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Fractional Brownian Motion and its Application in Financial Mathematics

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Abstract

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1 Introduction

2 Gaussian Process and Brownian Motion

In this section we start off the general concept of probability spaces and stochastic processes. Of this, a most important case we then discribe, is Gaussian process. It bring us to introduce the Brownian Motion as a fine example.

2.1 Probability Space and Stochastic Process

Definition 2.1. Let $\mathscr{A}$ be a collection of subsets of a set Ω. $\mathscr{A}$ is then a σ- *Algebra* on Ω if it satisfies the following conditions:

(i) $\Omega \in \mathscr{A}$.

(ii) For any set $F \in \mathscr{A}$, its complement $F^c \in \mathscr{A}$.

(iii) If a serie $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathscr{A}$, then $\cup_{n\in\mathbb{N}} F_n \in \mathscr{A}$.

Definition 2.2. A mapping $\mathcal{P}$ is said to be a *probability measure* from $\mathscr{A}$ to $\mathscr{B}(\mathbb{R}^n)$, if $\mathcal{P}\left[\sum_{n=1}^{\infty} F_n\right] = \sum_{n=1}^{\infty} \mathcal{P}[F_n]$ for any $\{F_n\}_{n\in\mathbb{N}}$ disjoint in $\mathscr{A}$ satisfying $\sum_{n=1}^{\infty} F_n \in \mathscr{A}$.

Definition 2.3. A *probability space* is defined as a triple $(\Omega, \mathscr{A}, \mathcal{P})$ of a set Ω, a σ-Algebra $\mathscr{A}$ of Ω and a measure $\mathcal{P}$ from $\mathscr{A}$ to $\mathscr{B}(\mathbb{R}^n)$.

The σ- Algebra generated of all open sets on $\mathbb{R}^n$ is called the *Borel σ- Algebra* which we denote as usual by $\mathscr{B}(\mathbb{R}^n)$. Let μ be a probability measure on $\mathbb{R}^n$. Indeed, $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mu)$ is a special case that probability space on $\mathbb{R}^n$. A function f mapping from $(\mathcal{D}, \mathscr{D}, \mu)$ into $(\mathcal{E}, \mathscr{E}, \nu)$ is *measurable* if its collection of the inverse image of $\mathscr{E}$ is a subset of $\mathscr{D}$. A *random variable* is a $\mathbb{R}^n$-valued measurable function on some probability space. Let $\mathcal{P}$ represent a probability measure, recall that in probability theory, for $B \in \mathscr{B}(\mathbb{R}^n)$ we call $\mathcal{P}[\{X \in B\}]$ the *distribution* of X. We write also $\mathcal{P}_X[\cdot]$ or $\mathcal{P}[X]$ for convenience of the notation above.

Definition 2.4. Let $(\Omega, \mathscr{A}, \mathcal{P})$ be a probability space. A n-dimensional *stochastic process* (X_t) is a family of random variable such that $X_t(\omega) : \Omega \longrightarrow \mathbb{R}^n, \forall t \in T$, where T denotes the set of Index of Time.

Definition 2.5. A stochastic process $(X_t)_{t\in T}$ is said to be *stationary*, if the joint distribution

$$\mathcal{P}[X_{t_1}, \ldots, X_{t_n}] = \mathcal{P}[X_{t_1+\tau}, \ldots, X_{t_n+\tau}]$$

for $t_1, \ldots, t_n$ and $t_1 + \tau, \ldots, t_n + \tau \in T$.

Remark that, definition 2.5 means the distribution of a stationary process is independent of a shift of time.

2.2 Normal Distribution and Gaussian Process

Definition 2.6 (1-dimensional normal distribution)**.** A $\mathbb{R}$-valued random variable X is said to be *standard normal distributed*, if its distribution can be discribed as

$$\mathcal{P}\left[X \leq x\right] = \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-\frac{u^2}{2}}\ du$$

for $x \in \mathbb{R}$.

