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**Fractional Brownian Motion and its
Application in Financial Mathematics**

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Abstract

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Contents

1	Introduction	1
2	Gaussian Process and Brownian Motion	2
2.1	Probability Space and Stochastic Process	2
2.2	Normal Distribution and Gaussian Process	3
2.3	Brownian Motion	8
3	Regularity for Brownian Motion and Itô Integral	10
3.1	Theorem of Kolmogorov Chentsov and Lévy Modulus of Continuity	10
4	Fractional Brownian Motion	11
4.1	Definition of fractional Brownian motion	11
5	Fractional Ornstein Uhlenbeck Process Model	12
6	Application in Financial Mathematics	13
7	Conclusion	14
	References	15

1 Introduction

2 Gaussian Process and Brownian Motion

In this section we start off by looking at some general concepts of probability spaces and stochastic processes. Of this, a most important case we then describe, is Gaussian process. It brings us to introduce the Brownian Motion as a fine example.

2.1 Probability Space and Stochastic Process

DEFINITION 2.1. Let \mathcal{A} be a collection of subsets of a set Ω . \mathcal{A} is then a σ -Algebra on Ω if it satisfies the following conditions:

- (i) $\Omega \in \mathcal{A}$.
- (ii) For any set $F \in \mathcal{A}$, its complement $F^c \in \mathcal{A}$.
- (iii) If a series $\{F_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$, then $\cup_{n \in \mathbb{N}} F_n \in \mathcal{A}$.

DEFINITION 2.2. A mapping \mathcal{P} is said to be a *probability measure* from \mathcal{A} to $\mathcal{B}(\mathbb{R}^n)$, if $\mathcal{P}[\sum_{n=1}^{\infty} F_n] = \sum_{n=1}^{\infty} \mathcal{P}[F_n]$ for any $\{F_n\}_{n \in \mathbb{N}}$ disjoint in \mathcal{A} satisfying $\sum_{n=1}^{\infty} F_n \in \mathcal{A}$.

DEFINITION 2.3. A *probability space* is defined as a triple $(\Omega, \mathcal{A}, \mathcal{P})$ of a set Ω , a σ -Algebra \mathcal{A} of Ω and a measure \mathcal{P} from \mathcal{A} to $\mathcal{B}(\mathbb{R}^n)$.

The σ -Algebra generated of all open sets on \mathbb{R}^n is called the *Borel σ -Algebra* which we denote as usual by $\mathcal{B}(\mathbb{R}^n)$. Let μ be a probability measure on \mathbb{R}^n . Indeed, $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu)$ is a special case that probability space on \mathbb{R}^n . A function f mapping from $(\mathcal{D}, \mathcal{D}, \mu)$ into $(\mathcal{E}, \mathcal{E}, \nu)$ is *measurable* if its collection of the inverse image of \mathcal{E} is a subset of \mathcal{D} . A *random variable* is a \mathbb{R}^n -valued measurable function on some probability space. Let \mathcal{P} represent a probability measure, recall that in probability theory, for $B \in \mathcal{B}(\mathbb{R}^n)$ we call $\mathcal{P}[\{X \in B\}]$ the *distribution* of X . We write also $\mathcal{P}_X[\cdot]$ or $\mathcal{P}[X]$ for convenience of the notation above.

DEFINITION 2.4. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. A n -dimensional *stochastic process* (X_t) is a family of random variable such that $X_t(\omega) : \Omega \rightarrow \mathbb{R}^n, \forall t \in T$, where T denotes the set of Index of Time.

DEFINITION 2.5. A stochastic process $(X_t)_{t \in T}$ is said to be *stationary*, if the joint distribution

$$\mathcal{P}[X_{t_1}, \dots, X_{t_n}] = \mathcal{P}[X_{t_1+\tau}, \dots, X_{t_n+\tau}]$$

for t_1, \dots, t_n and $t_1 + \tau, \dots, t_n + \tau \in T$.

Remark that, definition 2.5 means the distribution of a stationary process is independent of a shift of time.

2.2 Normal Distribution and Gaussian Process

DEFINITION 2.6 (1-dimensional normal distribution). A \mathbb{R} -valued random variable X is said to be *standard normal distributed*, if its distribution can be described as

$$\mathcal{P}[X \leq x] = \int_{-\infty}^x (2\pi)^{-\frac{1}{2}} e^{-\frac{u^2}{2}} du$$

for $x \in \mathbb{R}$.

