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Fractional Brownian Motion and its Application in Financial Mathematics

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2 Gaussian Process and Brownian Motion

In this section we start off the general concept of probability spaces and stochastic processes. Of this, a most important case we then discribe, is Gaussian process. It bring us to introduce the Brownian Motion as a fine example.

2.1 Probability Space and Stochastic Process

DEFINITION 2.1. Let $\mathscr{A}$ be a collection of subsets of a set Ω. $\mathscr{A}$ is then a σ- *Algebra* on Ω if it satisfies the following conditions:

(i) $\Omega \in \mathscr{A}$.

(ii) For any set $F \in \mathscr{A}$, its complement $F^c \in \mathscr{A}$.

(iii) If a serie $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathscr{A}$, then $\cup_{n\in\mathbb{N}} F_n \in \mathscr{A}$.

DEFINITION 2.2. A mapping $\mathcal{P}$ is said to be a *probability measure* from $\mathscr{A}$ to $\mathscr{B}(\mathbb{R}^n)$, if $\mathcal{P}\left[\sum_{n=1}^{\infty} F_n\right] = \sum_{n=1}^{\infty} \mathcal{P}\left[F_n\right]$ for any $\{F_n\}_{n\in\mathbb{N}}$ disjoint in $\mathscr{A}$ satisfying $\sum_{n=1}^{\infty} F_n \in \mathscr{A}$.

DEFINITION 2.3. A *probability space* is defined as a triple $(\Omega, \mathscr{A}, \mathcal{P})$ of a set Ω, a σ-Algebra $\mathscr{A}$ of Ω and a measure $\mathcal{P}$ from $\mathscr{A}$ to $\mathscr{B}(\mathbb{R}^n)$.

The σ- Algebra generated of all open sets on $\mathbb{R}^n$ is called the *Borel σ- Algebra* which we denote as usual by $\mathscr{B}(\mathbb{R}^n)$. Let μ be a probability measure on $\mathbb{R}^n$. Indeed, $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mu)$ is a special case that probability space on $\mathbb{R}^n$. A function f mapping from $(\mathcal{D}, \mathscr{D}, \mu)$ into $(\mathcal{E}, \mathscr{E}, \nu)$ is *measurable* if its collection of the inverse image of $\mathscr{E}$ is a subset of $\mathscr{D}$. A *random variable* is a $\mathbb{R}^n$-valued measurable function on some probability space. Let $\mathcal{P}$ represent a probability measure, recall that in probability theory, for $B \in \mathscr{B}(\mathbb{R}^n)$ we call $\mathcal{P}\left[\{X \in B\}\right]$ the *distribution* of X. We write also $\mathcal{P}_X\left[\bullet\right]$ or $\mathcal{P}\left[X\right]$ for convenience of the notation above.

DEFINITION 2.4. Let $(\Omega, \mathscr{A}, \mathcal{P})$ be a probability space. A n-dimensional *stochastic process* (X_t) is a family of random variable such that $X_t(\omega) : \Omega \longrightarrow \mathbb{R}^n, \forall t \in T$, where T denotes the set of Index of Time.

DEFINITION 2.5. A stochastic process $(X_t)_{t\in T}$ is said to be *stationary*, if the joint distribution

$$\mathcal{P}\left[X_{t_1}, \ldots, X_{t_n}\right] = \mathcal{P}\left[X_{t_1+\tau}, \ldots, X_{t_n+\tau}\right]$$

for $t_1, \ldots, t_n$ and $t_1 + \tau, \ldots, t_n + \tau \in T$.

Remark that, definition 2.5 means the distribution of a stationary process is independent of a shift of time.

2.2 Definition of Gaussian Process

DEFINITION 2.6 (1-dimensional normal distribution)**.** A $\mathbb{R}$-valued random variable X is said to be *standard normal distributed*, if its distribution can be discribed as

$$\mathcal{P}\left[X \leq x\right] = \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-\frac{u^2}{2}} \, du$$

for $x \in \mathbb{R}$.

DEFINITION 2.7. A $\mathbb{R}$-valued random variable X is said to be *normal distributed* with a *mean* μ and a *variance* σ^2, if

$$(X - \mu)/\sigma$$

is standard normal distributed.

We use a notation $X \sim Y$, which means X and Y have the same distribution. In similar way it is denoted by $X \sim (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \, dx$, if it is standard normal distributed. In order to identifing a normal distributed random variable we recall the characteristic function in probability theory, see[1].

PROPOSITION 2.8. Let X be a $\mathbb{R}$-valued standard normal distributed random variable. The characteristic function of X

$$\Psi_X(\xi) := \int_{\mathbb{R}} e^{ix\xi} \mathcal{P}\left[X \in dx\right] = e^{-\frac{\xi^2}{2}} \tag{2.1}$$

for $\xi \in \mathbb{R}$.

Proof. According the definion of characteristic function

$$\Psi_X(\xi) = \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} \, dx,$$

take differentiating both sides of the equation by ξ, then

$$\begin{aligned}
\Psi'_X(\xi) &= \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} ix \, dx \\
&= (-i) \cdot \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} \left(\frac{d}{dx} e^{-\frac{x^2}{2}}\right) e^{ix\xi} \, dx \\
&\overset{part.int.}{=} - \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} \xi \, dx \\
&= -\xi \Phi_X(\xi).
\end{aligned}$$

Obviously, $\Psi(\xi) = \Psi(0) e^{-\frac{\xi^2}{2}}$ is the solution of the partial differential equation above, and $\Psi(0)$ is equal to 1. $\square$

In particular, the characteristic function of a normal distributed random variable with a mean μ and a variance σ^2, which denoted by $\Psi_{X_{\mu,\sigma^2}}(\xi)$, is $e^{i\mu\xi - \frac{1}{2}(\sigma\xi)^2}$. To achieve this result, we just need to substitute x by $(x - \mu)/\sigma$ in the calculation before.

3 Fractional Brownian Motion

4 Fractional Ornstein Uhlenbeck Process Model

5 Application in Financial Mathematics

6 Conclusion

References

[1] Bauer, H. (2002). Wahrscheinlichkeitstheorie(durchges. und verb. Aufl.). Berlin: W. de Gruyter