Definition 2.7. A $\mathbb{R}$-valued random variable X is said to be *normal distributed* with a *mean* μ and a *variance* σ^2, if

$$(X-\mu)/\sigma$$

is standard normal distributed.

We use a notation $X \sim Y$, which means X and Y have the same distribution. In similar way it is denoted by $X \sim (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}\ dx$, if it is standard normal distributed. In order to identifing the behaviour of a normal distributed random variable we recall the characteristic function in probability theory, see[1].

Proposition 2.8. Let X be a $\mathbb{R}$-valued standard normal distributed random variable. The characteristic function of X

$$\Psi_X(\xi) := \int_{\mathbb{R}} e^{ix\xi} \mathcal{P}\left[X \in dx\right] = e^{-\frac{\xi^2}{2}} \tag{2.1}$$

for $\xi \in \mathbb{R}$.

Proof. According to the definion of characteristic function

$$\Psi_X(\xi) = \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi}\ dx,$$

take differentiating both sides of the equation by ξ, then

$$\begin{aligned}
\Psi'_X(\xi) &= \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} ix\ dx \\
&= (-i) \cdot \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} \left(\frac{d}{dx} e^{-\frac{x^2}{2}}\right) e^{ix\xi}\ dx \\
&\overset{part.int.}{=} -\int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} \xi\ dx \\
&= -\xi \Psi_X(\xi).
\end{aligned}$$

Obviously, $\Psi(\xi) = \Psi(0) e^{-\frac{\xi^2}{2}}$ is the solution of the partial differential equation above, and $\Psi(0)$ is equal to 1. $\square$

In particular, the characteristic function of a normal distributed random variable with a mean μ and a variance σ^2, which denoted by $\Psi_{X_{\mu,\sigma^2}}(\xi)$, is $e^{i\mu\xi-\frac{1}{2}(\sigma\xi)^2}$. To achieve this result, we just need to substitute x by $(x-\mu)/\sigma$ in the calculation before.

Definition 2.9. Let X be a $\mathbb{R}^n$-valued random variable. X is said to be *normal distributed*, if for any $d \in \mathbb{R}^n$ such that $d^T X$ is normal distributed on $\mathbb{R}$.

Proposition 2.10. Let X be a $\mathbb{R}^n$-valued normal distributed. Then there exist $m \in \mathbb{R}^n$ and a positive definite symmetric matrix $\Sigma \in \mathbb{R}^{n\times n}$ such that,

$$\mathrm{E}\, e^{i\xi^T X} = e^{i\xi^T m-\frac{1}{2}\xi^T\Sigma\xi} \tag{2.2}$$

For $\xi \in \mathbb{R}^n$. Furthermore, the density function of X is

$$(2\pi)^{-\frac{d}{2}} (\det\Sigma)^{-\frac{1}{2}}\, e^{-\frac{1}{2}(x-m)^T\Sigma^{-1}(x-m)}\ dx\,. \tag{2.3}$$

Remark, the equation (2.2) can also be as definition of characteristic function of a n-dimensional normal distributed random variable. I.e., any normal distributed random variable can be characterized by form of the equation (2.2).

Proof. Since X normal distributed on $\mathbb{R}^n$, then $\xi^T X$ is normal distributed on $\mathbb{R}$. Due to the proposition 2.8 there is

$$\begin{aligned} \mathrm{E}e^{i\xi^T X} &= \mathrm{E}e^{i\cdot 1\cdot\xi^T X} \\ &= e^{i\mathrm{E}\left[\xi^T X\right]-\frac{1}{2}\mathrm{Var}\left[\xi^T X\right]} \\ &= e^{i\xi^T\mathrm{E}[X]-\frac{1}{2}\xi^T\mathrm{Var}[X]\xi}. \end{aligned}$$

According to the uniqueness theorem of characteristic function (Satz 23.4 in [1]), then we can deduce the density function of the equation (2.3). □

A normal distributed normal random variable can be characterized by its mean and variance respectively mean vector and covariance vector because of the characteristic function.

Corollary 2.11. A linear combination of independent normal distributed random variables has normal distribution.

