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**Fractional Brownian Motion and
Applications in Financial Mathematics**

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Abstract

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1 Introduction

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2 Gaussian Process and Brownian Motion

In this section we start off by looking at some general concepts of probability spaces and stochastic processes. Of this, a most important case we then describe is Gaussian process. Within the framework of Gaussian processes, one could specify a stationary and independent behaviour of increments of it. This leads us to introduce the Brownian motion as a fine example.

2.1 Probability Space and Stochastic Process

DEFINITION 2.1. Let \mathcal{A} be a collection of subsets of a set Ω . \mathcal{A} is said to be a σ -Algebra on Ω , if it satisfies the following conditions:

- (i) $\Omega \in \mathcal{A}$.
- (ii) For any set $F \in \mathcal{A}$, its complement $F^c \in \mathcal{A}$.
- (iii) If there is a sequence such that $\{F_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$, then $\cup_{n \in \mathbb{N}} F_n \in \mathcal{A}$.

DEFINITION 2.2. A mapping \mathcal{P} is said to be a *probability measure* from \mathcal{A} to $\mathcal{B}(\mathbb{R}^n)$, if $\mathcal{P}[\sum_{n=1}^{\infty} F_n] = \sum_{n=1}^{\infty} \mathcal{P}[F_n]$ for any $\{F_n\}_{n \in \mathbb{N}}$ disjoint in \mathcal{A} satisfying $\sum_{n=1}^{\infty} F_n \in \mathcal{A}$.

DEFINITION 2.3. A *probability space* is defined as a triple $(\Omega, \mathcal{A}, \mathcal{P})$ of a set Ω , a σ -Algebra \mathcal{A} of Ω and a measure \mathcal{P} from \mathcal{A} to $\mathcal{B}(\mathbb{R}^n)$.

The σ -Algebra generated of all open sets on \mathbb{R}^n is called the *Borel σ -Algebra* which we denote as usual by $\mathcal{B}(\mathbb{R}^n)$. Let μ be a probability measure on \mathbb{R}^n . Indeed, $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu)$ is a special case that probability space on \mathbb{R}^n . A function f mapping from $(\mathcal{D}, \mathcal{D}, \mu)$ into $(\mathcal{E}, \mathcal{E}, \nu)$ is *measurable*, if its collection of the inverse image of \mathcal{E} is a subset of \mathcal{D} . A *random variable* is a \mathbb{R}^n -valued measurable function on some probability space. Let \mathcal{P} represent a probability measure, recall that in probability theory, for $B \in \mathcal{B}(\mathbb{R}^n)$ we call $\mathcal{P}[\{X \in B\}]$ the *distribution* of X . We write also $\mathcal{P}_X[\cdot]$ or $\mathcal{P}[X]$ for convenience for those notations.

DEFINITION 2.4. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. A *n-dimensional stochastic process* $(X_t)_{t \in T}$ is a family of random variable such that $X_t(\omega) : \Omega \rightarrow \mathbb{R}^n, \forall t \in T$, where T denotes the set of Index of Time.

Some basic definitions, which are needed in following sections, are given.

DEFINITION 2.5. A stochastic process $(X_t)_{t \in T}$ is said to be *stationary*, if the joint distribution

$$\mathcal{P}[X_{t_1}, \dots, X_{t_n}] = \mathcal{P}[X_{t_1+\tau}, \dots, X_{t_n+\tau}]$$

for t_1, \dots, t_n and $t_1 + \tau, \dots, t_n + \tau \in T$.

Remark that, Definition 2.5 means the distribution of a stationary process is independent of a shift of time.

DEFINITION 2.6. Let $(X_t)_t$ be a stochastic process.

$$\varsigma_X(t, s) := \text{Cov}(X_t, X_s)$$

is called *autocovariance* between s, t and

$$\eta_X(t, s) := \frac{\text{Cov}[X_t, X_s]}{\sqrt{\text{Var}[X_t]\text{Var}[X_s]}}$$

is called *autocorrelation* between s, t .

If $(X_t)_t$ is stationary process, we write $\varsigma_X(\tau)$ for $\varsigma_X(t, t + \tau)$ for any t . $\eta_X(\tau)$ is used in the same way.

We use a notation $X \sim Y$ represents X equals Y in distribution.

DEFINITION 2.7. A stochastic process $(X_t)_{t \in T}$ is said to be α -self similar if $(X_{ct_1}, \dots, X_{ct_k}) \sim (c^\alpha X_{t_1}, \dots, c^\alpha X_{t_k})$ for any $t_1, \dots, t_k, ct_1, \dots, ct_k \in T$ and $c > 0$.

2.2 Normal Distribution and Gaussian Process

DEFINITION 2.8 (1-dimensional normal distribution). A \mathbb{R} -valued random variable X is said to be *standard normal distributed* or *standard Gaussian*, if its distribution can be described as

$$\mathcal{P}[X \leq x] = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du \quad (2.1)$$

for $x \in \mathbb{R}$.

The integrand of (2.1) is also called *density function* of a Gaussian random variable.

DEFINITION 2.9. A \mathbb{R} -valued random variable X is said to be *normal distributed* or *Gaussian* with a *expected value* μ and a *variance* σ^2 , if

$$(X - \mu)/\sigma$$

is standard Gaussian.

PROPOSITION 2.10. Let X be a \mathbb{R} -valued Gaussian random variable with expected value μ and variance σ^2 , then it is distributed as

$$\mathcal{P}[X \leq x] = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^x e^{-\frac{(u-\mu)^2}{2\sigma^2}} du$$

Proof. Suppose $X = \sigma Y + \mu$ with Y standard Gaussian. We denote this mapping by $g(y) : y \rightarrow \sigma y + \mu$ and give the inverse $g^{-1}(x) : x \rightarrow \frac{(x-\mu)}{\sigma}$. The distribution function of X is

$$\begin{aligned} \int_{\Omega} \mathcal{P}[X \in dx] &= \int_{\Omega} \mathcal{P}[Y \circ g \in dy] \\ &= \int_{\mathbb{R} \circ g} f_X \circ g^{-1}(y) dy \\ &= \int_{\mathbb{R}} \sigma \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left(\frac{(y-\mu)}{\sigma}\right)^2}{2}\right\} dy \\ &= (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} dy, \end{aligned}$$

where f_X is density function of X . □

It is denoted by $X \sim (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} dx$, if X is standard Gaussian. In order to verifying the behaviour of a normal distributed random variable we use the characteristic function in probability theory, Cf.[1].

THEOREM 2.11. Let X be a \mathbb{R} -valued Gaussian random variable. The characteristic function of X

$$\Psi_X(\xi) := \int_{\mathbb{R}} e^{ix\xi} \mathcal{P}[X \in dx] = e^{i\mu\xi - \frac{1}{2}(\sigma\xi)^2} \quad (2.2)$$

for $\xi \in \mathbb{R}$.

Proof. Cf.[10]. We assume firstly X is standard Gaussian. In terms of the Definition of characteristic function of a standard Gaussian X , integrating its density function over \mathbb{R} we get

$$\Psi_X(\xi) = \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} dx,$$

take differentiating both sides of the equation by ξ , then

$$\begin{aligned} \Psi'_X(\xi) &= \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} ix dx \\ &= (-i) \cdot \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} \left(\frac{d}{dx} e^{-\frac{x^2}{2}}\right) e^{ix\xi} dx \\ &\stackrel{part.int.}{=} - \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} \xi dx \\ &= -\xi \Psi_X(\xi). \end{aligned}$$

Obviously, $\Psi(\xi) = \Psi(0)e^{-\frac{\xi^2}{2}}$ is the solution of the partial differential equation, and $\Psi(0)$ equals 1, hence $\Psi(\xi) = e^{-\frac{\xi^2}{2}}$. In particular, the characteristic function of a Gaussian random variable with a expected value μ and a variance σ^2 , which denoted by $\Psi_{X_{\mu,\sigma^2}}(\xi)$, is $e^{i\mu\xi - \frac{1}{2}(\sigma\xi)^2}$. To achieve this result, we just need to substitute x by $(x - \mu)/\sigma$ in the previous calculation. □

DEFINITION 2.12. Let X be a \mathbb{R}^n -valued random vector. X is said to be *normal distributed* or *Gaussian*, if for any $d \in \mathbb{R}^n$ such that $d^T X$ is Gaussian in \mathbb{R} .

DEFINITION 2.13. A stochastic process $(X_t)_{t \in T}$ is said to be *Gaussian process* if the joint distribution of any finite instance is Gaussian, that means $(X_{t_1}, \dots, X_{t_n})$ has joint Gaussian distribution in \mathbb{R}^n for $t_1, \dots, t_n \in T$.

The definition immediately shows every instance X_t in Gaussian process is Gaussian.

COROLLARY 2.14. Let $(X_t)_{t \in T}$ be a stochastic process. The following condition is equivalent to Definition 2.13.

$$\sum_j^n c_{t_j} X_{t_j} \quad (2.3)$$

is Gaussian for any $t_1, \dots, t_n \in T$.

Proof. It is clear due to Definition 2.12. □

LEMMA 2.15. Let X be a \mathbb{R}^n -valued normal distributed random vector. Then its characteristic function is

$$\mathbb{E} e^{i\xi^T X} = e^{i\xi^T m - \frac{1}{2}\xi^T \Sigma \xi}. \quad (2.4)$$

For $\xi \in \mathbb{R}^n$. Where $m \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n \times n}$ are *mean vector*, *covariance matrix* of X respectively. Furthermore, the density function of X is

$$(2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-m)^T \Sigma^{-1}(x-m)}. \quad (2.5)$$

Remark, the equation (2.4) can also be as definition of characteristic function of a n -dimensional normal distributed random variable. I.e., any normal distributed random variable can be characterized by form of the equation (2.4).

