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# Fractional Brownian Motion and Applications in Financial Mathematics

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# Abstract

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# 1 Introduction

Fractional Brownian motion(FBM) introduced by Kolmogorov is a centered Gaussian process $U_H$ which has covariance function as follows

$$\mathrm{Cov}[U_H(t), U_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

where $H$ is real number in $(0, \frac{1}{2})$. Mandelbrot and Van Ness recognized there is a integration representation

$$U_H(t) - U_H(s) = \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{\mathbb{R}} \mathbb{1}_{\{t\geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{s\geq u\}}(-u)^{H-\frac{1}{2}}\ dB_u\right), \quad (1.1)$$

which could fulfil the preceeding covariance property.
The main tasks of this thesis are to study FBM by its integration representation and to discuss its applications in financial mathematics. It is arranged as follows. In Section 2 we shall give some basic concepts of Gaussion processes and Brownian motion. To define the integral of (1.1), we introduce in Section 3 the stable integrals. In terms of (1.1), FBM could be a self-similar process and has stationary increments satisfying

$$U_H(t+\tau) - U_H(t) \sim \kappa\tau^H U_H(1). \quad (1.2)$$

We deal with it in detail in Section 4 and show FBM is not a semi-martingal when $H \neq \frac{1}{2}$. From Section 5 up to the end we'll turn our attension on applications of FBM.

is called Ornstein-Uhlenbeck process which has a stationary and long memorial solution. In Section 6 are presented financial modelling driven by FBM. It will be split into two parts to discuss, the FSV model with $H > \frac{1}{2}$, and RFSV model with $H < \frac{1}{2}$. Each one has its advantages and disvantages. Howerver, we give a mix of them in the end of this section. It may provide new research direction on the future work.

# 2 Gaussian Processes and Brownian Motion

In this section we start off by looking at some general concepts of probability spaces and stochastic processes. Of this, a most important case we then discribe is Gaussian process. Within the framework of Gaussian processes, one could specific a stationary and independent behaviour of increments of it. This lead us to introduce the Brownian motion as a fine example.

## 2.1 Probability Spaces and Stochastic Processes

**Definition 2.1.** Let $\mathscr{A}$ be a collection of subsets of a set $\Omega$. $\mathscr{A}$ is said to be a $\sigma$- *Algebra* on $\Omega$, if it satisfies the following conditions:

(i) $\Omega \in \mathscr{A}$ .

(ii) For any set $F \in \mathscr{A}$, its complement $F^c \in \mathscr{A}$ .

(iii) If there is a series $\{F_n\}_{n\in\mathbb{N}}$ such that $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathscr{A}$, then $\cup_{n\in\mathbb{N}} F_n \in \mathscr{A}$ .

**Definition 2.2.** A mapping $\mathcal{P}$ is said to be a *probability measure* from $\mathscr{A}$ to $\mathscr{B}(\mathbb{R}^n)$, if $\mathcal{P}\left[\sum_{n=1}^{\infty} F_n\right] = \sum_{n=1}^{\infty} \mathcal{P}\left[F_n\right]$ for any $\{F_n\}_{n\in\mathbb{N}}$ disjoint in $\mathscr{A}$ satisfying $\sum_{n=1}^{\infty} F_n \in \mathscr{A}$.

**Definition 2.3.** A *probability space* is defined as a triple $(\Omega, \mathscr{A}, \mathcal{P})$ of a set $\Omega$, a $\sigma$-Algebra $\mathscr{A}$ of $\Omega$ and a measure $\mathcal{P}$ from $\mathscr{A}$ to $\mathscr{B}(\mathbb{R}^n)$.

The $\sigma$- Algebra generated of all open sets on $\mathbb{R}^n$ is called the *Borel $\sigma$- Algebra* which we denote as usual by $\mathscr{B}(\mathbb{R}^n)$. Let $\mu$ be a probability measure on $\mathbb{R}^n$. Indeed, $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mu)$ is a special case that probability space on $\mathbb{R}^n$. A function $f$ mapping from $(\mathcal{D}, \mathscr{D}, \mu)$ into $(\mathcal{E}, \mathscr{E}, \nu)$ is *measurable*, if its collection of the inverse image of $\mathscr{E}$ is a subset of $\mathscr{D}$. A *random variable* is a $\mathbb{R}^n$-valued measurable function on some probability space. Let $\mathcal{P}$ represent a probability measure, recall that in probability theory, for $B \in \mathscr{B}(\mathbb{R}^n)$ we call $\mathcal{P}\left[\{X \in B\}\right]$ the *distribution* of $X$. We write also $\mathcal{P}_X\left[\cdot\right]$ or $\mathcal{P}\left[X\right]$ for convenience for those notations .

**Definition 2.4.** Let $(\Omega, \mathscr{A}, \mathcal{P})$ be a probability space. A $n$-dimensional *stochastic process* $(X_t)_t$ is a family of random variable such that $X_t(\omega) : \Omega \longrightarrow \mathbb{R}^n, \forall t \in T$, where $T$ denotes the set of Index of Time.

We set $T = \mathbb{R}$ for without specificing. Some basically definitions, which are needed in following sections, are given.

**Definition 2.5.** A stochastic process $(X_t)_{t\in T}$ is said to be *stationary*, if the joint distribution

$$\mathcal{P}\left[X_{t_1}, \ldots, X_{t_n}\right] = \mathcal{P}\left[X_{t_1+\tau}, \ldots, X_{t_n+\tau}\right]$$

for $t_1, \dots, t_n$ and $t_1 + \tau, \dots, t_n + \tau \in \mathbb{R}$.