Proof. In general case, we suppose $Y_1, \cdots, Y_m$ are independent random variables on $\mathbb{R}^n$, for $c_1, \cdots, c_m \in \mathbb{R}$. Let have a look at the chracteristic function of it,

$$\begin{aligned}
\mathrm{E}e^{i\xi^T \sum_{j=1}^m (c_j X_j)} &\overset{independent}{=} \prod_{j=1}^m \mathrm{E}e^{i\xi^T (c_j X_j)} \\
&= \prod_{j=1}^m \exp\left(i\xi^T \mathrm{E}[c_j X_j] - \frac{1}{2}\xi^T \mathrm{Var}[c_j X_j]\xi\right) \\
&= \exp\left(i\xi^T \mathrm{E}[\sum_{j=1}^m c_j X_j] - \frac{1}{2}\xi^T \sum\nolimits_{j=1}^m Var[c_j X_j]\xi\right) \\
&\overset{independent}{=} \exp\left(i\xi^T \mathrm{E}[\sum_{j=1}^m c_j X_j] - \frac{1}{2}\xi^T \mathrm{Var}[\sum_{j=1}^m c_j X_j]\xi\right),
\end{aligned}$$

which is a form of characteristc function of normal distribution. That means $\sum_{j=1}^m c_j X_j$ is normal distributed. $\square$

Example 2.12 (Bivariate Normal Distribution)**.** Suppose S_1, S_2 are independent random variables on $\mathbb{R}$ and have standard normal distributions. $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ has standard normal joint distribution since they are independent. We define

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \sigma_1, & 0 \\ \sigma_2\rho, & \sigma_2(1-\rho^2)^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \tag{2.4}$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 \in \mathbb{R}, -1 \leq \rho \leq 1$. Again, Y_1, Y_2 are normal distributed and the joint distribution $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is normal. We set $\mathrm{E}[Y_1] = \mu_1, \mathrm{E}[Y_2] = \mu_2$ for short. Since S_1, S_2 are independent,

$$\begin{aligned}
\mathrm{Var}[Y_1] &= \mathrm{Var}[\sigma_1 S_1] \\
&= \sigma_1^2, \\
\mathrm{Var}[Y_2] &= \mathrm{Var}[\sigma_2\rho S_1] + \mathrm{Var}[\sigma_2(1-\rho^2)^{\frac{1}{2}} S_2] \\
&= \sigma_2^2\rho^2 + \sigma_2^2(1-\rho^2) \\
&= \sigma_2^2, \\
\mathrm{Cov}[Y_1, Y_2] &= \mathrm{E}[(Y_1 - \mathrm{E}[Y_1])(Y_2 - \mathrm{E}[Y_2])] \\
&= \mathrm{E}[Y_1 Y_2 - \mu_1 Y_2 - \mu_2 Y_1 + \mu_1\mu_2] \\
&= \mathrm{E}[(\sigma_1 S_1 + \mu_1)(\sigma_2\rho S_1 + \sigma_2(1-\rho^2)^{\frac{1}{2}} S_2 + \mu_2)] - \mu_1\mu_2 \\
&= \sigma_1\sigma_2 \underbrace{\mathrm{E}[S_1^2]}_{=1} \rho + \mu_1\sigma_2\rho \underbrace{\mathrm{E}[S_1]}_{=0} + \sigma_1\sigma_2(1-\rho^2)^{\frac{1}{2}} \underbrace{\mathrm{E}[S_1 S_2]}_{=\mathrm{E}[S_1]\mathrm{E}[S_2]=0} \\
&+ \mu_1\sigma_2(1-\rho^2)^{\frac{1}{2}} \underbrace{\mathrm{E}[S_2]}_{=0} + \sigma_1 \underbrace{\mathrm{E}[S_1]}_{=0} \mu_2 + \mu_1\mu_2 - \mu_1\mu_2 \\
&= \rho\sigma_1\sigma_2,
\end{aligned}$$

that means the corrlation of Y_1, Y_2 is ρ. Because of the equation (2.3), the joint density function

$$f_{Y_1,Y_2}(y_1,y_2) \quad = \quad (2\pi)^{-1}(\det(\Sigma))^{-\frac{1}{2}} \exp\left((y_1-\mu_1)\Sigma^{-1}(y_2-\mu_2)\right),$$

where $\Sigma = \begin{pmatrix} \sigma_1^2, & 0 \\ \sigma_2^2\rho^2, \sigma_2^2(1-\rho^2) \end{pmatrix}$