Proof. Since X normal distributed on \mathbb{R}^n , then $\xi^T X$ is normal distributed on \mathbb{R} . Due to the Theorem 2.11, there is

$$\begin{aligned} \mathbb{E} e^{i\xi^T X} &= \mathbb{E} e^{i \cdot 1 \cdot \xi^T X} \\ &= e^{i\mathbb{E}[\xi^T X] - \frac{1}{2}\text{Var}[\xi^T X]} \\ &= e^{i\xi^T \mathbb{E}[X] - \frac{1}{2}\xi^T \text{Var}[X]\xi} \\ &= e^{i\xi^T m - \frac{1}{2}\xi^T \Sigma \xi}. \end{aligned}$$

2.2 Normal Distribution and Gaussian Process

Moreover, since Σ symmetric and positive definite, there exist Σ^{-1} , $\Sigma^{\frac{1}{2}}$ and $\Sigma^{-\frac{1}{2}}$.

$$\begin{aligned}
& (2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{ix^T \xi} e^{-\frac{1}{2}(x-m)^T \Sigma^{-1}(x-m)} dx \\
= & (2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{ix^T \xi} e^{i(x-m)^T \xi} e^{-\frac{1}{2}(x-m)^T \Sigma^{-1}(x-m)} dx \\
= & (2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} e^{im^T \xi} \int_{\mathbb{R}^n} e^{i(x-m)^T \xi} e^{i(x-m)^T \xi} e^{-\frac{1}{2}(x-m)^T \Sigma^{-1}(x-m)} dx \\
\stackrel{y=\Sigma^{-\frac{1}{2}}x}{=} & (2\pi)^{-\frac{n}{2}} e^{im^T \xi} \int_{\mathbb{R}^n} e^{i(\Sigma^{\frac{1}{2}}y)^T \xi} e^{-\frac{1}{2}|y|^2} dy \\
= & (2\pi)^{-\frac{n}{2}} e^{im^T \xi} \int_{\mathbb{R}^n} e^{iy^T (\Sigma^{\frac{1}{2}} \xi)} e^{-\frac{1}{2}|y|^2} dy \\
\stackrel{\text{Fourier transformation}}{=} & e^{im^T \xi} e^{-\frac{1}{2}|\Sigma^{\frac{1}{2}} \xi|^2} \\
= & e^{im^T \xi} e^{-\frac{1}{2} \xi^T \Sigma \xi}
\end{aligned}$$

In terms of the uniqueness theorem of characteristic function (in [1], p.199, Satz 23.4), then we can deduce (2.5) is density function of X . \square

THEOREM 2.16. A linear combination of independent normal distributed random variable (or vector) is Gaussian.

Proof. We suppose X_1, \dots, X_m are independent random vectors on \mathbb{R}^n and $c_1, \dots, c_m \in \mathbb{R}$. Let have a look at the characteristic function of it,

$$\begin{aligned}
\mathbb{E} e^{i\xi^T \sum_{j=1}^m (c_j X_j)} & \stackrel{\text{independent}}{=} \prod_{j=1}^m \mathbb{E} e^{i\xi^T (c_j X_j)} \\
& = \prod_{j=1}^m \exp \left(i\xi^T \mathbb{E}[c_j X_j] - \frac{1}{2} \xi^T \text{Var}[c_j X_j] \xi \right) \\
& = \exp \left(i\xi^T \mathbb{E} \left[\sum_{j=1}^m c_j X_j \right] - \frac{1}{2} \xi^T \sum_{j=1}^m \text{Var}[c_j X_j] \xi \right) \\
& \stackrel{\text{independent}}{=} \exp \left(i\xi^T \mathbb{E} \left[\sum_{j=1}^m c_j X_j \right] - \frac{1}{2} \xi^T \text{Var} \left[\sum_{j=1}^m c_j X_j \right] \xi \right),
\end{aligned}$$

which is a form of characteristic function of normal distribution. That means $\sum_{j=1}^m c_j X_j$ is Gaussian. \square

EXAMPLE 2.17 (Bivariate Normal Distribution). Cf.[7], p.241, Example 8.6. Suppose S_1, S_2 are independent random variables and have standard normal distributions. $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ has standard normal joint distribution since they are independent. We define

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 \rho & \sigma_2(1-\rho^2)^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad (2.6)$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 \in \mathbb{R}, -1 \leq \rho \leq 1$. Again, Y_1, Y_2 are Gaussian and the joint distribution $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is also Gaussian. We set $E[Y_1] = \mu_1, E[Y_2] = \mu_2$ for short. Since S_1, S_2 are independent,

$$\begin{aligned}
\text{Var}[Y_1] &= \text{Var}[\sigma_1 S_1] \\
&= \sigma_1^2, \\
\text{Var}[Y_2] &= \text{Var}[\sigma_2 \rho S_1] + \text{Var}[\sigma_2(1 - \rho^2)^{\frac{1}{2}} S_2] \\
&= \sigma_2^2 \rho^2 + \sigma_2^2(1 - \rho^2) \\
&= \sigma_2^2, \\
\text{Cov}[Y_1, Y_2] &= E[(Y_1 - E[Y_1])(Y_2 - E[Y_2])] \\
&= E[Y_1 Y_2 - \mu_1 Y_2 - \mu_2 Y_1 + \mu_1 \mu_2] \\
&= E[(\sigma_1 S_1 + \mu_1)(\sigma_2 \rho S_1 + \sigma_2(1 - \rho^2)^{\frac{1}{2}} S_2 + \mu_2)] - \mu_1 \mu_2 \\
&= \sigma_1 \sigma_2 \underbrace{E[S_1^2]}_{=1} \rho + \mu_1 \sigma_2 \rho \underbrace{E[S_1]}_{=0} + \sigma_1 \sigma_2 (1 - \rho^2)^{\frac{1}{2}} \underbrace{E[S_1 S_2]}_{=E[S_1]E[S_2]=0} \\
&\quad + \mu_1 \sigma_2 (1 - \rho^2)^{\frac{1}{2}} \underbrace{E[S_2]}_{=0} + \sigma_1 \underbrace{E[S_1]}_{=0} \mu_2 + \mu_1 \mu_2 - \mu_1 \mu_2 \\
&= \rho \sigma_1 \sigma_2,
\end{aligned}$$

that means the correlation of Y_1, Y_2 is ρ . Because of the equation (2.5), the joint density function

$$f_{Y_1, Y_2}(y_1, y_2) = (2\pi)^{-1} (\det(\Sigma))^{-\frac{1}{2}} \exp((y_1 - \mu_1) \Sigma^{-1} (y_2 - \mu_2)),$$

where $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ \sigma_2^2 \rho & \sigma_2^2(1 - \rho^2) \end{pmatrix}$

Indeed,

$$\det(\Sigma) = (1 - \rho^2) \sigma_1^2 \sigma_2^2$$

and

$$\Sigma^{-1} = \frac{\begin{pmatrix} \sigma_2^2(1 - \rho^2) & 0 \\ -\sigma_2^2 \rho & \sigma_1^2 \end{pmatrix}}{(1 - \rho^2) \sigma_1^2 \sigma_2^2}.$$

Namely,

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2\pi(1 - \rho^2)^{\frac{1}{2}} \sigma_1 \sigma_2} \exp\left(-\frac{1}{2(1 - \rho^2)}(z_1^2 - 2\rho z_1 z_2 + z_2^2)\right) \quad (2.7)$$

where $z_1 = \frac{y_1 - \mu_1}{\sigma_1}, z_2 = \frac{y_2 - \mu_2}{\sigma_2}$.

COROLLARY 2.18. Let Y_1, Y_2 be \mathbb{R} -valued random variables and $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ has a joint normal distribution, then the conditional expected value of Y_2 given Y_1

$$E[Y_2|Y_1 = y_1] = E[Y_2] + \rho(y_1 - E[Y_1])\frac{\sigma_2}{\sigma_1},$$

and the conditional variance of Y_2 given Y_1

$$\text{Var}[Y_2|Y_1 = y_1] = \sigma_2^2(1 - \rho^2).$$

Where σ_1, σ_2 are standard deviations of Y_1, Y_2 and ρ is the correlation of Y_1, Y_2 .

Proof. Recall the equation (2.7), we can specify the joint density function if σ_1, σ_2, ρ are known. As result of this, $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ has a form of the equation (2.6). Suppose S_1, S_2 are independent standard normal distributed random variables. Now we have

$$\begin{aligned} S_1 &\sim \frac{(Y_1 - E[Y_1])}{\sigma_1} \\ Y_2 &\sim \sigma_2 \rho S_1 + \sigma_2(1 - \rho^2)^{\frac{1}{2}} S_2 + E[Y_2], \end{aligned}$$

more precisely,

$$Y_2 \sim \sigma_2 \rho \frac{(Y_1 - E[Y_1])}{\sigma_1} + \sigma_2(1 - \rho^2)^{\frac{1}{2}} S_2 + E[Y_2].$$

Take expectation of both sides,

$$E[Y_2|Y_1 = y_1] = \sigma_2 \rho \frac{(y_1 - E[Y_1])}{\sigma_1} + E[Y_2].$$

Now consider

$$\begin{aligned} \text{Var}[Y_2|Y_1 = y_1] &= E[(Y_2 - \mu_{Y_2|Y_1})^2|Y_1 = y_1] \\ &= \int_{-\infty}^{\infty} (y_2 - \mu_{Y_2|Y_1})^2 f_{Y_2|Y_1}(y_2, y_1) dy_2 \\ &= \int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1) \right]^2 f_{Y_2|Y_1}(y_2, y_1) dy_2, \end{aligned}$$

After multiplying both sides by the density function of Y_1 and integrating it by y_1 , we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \text{Var}[Y_2|Y_1 = y_1] f_{Y_1}(y_1) dy_1 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1) \right]^2 \underbrace{f_{Y_2|Y_1}(y_2, y_1) f_{Y_1}(y_1)}_{f_{Y_1, Y_2}(y_1, y_2)} dy_2 dy_1 \\ &\iff \\ &\quad \text{Var}[Y_2|Y_1 = y_1] \underbrace{\int_{-\infty}^{\infty} f_{Y_1}(y_1) dy_1}_1 \\ &= E \left[\left(Y_2 - \mu_2 - \left(\frac{\rho\sigma_2}{\sigma_1} \right) (Y_1 - \mu_1) \right)^2 \right] \end{aligned}$$

multiplying right side out, we see

$$\begin{aligned}
 \text{Var}[Y_2|Y_1 = y_1] &= \underbrace{\text{E}[(Y_2 - \mu_2)^2]}_{\sigma_2^2} - 2 \frac{\rho\sigma_2}{\sigma_1} \underbrace{\text{E}[(Y_1 - \mu_1)(Y_2 - \mu_2)]}_{\rho\sigma_1\sigma_2} \\
 &\quad + \frac{\rho^2\sigma_2^2}{\sigma_1^2} \underbrace{\text{E}[(Y_1 - \mu_1)^2]}_{\sigma_1^2} \\
 &= \sigma_2^2 - 2\rho^2\sigma^2 + \rho^2\sigma_2^2 \\
 &= \sigma_2^2 - \rho^2\sigma_2^2.
 \end{aligned}$$

□

THEOREM 2.19. Let X be a Gaussian random variable, then

$$\text{E}[\exp(\beta X)] = \exp(\beta\mu + \frac{1}{2}\beta^2\sigma^2). \quad (2.8)$$

Where μ and σ are $\text{E}[X]$ and $\text{Var}[X]$ respectively.