**Definition 2.6.** Let $(X_t)_t$ be a stochastic process.

$$\varsigma_X(t, s) := \mathrm{Cov}(X_t, X_s)$$

is called *autocovariance* between $s, t$ and

$$\eta_X(t, s) := \frac{\mathrm{Cov[X_t, X_s]}}{\sqrt{\mathrm{Var}[X_t]\mathrm{Var}[X_s]}}$$

is called *autocorrelation* betwenn $s, t$.

**Definition 2.7.** A stochastic process $(X_t)_t$ is said to be *weak stationary* if

$$\mathrm{E}[X_t] = \mathrm{E}[X_{t+\tau}]$$

and

$$\varsigma_X(t, s) = \varsigma_X(t - s, 0)$$

for $\tau, s \in \mathbb{R}$.

Remark that, weak stationarity is more general as stationarity. If $(X_t)_t$ is weak stationary process, we write $\varsigma_X(\tau)$ for $\varsigma_X(t + \tau, t)$ for any $t$. $\eta_X(\tau)$ is used in the same way.

We use a notation $X \sim Y$ represents $X$ equals $Y$ *in distribution.*

**Definition 2.8.** A stochastic process $(X_t)_t$ is said to be $\alpha$*-self similar* if $(X_{ct_1}, \dots, X_{ct_k}) \sim (c^\alpha X_{t_1}, \dots, c^\alpha X_{t_k})$ for any $t_1, \dots, t_k \in \mathbb{R}$ and $c \in \mathbb{R}_+$.

## 2.2 Normal Distribution and Gaussian Process

**Definition 2.9** (1-dimensional normal distribution)**.** A $\mathbb{R}$-valued random variable $X$ is said to be *standard normal distributed* or *standard Gaussian*, if its distribution can be described as

$$\mathcal{P}\left[X \leq x\right] = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du \tag{2.1}$$

for $x \in \mathbb{R}$.

The integrand of (2.1) is also called *density function* of a standard Gaussian random variable.

**Definition 2.10.** A $\mathbb{R}$-valued random variable $X$ is said to be *normally distributed* or *Gaussian* with a *expected value* $\mu$ and a *variance* $\sigma^2$, if

$$(X-\mu)/\sigma$$

is standard Gaussian for $\sigma > 0$.

**Proposition 2.11.** Let $X$ be a $\mathbb{R}$-valued Gaussian random variable with expected value $\mu$ and variance $\sigma^2$, then it is distributed as

$$\mathcal{P}[X \leq x] = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{(u-\mu)^2}{2\sigma^2}}\, du$$

**Proof.** Suppose $X = \sigma Y + \mu$ with $Y$ standard Gaussian. We denote this mapping by $g(y) : y \to \sigma y + \mu$ and give the inverse $g^{-1}(x) : x \to \frac{(x-\mu)}{\sigma}$. The distribution function of $X$ is

$$\begin{aligned}
\int_{\Omega} \mathcal{P}[X \in dx] &= \int_{\Omega} \mathcal{P}[Y \circ g \in dx] \\
&= \int_{\mathbb{R}\circ g} f_Y \circ g^{-1}(x)\, dx \\
&= \int_{\mathbb{R}} \sigma \frac{1}{\sqrt{2\pi}} \exp\{\frac{(\frac{(x-\mu)}{\sigma})^2}{2}\}\, dy \\
&= (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}\, dy,
\end{aligned}$$

where $f_Y$ is density function of $Y$. $\square$

It is denoted by $X \sim (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}\, dx$, if $X$ is standard Gaussian. In order to verifying the behaviour of a normal distributed random variable we use the characteristic function in probability theory, Cf.[1].

**Theorem 2.12.** Let $X$ be a $\mathbb{R}$-valued Gaussian random variable with expected value $\mu$ and variance $\sigma^2$. The characteristic function of $X$

$$\Psi_X(\xi) := \int_{\mathbb{R}} e^{ix\xi} \mathcal{P}\left[X \in dx\right] = e^{i\mu\xi - \frac{1}{2}(\sigma\xi)^2} \tag{2.2}$$

for $\xi \in \mathbb{R}$.

**Proof.** Cf.[17]. We assume firstly $Y$ is standard Gaussian. In terms of the Definion of characteristic function of a standard Gaussian $Y$, integrating its density function over $\mathbb{R}$ we get

$$\Psi_Y(\xi) = \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} e^{iy\xi}\, dy,$$

take differentiating both sides of the equation by $\xi$, then

$$\begin{aligned}
\Psi_Y'(\xi) &= \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} e^{iy\xi} ix \; dy \\
&= (-i) \cdot \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} (\frac{d}{dx} e^{-\frac{y^2}{2}}) e^{iy\xi} \; dy \\
&\overset{part.int.}{=} -\int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} e^{iy\xi} \xi \; dy \\
&= -\xi \Psi_Y(\xi).
\end{aligned}$$

for $\xi \in \mathbb{R}$. Obviously, $\Psi(\xi) = \Psi(0) e^{-\frac{\xi^2}{2}}$ is the solution of the partial differential equation, and $\Psi(0)$ equals 1, hence $\Psi(\xi) = e^{-\frac{\xi^2}{2}}$.