Indeed,

$$\det(\Sigma) = (1-\rho^2)\sigma_1^2\sigma_2^2$$

and

$$\Sigma^{-1} = \frac{\begin{pmatrix} \sigma_2^2(1-\rho^2), & 0 \\ -\sigma_2^2\rho, & \sigma_1^2 \end{pmatrix}}{(1-\rho^2)\sigma_1^2\sigma_2^2}.$$

Namely,

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}\sigma_1\sigma_2} \exp\left(-\frac{1}{2(1-\rho^2)}(z_1^2 - 2\rho z_1 z_2 + z_2^2)\right) \tag{2.5}$$

where $z_1 = \frac{y_1-\mu_1}{\sigma_1}, z_2 = \frac{y_2-\mu_2}{\sigma_2}$.

Corollary 2.13. Let Y_1, Y_2 be $\mathbb{R}$-valued normal distributed random variables and $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ has a joint normal distribution, then the conditional mean of Y_2 given Y_1

$$\mathrm{E}[Y_2|Y_1 = y_1] = \mathrm{E}[Y_2] + \rho(y_1 - \mathrm{E}[Y_1])\frac{\sigma_2}{\sigma 1},$$

and the conditional variance of Y_2 given Y_2

$$\mathrm{Var}[Y_2|Y_1 = y_1] = \sigma_1^2(1-\rho^2).$$

Where σ_1, σ_2 are standard deviations of Y_1, Y_2 and ρ is the correlation of Y_1, Y_2.

Proof. Recall the equation (2.5), we can specify the joint density function if σ_1, σ_2, ρ are known. As result of this, $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ has a form of the equation (2.4). Suppose S_1, S_2 are independent standard normal distributed random variables. Now we have

$$\begin{aligned} S_1 &\sim \frac{(Y_1 - \mathrm{E}[Y_1])}{\sigma_1} \\ Y_2 &\sim \sigma_2\rho S_1 + \sigma_2(1-\rho^2)^{\frac{1}{2}}S_2 + \mathrm{E}[Y_2], \end{aligned}$$

more precisely,

$$Y_2 \sim \sigma_2\rho\frac{(Y_1 - \mathrm{E}[Y_1])}{\sigma_1} + \sigma_2(1-\rho^2)^{\frac{1}{2}}S_2 + \mathrm{E}[Y_2].$$

Take expectation of both sides,

$$\mathrm{E}[Y_2|Y_1 = y_1] = \sigma_2 \rho \frac{(y_1 - \mathrm{E}[Y_1])}{\sigma_1} + \mathrm{E}[Y_2].$$

Now consider

$$\begin{aligned}
\mathrm{Var}[Y_2|Y_1 = y_1] &= \mathrm{E}[(Y_2 - \mu_{Y_2|Y_1})^2 | Y_1 = y_1] \\
&= \int_{-\infty}^{\infty} (y_2 - \mu_{Y_2|Y_1})^2 f_{Y_2|Y_1}(y_2, y_1)\ dy_2 \\
&= \int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1) \right]^2 f_{Y_2|Y_1}(y_2, y_1)\ dy_2,
\end{aligned}$$

multiply both sides by the density function of Y_1 and integral it over by y_1, we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \mathrm{Var}[Y_2|Y_1 = y_1] f_{Y_1}(y_1)\, dy_1 \\
= \quad & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1) \right]^2 \underbrace{f_{Y_2|Y_1}(y_2, y_1)\, f_{Y_1}(y_1)}_{f_{Y_1,Y_2}(y_1,y_2)}\, dy_2\ dy_1 \\
\Longleftrightarrow \quad & \\
& \mathrm{Var}[Y_2|Y_1 = y_1] \underbrace{\int_{-\infty}^{\infty} f_{Y_1}(y_1)\, dy_1}_{1} \\
= \quad & \mathrm{E}\left[(Y_2 - \mu_2) - (\frac{\rho\sigma_2}{\sigma_1})(Y_1 - \mu_1) \right]^2
\end{aligned}$$