DEFINITION 2.7. A \mathbb{R} -valued random variable X is said to be *normal distributed* with a mean μ and a variance σ^2 , if

$$(X - \mu)/\sigma$$

is standard normal distributed.

We use a notation $X \sim Y$, which means X and Y have the same distribution. In similar way it is denoted by $X \sim (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} dx$, if it is standard normal distributed. In order to identifying the behaviour of a normal distributed random variable we recall the characteristic function in probability theory, see[1].

PROPOSITION 2.8. Let X be a \mathbb{R} -valued standard normal distributed random variable. The characteristic function of X

$$\Psi_X(\xi) := \int_{\mathbb{R}} e^{ix\xi} \mathcal{P}[X \in dx] = e^{-\frac{\xi^2}{2}} \quad (2.1)$$

for $\xi \in \mathbb{R}$.

Proof. According to the definion of characteristic function

$$\Psi_X(\xi) = \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} dx,$$

take differentiating both sides of the equation by ξ , then

$$\begin{aligned} \Psi'_X(\xi) &= \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} ix dx \\ &= (-i) \cdot \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} \left(\frac{d}{dx} e^{-\frac{x^2}{2}} \right) e^{ix\xi} dx \\ &\stackrel{part.int.}{=} - \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} \xi dx \\ &= -\xi \Psi_X(\xi). \end{aligned}$$

Obviously, $\Psi(\xi) = \Psi(0)e^{-\frac{\xi^2}{2}}$ is the solution of the partial differential equation above, and $\Psi(0)$ is equal to 1. □

2.2 Normal Distribution and Gaussian Process

In particular, the characteristic function of a normal distributed random variable with a mean μ and a variance σ^2 , which denoted by $\Psi_{X_{\mu,\sigma^2}}(\xi)$, is $e^{i\mu\xi - \frac{1}{2}(\sigma\xi)^2}$. To achieve this result, we just need to substitute x by $(x - \mu)/\sigma$ in the calculation before.

DEFINITION 2.9. Let X be a \mathbb{R}^n -valued random variable. X is said to be *normal distributed*, if for any $d \in \mathbb{R}^n$ such that $d^T X$ is normal distributed on \mathbb{R} .

PROPOSITION 2.10. Let X be a \mathbb{R}^n -valued normal distributed. Then there exist $m \in \mathbb{R}^n$ and a positive definite symmetric matrix $\Sigma \in \mathbb{R}^{n \times n}$ such that,

$$\mathbb{E} e^{i\xi^T X} = e^{i\xi^T m - \frac{1}{2}\xi^T \Sigma \xi} \quad (2.2)$$

For $\xi \in \mathbb{R}^n$. Furthermore, the density function of X is

$$(2\pi)^{-\frac{d}{2}} (\det \Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-m)^T \Sigma^{-1}(x-m)} dx. \quad (2.3)$$

Remark, the equation (2.2) can also be as definition of characteristic function of a n -dimensional normal distributed random variable. I.e., any normal distributed random variable can be characterized by form of the equation (2.2).

Proof. Since X normal distributed on \mathbb{R}^n , then $\xi^T X$ is normal distributed on \mathbb{R} . Due to the proposition 2.8 there is

$$\begin{aligned} \mathbb{E} e^{i\xi^T X} &= \mathbb{E} e^{i \cdot 1 \cdot \xi^T X} \\ &= e^{i\mathbb{E}[\xi^T X] - \frac{1}{2}\text{Var}[\xi^T X]} \\ &= e^{i\xi^T \mathbb{E}[X] - \frac{1}{2}\xi^T \text{Var}[X]\xi}. \end{aligned}$$

According to the uniqueness theorem of characteristic function (Satz 23.4 in [1]), then we can deduce the density function of the equation (2.3). \square

A normal distributed normal random variable can be characterized by its mean and variance respectively mean vector and covariance vector because of the characteristic function.

COROLLARY 2.11. A linear combination of independent normal distributed random variables has normal distribution.