Let $X = \sigma Y + \mu$

$$\begin{aligned}
& \Psi_X(\xi) \\
=\; & \mathrm{E}[e^{i\xi X}] \\
=\; & \mathrm{E}[e^{i\xi(\sigma Y + \mu)}] \\
=\; & e^{i\xi\mu} \mathrm{E}[e^{iy\xi(\sigma Y)}] \\
=\; & e^{i\xi\mu} \mathrm{E}[e^{iy(\xi\sigma)Y}] \\
=\; & e^{i\xi\mu} \Psi_Y(\xi\sigma) \\
=\; & e^{i\mu\xi - \frac{1}{2}(\sigma\xi)^2}
\end{aligned}$$

□

**Definition 2.13.** Let $X$ be a $\mathbb{R}^n$-valued random vector. $X$ is said to be *normally distributed* or *Gaussian*, if for any $d \in \mathbb{R}^n$ such that $d^T X$ is Gaussian in $\mathbb{R}$.

**Definition 2.14.** A stochastic process $(X_t)_{t \in T}$ is said to be *Gaussian process* if the joint distribution of any finite instance is Gaussian, that means $(X_{t_1}, \ldots, X_{t_n})$ has joint Gaussian distribution in $\mathbb{R}^n$ for $t_1, \ldots, t_n \in T$.

The definition immediately shows every instances $X_t$ in Gaussian process is Gaussian.

**Corollary 2.15.** Let $(X_t)_{t \in T}$ be a stochastic process. The following condition is equivalent to Definition 2.14.

$$\sum_j^n c_{t_j} X_{t_j}$$

is Gaussian for any $t_1, \ldots, t_n \in T, c_{t_j} \in \mathbb{R}$ for $j \in 1, \ldots, n$.

**Proof.** It is clear due to Definition 2.13. □

**Lemma 2.16.** Let $X$ be a $\mathbb{R}^n$-valued normally distributed ramdon vector. Then its characteristic function is

$$\mathrm{E}\, e^{i\xi^T X} = e^{i\xi^T m - \frac{1}{2}\xi^T \Sigma \xi}. \tag{2.3}$$

For $\xi \in \mathbb{R}^n$. Where $m \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n\times n}$ are *mean vector* , *covariance matrix* of $X$ respectively. Furthermore, the density function of $X$ is

$$(2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}}\, e^{-\frac{1}{2}(x-m)^T\Sigma^{-1}(x-m)}. \tag{2.4}$$

Remark, the equation (2.3) can also be as definition of characteristic function of a n-dimensional normally distributed random variable. I.e., any normally distributed random variable can be characterized by form of the equation (2.3).

**Proof.** Since $X$ normally distributed on $\mathbb{R}^n$, then $\xi^T X$ is normally distributed on $\mathbb{R}$. Due to the Theorem 2.12, there is

$$\begin{aligned}
\mathrm{E}e^{i\xi^T X} &= \mathrm{E}e^{i\cdot 1\cdot \xi^T X} \\
&= e^{i\mathrm{E}\left[\xi^T X\right] - \frac{1}{2}\mathrm{Var}\left[\xi^T X\right]} \\
&= e^{i\xi^T \mathrm{E}[X] - \frac{1}{2}\xi^T \mathrm{Var}[X]\xi} \\
&= e^{i\xi^T m - \frac{1}{2}\xi^T\Sigma\xi}.
\end{aligned}$$

Moreover, since $\Sigma$ symmetirc and positive definit, there exist $\Sigma^{-1}, \Sigma^{\frac{1}{2}}$ and $\Sigma^{-\frac{1}{2}}$.

$$\begin{aligned}
& (2\pi)^{-\frac{n}{2}}(\det\Sigma)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{ix^T\xi} e^{-\frac{1}{2}(x-m)^T\Sigma^{-1}(x-m)}\, dx \\
= \;& (2\pi)^{-\frac{n}{2}}(\det\Sigma)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{ix^T\xi} e^{i(x-m)^T\xi} e^{-\frac{1}{2}(x-m)^T\Sigma^{-1}(x-m)}\, dx \\
= \;& (2\pi)^{-\frac{n}{2}}(\det\Sigma)^{-\frac{1}{2}} e^{im^T\xi} \int_{\mathbb{R}^n} e^{i(x-m)^T\xi} e^{i(x-m)^T\xi} e^{-\frac{1}{2}(x-m)^T\Sigma^{-1}(x-m)}\, dx \\
\overset{y=\Sigma^{-\frac{1}{2}}x}{=} \;& (2\pi)^{-\frac{n}{2}} e^{im^T\xi} \int_{\mathbb{R}^n} e^{i(\Sigma^{\frac{1}{2}}y)\xi}\, e^{-\frac{1}{2}|y|^2}\, dy \\
= \;& (2\pi)^{-\frac{n}{2}} e^{im^T\xi} \int_{\mathbb{R}^n} e^{iy^T(\Sigma^{\frac{1}{2}}\xi)}\, e^{-\frac{1}{2}|y|^2}\, dy \\
\overset{\text{Fourier transformation}}{=} \;& e^{im^T\xi} e^{-\frac{1}{2}|\Sigma^{\frac{1}{2}}\xi|^2} \\
= \;& e^{im^T\xi} e^{-\frac{1}{2}\xi^T\Sigma\xi}
\end{aligned}$$

In terms of the uniqueness theorem of characteristic function (in [1], p.199, Satz 23.4), then we can deduce (2.4) is density function of $X$. □