Proof.

$$\begin{aligned}
 &\text{E}[\exp(\beta X)] \\
 &= (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp(\beta x) \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) dx \\
 &= (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp(\beta x) \exp\left(-\frac{(x^2 - 2x\mu + \mu^2)}{2\sigma^2}\right) dx \\
 &= (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{x^2 - 2(\beta\sigma^2 + \mu)x + \mu^2}{2\sigma^2}\right) dx \\
 &= (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{x^2 - 2(\beta\sigma^2 + \mu)x + (\beta\sigma^2 + \mu)^2 - (\beta\sigma^2 + \mu)^2 + \mu^2}{2\sigma^2}\right) dx \\
 &= (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{(x - (\beta\sigma^2 + \mu))^2 + \mu^2 - (\beta\sigma^2 + \mu)^2}{2\sigma^2}\right) dx \\
 &= \exp\left(\frac{(\beta\sigma^2 + \mu)^2 - \mu^2}{2\sigma^2}\right) \underbrace{(2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{(x - (\beta\sigma^2 + \mu))^2}{2\sigma^2}\right) dx}_1 \\
 &= \exp\left(\frac{\beta^2\sigma^4 + 2\mu\beta\sigma^2}{2\sigma^2}\right) \\
 &= \exp(\mu\beta + \frac{1}{2}\beta^2\sigma^2)
 \end{aligned}$$

□

2.3 Brownian Motion

The Brownian motion was first introduced by Bachelier in 1900 in his PhD thesis. We now give the common definition of it.

2.3 Brownian Motion

DEFINITION 2.20. Let $(B_t)_{t \geq 0}$ be a \mathbb{R}^n -valued stochastic process. (B_t) is called *Brownian motion* if it satisfies the following conditions:

- (i) $B_0 = 0$ a.s. .
- (ii) $(B_{t_1} - B_{t_0}), \dots, (B_{t_n} - B_{t_{n-1}})$ are independent for $0 = t_0 < t_1 < \dots < t_n$ and $n \in \mathbb{N}$.
- (iii) $B_t - B_s \sim B_{t-s}$, for $0 \leq s \leq t < \infty$.
- (iv) $B_t - B_s \sim \mathcal{N}(0, t - s)^{\otimes n}$.
- (v) B_t is continuous in t a.s. .

A usual saying for (ii) and (iii) is the Brownian motion has independent, stationary increments. In (iv), N represent a random variable which has a normal distribution. B_t is normal distributed due to (ii). It is clear that the increments of Brownian motion is stationary.

PROPOSITION 2.21. Let (B_t) be \mathbb{R} -valued Brownian motion. Then the covariance of B_m, B_n for $m, n \geq 0$ is $m \wedge n$.

Proof. Without loss of generality, we assume that $m \geq n$, then

$$\begin{aligned} \mathbb{E}[B_m B_n] &= \mathbb{E}[(B_m - B_n)B_n] + \mathbb{E}[B_n^2] \\ &= \mathbb{E}[B_m - B_n]\mathbb{E}[B_n] + n \\ &= n. \end{aligned}$$

□

PROPOSITION 2.22. Let (B_t) be \mathbb{R} -valued Brownian motion. Then $B_{cm} \sim c^{\frac{1}{2}} B_m$.

Proof. Because B_m is normal distributed for any $m > 0$, we then get

$$\begin{aligned} \mathbb{E}[e^{i\xi B_{cm}}] &= e^{-\frac{1}{2}cm\xi^2} \\ &= e^{-\frac{1}{2}(c(m)^{\frac{1}{2}}\xi)^2} \\ &= \mathbb{E}[e^{i\xi c^{\frac{1}{2}} B_m}]. \end{aligned}$$

□

THEOREM 2.23. A \mathbb{R} -valued Brownian motion is a Gaussian process.

Proof. The following idea using the independence of increments to prove the claim come from [10]. We choose $0 = t_0 < t_1 < \dots < t_n$, for $n \in \mathbb{N}$. Define $V = (B_{t_1}, \dots, B_{t_n})^T$,

$K = (B_{t_1} - B_{t_0}, \dots, B_{t_n} - B_{t_{n-1}})^T$ and $A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$. Let us see the characteristic function of V ,

$$\begin{aligned}
 & \mathbb{E}[e^{i\xi^T V}] \\
 = & \mathbb{E}[e^{i\xi^T AK}] \\
 = & \mathbb{E}[e^{iA^T \xi K}] \\
 = & \mathbb{E}[\exp(i(\xi^{(1)} + \dots + \xi^{(n)}, \xi^{(2)} + \dots + \xi^{(n)}, \dots, \xi^{(n)}) \\
 & \cdot (B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})^T) \\
 \stackrel{\text{ind.increments}}{=} & \prod_{j=1}^n \mathbb{E}[\exp(i(\xi^{(j)} + \dots + \xi^{(n)})(B_{t_j} - B_{t_{j-1}}))] \\
 \stackrel{\text{stat.increments}}{=} & \prod_{j=1}^n \exp(-\frac{1}{2}(t_j - t_{j-1})(\xi^{(j)} + \dots + \xi^{(n)})^2) \\
 = & \exp\left(-\frac{1}{2} \sum_{j=1}^n (t_j - t_{j-1})(\xi^{(j)} + \dots + \xi^{(n)})^2\right) \\
 = & \exp\left(-\frac{1}{2} \left(\sum_{j=1}^n t_j (\xi^{(j)} + \dots + \xi^{(n)})^2 - \sum_{j=1}^n t_{j-1} (\xi^{(j)} + \dots + \xi^{(n)})^2 \right)\right) \\
 = & \exp\left(-\frac{1}{2} \left(\sum_{j=1}^{n-1} t_j ((\xi^{(j)} + \dots + \xi^{(n)})^2 - (\xi^{(j+1)} + \dots + \xi^{(n)})^2) + t_n (\xi^{(n)})^2 \right)\right) \\
 = & \exp\left(-\frac{1}{2} \left(\sum_{j=1}^{n-1} t_j \xi^{(j)} (\xi^{(j)} + 2\xi^{(j+1)} + \dots + 2\xi^{(n)}) + t_n (\xi^{(n)})^2 \right)\right) \\
 = & \exp\left(-\frac{1}{2} \left(\sum_{j,h=1}^n (t_j \wedge t_h) \xi^{(j)} \xi^{(h)} \right)\right).
 \end{aligned}$$

Recall with Proposition 2.3, $(t_j \wedge t_h)_{j,h=1,\dots,n}$ is the covariance matrix of V and therefore it is symmetric and positive definit. The mean vector of it is zero, then we have been proved that the characteristic function is a form of some normal distributed random vector, i.e., V is Gaussian. \square

Schilling gave in his lecture [10] the relationship between a one-dimensional Brownian motion and a n -dimensional Brownian motion. In fact, $(B_t^{(l)})_{l=1,\dots,n}$ is Brownian motion if and only if $B_t^{(l)}$ is Brownian motion and all of the component are independent. Using this independence and the theorem of Fubini in the characteristic function for high dimensional Brownian motion we can say a n -dimensional Brownian motion is also a Gaussian process.

2.3 Brownian Motion

DEFINITION 2.24. Let $(X_t)_{t \in T}$ be a stochastic process. $(Y_t)_{t \in T}$ is defined on the same probability space as $(X_t)_{t \in T}$ and said to be *modification* of $(X_t)_{t \in T}$, if

$$\mathcal{P}[X_t = Y_t] = 1 \quad \forall \quad t \in T.$$

THEOREM 2.25 (Kolmogorov Chentsov). Let $(X_t)_{t \geq 0}$ be a stochastic process on \mathbb{R}^n such that

$$[|X_j - X_k|^\alpha] \leq c|j - k|^{1+\beta} \quad \forall \quad j, k \geq 0 \quad \text{and} \quad j \neq k,$$

for $\alpha, \beta > 0, c < \infty$. Then $(X_t)_t$ has a modification $(Y_t)_t$ with continuous sample path such that

$$\mathbb{E}\left[\left(\frac{|Y_j - Y_k|}{|j - k|^\gamma}\right)^\alpha\right] < \infty$$

for all $\gamma \in (0, \frac{\beta}{\alpha})$.

Proof. See [5], p.519. □

LEMMA 2.26. Let $(B_t)_{t \geq 0}$ be Brownian motion. Then

$$\mathbb{E}[B_t^{2k}] = (2k - 1)!! t^k$$

for $k \in \mathbb{N}_0$.

Proof. Cf.[10]. Taking expectation of B_t^{2k} , we get

$$\begin{aligned} \mathbb{E}[B_t^{2k}] &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2t}} dx \\ &\stackrel{x=\sqrt{2ty}}{=} \frac{2^k t^k}{\sqrt{\pi}} \int_0^{\infty} y^{k-\frac{1}{2}} e^{-y} dy \\ &= \frac{2^k t^k}{\sqrt{\pi}} \int_0^{\infty} y^{k+\frac{1}{2}-1} e^{-y} dy \\ &= \frac{2^k t^k}{\sqrt{\pi}} \Gamma(k + \frac{1}{2}) \\ &= \frac{2^k t^k}{\sqrt{\pi}} \Gamma(\frac{1}{2}) \prod_{j=1}^k (j - \frac{1}{2}) \\ &= 2^k t^k \prod_{j=1}^k \left(\frac{2j-1}{2}\right) \\ &= (2k-1)!! \cdot t^k \end{aligned}$$

□

COROLLARY 2.27. Let $(B_t)_{t \geq 0}$ be Brownian motion. Then B_t is γ -Hölder continuous on a compact set of time almost surely for all $\gamma < \frac{1}{2}$.

Proof. Because of Lemma 2.26, we have

$$\begin{aligned} \mathbb{E}[(B_t - B_s)^{2k}] &= \mathbb{E}[B_{t-s}^{2k}] \\ &= (2k-1)!! \cdot |t-s|^k. \end{aligned}$$

In terms of the Theorem of Kolmogorov Chenstov, B_t is γ -Hölder continuous a.s. for $\gamma \in (0, \frac{k}{2k})$. □

3 Stable Measures and Stable Integrals

In order to represent a integration form of fractional Brownian motion, we deal with the stable integral in this section. In fact, fractional Brownian motion is a Gaussian process with zero mean. To show Gaussian properties of it, we define it by a stable integral which can imaged as stochastic process of stable variables on time.

