Fractional Brownian Motion and its Application in financial mathematics

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Centered Gaussian process $(U_H(t))$

$\mathrm{Cov}[U_H(t), U_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$ with Hurst exponent $H \in (0, 1)$.

Integral Representation

Mandelbrot and Van Ness

Definition

Let $(U_H(t))_{t\in\mathbb{R}}$ be a $\mathbb{R}$-valued stochatstic process and H be a real number such that $0 < H < 1$. $(U_H(t))$ is said to be *fractional Brownian motion* if

$$\begin{aligned} & U_H(t) - U_H(s) \\ = \; & \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{\mathbb{R}} \mathbb{1}_{\{t>u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{s>u\}}(-u)^{H-\frac{1}{2}} \; dB_u\right) \quad (1) \end{aligned}$$

for $t \geq s, t, s \in \mathbb{R}$, where (B_u) is defined as two-sides Brownian motion and the integral is defined in the sense of stable integral.

Theorem

$U_H(0) = 0$, *then*

$$U_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{\mathbb{R}} \underbrace{\mathbb{1}_{\{t>u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u<0\}}(-u)^{H-\frac{1}{2}}}_{f_t(u)}\, dB_u\right) \quad (2)$$

$$J(f_t) : f_t \to \int_{\mathbb{R}} f_t \, dB_u$$

If the integrand f is quadratic integrable then (1) is well-defined in the sense of stable integral.

Theorem

$U_H(t) \sim \mathcal{N}(0, \frac{1}{\Gamma(H+\frac{1}{2})^2}(\int_{\mathbb{R}} |f_t(u)^2| \, du))$

Proposition

The stable integral is linear:

$$J(f_t + f_s) \overset{a.s.}{=} J(f_t) + J(f_s)$$

Theorem

$U_H(t) - U_H(s) \sim \mathcal{N}(0, \frac{1}{\Gamma(H+\frac{1}{2})^2}(\int_{\mathbb{R}} |f_t(u) - f_s(u)|^2 \, du)$

Theorem

The variance of $U_H(t)$ *is* $\frac{1}{(\Gamma(H+\frac{1}{2}))^2} t^{2H} \mathrm{E} U_H^2(1)$ *for any* $t \in \mathbb{R}$.

Divide with $\frac{1}{(\Gamma(H+\frac{1}{2}))^2} \mathrm{E} U_H^2(1)$ then,

$$\mathrm{Var}[U_H(t)] = t^{2H}.$$

What is a fBm ?

Theorem

Let $(U_H(t))_t$ *be a fBm. The covariance of* $U_H(t)$ *and* $U_H(s)$ *is* $\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ *for* $t, s \in \mathbb{R}$.

Proof.

$$\begin{aligned} \mathrm{Cov}[U_H(t), U_H(s)] &= \mathrm{E}[U_H(t)U_H(s)] \\ &= \frac{1}{2}(\mathrm{E}[U_H(t)^2] + \mathrm{E}[U_H(s)^2] \\ &- \mathrm{E}[(U_H(t) - U_H(s))^2]) \\ &= \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \end{aligned} \tag{3}$$

$\square$

Theorem

$(U_H(t))_t$ is Gaussian process.

Theorem

Let $(U_H(t))_t$ be a fBm, then $(U_H(t))_t$ has stationary and H-self similar increments .

Theorem

$$(U_H(t_1), \dots, U_H(t_n)) \sim (U_H(t_1 + \tau), \dots, U_H(t_n + \tau))$$

Theorem

$$(U_H(ct_1), \dots, U_H(ct_k)) \sim (c^H U_H(t_1), \dots, c^H U_H(t_k))$$

Definition

A stationary stochastic process $(X_t)_t$ is said to have *long memory* if its autocovariance $\varsigma_X(\tau)$ tends to 0 so slowly such that $\sum_{\tau=0}^{\infty} \varsigma_X(\tau)$ diverges.

$S_H(k) = U_H(k+1) - U_H(k)$ for $k \in \mathbb{R}$.

Theorem

The fractional Brownian noise $S_H(k)$ with $H \in (\frac{1}{2}, 1)$ has long memory.

fBm is not semimartingal for $H \neq \frac{1}{2}$.

$$dX_t = -aX_t\,dt + \gamma\,dU_H(t),$$

where $(X_t)_{t\geq 0}$ is a stochastic process, $a, \gamma \in \mathbb{R}_+$ and $(U_H(t))_{t\geq 0}$ fBm with Hurst exponent H. In fact, given an initial condition $X_0(\omega) = b(\omega)$, then in the theory of SDE, (4) is understood as

$$X_t(\omega) = b(\omega) - a\int_0^t X_u(\omega)\,du + \gamma U_H(t)(\omega)$$

for $t \geq 0$.

In order to have a stationary solution, we assume that the initial value is centered Gaussian that
$\hat{X}_{H,t} := \hat{X}_t^{\gamma\int_{-\infty}^0 e^{au}\,dU_H(u),H} := e^{-at}\left(\gamma\int_{-\infty}^t e^{au}\,dU_H(u)\right).$

Theorem

$(\hat{X}_{H,t})_{t\geq 0}$ *has long memory for* $H \in (\frac{1}{2}, 1)$.