**Theorem 2.17.** A linear combination of independent normally distributed random variable (or vector) is Gaussian.

**Proof.** We suppose $X_1, \cdots, X_m$ are independent random vectors on $\mathbb{R}^n$ and $c_1, \cdots, c_m \in \mathbb{R}$. Let have a look at the characteristic function of it,

$$\begin{aligned}
\mathrm{E}e^{i\xi^T \sum_{j=1}^m (c_j X_j)} &\overset{independent}{=} \prod_{j=1}^m \mathrm{E}e^{i\xi^T (c_j X_j)} \\
&= \prod_{j=1}^m \exp\left( i\xi^T \mathrm{E}[c_j X_j] - \frac{1}{2}\xi^T \mathrm{Var}[c_j X_j]\xi \right) \\
&= \exp\left( i\xi^T \mathrm{E}[\sum_{j=1}^m c_j X_j] - \frac{1}{2}\xi^T \sum\nolimits_{j=1}^m Var[c_j X_j]\xi \right) \\
&\overset{independent}{=} \exp\left( i\xi^T \mathrm{E}[\sum_{j=1}^m c_j X_j] - \frac{1}{2}\xi^T \mathrm{Var}[\sum_{j=1}^m c_j X_j]\xi \right),
\end{aligned}$$

which is a form of characteristc function of normal distribution. That means $\sum_{j=1}^m c_j X_j$ is Gaussian. □

**Example 2.18** (Bivariate Normal Distribution). Cf.[14], p.241, Example 8.6. Suppose $S_1, S_2$ are independent random variables and have standard normal distributions. $\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ has standard normal joint distribution since they are independent. We define

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \sigma_1, & 0 \\ \sigma_2\rho, \sigma_2(1-\rho^2)^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \tag{2.5}$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 \in \mathbb{R}, -1 \leq \rho \leq 1$. Again, $Y_1, Y_2$ are Gaussian and the joint distribution $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is also Gaussian. Since $S_1, S_2$ are independent,

$$\begin{aligned}
\mathrm{Var}[Y_1] &= \mathrm{Var}[\sigma_1 S_1] \\
&= \sigma_1^2, \\
\mathrm{Var}[Y_2] &= \mathrm{Var}[\sigma_2\rho S_1] + \mathrm{Var}[\sigma_2(1-\rho^2)^{\frac{1}{2}} S_2] \\
&= \sigma_2^2\rho^2 + \sigma_2^2(1-\rho^2) \\
&= \sigma_2^2, \\
\mathrm{Cov}[Y_1, Y_2] &= \mathrm{E}[(Y_1 - \mathrm{E}[Y_1])(Y_2 - \mathrm{E}[Y_2])] \\
&= \mathrm{E}[Y_1 Y_2 - \mu_1 Y_2 - \mu_2 Y_1 + \mu_1\mu_2] \\
&= \mathrm{E}[(\sigma_1 S_1 + \mu_1)(\sigma_2\rho S_1 + \sigma_2(1-\rho^2)^{\frac{1}{2}} S_2 + \mu_2)] - \mu_1\mu_2 \\
&= \sigma_1\sigma_2 \underbrace{\mathrm{E}[S_1^2]}_{=1} \rho + \mu_1\sigma_2\rho \underbrace{\mathrm{E}[S_1]}_{=0} + \sigma_1\sigma_2(1-\rho^2)^{\frac{1}{2}} \underbrace{\mathrm{E}[S_1 S_2]}_{=\mathrm{E}[S_1]\mathrm{E}[S_2]=0} \\
&+ \mu_1\sigma_2(1-\rho^2)^{\frac{1}{2}} \underbrace{\mathrm{E}[S_2]}_{=0} + \sigma_1 \underbrace{\mathrm{E}[S_1]}_{=0} \mu_2 + \mu_1\mu_2 - \mu_1\mu_2 \\
&= \rho\sigma_1\sigma_2,
\end{aligned}$$

that means the corrlation of $Y_1, Y_2$ is $\rho$. Because of the (2.4), the joint density function

$$f_{Y_1,Y_2}(y_1, y_2) \quad = \quad (2\pi)^{-1}(\det(\Sigma))^{-\frac{1}{2}} \exp\left((y_1 - \mu_1)\Sigma^{-1}(y_2 - \mu_2)\right),$$

where $\Sigma = \begin{pmatrix} \sigma_1^2, & 0 \\ \sigma_2^2\rho^2, \sigma_2^2(1-\rho^2) \end{pmatrix}$

Indeed,

$$\det(\Sigma) = (1-\rho^2)\sigma_1^2\sigma_2^2$$

and

$$\Sigma^{-1} = \frac{\begin{pmatrix} \sigma_2^2(1-\rho^2), & 0 \\ -\sigma_2^2\rho, & \sigma_1^2 \end{pmatrix}}{(1-\rho^2)\sigma_1^2\sigma_2^2}.$$

Namely,

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}\sigma_1\sigma_2} \exp\left(-\frac{1}{2(1-\rho^2)}(z_1^2 - 2\rho z_1 z_2 + z_2^2)\right) \tag{2.6}$$

where $z_1 = \frac{y_1-\mu_1}{\sigma_1}$, $z_2 = \frac{y_2-\mu_2}{\sigma_2}$.

**Corollary 2.19.** Let $Y_1, Y_2$ be $\mathbb{R}$-valued random variables and $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ has a joint normal distribution, then the conditional expected value of $Y_2$ given $Y_1$

$$\mathrm{E}[Y_2|Y_1 = y_1] = \mathrm{E}[Y_2] + \rho(y_1 - \mathrm{E}[Y_1])\frac{\sigma_2}{\sigma 1},$$

and the conditional variance of $Y_2$ given $Y_2$

$$\mathrm{Var}[Y_2|Y_1 = y_1] = \sigma_2^2(1-\rho^2).$$

Where $\sigma_1, \sigma_2$ are standard deviations of $Y_1, Y_2$ and $\rho$ is the correlation of $Y_1, Y_2$.