3.1 Stable Variables

DEFINITION 3.1. Let X be a random variable. X is said to have a stable distribution, if there exist $0 < \gamma \leq 2, \delta \geq 0, -1 \leq \kappa \leq 1, \theta \in \mathbb{R}$ such that its characteristic function can be described as following

$$\mathbb{E}[\exp i\xi X] = \begin{cases} \exp\{i\xi\theta - |\delta\xi|^\gamma(1 - i\kappa \cdot \text{sgn}(\xi) \tan \frac{\gamma\pi}{2})\}, & \text{if } \gamma \in (0, 1) \cup (1, 2], \\ \exp\{i\xi\theta - |\delta\xi|(1 + i\frac{2}{\pi}\kappa \cdot \text{sgn}(\xi) \ln |\xi|)\}, & \text{if } \gamma = 1. \end{cases} \quad (3.9)$$

Where

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases}$$

Notice, we write $\Lambda(\gamma, \kappa, \theta, \delta)$ as one random variable whose characteristic function equals (3.9).

THEOREM 3.2. X is Gaussian if and only $X \sim \Lambda(\gamma, \kappa, \theta, \delta)$ with $\gamma = 2$.

Proof. In one hand, if X is Gaussian, indeed, γ must equal 2. On the other hand, if $\gamma = 2$, then $i\kappa \cdot \text{sgn}(\xi) \tan \frac{\gamma\pi}{2}$ vanishes since $\tan(\pi) = 0$. Therefore, X is Gaussian because $\mathbb{E}[\exp i\xi X] = \exp\{i\xi\theta - |\delta\xi|^2\}$. \square

Remark, if $\gamma = 2$, then κ is irrelevant in Definition. We specific $\kappa = 0$ without loss of generality. For instance, $B_t \sim \Lambda(2, 0, 0, \frac{\sqrt{t}}{2})$ when $(B_t)_t$ is Brownian motion.

DEFINITION 3.3. A random variable X is said to be *symmetric* if X and $-X$ have the same distribution.

PROPOSITION 3.4. Let X be have a stable distribution. X is *symmetric* if and only if $X \sim \Lambda(\gamma, 0, 0, \delta)$. I.e. its characteristic function has the form

$$\mathbb{E}[\exp\{i\xi X\}] = \exp\{-|\delta\xi|^\gamma\} \quad (3.10)$$

Proof. The Definition of symmetricity implies

$$\begin{aligned}
& \exp\{i\xi\theta - |\delta\xi|^\gamma(1 - i\kappa \cdot \operatorname{sgn}(\xi) \tan \frac{\gamma\pi}{2})\} \\
&= \mathbb{E}[i\xi X] \\
&= \mathbb{E}[i\xi(-X)] \\
&= \mathbb{E}[i(-\xi)X] \\
&= \exp\{i(-\xi)\theta - |\delta\xi|^\gamma(1 - i\kappa \cdot \operatorname{sgn}(-\xi) \tan \frac{\gamma\pi}{2})\},
\end{aligned}$$

this requires $\theta = \kappa = 0$. □

COROLLARY 3.5. Let $(B_t)_t$ be Brownian motion, then B_t has a symmetric stable distribution.

Proof. It is clear due to the previous Proposition. □

3.2 Stable Random Measures

In this subsection we suppose (D, \mathcal{D}) and (E, \mathcal{E}) are probability spaces, $\kappa(\cdot) : \Omega \rightarrow [-1, 1]$ is a measurable function.

DEFINITION 3.6. Let ν be a measure such that

$$\nu : \mathcal{D} \rightarrow \mathcal{E}.$$

ν is said to be *independently scattered*, if $\nu[D_1], \dots, \nu[D_n]$ are independent for any D_1, \dots, D_n disjoint $\in \mathcal{D}$.

For the next definition we need a notation

$$\mathcal{G} = \{D \in \mathcal{D} : \mu[D] < \infty, \mu : \mathcal{D} \rightarrow \mathcal{E}\}. \quad (3.11)$$

DEFINITION 3.7. Let ν be an independent cattered and σ -additive set function such that

$$\nu : \mathcal{G} \rightarrow L^\infty(\Omega, \mathcal{A}, \mathcal{P}).$$

ν is said to be *stable random measure* on (D, \mathcal{D}) with control measure μ , degree γ and skewness intensity $\kappa(\cdot)$ if

$$\nu[F] \sim \Lambda\left(\gamma, \frac{\int_F \kappa(x) \mu[dx]}{\mu[F]}, 0, (\mu[F])^{\frac{1}{\gamma}}\right) \quad (3.12)$$

for $F \in \mathcal{D}$.

3.3 Stable Integrals

Samorodnitsky and Taqqu show the existence of stable measures, see [6], pp.119~120.

EXAMPLE 3.8. Suppose $[0, T]$ is a index set and $0 = t_0, t_1, \dots, t_k \in [0, T]$ for $k \in \mathbb{N}$. We show the mapping $\nu : \mathcal{B}([0, T]) \rightarrow \mathcal{B}(\mathbb{R})$, where $\nu[A_j](\omega) := B_{t_{j+1}}(\omega) - B_{t_j}(\omega)$, $A_j = [t_j, t_{j+1})$.

Firstly, we show ν is independently scattered and σ -additive. We take $\{A_j\}$ such that $\cup_{j=1}^{\infty} A_j = [0, T]$. $\{\nu[A_k]\}_{k=1}^{\infty}$ has independent elements since $B_{t_1} - B_{t_0}, \dots, B_{t_{j+1}} - B_{t_j}$ are independent.

Secondly,

$$\begin{aligned} \nu[(\cup_{j=1}^{\infty} A_j)] &= B_T - B_1 \\ &= \sum_{j=1}^{\infty} (B_{t_{j+1}} - B_{t_j}) \\ &= \sum_{j=1}^{\infty} \nu[A_j]. \end{aligned}$$

Finally,

$$\begin{aligned} \mathbb{E}[\exp(i\xi \nu[A_j])] &= \mathbb{E}[\exp(i\xi (B_{t_{j+1}} - B_{t_j}))] \\ &= \exp(-\frac{(t_{j+1} - t_j)\xi^2}{2}) \end{aligned}$$

Comparing with (3.9), we deduce the control measure must be $\frac{|\cdot|}{2}$. In fact, $\nu[A_j] \sim \Lambda(2, 0, 0, \frac{|t_{j+1} - t_j|}{2})$.

3.3 Stable Integrals

Samorodnitsky and Taqqu defined an Integral with respect to stable measure as stochastic process in [6]. The stable Integral is given as

$$\int_F f(x) \nu(dx). \quad (3.13)$$

Where $f : F \rightarrow \mathbb{R}$ is a measurable function such that

$$\begin{cases} \int_F |f(x)|^\gamma \mu(dx) < \infty, & \text{if } \gamma \in (0, 1) \cup (1, 2], \\ \int_F |\kappa(x) f(x) \ln |f(x)|| \mu(dx) < \infty, & \text{if } \gamma = 1, \end{cases} \quad (3.14)$$

, γ, μ, κ are, respectively, degree, control measure and skewness intensity of the stable measure ν .

Some properties of the stable function are given by Samorodnitsky and Taqqu.

PROPOSITION 3.9. Let $J(f)$ be a stable integral as form of (3.13). Then

$$J(f) \sim \Lambda(\gamma, \kappa, \theta, \delta)$$

for the degree, control measure, skewness intensity, respectively,

$$\begin{aligned} \gamma &\in (0, 2], \\ \kappa &= \frac{\int_F \kappa(x) |f(x)|^\gamma \cdot \text{sgn}(f(x)) \mu(dx)}{\int_F |f(x)|^\gamma \mu(dx)}, \\ \theta &= \begin{cases} 0, & \text{if } \gamma \in (0, 1) \cup (1, 2], \\ -\frac{2}{\pi} \int_F \kappa(x) f(x) \ln |f(x)| \mu(dx), & \text{if } \gamma = 1, \end{cases} \\ \delta &= \left(\int_F |f(x)|^\gamma \mu(dx) \right)^{\frac{1}{\gamma}}, \end{aligned}$$

of the stable measure ν .

Proof. See [6], p.124, Proposition 3.4.1 . □

PROPOSITION 3.10. The stable integral is linear, in fact,

$$J(c_1 f_1 + c_2 f_2) \stackrel{a.s.}{=} c_1 J(f_1) + c_2 J(f_2) \tag{3.15}$$

for any f_1, f_2 integrable with respect to some stable measure and real numbers c_1, c_2 .

Proof. See [6], p.117, Property 3.2.3 . □

4 Fractional Brownian Motion

The fractional Brownian motion (FBM) was defined by Kolmogorov primitively. After that Mandelbrot and Van Ness has present the work in detail. This section is concerned with the definition and some properties of it.

4.1 Definition of Fractional Brownian Motion

Mandelbrot and Van Ness [8] gave a integration presentation of the definion of FBM.

DEFINITION 4.1. Let $(U_H(t))_{t \in \mathbb{R}}$ be a \mathbb{R} -valued stochatstic process and H be such that $0 < H < 1$. $(U_H(t))$ is said to be *fractional Brownian motion* if

$$U_H(t) - U_H(s) = \frac{1}{\Gamma(H + \frac{1}{2})} \left(\int_{\mathbb{R}} \mathbb{1}_{\{t \geq u\}} \cdot (t - u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u \leq s\}} (-u)^{H-\frac{1}{2}} dB_u \right) \quad (4.1)$$

for $t \geq s, t, s \in \mathbb{R}$. Where (B_u) is defined as two-sides Brownian motion, the integral is in sense of stable integral as in previous section. H is called Hurst exponent or Hurst index of FBM.

As usual, we set $U_H(0) = 0$, then equation (4.1) is equivalent to

$$U_H(t) = \frac{1}{(\Gamma(H + \frac{1}{2}))^2} \left(\int_{\mathbb{R}} \mathbb{1}_{\{t \geq u\}} \cdot (t - u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u \leq 0\}} (-u)^{H-\frac{1}{2}} dB_u \right). \quad (4.2)$$

LEMMA 4.2. The equation (4.2) is well-defined, $U_H(t)$ has stable distribution and

$$U_H(t) \sim \Lambda(2, 0, 0, \frac{1}{\Gamma(H + \frac{1}{2})} (\int_{\mathbb{R}} |f(u)|^2 \frac{du}{2})^{\frac{1}{2}}),$$

Where $f(x)$ is the integrand of integral in (4.2).