Proof. In general case, we suppose Y_1, \dots, Y_m are independent random variables on \mathbb{R}^n ,

for $c_1, \dots, c_m \in \mathbb{R}$. Let have a look at the characteristic function of it,

$$\begin{aligned}
 \mathbb{E} e^{i\xi^T \sum_{j=1}^m (c_j X_j)} &\stackrel{\text{independent}}{=} \prod_{j=1}^m \mathbb{E} e^{i\xi^T (c_j X_j)} \\
 &= \prod_{j=1}^m \exp \left(i\xi^T \mathbb{E}[c_j X_j] - \frac{1}{2} \xi^T \text{Var}[c_j X_j] \xi \right) \\
 &= \exp \left(i\xi^T \mathbb{E} \left[\sum_{j=1}^m c_j X_j \right] - \frac{1}{2} \xi^T \sum_{j=1}^m \text{Var}[c_j X_j] \xi \right) \\
 &\stackrel{\text{independent}}{=} \exp \left(i\xi^T \mathbb{E} \left[\sum_{j=1}^m c_j X_j \right] - \frac{1}{2} \xi^T \text{Var} \left[\sum_{j=1}^m c_j X_j \right] \xi \right),
 \end{aligned}$$

which is a form of characteristic function of normal distribution. That means $\sum_{j=1}^m c_j X_j$ is normal distributed. \square

EXAMPLE 2.12 (Bivariate Normal Distribution). Suppose S_1, S_2 are independent random variables on \mathbb{R} and have standard normal distributions. $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ has standard normal joint distribution since they are independent. We define

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 \rho & \sigma_2 (1 - \rho^2)^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad (2.4)$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 \in \mathbb{R}, -1 \leq \rho \leq 1$. Again, Y_1, Y_2 are normal distributed and the joint distribution $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is normal. We set $\mathbb{E}[Y_1] = \mu_1, \mathbb{E}[Y_2] = \mu_2$ for short. Since S_1, S_2 are independent,

$$\begin{aligned}
 \text{Var}[Y_1] &= \text{Var}[\sigma_1 S_1] \\
 &= \sigma_1^2, \\
 \text{Var}[Y_2] &= \text{Var}[\sigma_2 \rho S_1] + \text{Var}[\sigma_2 (1 - \rho^2)^{\frac{1}{2}} S_2] \\
 &= \sigma_2^2 \rho^2 + \sigma_2^2 (1 - \rho^2) \\
 &= \sigma_2^2, \\
 \text{Cov}[Y_1, Y_2] &= \mathbb{E}[(Y_1 - \mathbb{E}[Y_1])(Y_2 - \mathbb{E}[Y_2])] \\
 &= \mathbb{E}[Y_1 Y_2 - \mu_1 Y_2 - \mu_2 Y_1 + \mu_1 \mu_2] \\
 &= \mathbb{E}[(\sigma_1 S_1 + \mu_1)(\sigma_2 \rho S_1 + \sigma_2 (1 - \rho^2)^{\frac{1}{2}} S_2 + \mu_2)] - \mu_1 \mu_2 \\
 &= \underbrace{\sigma_1 \sigma_2 \mathbb{E}[S_1^2]}_{=1} \rho + \underbrace{\mu_1 \sigma_2 \rho \mathbb{E}[S_1]}_{=0} + \underbrace{\sigma_1 \sigma_2 (1 - \rho^2)^{\frac{1}{2}} \mathbb{E}[S_1 S_2]}_{=\mathbb{E}[S_1] \mathbb{E}[S_2]=0} \\
 &\quad + \underbrace{\mu_1 \sigma_2 (1 - \rho^2)^{\frac{1}{2}} \mathbb{E}[S_2]}_{=0} + \underbrace{\sigma_1 \mathbb{E}[S_1] \mu_2}_{=0} + \mu_1 \mu_2 - \mu_1 \mu_2 \\
 &= \rho \sigma_1 \sigma_2,
 \end{aligned}$$

2.2 Normal Distribution and Gaussian Process

that means the correlation of Y_1, Y_2 is ρ . Because of the equation (2.3), the joint density function

$$f_{Y_1, Y_2}(y_1, y_2) = (2\pi)^{-1} (\det(\Sigma))^{-\frac{1}{2}} \exp\left((y_1 - \mu_1)\Sigma^{-1}(y_2 - \mu_2)\right),$$

where $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ \sigma_2^2 \rho^2 & \sigma_2^2(1 - \rho^2) \end{pmatrix}$