**Proof.** Recall the equation (2.6), we can specify the joint density function if $\sigma_1, \sigma_2, \rho$ are known. As result of this, $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ has a form of the equation (2.5). Suppose $S_1, S_2$ are independent standard normal distributed random variables. Now we have

$$\begin{aligned} S_1 &\sim \frac{(Y_1 - \mathrm{E}[Y_1])}{\sigma_1} \\ Y_2 &\sim \sigma_2\rho S_1 + \sigma_2(1-\rho^2)^{\frac{1}{2}}S_2 + \mathrm{E}[Y_2], \end{aligned}$$

more precisely,

$$Y_2 \sim \sigma_2\rho\frac{(Y_1 - \mathrm{E}[Y_1])}{\sigma_1} + \sigma_2(1-\rho^2)^{\frac{1}{2}}S_2 + \mathrm{E}[Y_2].$$

Take expectation of both sides,

$$\mathrm{E}[Y_2|Y_1 = y_1] = \sigma_2\rho\frac{(y_1 - \mathrm{E}[Y_1])}{\sigma_1} + \mathrm{E}[Y_2].$$

Now consider

$$\begin{aligned}
\mathrm{Var}[Y_2|Y_1 = y_1] &= \mathrm{E}[(Y_2 - \mu_{Y_2|Y_1})^2|Y_1 = y_1] \\
&= \int_{-\infty}^{\infty} (y_2 - \mu_{Y_2|Y_1})^2 f_{Y_2|Y_1}(y_2, y_1)\ dy_2 \\
&= \int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1)\right]^2 f_{Y_2|Y_1}(y_2, y_1)\ dy_2,
\end{aligned}$$

After multiplying both sides by the density function of $Y_1$ and integrating it by $y_1$, we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \mathrm{Var}[Y_2|Y_1 = y_1] f_{Y_1}(y_1)\, dy_1 \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[y_2 - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1)\right]^2 \underbrace{f_{Y_2|Y_1}(y_2, y_1)\, f_{Y_1}(y_1)}_{f_{Y_1,Y_2}(y_1,y_2)}\, dy_2\ dy_1 \\
\Longleftrightarrow & \\
& \mathrm{Var}[Y_2|Y_1 = y_1] \underbrace{\int_{-\infty}^{\infty} f_{Y_1}(y_1)\, dy_1}_{1} \\
= & \quad \mathrm{E}\left[(Y_2 - \mu_2) - (\frac{\rho\sigma_2}{\sigma_1})(Y_1 - \mu_1)\right]^2
\end{aligned}$$

multiplying right side out, we see

$$\begin{aligned}
\mathrm{Var}[Y_2|Y_1 = y_1] &= \underbrace{\mathrm{E}[(Y_2 - \mu_2)^2]}_{\sigma_2^2} - 2\frac{\rho\sigma_2}{\sigma_1}\underbrace{\mathrm{E}[(Y_1 - \mu_1)(Y_2 - \mu_2)]}_{\rho\sigma_1\sigma_2} \\
&+ \frac{\rho^2\sigma_2^2}{\sigma_1^2}\underbrace{\mathrm{E}[(Y_1 - \mu_1)^2]}_{\sigma_1^2} \\
&= \sigma_2^2 - 2\rho^2\sigma^2 + \rho^2\sigma_2^2 \\
&= \sigma_2^2 - \rho^2\sigma_2^2 \\
&= \sigma_2^2(1 - \rho^2).
\end{aligned}$$

□

**Theorem 2.20.** Let $X$ be a Gaussian random variable, then

$$\mathrm{E}[\exp(\beta X)] = \exp(\beta\mu + \frac{1}{2}\beta^2\sigma^2). \tag{2.7}$$

Where $\mu$ and $\sigma$ are $\mathrm{E}[X]$ and $\mathrm{Var}[X]$ respectively.

**Proof.**

$$
\begin{aligned}
& \mathrm{E}[\exp(\beta X)] \\
= \;& (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp(\beta x) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
= \;& (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp(\beta x) \exp\left(-\frac{(x^2-2x\mu+\mu^2)}{2\sigma^2}\right) dx \\
= \;& (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{x^2-2(\beta\sigma^2+\mu)x+\mu^2}{2\sigma^2}\right) dx \\
= \;& (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{x^2-2(\beta\sigma^2+\mu)x+(\beta\sigma^2+\mu)^2-(\beta\sigma^2+\mu)^2+\mu^2}{2\sigma^2}\right) dx \\
= \;& (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{(x-(\beta\sigma^2+\mu))^2+\mu^2-(\beta\sigma^2+\mu)^2}{2\sigma^2}\right) dx \\
= \;& \exp\left(\frac{(\beta\sigma^2+\mu)^2-\mu^2}{2\sigma^2}\right) \underbrace{(2\pi\sigma^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{(x-(\beta\sigma^2+\mu))^2}{2\sigma^2}\right) dx}_{1} \\
= \;& \exp\left(\frac{\beta^2\sigma^4+2\mu\beta\sigma^2}{2\sigma^2}\right) \\
= \;& \exp(\mu\beta+\frac{1}{2}\beta^2\sigma^2)
\end{aligned}
$$

□

## 2.3 Brownian Motion

The Brownian motion was first introduced by Bachelier in 1900 in his PhD thesis. Now we give the common definition of it.

**Definition 2.21.** Let $(B_t)_{t\geq 0}$ be a $\mathbb{R}^n$-valued stochastic process. $(B_t)$ is called *Brownian motion* if it satisfies the following conditions:

(i) $B_0 = 0$ a.s. .