Proof. Firstly, B_t is Gaussian and symmetric stable measure with zero mean and $\frac{|t|}{2}$ as the control measure of it shown in Example 3.8.

Secondly, the well-definition was refered to by Samorodnitsky and Taqqu in [6], p.321, Proposition 7.2.6 for not only $H = \frac{1}{2}$, but also $H \neq \frac{1}{2}$. Which satisfies, in other words, the condition $\int_{-\infty}^{\infty} f^2(u) \frac{du}{2} < \infty$.

Finally, in terms of Proposition 3.9, we get the claim. \square

Remark that, if we take $H = \frac{1}{2}$ choosing a restriction of the integrand on \mathbb{R}_+ , $(U_2(t))_{t \geq 0}$ is a Brownian motion.

THEOREM 4.3. Let $(U_H(t))_t$ be a FBM. Then $U_H(t) \sim \mathcal{N}(0, \frac{1}{\Gamma(H + \frac{1}{2})^2} (\int_{\mathbb{R}} |f(u)|^2 du))$.

Proof. In terms of Lemma 4.2, $E[i\xi U_H(t)] = \exp\{-\xi^2 \frac{1}{2\Gamma(H+\frac{1}{2})^2} (\int_{\mathbb{R}} |f(u)|^2 du)\}$. The rest is clear thanks to the form of characteristic function of a Gaussian random variable. \square

LEMMA 4.4. Let $(U_H(t))_t$ be a FBM. Then $U_H(t)$ has an expected value 0 and variance $t^{2H} EU_H^2(1)$ for any $t \in \mathbb{R}$.

Proof. It is clear that U_H is Gaussian with zero mean due to Lemma 4.2. We suppose that $t \geq s \geq 0$, $c(H) = \frac{1}{(\Gamma(H+\frac{1}{2}))^2}$.

$$\begin{aligned}
 & E[(U_H(t) - U_H(s))^2] \\
 = & c(H) E\left[\left(\int_{\mathbb{R}} [\mathbb{1}_{\{t \geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{s \geq u\}} \cdot (s-u)^{H-\frac{1}{2}}] U_H(u) du\right)^2\right] \\
 \stackrel{\text{Theorem 4.3}}{=} & c(H) E\left[\int_{\mathbb{R}} \left(\mathbb{1}_{\{t \geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{s \geq u\}} \cdot (s-u)^{H-\frac{1}{2}}\right)^2 du\right] \\
 = & c(H) E\left[\int_{\mathbb{R}} \left(\mathbb{1}_{\{t-s \geq u\}} \cdot (t-s-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{0 \geq u\}} \cdot (-u)^{H-\frac{1}{2}}\right)^2 du\right] \\
 \stackrel{m=t-s}{=} & c(H) E\left[\int_{\mathbb{R}} \left(\mathbb{1}_{\{m \geq u\}} \cdot (m-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{0 \geq u\}} \cdot (-u)^{H-\frac{1}{2}}\right)^2 du\right] \\
 \stackrel{u=ml}{=} & c(H) E\left[\int_{\mathbb{R}} \left(\mathbb{1}_{\{m \geq ml\}} \cdot (m-ml)^{H-\frac{1}{2}} - \mathbb{1}_{\{0 \geq ml\}} \cdot (-ml)^{H-\frac{1}{2}}\right)^2 m \cdot dl\right] \\
 = & c(H) E\left[\int_{\mathbb{R}} \left(\mathbb{1}_{\{1 \geq l\}} \cdot (1-l)^{H-\frac{1}{2}} - \mathbb{1}_{\{0 \geq l\}} \cdot (-l)^{H-\frac{1}{2}}\right)^2 \cdot m^{2H-1} \cdot m \cdot dl\right] \\
 = & c(H) m^{2H} E[U_H(1)^2] \\
 = & c(H) (t-s)^{2H} E[U_H(1)^2]
 \end{aligned} \tag{4.3}$$

Using the same calculation, we get

$$E[(U_H(t))^2] = c(H) t^{2H} E[U_H(1)^2]. \tag{4.4}$$

(4.4) is variance of $U_H(t)$ due to $E[U_H(t)] = 0$. \square

To normalize the variance, a definition of standard FBM is given.

DEFINITION 4.5. A stochastic process $(U_H(t))_t$ is said to be a *standard fractional Brownian motion* (sFBM) if

$$U_H(t) = \hat{c}(H) \int_{\mathbb{R}} \mathbb{1}_{\{t \geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u \leq 0\}} (-u)^{H-\frac{1}{2}} dB_u. \tag{4.5}$$

Where $\hat{c}(H) = \frac{1}{E[U_H(1)^2]}$.

We consider from now on sFBM as FBM.

THEOREM 4.6. Let $(U_H(t))_t$ be a FBM. The Covariance function of $U_H(t), U_H(s)$ is $\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ for $t, s \in \mathbb{R}$.

Proof. Cf.[8], Theorem 5.3 .

$$\begin{aligned}
 \text{Cov}[U_H(t), U_H(s)] &= \mathbb{E}[U_H(t)U_H(s)] \\
 &= \frac{1}{2} (\mathbb{E}[U_H(t)^2] + \mathbb{E}[U_H(s)^2] - \mathbb{E}[(U_H(t) - U_H(s))^2]) \\
 &\stackrel{(4.4)}{=} \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})
 \end{aligned} \tag{4.6}$$

□

THEOREM 4.7. $(U_H(t))_t$ is Gaussian process.

Proof. We just need to prove that for an arbitrary finite linear combination of values of time is Gaussian. We take $t_1, \dots, t_k \in T, c_1, \dots, c_k \in \mathbb{R}$, and the stable integral $J(f)$ is a linear functional with $\gamma = 2, \kappa = 0, \theta = 0, \delta = (\frac{1}{2} \int_{-\infty}^{\infty} f^2(u) du)^{\frac{1}{2}}$ due to Corollary 3.15. Suppose f_1, \dots, f_k are integrands of the form of stable integral of $U_H(t_1), \dots, U_H(t_k)$.

Consider now, according to the Minkowski inequality,

$$\begin{aligned}
 \int_{-\infty}^{\infty} (\sum_{j=1}^k c_j f_j)^2 du &\leq \sum_{j=1}^k \underbrace{\int_{-\infty}^{\infty} (c_j f_j)^2 du}_{< \infty} \\
 &< \infty.
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 \sum_{j=1}^k c_j U_H(t_j) &= \sum_{j=1}^k c_j J(f_j) \\
 &= J(\sum_{j=1}^k c_j f_j) \\
 &\sim \Lambda(2, 0, 0, (\frac{1}{2} \int_{-\infty}^{\infty} (\sum_{j=1}^k c_j f_j)^2 du)^{\frac{1}{2}})
 \end{aligned}$$

is Gaussian and the rest is clear. □

COROLLARY 4.8. Let $(U_H(t))_t$ be a FBM, then $(U_H(t))_t$ has stationary and H-self similar increments .

Proof. Assume that $s \geq u$. Because the joint distribution of $(U_H(s), U_H(u))^T$ is Gaussian, $(1, -1) \cdot (U_H(s), U_H(u))^T$ is Gaussian. In other words, $U_H(s) - U_H(u) \sim \mathcal{N}(0, (s - u)^{2H})$ which is only dependent on $(s - u)$ and $(U_H(t))$ has therefore stationary increments.

$(U_H(t))$ has zero mean and $\text{Var}[U_H(s)] = s^{2H} \text{Var}[U_H(1)]$ we get $U_H(s) \sim s^H U_H(1)$ due to it is Gaussian. To show FBM has H-self similar increments, we have to prove

$(U_H(z t_1), U_H(z t_2), \dots, U_H(z t_n)) \sim (z^H U_H(t_1), z^H U_H(t_2), \dots, z^H U_H(t_n))$ for any $z > 0$.

Obviously, the former and the latter of the term are Gaussian and $\text{Var}[U_H(z t_i), U_H(z t_j)] = \text{Var}[z^H U_H(t_i), z^H U_H(t_j)] = \frac{1}{2} z^{2H} (t_i^{2H} + t_j^{2H} - |t_i - t_j|^{2H})$. Thus they have the same covariance matrix and zero mean in their characteristic function. Using uniqueness theorem we can prove the claim. \square

4.2 Regularity

THEOREM 4.9 (Kolmogorov Chentsov). FBM has almost surely continuous sample path.

Proof. Cf.[8] Proposition 4.1 . Let $(U_H(t))_t$ be FBM with Hurst index H . Fix α such that $1 < \alpha H$. Let look at the expectation of $(U_H(t) - U_H(s))^\alpha$ using the same calculation in (4.3)

$$\begin{aligned} \mathbb{E}[(U_H(t) - U_H(s))^\alpha] &= |t - s|^{\alpha H} \cdot \underbrace{\mathbb{E} \left(\int_{\mathbb{R}} \mathbb{1}_{\{1 \geq u\}} \cdot (1 - u)^{H - \frac{1}{2}} - \mathbb{1}_{\{u \leq 0\}} (-u)^{H - \frac{1}{2}} dB_u \right)^\alpha}_{c(\alpha, H)} \\ &= c(\alpha, H) \cdot |t - s|^{\alpha H}. \end{aligned} \quad (4.7)$$

We choose $\beta = \alpha H - 1$ and $\gamma \in (0, H - \frac{1}{\alpha})$ then the rest follows from Theorem 2.25 . \square

Remark, $U_H(t)$ is, in fact, γ -Hölder continuous with $\gamma < H$ almost surely.

THEOREM 4.10. The sample path of FBM is almost surely not differentiable.

Proof. Cf. [8] Proposition 4.2 . Fix $\omega \in \Omega$, we assume $c > 0, t_j \rightarrow s$.

$$\begin{aligned} &\mathcal{P}[\limsup_{t \rightarrow s} \left| \frac{U_H(t) - U_H(s)}{t - s} \right| > c] \\ &= \mathcal{P}[\lim_{j \rightarrow \infty} \sup_{t_j \neq s} \left| \frac{U_H(t_j) - U_H(s)}{t_j - s} \right| > c] \end{aligned} \quad (4.8)$$

Since continuity of measures from above, then

$$\begin{aligned} (4.8) &= \lim_{j \rightarrow \infty} \mathcal{P}[\sup_{t_j \neq s} \left| \frac{U_H(t_j) - U_H(s)}{t_j - s} \right| > c] \\ &\geq \lim_{j \rightarrow \infty} \mathcal{P}[\left| \frac{U_H(t_j) - U_H(s)}{t_j - s} \right| > c] \\ &= \lim_{j \rightarrow \infty} \mathcal{P}[\left| \frac{(t_j - s)^H U_H(1)}{t_j - s} \right| > c] \\ &= \lim_{j \rightarrow \infty} \mathcal{P}[|(t_j - s)^{H-1} U_H(1)| > c] \\ &= \lim_{j \rightarrow \infty} \mathcal{P}[|U_H(1)| > \underbrace{|t_j - s|^{1-H}}_{\xrightarrow{j \rightarrow \infty} 0} c] \\ &\xrightarrow{j \rightarrow \infty} 1 \end{aligned}$$

□

THEOREM 4.11. Let $(U_H(k))_k$ be FBM. The conditional expectation of $U_H(s)$ given $U_H(t) = x$ is

$$\frac{|\frac{s}{t}|^{2H} + 1 - |\frac{s}{t} - 1|^{2H}}{2} \cdot x$$

for all $s < t$.

