Fractional Brownian Motion and ist Application in financial mathematics

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July 22, 2015

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Introduction of fBm

What is a fBm ?

Classical Definition

A centered Gaussian process $(U_H(t))_{t\in\mathbb{R}}$ with real number $0 < H < 1$ such that $\mathrm{E}[U_H(t)U_H(s)] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$

Motivation

Problem: Is there a Representation of fBM ?

Motivation

Problem: Is there a Representation of fBM ?

Yes, Integral Representation.

Integral Representation

Mandelbrot and Van Ness[1]

Definition

$(U_H(t))$ is said to be *fractional Brownian motion* if

$$\begin{aligned} & U_H(t) - U_H(s) \\ = & \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{\mathbb{R}} \mathbb{1}_{\{t>u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{s>u\}}(-u)^{H-\frac{1}{2}}\, dB_u\right) \end{aligned}$$

for $t \geq s, t, s \in \mathbb{R}$, where (B_u) is defined as two-sides Brownian motion and the integral is defined in the sense of stable integral.

Theorem

$U_H(0) = 0$, *then*

$$U_H(t)$$

$$= \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{\mathbb{R}} \underbrace{\mathbb{1}_{\{t>u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u<0\}}(-u)^{H-\frac{1}{2}}}_{f_t(u)}\, dB_u\right).$$

$$J(f_t) : f_t \to \int_{\mathbb{R}} f_t \, dB_u$$

If the integrand f_t is quadratic integrable then integral is well-defined in the sense of stable integral.

Introduction of fBm

Proposition

The stable integral is linear[2] :

$$J(af_t + bf_s) \overset{a.s.}{=} aJ(f_t) + bJ(f_s)$$

Theorem

$$U_H(t) \sim \mathcal{N}(0, \frac{1}{\Gamma(H+\frac{1}{2})^2}(\int_{\mathbb{R}} |f_t(u)^2| \; du))$$

Introduction of fBm

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Theorem

$$U_H(t) - U_H(s) \sim \mathcal{N}(0, \frac{1}{\Gamma(H+\frac{1}{2})^2}(\int_{\mathbb{R}} |f_t(u) - f_s(u)|^2\ du)$$

Standardization

Corollary

The variance of $U_H(t)$ *is* $\frac{1}{(\Gamma(H+\frac{1}{2}))^2}\mathrm{E}U_H^2(1)t^{2H}$ *for any* $t \in \mathbb{R}$.

Divide with $\frac{1}{(\Gamma(H+\frac{1}{2}))^2}\mathrm{E}U_H^2(1)$ then,

$$\mathrm{Var}[U_H(t)] = t^{2H}.$$

fBm properties

Theorem

Let $(U_H(t))_t$ be a fBm. The covariance of $U_H(t)$ and $U_H(s)$ is $\frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$ for $t, s \in \mathbb{R}$.

fBm properties

Theorem

Let $(U_H(t))_t$ be a fBm. The covariance of $U_H(t)$ and $U_H(s)$ is $\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ for $t, s \in \mathbb{R}$.

Proof.

$$\begin{aligned}
\mathrm{Cov}[U_H(t), U_H(s)] &= \mathrm{E}[U_H(t)U_H(s)] \\
&= \frac{1}{2}(\mathrm{E}[U_H(t)^2] + \mathrm{E}[U_H(s)^2] \\
&- \mathrm{E}[(U_H(t) - U_H(s))^2]) \\
&= \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})
\end{aligned}$$

□

fBm properties

Theorem

$(U_H(t))_t$ is Gaussian process.

Theorem

$(U_H(t))_t$ has stationary and H-self similar increments .

fBm properties

Theorem

$$
\begin{aligned}
& ((U_H(t_1+\tau)-U_H(t_0+\tau)),\dots,(U_H(t_n+\tau)-U_H(t_{n-1}+\tau))) \\
\sim\ & (U_H(t_1-t_0),\dots,U_H(t_n-t_{n-1}))
\end{aligned}
$$

Theorem

$$
\begin{aligned}
& (U_H(c(t_1-t_0)),\dots,U_H(c(t_n-t_{n-1}))) \\
\sim\ & (c^H U_H(t_1-t_0),\dots,c^H U_H(t_n-t_{n-1}))
\end{aligned}
$$

for $c > 0$.

Hurst exponent

H plays role.

Hurst exponent

$\boxed{\text{Case } H = \frac{1}{2}}$

fBM is BM.

Hurst exponent

Definition

A stationary stochastic process $(X_t)_t$ is said to have *long memory* if its autocovariance $\varsigma_X(\tau)$ tends to 0 so slowly such that $\sum_{\tau=0}^{\infty} \varsigma_X(\tau)$ diverges.

Hurst exponent

$\boxed{\text{Case } H > \frac{1}{2}}$

$\boxed{S_H(k) = U_H(k+1) - U_H(k) \text{ for } k \in \mathbb{R}}$

Theorem

The fractional Brownian noise $S_H(k)$ with $H \in (\frac{1}{2}, 1)$ has long memory.

Hurst exponent

$\boxed{\text{Case } H \neq \frac{1}{2}}$

Theorem

fBm is not semimartingale for $H \neq \frac{1}{2}$.

fOU

$$dX_t = -aX_t\,dt + \gamma\,dU_H(t),$$

where $(X_t)_{t\geq 0}$ is a stochastic process, $a, \gamma \in \mathbb{R}_+$ and $(U_H(t))_{t\geq 0}$ fBm with Hurst exponent H.

$$X_t(\omega) = X_0(\omega) - a\int_0^t X_u(\omega)\,du + \gamma U_H(t)(\omega)$$

for $t \geq 0$.
Stationary solution: $\hat{X}_{H,t} := e^{-at}\left(\gamma \int_{-\infty}^t e^{au}\,dU_H(u)\right)$.

fOU

Theorem

$(\hat{X}_{H,t})_{t\geq 0}$ *is centered Gaussian process.*

Theorem

$(\hat{X}_{H,t})_{t\geq 0}$ *has long memory for* $H \in (\frac{1}{2}, 1)$.

