\geq 0}$ be a stochastic process continueous in t. If (X_t) is a modification of the process

$$\left(\int_0^t (t-u)^{H-\frac{1}{2}}\, dB_u\right)_{t\geq 0}$$

for $(B_t)_{t\geq 0}$ a Brownian motion and $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, then

$$\mathcal{P}[\sup_{t\in[a,b]} X_t \leq -c] > 0$$

for $c \geq 0$ and $0 < a \leq b$.

Proof. See [3], p.15, Lemma 4.2 . □

TODO: wiener sample path space. . . . .

Theorem 6.7. Let $(S_t)_{t\in[0,T]}$ be a stochastic process such that

$$\tilde{S}_t = \exp\left(\mu(t) + \sigma U_H(t)\right), \tag{6.3}$$

where μ, σ are as in (6.2), $U_H(t)$ is a FBM. If there exist

$$\xi_t^1 = f_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{n-1} f_k \mathbb{1}_{(\tau_k, \tau_{k+1}]}(t)$$

where $t \in [0,T], \{f_k\}_0^{n-1}$ is family of $\mathcal{F}_k^{U_H}$-measurable function, $0 = \tau_1 < \cdots < \tau_n = T$ are stopping times respect to $\mathcal{F}_t^{U_H}$ respectively, with $\tau_{k+1} - \tau_k \geq m$ and $\mathcal{P}[f_k \neq 0] > 0$, then

$$\mathcal{P}[(\xi^1 \cdot \tilde{S})_T < 0] > 0.$$

Proof. Cf.[3], p.18, Theorem 4.3 . ξ^1 is predictable. Assume $\mathcal{P}[(\xi^1 \cdot \tilde{S})_T < 0] = 0$, then there exist

$$l = \min\{j : \mathcal{P}[f_j \neq 0] > 0, \quad \mathcal{P}[\sum_{k=1}^{j} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) \geq 0] = 1\}$$

This leads to

$$\mathcal{P}\left[\left(\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}\right) \leq 0\right] = 1$$

Ignoring constant term, we define

$$U_H(t)(\omega) = \int_{\mathbb{R}} \mathbb{1}_{\{t \geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u \leq 0\}}(-u)^{H-\frac{1}{2}} \; d\omega(u) \, .$$

where $\omega(u) := B_u(\omega)$ for all $\omega \in \Omega^w$. We give the filtration $(\mathcal{F}_t^{\Omega^w})$ denoting by

$$\mathcal{F}_t^{\Omega^w} := \sigma(\{\{w \in \Omega^w : \omega(u) \in \mathbb{R}\} : -\infty < u \leq t, c \in \mathbb{R}\}).$$

Then τ_k is also stopping time of $\mathcal{F}_t^{\Omega^w}$ due to

$$\mathcal{F}_t^{U_H} \subset \mathcal{F}_t^{\Omega^w} . t \in \mathbb{R}$$

For $\omega \in \Omega^w$, we split it into two parts as follows

$$\psi_\omega(u) := \omega(u)\mathbb{1}_{(-\infty, \tau_l(\omega)]}, u \in \mathbb{R}$$
$$\phi_\omega(u) := \omega(\tau_l(\omega) + m) - \omega(\tau_l(\omega)), u \geq 0.$$

Corresponding to each, we define

$$\Omega^1 := \{\psi_\omega \in \mathcal{C}(\mathbb{R}) : \omega \in \Omega^w\}$$
$$\Omega^2 := \{\phi_\omega \in \mathcal{C}([0, \infty)) : \omega \in \Omega^w\}$$

And for the smallst σ-algebra of all subsets, respectively, of Ω^1, Ω^2 denoted by $\mathscr{B}^1, \mathscr{B}^2$. Notice that

$$\begin{aligned} \{\tau_l \leq t\} \cap \{\psi_\omega \in \Omega^w\} &= \{\{\omega \in \Omega^w : \omega(u) \in \mathbb{R}\} : -\infty < u \leq t\} \\ &\in \mathcal{F}_t^{\Omega^w}, \end{aligned}$$

therefore is ψ_ω a $\mathcal{F}_{\tau_l}^{\Omega^w}$- measurable mapping. Moreover, since the strong Markovian property of Brownian motion, ϕ_ω is independent of $\mathcal{F}_{\tau_l}^{\Omega^w}$ and it must be a Brownian motion.