Fractional Black-Scholes model

$$\begin{aligned} A_t &= \exp(rt) \\ S_t &= \exp(rt + \mu(t) + \sigma U_H(t)), t \in [0, T], \end{aligned}$$

where $r \in \mathbb{R}, \sigma \in \mathbb{R}_+, \sup\limits_{t \in [0,T]} \mu(t) < \infty.$

Theorem

Let $(S_t)_{t\in[0,T]}$ *be a stochastic process such that*

$$\tilde{S}_t = \exp\left(\mu(t) + \sigma U_H(t)\right), \tag{4}$$

where μ, σ *are as in (4),* $U_H(t)$ *is a fBm. If there exist*

$$\xi_t^1 = f_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{n-1} f_k \mathbb{1}_{(\tau_k,\tau_{k+1}]}(t)$$

where $t \in [0, T], f_k$ *is family of* $\mathcal{F}_k^{U_H}$*-measurable function for* $k \in \{1, \ldots, n-1\}$. $0 = \tau_1 < \cdots < \tau_n = T$ *are stopping times with respect to* $\mathcal{F}_{\tau_k}^{U_H}$ *respectively, with* $\tau_{k+1} - \tau_k \geq m$ *for some* $m > 0$*. If there exists a* $k \in \{0, \ldots, n-1\}$ *such that* $\mathcal{P}[f_k \neq 0] > 0$ *, then*

$$\mathcal{P}[(\xi^1 \cdot \tilde{S})_T < 0] > 0,$$

where $(\xi^1 \cdot \tilde{S})_T := \sum_{k=1}^n \xi_{\tau_k}^1 (\tilde{S}_k - \tilde{S}_{k-1})$.

The market (4) is arbitrage-free, if there exists a minimal amount of time between two successive transactions,.

$$
\begin{aligned}
dS_t &= r_t S_t \, dt + \sigma_t S_t \, dB_t, && (5) \\
\sigma_t &= \exp\{X_t\} \\
dX_t &= -aX_t \, dt + \gamma \, dU_H(t), && (6)
\end{aligned}
$$

where $a, \gamma \in \mathbb{R}_+$. In the proceeding section, we have a stationary solution

$$
\hat{X}_{H,t} = e^{-at}\gamma \int_{-\infty}^{t} e^{au} \, dU_H(u) \tag{7}
$$

Theorem

$(\hat{\sigma}_{H,t})$ *has long memory for* $H \in (\frac{1}{2}, 1)$.

Smoothness of σ_t.

$$s(\tau,\sigma) = \frac{1}{N}\sum_{k=1}^{N} |\log(\sigma_{k\tau}) - \log(\sigma_{(k-1)\tau})|^2,$$

Theorem

$$\mathrm{Var}[\hat{X}_{H,t,a}] - \mathrm{Cov}[\hat{X}_{H,t,a}, \hat{X}_{H,t+\tau,a}] \to \frac{1}{2}\gamma^2\tau^{2H} \tag{8}$$

as a goes to zero, for $t > 0, \tau > 0$.

Definition

A *mixed fractional Brownian motion* is defined as follows

$$M_{\alpha,\beta,H_1,H_2}(t) = \alpha U_{H_1}(t) + \beta U_{H_2}(t) \tag{9}$$

for $t \in \mathbb{R}$, where α, β are real numbers and U_{H_1}, U_{H_2} are two independent fBm's with Hurst exponents $H_1 \in (0, \frac{1}{2}), H_2 \in (\frac{1}{2}, 1)$ respectively.

$$X_{\alpha,\beta,H_1,H_2}(t) = X_{\alpha,\beta,H_1,H_2}(0) - a\int_0^t X_u\,du + \gamma M_{\alpha,\beta,H_1,H_2}(t).$$

$$\hat{X}_{\alpha,\beta,H_1,H_2}(t) = \alpha\gamma e^{-at}\int_{-\infty}^{t} e^{au}\,dU_{H_1} + \beta\gamma e^{-at}\int_{-\infty}^{t} e^{au}\,dU_{H_2}\,.$$

Proposition

$\hat{X}_t$ satisfies following properties:

(i) $(\hat{X}_t)_{t\geq 0}$ is a centered Gaussian stationary process.

(ii) $\hat{X}_t$ has long memory.

Proposition

Let M_{α,β,H_1,H_2} *be a weighted fractional brownian motion with respect to* U_{H_1} *and* U_{H_2}*.* $T, \tau > 0$*,* a, γ *are defined by (??).* J_{H_1}, J_{H_2} *are defined by (10).* $\phi = H_1 - \frac{1}{2}, \psi = H_2 - \frac{1}{2}$*. Then, for* $t \in [0, T]$*,*

(i) $\mathrm{E}[\sup_{t \in [0,T]} |\hat{X}_{\alpha,\beta,H_1,H_2}(t) - U_{H_1}(t)|] \to 0$ *as* $a \to 0, \alpha \to 1.$

(ii) $\mathrm{E}[|\hat{X}_{\alpha,\beta,H_1,H_2}(t+\tau) - \hat{X}_{\alpha,\beta,H_1,H_2}(t)|^2] \to \gamma^2 \tau^{2H}$ *as* $a \to 0, \alpha \to 1.$