Fractional Black-Scholes

$$
\begin{aligned}
A_t &= \exp(rt) \\
S_t &= \exp(rt + \mu(t) + \sigma U_H(t)), t \in [0, T],
\end{aligned}
$$

$$\text{where } r \in \mathbb{R}, \sigma \in \mathbb{R}_+, \sup_{t \in [0,T]} \mu(t) < \infty.$$

Theorem

$$\tilde{S}_t = \exp\left(\mu(t) + \sigma U_H(t)\right),$$

if there exists

$$\xi_t^1 = f_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{n-1} f_k \mathbb{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where $t \in [0, T]$, f_k *is family of* $\mathcal{F}_k^{U_H}$*-measurable function for* $k \in \{1, \dots, n-1\}$. $0 = \tau_1 < \dots < \tau_n = T$ *are stopping times with respect to* $\mathcal{F}_{\tau_k}^{U_H}$ *respectively, with* $\tau_{k+1} - \tau_k \geq m$ *for some* $m > 0$. *If there exists a* $k \in \{0, \dots, n-1\}$ *such that* $\mathcal{P}[f_k \neq 0] > 0$ *, then*

$$\mathcal{P}[(\xi^1 \cdot \tilde{S})_T < 0] > 0.$$

Fractional Black Scholes

The fractional Black-Scholes market is arbitrage-free, if there exists a minimal amount of time between two successive transactions,.

FSV

$\boxed{H > \frac{1}{2}}$

$$\begin{aligned} dS_t &= r_t S_t\, dt + \sigma_t S_t\, dB_t, \\ \sigma_t &= \exp\{X_t\} \\ dX_t &= -aX_t\, dt + \gamma\, dU_H(t), \end{aligned}$$

where $a, \gamma \in \mathbb{R}_+$.

Stationary solution :

$$\hat{X}_{H,t} = e^{-at} \gamma \int_{-\infty}^{t} e^{au}\, dU_H(u)$$

FSV

$$\hat{\sigma}_{H,t} = \exp\{\hat{X}_{H,t}\}$$

Theorem

$(\hat{\sigma}_{H,t})$ *has long memory for* $H \in (\frac{1}{2}, 1)$.

RFSV

$$\boxed{H < \frac{1}{2}}$$

$$\begin{aligned} dS_t &= r_t S_t \, dt + \sigma_t S_t \, dB_t, \\ \sigma_t &= \exp\{X_t\} \\ dX_t &= -aX_t \, dt + \gamma \, dU_H(t), \end{aligned}$$

RFSV

Smoothness of σ_t.

$$s(\tau, \sigma) = \frac{1}{N} \sum_{k=1}^{N} |\log(\sigma_{k\tau}) - \log(\sigma_{(k-1)\tau})|^2,$$

Empirical result:

$$s(\tau, \sigma) = k\tau^z.$$

RFSV

Gatheral et la.[3]

Theorem

$$s(\tau, \hat{X}_H) \to \gamma^2 \tau^{2H}$$

as a goes to zero, for $t > 0, \tau > 0$.

Motivation

Find a model which combine advatages on both FSV and RFSV.

Weighted fractional Brownian motion

Definition

A *weighted fractional Brownian motion* is defined as follows

$$M_{\alpha,\beta,H_1,H_2}(t) = \alpha U_{H_1}(t) + \beta U_{H_2}(t)$$

for $t \in \mathbb{R}$, where $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha^2 + \beta^2 = 1$ and U_{H_1}, U_{H_2} are two independent fBm's with Hurst exponents $H_1 \in (0, \frac{1}{2}), H_2 \in (\frac{1}{2}, 1)$ respectively.

Weighted fractional Brownian motion

$$\boxed{H_1 < \tfrac{1}{2}, H_2 > \tfrac{1}{2}}$$

$$\hat{X}_{\alpha,\beta,H_1,H_2}(t) = \alpha\gamma e^{-at}\int_{-\infty}^{t} e^{au}\, dU_{H_1} + \beta\gamma e^{-at}\int_{-\infty}^{t} e^{au}\, dU_{H_2}\,.$$

Weighted fractional Brownian motion

Proposition

$\hat{X}_t$ satisfies following properties:
(i) $(\hat{X}_t)_{t\geq 0}$ is a centered Gaussian stationary process.
(ii) $(\hat{X}_t)_{t\geq 0}$ has long memory.

Weighted fractional Brownian motion

Proposition

$\hat{X}_t$ satisfies following properties:

(i) $\mathrm{E}[\sup_{t\in[0,T]} |\hat{X}_{\alpha,\beta,H_1,H_2}(t) - U_{H_1}(t)|] \to 0$ *as* $a \to 0, \alpha \to 1$.

(ii) $\mathrm{E}[|\hat{X}_{\alpha,\beta,H_1,H_2}(t+\tau) - \hat{X}_{\alpha,\beta,H_1,H_2}(t)|^2] \to \gamma^2\tau^{2H_1}$ *as* $a \to 0, \alpha \to 1$.

Summary

FSV ensures the volatility process has long memory.

RFSV demonstrates a reasonable smoothness of volatility.

Weighted-FSV inherits long memory of FSV and 'good' smoothness of RFSV.

Thank you for your Attention !
Vielen Dank für Ihre Aufmerksamekeit !

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