Indeed,

$$\det(\Sigma) = (1 - \rho^2)\sigma_1^2\sigma_2^2$$

and

$$\Sigma^{-1} = \frac{\begin{pmatrix} \sigma_2^2(1 - \rho^2) & 0 \\ -\sigma_2^2\rho & \sigma_1^2 \end{pmatrix}}{(1 - \rho^2)\sigma_1^2\sigma_2^2}.$$

Namely,

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2\pi(1 - \rho^2)^{\frac{1}{2}}\sigma_1\sigma_2} \exp\left(-\frac{1}{2(1 - \rho^2)}(z_1^2 - 2\rho z_1 z_2 + z_2^2)\right) \quad (2.5)$$

where $z_1 = \frac{y_1 - \mu_1}{\sigma_1}$, $z_2 = \frac{y_2 - \mu_2}{\sigma_2}$.

COROLLARY 2.13. Let Y_1, Y_2 be \mathbb{R} -valued normal distributed random variables and $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ has a joint normal distribution, then the conditional mean of Y_2 given Y_1

$$\mathbb{E}[Y_2|Y_1 = y_1] = \mathbb{E}[Y_2] + \rho(y_1 - \mathbb{E}[Y_1])\frac{\sigma_2}{\sigma_1},$$

and the conditional variance of Y_2 given Y_1

$$\text{Var}[Y_2|Y_1 = y_1] = \sigma_2^2(1 - \rho^2).$$

Where σ_1, σ_2 are standard deviations of Y_1, Y_2 and ρ is the correlation of Y_1, Y_2 .

Proof. Recall the equation (2.5), we can specify the joint density function if σ_1, σ_2, ρ are known. As result of this, $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ has a form of the equation (2.4). Suppose S_1, S_2 are independent standard normal distributed random variables. Now we have

$$\begin{aligned} S_1 &\sim \frac{(Y_1 - \mathbb{E}[Y_1])}{\sigma_1} \\ Y_2 &\sim \sigma_2 \rho S_1 + \sigma_2(1 - \rho^2)^{\frac{1}{2}} S_2 + \mathbb{E}[Y_2], \end{aligned}$$

more precisely,

$$Y_2 \sim \sigma_2 \rho \frac{(Y_1 - \mathbb{E}[Y_1])}{\sigma_1} + \sigma_2(1 - \rho^2)^{\frac{1}{2}} S_2 + \mathbb{E}[Y_2].$$

Take expectation of both sides,

$$\mathbb{E}[Y_2|Y_1 = y_1] = \sigma_2 \rho \frac{(y_1 - \mathbb{E}[Y_1])}{\sigma_1} + \mathbb{E}[Y_2].$$

Now consider

$$\begin{aligned} \text{Var}[Y_2|Y_1 = y_1] &= \mathbb{E}[(Y_2 - \mu_{Y_2|Y_1})^2|Y_1 = y_1] \\ &= \int_{-\infty}^{\infty} (y_2 - \mu_{Y_2|Y_1})^2 f_{Y_2|Y_1}(y_2, y_1) dy_2 \\ &= \int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1) \right]^2 f_{Y_2|Y_1}(y_2, y_1) dy_2, \end{aligned}$$

multiply both sides by the density function of Y_1 and integral it over by y_1 , we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \text{Var}[Y_2|Y_1 = y_1] f_{Y_1}(y_1) dy_1 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1) \right]^2 \underbrace{f_{Y_2|Y_1}(y_2, y_1) f_{Y_1}(y_1)}_{f_{Y_1, Y_2}(y_1, y_2)} dy_2 dy_1 \\ &\iff \\ &\text{Var}[Y_2|Y_1 = y_1] \underbrace{\int_{-\infty}^{\infty} f_{Y_1}(y_1) dy_1}_1 \\ &= \mathbb{E} \left[(Y_2 - \mu_2) - \left(\frac{\rho\sigma_2}{\sigma_1} \right) (Y_1 - \mu_1) \right]^2 \end{aligned}$$

Also

$$\begin{aligned} \text{Var}[Y_2|Y_1 = y_1] &= \underbrace{\mathbb{E}[(Y_2 - \mu_2)^2]}_{\sigma_2^2} - 2 \frac{\rho\sigma_2}{\sigma_1} \underbrace{\mathbb{E}[(Y_1 - \mu_1)(Y_2 - \mu_2)]}_{\rho\sigma_1\sigma_2} \\ &\quad + \frac{\rho^2\sigma_2^2}{\sigma_1^2} \underbrace{\mathbb{E}[(Y_1 - \mu_1)^2]}_{\sigma_1^2} \\ &= \sigma_2^2 - 2\rho^2\sigma^2 + \rho^2\sigma_2^2 \\ &= \sigma_2^2 - \rho^2\sigma_2^2. \end{aligned}$$