Also

$$\begin{aligned}
\mathrm{Var}[Y_2|Y_1 = y_1] &= \underbrace{\mathrm{E}[(Y_2 - \mu_2)^2]}_{\sigma_2^2} - 2\frac{\rho\sigma_2}{\sigma_1} \underbrace{\mathrm{E}[(Y_1 - \mu_1)(Y_2 - \mu_2)]}_{\rho\sigma_1\sigma_2} \\
&+ \frac{\rho^2\sigma_2^2}{\sigma_1^2} \underbrace{\mathrm{E}[(Y_1 - \mu_1)^2]}_{\sigma_1^2} \\
&= \sigma_2^2 - 2\rho^2\sigma^2 + \rho^2\sigma_2^2 \\
&= \sigma_2^2 - \rho^2\sigma_2^2.
\end{aligned}$$

□

Definition 2.14. Let $(X_t)_{t \in T}$ be a $\mathbb{R}^n$-valued stochastic process. (X_t) is said to be a *gaussian process* if $X_{t_1}, \ldots, X_{t_n}$ has a joint normal distribution for any $t_1 \ldots t_n \in T$ and $n \in \mathbb{N}$.

The definition immediately shows for every X_t in gaussian process has a normal distribution. Therefore the prior corollary is applicable to a gaussian process.

2.3 Brownian Motion

The brownian motion was first introduced by Bachelier in 1900 in his PhD thesis. We now give the common definition of it.

Definition 2.15. Let $(B_t)_{t\geq 0}$ be a $\mathbb{R}^n$-valued stochastic process. (B_t) is called *brownian motion* if it satisfies the following conditions:

(i) $B_0 = 0$ a.s. .

(ii) $(B_{t_1} - B_{t_0}), \ldots, (B_{t_n} - B_{t_{n_1}})$ are independet for $0 = t_0 < t_1 < \cdots < t_n$ and $n \in \mathbb{N}$.

(iii) $B_t - B_s \sim B_{t-s}$, for $0 \leq s \leq t < \infty$.

(iv) $B_t - B_s \sim \mathcal{N}(0, t-s)^{\otimes n}$.

(v) B_t is continuous in t a.s. .

A usual saying for *(ii)* and *(iii)* is the brownian motion has independent, stationary increments. In (iv), N represent a random variable which has a normal distribution. B_t is normal distributed due to (ii). It is clear that the increments of brownian motion is stationary.

Proposition 2.16. Let (B_t) be a one-dimensional brownian motion. Then the covarice of B_m, B_n for $m, n \geq 0$ is $m \wedge n$.

Proof. WLOG, we assume that $m \geq n$, then

$$\begin{aligned} \mathrm{E}[B_m B_n] &= \mathrm{E}[(B_m - B_n)B_n] + \mathrm{E}[B_n^2] \\ &= \mathrm{E}[B_m - B_n]\mathrm{E}[B_n] + n \\ &= n. \end{aligned}$$

□

Proposition 2.17. Let (B_t) be a one-dimensional brownian motion. Then $B_{cm} \sim c^{\frac{1}{2}} B_m$.

Proof. Because B_m is normal distributed for any $m > 0$, we then get

$$\begin{aligned} \mathrm{E}[e^{i\xi B_{cm}}] &= e^{-\frac{1}{2}cm\xi^2} \\ &= e^{-\frac{1}{2}(c(m)^{\frac{1}{2}}\xi)^2} \\ &= \mathrm{E}[e^{i\xi c^{\frac{1}{2}} B_m}]. \end{aligned}$$

□

Theorem 2.18. A one-dimensional brownian motion is a gaussian process.