Proof. Cf. [8] Theorem 5.3. Taking conditional expectation of $U_H(s)$ given $U_H(t)$,

$$\begin{aligned} & \mathbb{E}[U_H(s)|U_H(t)] \\ \stackrel{\text{Corollary 2.18}}{=} & \mu_s + \rho_{s,t} \left(\frac{\sigma_s}{\sigma_t} U_H(t) - \mu_t \right) \\ = & \rho \frac{\sigma_s}{\sigma_t} U_H(t) \\ = & \frac{\rho \cdot \sigma_s \sigma_t \cdot U_H(t)}{\sigma_t^2} \\ = & \frac{\mathbb{E}[U_H(s)U_H(t)]}{\mathbb{E}[U_H^2(t)]} \cdot U_H(t) \\ \stackrel{(4.6)}{=} & \frac{s^{2H} + t^{2H} - |s - t|^{2H}}{2\mathbb{E}[U_H^2(t)]} \cdot U_H(t) \\ = & \frac{s^{2H} + t^{2H} - |s - t|^{2H}}{2t^{2H}} \cdot U_H(t) \\ = & \frac{|\frac{s}{t}|^{2H} + 1 - |\frac{s}{t} - 1|^{2H}}{2} \cdot U_H(t) \end{aligned}$$

□

4.3 Fractional Brownian Noise

DEFINITION 4.12. Let $(U_H(t))_t$ be a FBM. The *fractional Brownian noise* is a sequence $(S_k)_k$ forms

$$S_H(k) = U_H(k+1) - U_H(k)$$

for $k \in \mathbb{R}$.

PROPOSITION 4.13. Fractional Brownian noise is stationary and its autocovariance is

$$\varsigma_{S_H}(\tau) = \frac{1}{2}(|\tau+1|^{2H} - 2|\tau|^{2H} + |\tau-1|^{2H}) \quad (4.9)$$

for $\tau \in \mathbb{R}$.

Proof. Cf. [9], p.333, Proposition 7.2.9 . The first part of the claim is clear due to FBM has stationary increments.

In terms of definition of fractional Brownian noise we have

$$\begin{aligned}
\varsigma_{S_H}(\tau) &= \mathbb{E}[S_H(0)S_H(\tau)] \\
&= \mathbb{E}[(U_H(1) - U_H(0))(U_H(\tau + 1) - U_H(\tau))] \\
&= \mathbb{E}[(U_H(1) - U_H(0))(U_H(\tau + 1) - U_H(\tau))] \\
&= \mathbb{E}[U_H(1)(U_H(\tau + 1) - U_H(\tau))] \\
&= \mathbb{E}[U_H(1)U_H(\tau + 1)] - \mathbb{E}[U_H(1)U_H(\tau)] \\
&= \varsigma_{U_H}(1, \tau + 1) - \varsigma_{U_H}(1, \tau) \\
&\stackrel{(4.6)}{=} \frac{1}{2}(1 + |1 + \tau|^{2H} - \tau^{2H}) - \frac{1}{2}(1 + \tau^{2H} - |1 - \tau|^{2H}) \\
&= \frac{1}{2}(|1 + \tau|^{2H} - 2|\tau|^{2H} + |1 - \tau|^{2H})
\end{aligned}$$

for $\tau \in \mathbb{R}$. □

DEFINITION 4.14. A stationary stochastic process $(X_t)_t$ is said to have *long memory* if its autocovariance $\varsigma_X(\tau)$ tend to 0 with so slowly such that $\sum_{\tau=-\infty}^{\infty} \varsigma_X(\tau)$ diverges for $\tau \in \mathbb{Z}$.

THEOREM 4.15. The fractional Brownian noise has long memory.

Proof. Cf. [9], p.335, Proposition 7.2.10 .

Without loss of generality, we suppose $\tau \in \mathbb{Z}_0^+$. For $\tau \in \mathbb{Z}^-$, we deal with it in a similar way and $\tau = 0$ is clear. Following (4.9), we have

$$\begin{aligned}
&\varsigma(\tau) \\
&= \frac{1}{2}\tau^{2H-2}\left\{\tau^2\left[\left(1 + \frac{1}{\tau}\right)^{2H} - 2 + \left(1 - \frac{1}{\tau}\right)^{2H}\right]\right\} \\
&= \frac{1}{2}\tau^{2H-2}\left\{\frac{\left(1 + \frac{1}{\tau}\right)^{2H} - 1}{\frac{1}{\tau^2}} - \frac{1 - \left(1 - \frac{1}{\tau}\right)^{2H}}{\frac{1}{\tau^2}}\right\}
\end{aligned}$$

We deal with the former of the content in $\{ \}$ with L'Hôpital's rule as τ tend to infinity.

$$\begin{aligned}
&\lim_{\tau \rightarrow \infty} \frac{\left(1 + \frac{1}{\tau}\right)^{2H} - 1}{\frac{1}{\tau^2}} \\
&= \lim_{\tau \rightarrow \infty} \frac{2H\left(1 + \frac{1}{\tau}\right)^{2H-1}\left(-\frac{1}{\tau^2}\right)}{-\frac{4}{\tau^3}} \\
&= \lim_{\tau \rightarrow \infty} \frac{H\left(1 + \frac{1}{\tau}\right)^{2H-1}}{\frac{1}{\tau}}.
\end{aligned}$$

4.4 FBM is not Semimartingale for $H \neq \frac{1}{2}$

We calculate the Latter of the content in $\{ \}$ in a similar way. Then

$$\begin{aligned}
& \lim_{\tau \rightarrow \infty} \varsigma(\tau) \\
&= \lim_{\tau \rightarrow \infty} \tau^{2H-2} \frac{1}{2} \left\{ \frac{H(1 + \frac{1}{\tau})^{2H-1}}{\frac{1}{\tau}} - \frac{H(1 - \frac{1}{\tau})^{2H-1}}{\frac{1}{\tau}} \right\} \\
&\stackrel{\text{L'Hôpital}}{=} \lim_{\tau \rightarrow \infty} \frac{1}{2} \tau^{2H-2} \left\{ \frac{H(2H-1)(1 + \frac{1}{\tau})^{2H-2}(-\frac{1}{\tau^2})}{-\frac{1}{\tau^2}} - \frac{H(2H-1)(1 - \frac{1}{\tau})^{2H-2}(-\frac{1}{\tau^2})}{-\frac{1}{\tau^2}} \right\} \\
&= \lim_{\tau \rightarrow \infty} \frac{1}{2} \tau^{2H-2} \left\{ H(2H-1)(1 + \frac{1}{\tau})^{2H-2} + H(2H-1)(1 - \frac{1}{\tau})^{2H-2} \right\} \\
&= \lim_{\tau \rightarrow \infty} \frac{1}{2} \tau^{2H-2} 2H(2H-1) \\
&= \lim_{\tau \rightarrow \infty} H(2H-1) \tau^{2H-2}
\end{aligned}$$

Convergence of $\sum_{\tau=-n}^n \varsigma(\tau)$ as $n \rightarrow \infty$ requires $2H-2 < -1$, namely, $H < \frac{1}{2}$. Otherwise, if $\frac{1}{2} < H < 1$, $\sum_{\tau=-\infty}^{\infty} \varsigma(\tau)$ diverges. With $\lim_{\tau \rightarrow \infty} \varsigma(\tau) = 0$, we achieve the claim. \square

COROLLARY 4.16. Let S_H be fractional Brownian noise, $\tau \in \mathbb{Z}$ and $\varsigma_{S_H}(\cdot)$ its autocovariance. Then $\sum_{\tau \in \mathbb{Z}} \varsigma_{S_H}^2(\tau) < \infty$ if and only if $H < \frac{3}{4}$.

Proof. Cf. [9], p.72, Lemma 6.3. In Theorem 4.15, we have $\varsigma_{S_H}^2(\tau) \propto H^2(2H-1)^2 \tau^{4H-4}$. The sum of it is finite if and only if, according to the same reason as in Theorem 4.15, $\tau^{4H-4} < -1$. That means $H < \frac{3}{4}$. \square

4.4 FBM is not Semimartingale for $H \neq \frac{1}{2}$

Let look at our Integration representation for FBM, in the case of FBM with Hurst index $\frac{1}{2}$, it must be an ordinary Brownian motion. Except for this, we will show FBM is not a seimimartingal.

DEFINITION 4.17. The *Hermite polynomials* form as following

$$\eta_n(u) = (-1)^n e^{\frac{u^2}{2}} \left(\frac{\partial^n}{\partial u^n} e^{-\frac{u^2}{2}} \right), \quad (4.10)$$

for $u \in \mathbb{R}, n \in \mathbb{N}_0$.

PROPOSITION 4.18. Let $(\eta_n)_{n \in \mathbb{N}}$ be a family of Hermite polynomials, W a standard Gaussian variable, $f, g : \mathbb{R} \rightarrow \mathbb{R}$ continuously differentiable. It has then following properties

- (i) $\eta_{n+1}(u) = (n+1)\eta_n(u)$ and $\eta_{n+2}(u) = u \cdot \eta_{n+1}(u) - (n+1)\eta_n(u)$ for all $n \in \mathbb{N}_0, u \in \mathbb{R}$.

(ii) Let W, V be standard Gaussian distributed such that (W, V) have a disjoint Gaussian distribution. Then

$$\int_{\Omega} \eta_j(W) \cdot \eta_k(V) \mathcal{P} = \begin{cases} j! (\mathbb{E}[WV])^j & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Let W be standard Gaussian distributed, then

$$\frac{1}{j!} \int_{\Omega} \eta_j(W) \eta_k(W) \mathcal{P} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Remark that, (iii) means, in fact, $\{\frac{1}{\sqrt{j!}} \cdot \eta_j\}_{j=0}^{\infty}$ is an orthonormal basis in $\mathcal{L}^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), e^{-\frac{x^2}{2}} dx)$.