We replace the ω by ψ_ω, ϕ_ω calculating the value process

$$
\begin{aligned}
& \left(\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) + f_l(e^{U_H(\tau_l+m)} - e^{U_H(\tau_l)})\right)(\omega) \\
= & \underbrace{\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)})}_{:=J^1}(\omega) + f_l\left(e^{U_H(\tau_l)}\left((e^{U_H(\tau_l+m)} - e^{U_H(\tau_l)}) - 1\right)\right)(\omega) \\
= & J^1(\omega) + f_l(\exp\{\int_{\mathbb{R}} \mathbb{1}_{\{\tau_l \geq u\}}(\tau_l(\omega) + m - u)^{H-\frac{1}{2}} - \mathbb{1}_{\{\tau_l \geq u\}}(\tau_l(\omega) - u)^{H-\frac{1}{2}}\, d\omega(u) \\
+ & \int_{\mathbb{R}} \mathbb{1}_{(\tau_l, \tau_l+m]}(\tau_l + m - u)^{H-\frac{1}{2}}\, d\omega(u)\}) \\
= & J^1(\psi_\omega) + f_l \underbrace{\left(\exp\{\int_{-\infty}^{\tau_l(\psi_\omega)} (\tau_l(\psi_\omega) + h + u)^{H-\frac{1}{2}} - (\tau_l(\psi_\omega) - u)^{H-\frac{1}{2}}\, d\psi_\omega(u)\}\right)_{=:J^2(\psi_\omega, m)}} \\
\cdot & \underbrace{\left(\exp\{\int_0^m (m-u)^{H-\frac{1}{2}}\, d\phi_\omega(u)\} - 1\right)}_{=:J^3(\phi_\omega, m)} \\
= & J(\psi_\omega, \phi_\omega, m)
\end{aligned}
$$

where J is defined as

$$J(\psi, \phi, t) := J^1(\psi) + J^2(\psi, t) \quad \cdot \quad J^3(\phi, t)$$

for $\psi \in \Omega^1, \phi \in \Omega^2$.

Indeed, fixing $\psi \in \Omega^1, \phi \in \Omega^2$, $J(\psi, \phi, \cdot)$ has continueous path on $(\Omega^1 \times \Omega^2, \mathscr{B}^1 \otimes \mathscr{B}^2)$, then we can define a $\mathscr{B}^1 \otimes \mathscr{B}^2$- measurable set

$$E := \{(\psi, \phi) : \sup_{m \leq t \leq T} J(\psi, \phi, t) < 0\}$$

$$
\begin{aligned}
\mathrm{E}[\mathbb{1}_E(\omega_1, \phi_\omega) | \psi_\omega = \omega_1] &= \mathcal{P}^\omega[\sup_{m \leq t \leq T} J(\omega_1, \phi_\omega, t) < 0] \\
&\geq \mathcal{P}^w[J^1(\omega_1) + \sup_{m \leq t \leq T} J^2(\omega_1, t) \cdot \sup_{m \leq t \leq T} J^3(\phi_\omega, t) < 0]
\end{aligned}
$$

It is clear $J^2(\omega_1)$ is bounded for a fixed ω_1 on the compact set and $J^1(\omega_1)$ also. We set $J^1(\omega_1) = c_1$, $\sup_{m \leq t \leq T} J^2(\omega_1, t) = c_2$. Thanks to Lemma 6.6, there exist $D \subset \Omega^w$ such that $\mathcal{P}^w[C] > 0$ and

$$\int_0^m (m-u)^{H-\frac{1}{2}}\, d\phi_\omega(u) < c_3$$

for $\omega \in D$. where c_3 could be small enough that $c_2 \cdot e^{c_3 - 1} < 0$. Under assumtion at beginning, $c_1 \leq 0$ almost surely. All of this leads to

$$\mathcal{P}^w[\mathrm{E}[\mathbb{1}_E(\omega_1, \phi_\omega)|\psi_\omega = \omega_1]] > 0$$

for $\omega \in \Omega^w$. Then

$$\begin{aligned}
& \mathcal{P}^w[\sum_{k=1}^{l} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) < 0] \\
\geq \quad & \mathcal{P}^w[\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) + \sup_{m \leq t \leq T} f_l(e^{U_H(\tau_l + t)} - e^{U_H(\tau_l)}) < 0] \\
= \quad & \mathrm{E}[\mathrm{E}[\mathbb{1}_E(\omega_1, \phi_\omega)|\psi_\omega = \omega_1]] > 0
\end{aligned}$$

this contradicts our assumption. It must be that $\mathcal{P}[(\xi^1 \cdot S)_T < 0] > 0$. $\square$

Corollary 6.8. Let strategy (ξ^0, ξ^1) be such that, ξ^1 is given as in Theorem 6.4. Then the strategy has no arbitrage in our finance market.

Proof. Assume (ξ^0, ξ^1) is a self-financing strategy. In terms of Definition 6.4,

$$\begin{aligned}
\tilde{V}_T - \tilde{V}_0 \quad &= \quad \sum_{k=1}^{n} \frac{(\xi_k^0 A_k + \xi_k^1 S_k)}{A_k} - \frac{(\xi_k^0 A_{k-1} + \xi_k^1 S_{k-1})}{A_{k-1}} \\
&= \quad \sum_{k=1}^{n} \xi_k (\tilde{S_k} - \tilde{S_{k-1}})
\end{aligned}$$

It follows then from 6.4, $\mathcal{P}[(\tilde{V}_T - \tilde{V}_0) < 0] > 0$, since and therefore there is no arbitrage in this sense. $\square$

6.2 Fractional Volatility Model

7 Conclusion

blah

blah

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