Proof. The following idea using the independence of increments to prove the claim come from [2]. We choose $0 = t_0 < t_1 < \cdots < t_n$, for $n \in \mathbb{N}$. Define $V = (B_{t_1}, \ldots, B_{t_n})^T$, $K = (B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}})^T$ and $A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$. Let us see the characteristic function of V,

$$
\begin{aligned}
\mathrm{E}[e^{i\xi^T V}] \quad &= \quad \mathrm{E}[e^{i\xi^T AK}] \\
&= \quad \mathrm{E}[e^{iA^T \xi K}] \\
&= \quad \mathrm{E}[\exp(i(\xi^{(1)} + \cdots + \xi^{(n)}, \xi^{(2)} + \cdots + \xi^{(n)}, \cdots, \xi^{(n)})] \\
&\cdot \quad (B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})^T) \\
&\overset{ind.increments}{=} \quad \prod_{j=1}^{n} \mathrm{E}[\exp(i(\xi^{(j)+\cdots+\xi^{(n)}})(B_{t_j} - B_{t_{t-1}}))] \\
&\overset{stat.increments}{=} \quad \prod_{j=1}^{n} \exp(-\frac{1}{2}(t_j - t_{j-1})(\xi^{(j)} + \cdots + \xi^{(n)})^2) \\
&= \quad \exp\left(-\frac{1}{2}\sum_{j=1}^{n}(t_j - t_{j-1})(\xi^{(j)} + \cdots + \xi^{(n)})^2\right) \\
&= \quad \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n} t_j(\xi^{(j)} + \cdots + \xi^{(n)})^2 - \sum_{j=1}^{n} t_{j-1}(\xi^{(j)} + \cdots + \xi^{(n)})^2\right)\right) \\
&= \quad \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n-1} t_j((\xi^{(j)} + \cdots + \xi^{(n)})^2 - (\xi^{(j+1)} + \cdots + \xi^{(n)})^2) + t_n(\xi^{(n)})^2\right)\right) \\
&= \quad \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n-1} t_j\xi^{(j)}(\xi^{(j)} + 2\xi^{(j+1)} + \cdots + 2\xi^{(n)}) + t_n(\xi^{(n)})^2\right)\right) \\
&= \quad \exp\left(-\frac{1}{2}\left(\sum_{j,h=1}^{n} (t_j \wedge t_h)\xi^{(j)}\xi^{(h)}\right)\right).
\end{aligned}
$$

Recall with proposition 2.3, $(t_j \wedge t_h)_{t,h=1,\ldots,n}$ is the covariance matrix of V. The mean vector of it is zero, then we have been proved that the characteristic function is a form of some normal distributed random vector, i.e., V is normal distributed. □

Shilling gave in his lecture [2] the relationship between a one-dimensional brownian motion and a n-dimensional brownian motion. $(B_t^{(l)})_{l=1,\ldots,n}$ is brownian motion if and only if $B_t^{(l)}$ is brownian motion and all of the component are independent. Using this independence and te theorem of fubini in the characteristic function for high-dimensional brownian motion we can say a n-dimensional brownian motion is also a gaussian process.

3 Regularity for Brownian Motion and Itó Integral

3.1 Lévy Modulus of Continuity

We consider now the one-dimensional brownian motion. In this section we need some notations, which are defined as followings

$$\Delta^{[0,T]} = \{t_1, \dots, t_n | 0 = t_0 < \cdots < t_n = T\}$$

$$|\Delta^{[0,T]}| = \max_{t_j \in \Delta^{[0,T]}} |t_j - t_{j-1}|$$

.

Lemma 3.1. Let B_t be a brownian motion. Then

$$\sum_{t_j \in \Delta^{[0,T]}} |B_{t_j} - B_{t_{j-1}}|^2 \xmapsto[L^2(\mathcal{P})]{|\Delta^{[0,T]}| \to 0} T$$

4 Fractional Brownian Motion

5 Fractional Ornstein Uhlenbeck Process Model

6 Application in Financial Mathematics

7 Conclusion

References

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