(ii) $(B_{t_1} - B_{t_0}), \dots, (B_{t_n} - B_{t_{n_1}})$ are independet for $0 = t_0 < t_1 < \cdots < t_n$ and $n \in \mathbb{N}$.

(iii) $B_t - B_s \sim B_{t-s}$, for $0 \leq s \leq t < \infty$.

(iv) $B_t - B_s \sim \mathcal{N}(0, t-s)^{\otimes n}$.

(v) $B_t$ is continuous in $t$ a.s. .

A usual saying for *(ii)* and *(iii)* is the Brownian motion has independent, stationary increments. In *(iv)*, $\mathcal{N}$ represent a random variable which has a normal distribution. $B_t$ is normally distributed due to *(ii)*. It is clear that the increments of Brownian motion is stationary.

**Proposition 2.22.** Let $(B_t)$ be $\mathbb{R}$-valued Brownian motion. Then the covariance of $B_m, B_n$ for $m, n \geq 0$ is $m \wedge n$.

**Proof.** Without loss of generality, we assume that $m \geq n$, then

$$\begin{aligned} \mathrm{E}[B_m B_n] &= \mathrm{E}[(B_m - B_n)B_n] + \mathrm{E}[B_n^2] \\ &= \mathrm{E}[B_m - B_n]\mathrm{E}[B_n] + n \\ &= n. \end{aligned}$$

□

**Proposition 2.23.** Let $(B_t)$ be $\mathbb{R}$-valued Brownian motion. Then $B_{cm} \sim c^{\frac{1}{2}} B_m$.

**Proof.** Because $B_m$ is normal distributed for any $m > 0$, we then get

$$\begin{aligned} \mathrm{E}[e^{i\xi B_{cm}}] &= e^{-\frac{1}{2}cm\xi^2} \\ &= e^{-\frac{1}{2}m(c^{\frac{1}{2}}\xi)^2} \\ &= \mathrm{E}[e^{i\xi c^{\frac{1}{2}} B_m}]. \end{aligned}$$

□

**Theorem 2.24.** A $\mathbb{R}$-valued Brownian motion is a Gaussian process.

**Proof.** The following idea using the independence of increments to prove the claim come from [17]. We choose $0 = t_0 < t_1 < \cdots < t_n$, for $n \in \mathbb{N}$. Define $V = (B_{t_1}, \ldots, B_{t_n})^T$, $K = (B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}})^T$ and $A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$. Let us see the characteristic function of $V$,

$$
\begin{array}{ll}
 & \mathrm{E}[e^{i\xi^T V}] \\
= & \mathrm{E}[e^{i\xi^T AK}] \\
= & \mathrm{E}[e^{iA^T \xi K}] \\
= & \mathrm{E}[\exp(i(\xi^{(1)} + \cdots + \xi^{(n)}, \xi^{(2)} + \cdots + \xi^{(n)}, \cdots, \xi^{(n)})] \\
\cdot & (B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})^T) \\
\overset{ind.increments}{=} & \prod_{j=1}^{n} \mathrm{E}[\exp(i(\xi^{(j)+\cdots+\xi^{(n)}})(B_{t_j} - B_{t_{t-1}}))] \\
\overset{stat.increments}{=} & \prod_{j=1}^{n} \exp(-\frac{1}{2}(t_j - t_{j-1})(\xi^{(j)} + \cdots + \xi^{(n)})^2) \\
= & \exp\left(-\frac{1}{2}\sum_{j=1}^{n}(t_j - t_{j-1})(\xi^{(j)} + \cdots + \xi^{(n)})^2\right) \\
= & \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n} t_j(\xi^{(j)} + \cdots + \xi^{(n)})^2 - \sum_{j=1}^{n} t_{j-1}(\xi^{(j)} + \cdots + \xi^{(n)})^2\right)\right) \\
= & \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n-1} t_j((\xi^{(j)} + \cdots + \xi^{(n)})^2 - (\xi^{(j+1)} + \cdots + \xi^{(n)})^2) + t_n(\xi^{(n)})^2\right)\right) \\
= & \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{n-1} t_j\xi^{(j)}(\xi^{(j)} + 2\xi^{(j+1)} + \cdots + 2\xi^{(n)}) + t_n(\xi^{(n)})^2\right)\right) \\
= & \exp\left(-\frac{1}{2}\left(\sum_{j,h=1}^{n} (t_j \wedge t_h)\xi^{(j)}\xi^{(h)}\right)\right).
\end{array}
$$

Recall with Proposition 2.3, $(t_j \wedge t_h)_{t,h=1,\ldots,n}$ is the covariance matrix of $V$ and therefore it is symmetric and positive definit. The mean vector of it is zero, then we have been proved that the characteristic function is a form of some normal distributed random vector, i.e., $V$ is Gaussian. $\square$

Schilling gave in his lecture [17] the relationship between a one-dimensional Brownian motion and a n-dimensional Brownian motion. In fact, $(B_t^{(l)})_{l=1,\ldots,n}$ is Brownian motion if and only if $B_t^{(l)}$ is Brownian motion and all of the component are independent. Using this independence and the theorem of fubini in the characteristic function for high dimensional Brownian motion we can say a n-dimensional Brownian motion is also a Gaussian process.