Proof. See [9] p.3, Propostion 1.3. □

LEMMA 4.19. Let $(U_H(t))_t$ be a FBM, W standrad Gaussian variable, and $f : \mathbb{R} \rightarrow \mathbb{R}$, Borel-measurable function such that $\mathbb{E}[f^2(G)] < \infty$. Then,

$$\frac{1}{n} \sum_{j=1}^n f(U_H(j) - U_H(j-1)) \xrightarrow{\mathcal{L}^2(\mathcal{P})} \mathbb{E}[f(W)],$$

as n tend to ∞ . In particular,

$$\sum_{j=1}^n |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\beta \xrightarrow{\mathcal{L}^2(\mathcal{P})} \begin{cases} 0 & \text{if } \beta > \frac{1}{H} \\ \infty & \text{if } \beta < \frac{1}{H} \end{cases} \quad (4.11)$$

as n tend to ∞ .

Proof. C.f. [9], p.17, Theorem 2.1 .

Firstly, because $\mathbb{E}[f^2(W)] < \infty$ one has $f \in \mathcal{L}^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), e^{-\frac{x^2}{2}} dx)$. In terms of Proposition 4.18(iii), taking expectation

$$\mathbb{E}[f(x)] = \mathbb{E}\left[\sum_{j=0}^{\infty} \frac{a_j H_j(x)}{\sqrt{j!}}\right],$$

for $x \in \mathbb{R}$. Notice $H_0 = 1$ due to (4.10). Setting $x = W$, Equating coefficients leads to

$a_0 = E[f(W)]$. Moreover,

$$\begin{aligned}
 & E\left[\left\{\frac{1}{n} \sum_{j=1}^n f(U_H(j) - U_H(j-1)) - E[f(W)]\right\}^2\right] \\
 = & E\left[\left\{\frac{1}{n} \sum_{j=1}^n (f(U_H(j) - U_H(j-1)) - E[f(W)])\right\}^2\right] \\
 = & E\left[\left\{\frac{1}{n} \sum_{j=1}^n \left(\sum_{k=0}^{\infty} \frac{a_k}{\sqrt{k!}} H_k(U_H(j) - U_H(j-1))\right) - E[f(W)]\right\}^2\right] \\
 = & E\left[\left\{\frac{1}{n} \sum_{j=1}^n \left(\sum_{k=1}^{\infty} \frac{a_k}{\sqrt{k!}} H_k(U_H(j) - U_H(j-1))\right)\right\}^2\right]
 \end{aligned}$$

Consider now

$$E[f^2(W)] < \infty,$$

which requires $\sum_{k=1}^{\infty} (a_k)^2 < \infty$. Then

$$\begin{aligned}
 & E\left[\left\{\frac{1}{n} \sum_{j=1}^n f(U_H(j) - U_H(j-1)) - E[f(W)]\right\}^2\right] \\
 = & \frac{1}{n^2} E\left[\sum_{k=1}^{\infty} \frac{a_k^2}{k!} \left(\sum_{j=1}^n H_k(U_H(j) - U_H(j-1))\right)^2\right] \\
 = & \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{a_k^2}{k!} \sum_{j=1, m=1}^n E[H_k(U_H(j) - U_H(j-1)) H_k(U_H(m) - U_H(m-1))] \\
 \stackrel{\text{Proposition 4.18(ii)}}{=} & \frac{1}{n^2} \sum_{k=1}^{\infty} a_k^2 \sum_{j=1, m=1}^n (E[(U_H(j) - U_H(j-1))(U_H(m) - U_H(m-1))])^k \\
 = & \frac{1}{n^2} \sum_{k=1}^{\infty} a_k^2 \sum_{j=1, m=1}^n (E[S_H(j-1) S_H(m-1)])^k \\
 = & \frac{1}{n^2} \sum_{k=1}^{\infty} a_k^2 \sum_{j=1, m=1}^n (\varsigma_{S_H}(j-m))^k
 \end{aligned}$$

And

$$\begin{aligned}
 |\varsigma_{S_H}(k)| &= |\varsigma_{S_H}(|k|)| \\
 &= E[(U_H(1) - U_H(0))(U_H(|x|+1) - U_H(|x|))] \\
 &\stackrel{\text{Cauchy Schwartz}}{\leq} \underbrace{\sqrt{E[U_H(1)^2]}}_{=1} \cdot \underbrace{\sqrt{E[U_H(|x|+1) - U_H(|x|)]^2}}_{=1} \\
 &= 1
 \end{aligned}$$

Consequently, $(\varsigma_{S_H}(j-m))^k \leq |\varsigma_{S_H}(j-m)|$. In fact,

$$\begin{aligned}
 & \mathbb{E}[\{\frac{1}{n} \sum_{j=1}^n f(U_H(j) - U_H(j-1)) - \mathbb{E}[f(W)]\}^2] \\
 & \leq \frac{1}{n^2} \sum_{\substack{k=1 \\ =:\alpha < \infty}}^{\infty} a_k^2 \sum_{j=1, m=1}^n |\varsigma_{S_H}(j-m)| \\
 & = \frac{\alpha}{n^2} \sum_{j=1, m=1}^n |\varsigma_{S_H}(j-m)| \\
 & = \frac{\alpha}{n^2} 2 \cdot \sum_{j=1}^n \sum_{m < j} |\varsigma_{S_H}(j-m)| \\
 & \leq \frac{\alpha}{n^2} 2n \sum_{k=1}^{n-1} |\varsigma_{S_H}(k)| \\
 & = \frac{2\alpha}{n} \sum_{k=0}^{n-1} |\varsigma_{S_H}(k)|,
 \end{aligned}$$

As in the proof in Theorem 4.15, $\sum_{k=0}^{n-1} |\varsigma_{S_H}(k)| \propto H(2H-1) \sum_{j=1}^{n-1} j^{2H-2} \propto H(2H-1)n \cdot$

$n^{2H-2} \propto n^{2H-1}$ and $\frac{2\alpha}{n} \sum_{k=0}^{n-1} |\varsigma_{S_H}(k)| \propto n^{2H-2}$ as k goes to infinity. This leads $\mathbb{E}[\{\frac{1}{n} \sum_{j=1}^n f(U_H(j) - U_H(j-1)) - \mathbb{E}[f(W)]\}^2] = 0$ due to $n^{2H-2} \rightarrow 0$ for $0 < H < 1$ as $n \rightarrow \infty$.

Secondly, we apply previous result for (4.11), In fact,

$$\begin{aligned}
 & \sum_{j=1}^n |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\beta \\
 & = \frac{1}{n^{\beta H}} \sum_j^n |U_H(j) - U_H(j-1)|^\beta \\
 & = \frac{1}{n^{\beta H-1}} \frac{1}{n} \sum_j^n |U_H(j) - U_H(j-1)|^\beta \\
 & \longrightarrow n^{1-\beta H} \mathbb{E}[|W|^\beta]
 \end{aligned}$$

Due to $\mathbb{E}[|W|^\beta] < \infty$, (4.11) holds as well as $n \rightarrow \infty$. □

THEOREM 4.20. FBM is not a semimartingale for $H \neq \frac{1}{2}$.

Proof. Without loss of generality, we set the Time index $T = [0, 1]$. Choosing $\beta = 2$ in (4.11), we suppose $U_H(t)$ were a semimartingale.

Case $H < \frac{1}{2}$. Then $\sum_{j=1}^n |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^2 = \infty$ contradicts that semimartingale has finite quadratic variation.

Case $H > \frac{1}{2}$. $\sum_{j=1}^n |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^2 = 0$. On the one hand, according to Doob-Meyer

4.4 FBM is not Semimartingale for $H \neq \frac{1}{2}$

decomposition, $U_H(t) = M(t) + A(t)$, where $M(t)$ is a local martingale and $A(t)$ is local finite variation process. Due to $A(t)$ is local finite, we have $0 = \langle U_H, U_H \rangle = \langle M, M \rangle$, where $\langle \cdot, \cdot \rangle$ is denoted for quadratic variation. Consequently, $M(t)$ is zero process due to Cauchy Schwarz inequality. In other words, $U_H(t) = A(t)$ which has finite variation. On the other Hand, choosing $1 < \gamma < \frac{1}{H}$, then $\sum_{j=1}^n |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\gamma \rightarrow \infty$. Precisely,

$$\begin{aligned} \infty &\leftarrow \sum_{j=1}^n |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\gamma \\ &\leq \underbrace{\sup_{1 \leq j \leq n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^{\gamma-1}}_{(\gamma-1)\text{-H\"older}_0} \cdot \sum_{j=1}^n |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|, \end{aligned}$$

this leads to $\sum_{j=1}^n |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})| \rightarrow \infty$ contradicts mentioned above that $U_H(t)$ has finite variation.

Given all that, FBM is not a semimartingale for $H \neq \frac{1}{2}$. □

5 Fractional Ornstein Uhlenbeck Process Model

6 Application in Financial Mathematics

6.1 Fractional Black-Scholes Model

blah...blah A finance market is modeled with two stochastic processes that a process of a riskless asset $(A_t)_t$ and a process of price of a stock $(S_t)_t$. The stock is assumed that it pays no dividends. We give our fractional Black-Scholes model as follows

$$\begin{aligned}dA_t &= rA_t dt \\dS_t &= \mu_t S_t dt + \sigma U_H(t)\end{aligned}\tag{6.1}$$

for $t \in [0, T]$.

Setting initial values $A_0 = 1, S_0 = 1$,

$$\begin{aligned}A_t &= \exp(rt) \\S_t &= \exp(\mu_t t + \sigma U_H(t)), t \in [0, T]\end{aligned}\tag{6.2}$$

is the solution of (6.1).

First we give some basic concepts in following. Through this section we denote \mathcal{F}_t^X the filtration of a stochastic process (X_t) . Our finance market consist of $\hat{X} := (X_t^0, \dots, X_t^n)$.

DEFINITION 6.1. A \mathbb{R}^{n+1} -valued stochastic process $(\hat{\xi}_t)_t := (\xi_t^0, \xi_t^1, \dots, \xi_t^n)_{t \in [0, T]}$ is said to be a *strategy*, if $\hat{\xi}_t \in \mathcal{F}_j^X$ for $0 \leq j \leq t$.