□

DEFINITION 2.14. Let $(X_t)_{t \in T}$ be a \mathbb{R}^n -valued stochastic process. (X_t) is said to be a *Gaussian process* if X_{t_1}, \dots, X_{t_n} has a joint normal distribution for any $t_1 \dots t_n \in T$ and $n \in \mathbb{N}$.

The definition immediately shows for every X_t in Gaussian process has a normal distribution. Therefore the prior corollary is applicable to a Gaussian process.

2.3 Brownian Motion

The Brownian motion was first introduced by Bachelier in 1900 in his PhD thesis. We now give the common definition of it.

DEFINITION 2.15. Let $(B_t)_{t \geq 0}$ be a \mathbb{R}^n -valued stochastic process. (B_t) is called *Brownian motion* if it satisfies the following conditions:

- (i) $B_0 = 0$ a.s. .
- (ii) $(B_{t_1} - B_{t_0}), \dots, (B_{t_n} - B_{t_{n-1}})$ are independent for $0 = t_0 < t_1 < \dots < t_n$ and $n \in \mathbb{N}$.
- (iii) $B_t - B_s \sim B_{t-s}$, for $0 \leq s \leq t < \infty$.
- (iv) $B_t - B_s \sim \mathcal{N}(0, t-s)^{\otimes n}$.
- (v) B_t is continuous in t a.s. .

A usual saying for (ii) and (iii) is the Brownian motion has independent, stationary increments. In (iv), N represent a random variable which has a normal distribution. B_t is normal distributed due to (ii). It is clear that the increments of Brownian motion is stationary.

PROPOSITION 2.16. Let (B_t) be a one-dimensional Brownian motion. Then the covariance of B_m, B_n for $m, n \geq 0$ is $m \wedge n$.

Proof. WLOG, we assume that $m \geq n$, then

$$\begin{aligned} \mathbb{E}[B_m B_n] &= \mathbb{E}[(B_m - B_n)B_n] + \mathbb{E}[B_n^2] \\ &= \mathbb{E}[B_m - B_n]\mathbb{E}[B_n] + n \\ &= n. \end{aligned}$$

□

PROPOSITION 2.17. Let (B_t) be a one-dimensional Brownian motion. Then $B_{cm} \sim c^{\frac{1}{2}} B_m$.

Proof. Because B_m is normal distributed for any $m > 0$, we then get

$$\begin{aligned} \mathbb{E}[e^{i\xi B_{cm}}] &= e^{-\frac{1}{2}cm\xi^2} \\ &= e^{-\frac{1}{2}(c(m)^{\frac{1}{2}}\xi)^2} \\ &= \mathbb{E}[e^{i\xi c^{\frac{1}{2}} B_m}]. \end{aligned}$$

□

THEOREM 2.18. A one-dimensional Brownian motion is a Gaussian process.

Proof. The following idea using the independence of increments to prove the claim come from [2]. We choose $0 = t_0 < t_1 < \dots < t_n$, for $n \in \mathbb{N}$. Define $V = (B_{t_1}, \dots, B_{t_n})^T$,

$$K = (B_{t_1} - B_{t_0}, \dots, B_{t_n} - B_{t_{n-1}})^T \text{ and } A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}. \text{ Let us see the characteristic}$$