**Definition 2.25.** Let $(X_t)_{t\in T}$ be a stochastic process. $(Y_t)_{t\in T}$ is defined on the same probability space as $(X_t)_{t\in T}$ and said to be *modification* of $(X_t)_{t\in T}$, if

$$\mathcal{P}[X_t = Y_t] = 1 \quad \forall \quad t \in T.$$

**Theorem 2.26** (Kolmogorov Chentsov). Let $(X_t)_{t\geq 0}$ be a stochastic process on $\mathbb{R}^n$ such that

$$[|X_j - X_k|^\alpha] \leq c|j-k|^{1+\beta} \quad \forall \quad j,k \geq 0 \quad \text{and} \quad j \neq k,$$

for $\alpha, \beta > 0, c < \infty$. Then $(X_t)_t$ has a modification $(Y_t)_t$ with continuous sample path such that

$$\mathrm{E}[(\frac{|Y_j - Y_k|}{|j-k|^\gamma})^\alpha] < \infty$$

for all $\gamma \in (0, \frac{\beta}{\alpha})$.

**Proof.** See [11], p.519. □

**Lemma 2.27.** Let $(B_t)_{t\geq 0}$ be Brownian motion. Then

$$\mathrm{E}[B_t^{2k}] = (2k-1)!!t^{2k}$$

for $k \in \mathbb{N}_0$.

**Proof.** Cf.[17]. Taking expectation of $B_t^{2k}$, we get

$$\begin{aligned}
\mathrm{E}[B_t^{2k}] &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2t}}\, dx \\
&\overset{x=\sqrt{2ty}}{=} \frac{2^k t^k}{\sqrt{\pi}} \int_0^\infty y^{k-\frac{1}{2}} e^{-y}\, dy \\
&= \frac{2^k t^k}{\sqrt{\pi}} \int_0^\infty y^{k+\frac{1}{2}-1} e^{-y}\, dy \\
&= \frac{2^k t^k}{\sqrt{\pi}} \Gamma(k+\frac{1}{2}) \\
&= \frac{2^k t^k}{\sqrt{\pi}} \Gamma(\frac{1}{2}) \prod_{j=1}^{k} (j - \frac{1}{2}) \\
&= 2^k t^k \prod_{j=1}^{k} (\frac{2j-1}{2}) \\
&= (2k-1)!! \cdot t^k
\end{aligned}$$

□

**Corollary 2.28.** Let $(B_t)_{t\geq 0}$ be Brownian motion. Then $B_t$ is $\gamma$-Hölder continuous on a compact scale almost surely for all $\gamma < \frac{1}{2}$.

**Proof.** Because of Lemma 2.27, we have

$$\begin{aligned}
\mathrm{E}[(B_t - B_s)^{2k}] &= \mathrm{E}[B_{t-s}^{2k}] \\
&= (2k-1)!! \cdot |t-s|^k.
\end{aligned}$$

In terms of the Theorem of Kolmogorov Chenstov, $B_t$ is $\gamma$-Hölder continuous a.s. for $\gamma \in (0, \frac{k}{2k})$. $\square$

# 3 Stable Measures and Stable Integrals

In order to represent a integration form of fractional Brownian motion, we deal with the stable integral in this section. In fact, fractional Brownian motion is a Gaussian process with zero mean. To show Gaussian properties of it, we define it by a stable integral which can imaged as stochastic process of stable variables on time.

## 3.1 Stable Variables

**Definition 3.1.** Let $X$ be a random variable. $X$ is said to have a stable distribution, if there exist $0 < \gamma \leq 2, \delta \geq 0, -1 \geq \kappa \geq 1, \theta \in \mathbb{R}$ such that its characteristic function can be described as following

$$\mathrm{E}[\exp i\xi X] = \begin{cases} \exp\{i\xi\theta - |\delta\xi|^\gamma(1 - i\kappa \cdot sgn(\xi)\tan\frac{\gamma\pi}{2})\}, & \text{if} \quad \gamma \in (0,1) \cup (1,2], \\ \exp\{i\xi\theta - |\delta\xi|(1 + i\frac{2}{\pi}\kappa \cdot sgn(\xi)\ln|\xi|)\}, & \text{if} \quad \gamma = 1. \end{cases} \tag{3.1}$$

Where

$$sgn(x) = \begin{cases} 1, & \text{if} \quad x > 1 \\ 0, & \text{if} \quad x = 0 \\ -1, & \text{if} \quad x < 0. \end{cases}$$

Notice, we write $\Lambda(\gamma, \kappa, \theta, \delta)$ for one random variable whose characteristic function equals (3.1).

**Theorem 3.2.** $X$ is Gaussian if and only $X \sim \Lambda(\gamma, \kappa, \theta, \delta)$ with $\gamma = 2$.

**Proof.** In the one hand, if $X$ is Gaussian, according to the charateric function of a Gaussian variable, $\gamma$ must equal 2. On the other hand, if $\gamma = 2$, then $i\kappa \cdot sgn(\xi)\tan\frac{\gamma\pi}{2}$ vanishes since $\tan(\pi) = 0$. Therefore, $X$ is Gaussian because $\mathrm{E}[\exp i\xi X] = \exp\{i\xi\theta - |\delta\xi|^2\}$. □

Remark, if $\gamma = 2$, then $\kappa$ is irrelevant in Definition. We specific $\kappa = 0$ without loss of generality. For instance, $B_t \sim \Lambda(2, 0, 0, \frac{\sqrt{2t}}{2})$ when $(B_t)_t$ is Brownian motion.