DEFINITION 6.2. A stochastic process $(V_t)_{t \in [0, T]}$ is said to be *value process* respect to strategy $\hat{\xi}_{t \in [0, T]}$, if

$$V_t = \sum_{k=0}^n \xi_t^k X_t^k =: \hat{\xi}_t \cdot \hat{X}_t.$$

for $t \in [0, T]$.

DEFINITION 6.3. A strategy $(\hat{\xi}_t)_{t \in [0, T]}$ is said to be *self-financing*, if

$$\hat{\xi}_t \cdot \hat{X}_t = \hat{\xi}_{t+\Delta} \cdot \hat{X}_t$$

for $t \in [0, T]$.

DEFINITION 6.4. A stochastic process $(\tilde{V}_{t \in [0, T]})$ is said to be *discounted value process* of a value process $(V_t)_{t \in [0, T]}$ respect to $\hat{\xi}_{t \in [0, T]}$, if

$$\tilde{V}_t = \frac{V_t}{\xi_t^0}$$

for $t \in [0, T]$.

DEFINITION 6.5. A self-financing strategy $\hat{\xi}$ is said to be *arbitrage*, if its discounted process satisfies following conditions

- (i) $\mathcal{P}[\tilde{V}_T - \tilde{V}_0] = 1$.
- (ii) $\mathcal{P}[\tilde{V}_T > 0] > 0$.

LEMMA 6.6. Let $(X_t)_{t \geq 0}$ be a stochastic process continuous in t . If (X_t) is a modification of the process

$$\left(\int_0^t (t-u)^{H-\frac{1}{2}} dB_u \right)_{t \geq 0}$$

for $(B_t)_{t \geq 0}$ a Brownian motion and $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, then

$$\mathcal{P}\left[\sup_{t \in [a, b]} X_t \leq -c\right] > 0$$

for $c \geq 0$ and $0 < a \leq b$.

Proof. See [3], p.15, Lemma 4.2 . □

THEOREM 6.7. Let $(S_t)_{t \in [0, T]}$ be a stochastic process such that

$$S_t = \exp(a(t) + \sigma U_H(t)), \quad (6.3)$$

where $U_H(t)$ is FBM and $a : [0, T] \rightarrow \mathbb{R}$ a measurable and bounded on its domain. If there exist

$$\xi^1 = f_0 \mathbb{1}_{\{0\}} + \sum_{k=1}^{n-1} f_k \mathbb{1}_{(\tau_k, \tau_{k+1}]}$$

where $\{f_k\}_0^n$ is family of real $\mathcal{F}_k^{U_H}$ -measurable function with $\mathcal{F}_k^{U_H}$ Filtration generated of $\{U_H(k)\}$, $0 < \tau_1 < \dots < \tau_n \leq T$ are stopping times respect to $\mathcal{F}_t^{U_H}$ respectively, with $\tau_{k+1} - \tau_k \geq m$ and $\mathcal{P}[f_k \neq 0] > 0$, then

$$\mathcal{P}[(\xi^1 \cdot S)_T < 0] > 0.$$

Proof. Cf.[3], p.18, Theorem 4.3 . We have

$$(\xi^1 \cdot S)_T = \sum_{k=1}^{n-1} f_k (e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}).$$

Assume $\mathcal{P}[(\xi^1 \cdot S)_T < 0] = 0$, then there exist

$$l = \min\{j : \mathcal{P}[f_j \neq 0] > 0, \quad \mathcal{P}\left[\sum_{k=1}^j f_k (e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) \geq 0\right] = 1\}$$

This leads to

$$\mathcal{P} \left[\left(\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) \right) \leq 0 \right] = 1$$

Ignoring constant term, we define

$$U_H(t)(\omega) = \int_{\mathbb{R}} \mathbb{1}_{\{t \geq u\}} \cdot (t - u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u \leq 0\}}(-u)^{H-\frac{1}{2}} d\omega(u).$$

where $\omega(u) := B_u(\omega)$ for all $\omega \in \Omega^w$. We give the filtration $(\mathcal{F}_t^{\Omega^w})$ denoting by

$$\mathcal{F}_t^{\Omega^w} := \sigma(\{\{w \in \Omega^w : \omega(u) \in \mathbb{R}\} : -\infty < u \leq t, c \in \mathbb{R}\}).$$

Then τ_k is also stopping time of $\mathcal{F}_t^{\Omega^w}$ due to

$$\mathcal{F}_t^{U_H} \subset \mathcal{F}_t^{\Omega^w}, t \in \mathbb{R}$$

For $\omega \in \Omega^w$, we split it into two parts as follows

$$\begin{aligned} \psi_\omega(u) &:= \omega(u) \mathbb{1}_{(-\infty, \tau_l(\omega)]}, u \in \mathbb{R} \\ \phi_\omega(u) &:= \omega(\tau_l(\omega) + m) - \omega(\tau_l(\omega)), u \geq 0. \end{aligned}$$

Corresponding to each, we define

$$\begin{aligned} \Omega^1 &:= \{\psi_\omega \in \mathcal{C}(\mathbb{R}) : \omega \in \Omega^w\} \\ \Omega^2 &:= \{\phi_\omega \in \mathcal{C}([0, \infty)) : \omega \in \Omega^w\} \end{aligned}$$

And for the smallest σ -algebra of all subsets, respectively, of Ω^1, Ω^2 denoted by $\mathcal{B}^1, \mathcal{B}^2$.

Notice that

$$\begin{aligned} \{\tau_l \leq t\} \cap \{\psi_\omega \in \Omega^w\} &= \{\{\omega \in \Omega^w : \omega(u) \in \mathbb{R}\} : -\infty < u \leq t\} \\ &\in \mathcal{F}_t^{\Omega^w}, \end{aligned}$$

therefore is ψ_ω a $\mathcal{F}_{\tau_l}^{\Omega^w}$ -measurable mapping. Moreover, since the strong Markovian property of Brownian motion, ϕ_ω is independent of $\mathcal{F}_{\tau_l}^{\Omega^w}$ and it must be a Brownian motion.

We replace the ω by ψ_ω, ϕ_ω calculating the value process

$$\begin{aligned}
 & \left(\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) + f_l(e^{U_H(\tau_l+m)} - e^{U_H(\tau_l)}) \right) (\omega) \\
 = & \underbrace{\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)})}_{:=J^1}(\omega) + f_l \left(e^{U_H(\tau_l)} \left((e^{U_H(\tau_l+m)} - e^{U_H(\tau_l)}) - 1 \right) \right) (\omega) \\
 = & J^1(\omega) + f_l \left(\exp \left\{ \int_{\mathbb{R}} \mathbb{1}_{\{\tau_l \geq u\}} (\tau_l(\omega) + m - u)^{H-\frac{1}{2}} - \mathbb{1}_{\{\tau_l \geq u\}} (\tau_l(\omega) - u)^{H-\frac{1}{2}} d\omega(u) \right. \right. \\
 & \left. \left. + \int_{\mathbb{R}} \mathbb{1}_{(\tau_l, \tau_l+m]} (\tau_l + m - u)^{H-\frac{1}{2}} d\omega(u) \right\} \right) \\
 = & J^1(\psi_\omega) + f_l \underbrace{\left(\exp \left\{ \int_{-\infty}^{\tau_l(\psi_\omega)} (\tau_l(\psi_\omega) + h + u)^{H-\frac{1}{2}} - (\tau_l(\psi_\omega) - u)^{H-\frac{1}{2}} d\psi_\omega(u) \right\} \right)}_{:=J^2(\psi_\omega, m)} \\
 & \cdot \underbrace{\left(\exp \left\{ \int_0^m (m - u)^{H-\frac{1}{2}} d\phi_\omega(u) \right\} - 1 \right)}_{:=J^3(\phi_\omega, m)} \\
 = & J(\psi_\omega, \phi_\omega, m)
 \end{aligned}$$

where J is defined as

$$J(\psi, \phi, t) := J^1(\psi) + J^2(\psi, t) \quad \cdot \quad J^3(\phi, t)$$

for $\psi \in \Omega^1, \phi \in \Omega^2$.

Indeed, fixing $\psi \in \Omega^1, \phi \in \Omega^2$, $J(\psi, \phi, \cdot)$ has continuous path on $(\Omega^1 \times \Omega^2, \mathcal{B}^1 \otimes \mathcal{B}^2)$, then we can define a $\mathcal{B}^1 \otimes \mathcal{B}^2$ -measurable set

$$E := \{(\psi, \phi) : \sup_{m \leq t \leq T} J(\psi, \phi, t) < 0\}$$

$$\begin{aligned}
 \mathbb{E}[\mathbb{1}_E(\omega_1, \phi_\omega) | \psi_\omega = \omega_1] &= \mathcal{P}^\omega \left[\sup_{m \leq t \leq T} J(\omega_1, \phi_\omega, t) < 0 \right] \\
 &\geq \mathcal{P}^\omega [J^1(\omega_1) + \sup_{m \leq t \leq T} J^2(\omega_1, t) \cdot \sup_{m \leq t \leq T} J^3(\phi_\omega, t) < 0]
 \end{aligned}$$

It is clear $J^2(\omega_1)$ is bounded for a fixed ω_1 on the compact set and $J^1(\omega_1)$ also. We set $J^1(\omega_1) = c_1$, $\sup_{m \leq t \leq T} J^2(\omega_1, t) = c_2$. Thanks to Lemma 6.6, there exist $D \subset \Omega^w$ such that $\mathcal{P}^w[C] > 0$ and

$$\int_0^m (m - u)^{H-\frac{1}{2}} d\phi_\omega(u) < c_3$$

6.1 Fractional Black-Scholes Model

for $\omega \in D$. where c_3 could be small enough that $c_2 \cdot e^{c_3-1} < 0$. Under assumption at beginning, $c_1 \leq 0$ almost surely. All of this leads to

$$\mathcal{P}^w[\mathbb{E}[\mathbb{1}_E(\omega_1, \phi_\omega)|\psi_\omega = \omega_1]] > 0$$

for $\omega \in \Omega^w$. Then

$$\begin{aligned} & \mathcal{P}^w\left[\sum_{k=1}^l f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) < 0\right] \\ \geq & \mathcal{P}^w\left[\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) + \sup_{m \leq t \leq T} f_l(e^{U_H(\tau_l+t)} - e^{U_H(\tau_l)}) < 0\right] \\ = & \mathbb{E}[\mathbb{E}[\mathbb{1}_E(\omega_1, \phi_\omega)|\psi_\omega = \omega_1]] > 0 \end{aligned}$$

this contradicts our assumption. It must be that $\mathcal{P}[(\xi^1 \cdot S)_T < 0] > 0$.

□

7 Conclusion

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