function of V ,

$$\begin{aligned} \mathbb{E}[e^{i\xi^T V}] &= \mathbb{E}[e^{i\xi^T AK}] \\ &= \mathbb{E}[e^{iA^T \xi K}] \\ &= \mathbb{E}[\exp(i(\xi^{(1)} + \dots + \xi^{(n)}, \xi^{(2)} + \dots + \xi^{(n)}, \dots, \xi^{(n)}) \\ &\quad \cdot (B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})^T) \\ &\stackrel{\text{ind.increments}}{=} \prod_{j=1}^n \mathbb{E}[\exp(i(\xi^{(j)} + \dots + \xi^{(n)})(B_{t_j} - B_{t_{j-1}}))] \\ &\stackrel{\text{stat.increments}}{=} \prod_{j=1}^n \exp(-\frac{1}{2}(t_j - t_{j-1})(\xi^{(j)} + \dots + \xi^{(n)})^2) \\ &= \exp\left(-\frac{1}{2} \sum_{j=1}^n (t_j - t_{j-1})(\xi^{(j)} + \dots + \xi^{(n)})^2\right) \\ &= \exp\left(-\frac{1}{2} \left(\sum_{j=1}^n t_j (\xi^{(j)} + \dots + \xi^{(n)})^2 - \sum_{j=1}^n t_{j-1} (\xi^{(j)} + \dots + \xi^{(n)})^2 \right)\right) \\ &= \exp\left(-\frac{1}{2} \left(\sum_{j=1}^{n-1} t_j ((\xi^{(j)} + \dots + \xi^{(n)})^2 - (\xi^{(j+1)} + \dots + \xi^{(n)})^2) + t_n (\xi^{(n)})^2 \right)\right) \\ &= \exp\left(-\frac{1}{2} \left(\sum_{j=1}^{n-1} t_j \xi^{(j)} (\xi^{(j)} + 2\xi^{(j+1)} + \dots + 2\xi^{(n)}) + t_n (\xi^{(n)})^2 \right)\right) \\ &= \exp\left(-\frac{1}{2} \left(\sum_{j,h=1}^n (t_j \wedge t_h) \xi^{(j)} \xi^{(h)} \right)\right). \end{aligned}$$

Recall with proposition 2.3, $(t_j \wedge t_h)_{j,h=1,\dots,n}$ is the covariance matrix of V . The mean vector of it is zero, then we have been proved that the characteristic function is a form of some normal distributed random vector, i.e., V is normal distributed. \square

Shilling gave in his lecture [2] the relationship between a one-dimensional Brownian motion and a n -dimensional Brownian motion. $(B_t^{(l)})_{l=1,\dots,n}$ is Brownian motion if and only if $B_t^{(l)}$ is Brownian motion and all of the component are independent. Using this independence and the theorem of Fubini in the characteristic function for high-dimensional Brownian motion we can say a n -dimensional Brownian motion is also a Gaussian process.

3 Regularity for Brownian Motion and Itô Integral

3.1 Theorem of Kolmogorov Chentsov and Lévy Modulus of Continuity

We consider now the one-dimensional Brownian motion. In this section we need some notations, which are defined as followings

$$\Delta^{[0,T]} = \{t_1, \dots, t_n | 0 = t_0 < \dots < t_n = T\}$$

$$|\Delta^{[0,T]}| = \max_{t_j \in \Delta^{[0,T]}} |t_j - t_{j-1}|$$

.

LEMMA 3.1. Let B_t be a Brownian motion. Then

$$\sum_{t_j \in \Delta^{[0,T]}} |B_{t_j} - B_{t_{j-1}}|^2 \xrightarrow[L^2(\mathcal{P})]{|\Delta^{[0,T]}| \rightarrow 0} T$$

4 Fractional Brownian Motion

The fractional Brownian motion(FBM) was defined by Kolmogorov primitively. After that Mandelbrot and Van Ness has present the work in detail. This section is concerned with the xxxxx of it.

4.1 Definition of fractional Brownian motion

Mandelbrot and Van Ness gave a integration presentation of the definition of FBM.

DEFINITION 4.1. Let $(U_t)_t$ be a \mathbb{R} -valued stochatstic process an H be such that $1 < H < 0$. (U_t) is said to be *fractional Brownian motion* if

$$\begin{aligned} U_t - U_s &= \int_{\mathbb{R}} \mathbb{1}_{\{t>u\}} (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u<0\}} (-u)^{H-\frac{1}{2}} dB_u \\ &\quad - \int_{\mathbb{R}} \mathbb{1}_{\{s>u\}} (s-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u<0\}} (-u)^{H-\frac{1}{2}} dB_u \end{aligned} \quad (4.1)$$

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5 Fractional Ornstein Uhlenbeck Process Model

6 Application in Financial Mathematics

7 Conclusion

References

- [1] BAUER, H. (2002). Wahrscheinlichkeitstheorie(5th. durchges. und verb. Aufl.). Berlin: W. de Gruyter
- [2] (2012, October 1). Stochastic Processes. Lecture conducted from SHILLING, R., Dresden.