**Definition 3.3.** A random variable $X$ is said to be *symmetric* if $X$ and $-X$ have the same distribution.

**Proposition 3.4.** Let $X$ be have a stable distribution. $X$ is *symmetric* if and only if $X \sim \Lambda(\gamma, 0, 0, \delta)$. I.e. its characteristic function has the form

$$\mathrm{E}[\exp\{i\xi X\}] = \exp\{-|\delta\xi|^\gamma\} \tag{3.2}$$

**Proof.** The Definition of symmetricity implies

$$\begin{aligned}
& \exp\{i\xi\theta - |\delta\xi|^\gamma(1 - i\kappa \cdot sgn(\xi)\tan\frac{\gamma\pi}{2})\} \\
= \; & \mathrm{E}[i\xi X] \\
= \; & \mathrm{E}[i\xi(-X)] \\
= \; & \mathrm{E}[i(-\xi)X] \\
= \; & \exp\{i(-\xi)\theta - |\delta\xi|^\gamma(1 - i\kappa \cdot sgn(-\xi)\tan\frac{\gamma\pi}{2})\},
\end{aligned}$$

for $\xi \in \mathbb{R}$. This requires $\theta = \kappa = 0$. □

**Corollary 3.5.** Let $(B_t)_t$ be Brownian motion, then $B_t$ has a symmetric stable distribution.

**Proof.** It is clear due to the previous Proposition. □

## 3.2 Stable Random Measures

In this subsection we suppose $(\Omega, \mathscr{A}, \mathcal{P}), (\mathrm{D}, \mathscr{D}, \mu)$ are probability spaces, $\kappa(\cdot) : \mathrm{D} \to [-1, 1]$ is a measurable function.

**Definition 3.6.** Let $\nu$ be a measure such that

$$\nu : \mathscr{D} \to \mathscr{E}.$$

$\nu$ is said to be *independently scattered*, if $\nu[D_1], \ldots, \nu[D_n]$ are independent for any $D_1, \ldots, D_n$ *disjoint* $\in \mathscr{D}$ .

For the next definition we need a notation

$$\mathscr{G} = \{D \in \mathscr{D} : \mu[D] < \infty\}. \tag{3.3}$$

**Definition 3.7.** Let $\nu$ be a set function such that

$$\nu : \mathscr{G} \to \mathcal{L}^0(\Omega).$$

$\nu$ is said to be *independently scattered*, if $\nu[D_1], \ldots, \nu[D_n]$ are independent for any $D_1, \ldots, D_n$ *disjoint* $\in \mathscr{D}$ .

**Definition 3.8.** Let $\nu$ be an independent cattered and $\sigma$-additive set function, $\nu$ is said to be *stable random measure* on $(D, \mathscr{D})$ with control measure $\mu$, degree $\gamma$ and skewness intensity $\kappa(\cdot)$ if

$$\nu[F] \sim \Lambda\left(\gamma, \frac{\int_F \kappa(x)\,\mu[dx]}{\mu[F]}, 0, (\mu[F])^{\frac{1}{\gamma}}\right) \tag{3.4}$$

for $F \in \mathscr{D}$.

Samorodnitsky and Taqqu show the existence of stable measures, see [13], pp.119∼120.

**Example 3.9.** Suppose $[0,T]$ is a index set and $0 = t_0, t_1, \ldots, t_k \in [0,T]$ for $k \in \mathbb{N}$. We show the mapping $\nu : \mathscr{B}([0,T]) \to \mathcal{L}^0(dt)$, where $\nu[A_j](\omega) := B_{t_{j+1}}(\omega) - B_{t_j}(\omega), A_j = [t_j, t_{j+1})$.
Firstly, we show $\nu$ is independently scattered and $\sigma$-additive. We take $\{A_j\}$ such that $\cup_{j=1}^{\infty} A_j = [0,T]$. $\{\nu[A_k]\}_{k=1}^{\infty}$ has independent elements since $B_{t_1} - B_{t_0}, \ldots, B_{t_{j+1}} - B_{t_j}$ are independent.
Secondly,

$$\begin{aligned}\nu[(\cup_{j=1}^{\infty} A_j)] &= B_T - B_1 \\ &= \sum_{j=1}^{\infty}(B_{t_{j+1}} - B_{t_j}) \\ &= \sum_{j=1}^{\infty} \nu[A_j].\end{aligned}$$

Finally,

$$\begin{aligned}\mathrm{E}[\exp(i\xi\nu[A_j])] &= \mathrm{E}[\exp(i\xi(B_{t_{j+1}} - B_{t_j})] \\ &= \exp(-\frac{(t_{j+1} - t_j)\xi^2}{2})\end{aligned}$$

Comparing with (3.1), we deduce the control measure must be $\frac{|\cdot|}{2}$. In fact, $\nu[A_j] \sim \Lambda(2, 0, 0, \frac{\sqrt{2|t_{j+1} - t_j|}}{2})$.

### 3.3 Stable Integrals

Samorodnitsky and Taqqu defined an Integral with respect to stable measure as stochastic process in [13]. The stable Integral is given as

$$\int_F f(x)\,\nu(dx)(\omega) \tag{3.5}$$

with $f : F \to \mathbb{R}$ is a measurable function such that

$$\begin{cases} \int_F |f(x)|^{\gamma} \mu(dx) < \infty, & \text{if} \quad \gamma \in (0,1) \cup (1,2], \\ \int_F |\kappa(x) f(x) \ln |f(x)|| \mu(dx) < \infty, & \text{if} \quad \gamma = 1, \end{cases} \tag{3.6}$$

, $\gamma, \mu, \kappa$ are, respectively, degree, control measure and skewness intensity of the stable measure $\nu$.

Some properties of the stable function are given by Samorodnitsky and Taqqu.

**Proposition 3.10.** Let $J(f)$ be a stable integral as form of (3.5). Then

$$J(f) \sim \Lambda(\gamma, \kappa, \theta, \delta)$$

with degree, control measure, skewness intensity, respectively,

$$\begin{aligned}
\gamma &\in (0, 2], \\
\kappa &= \frac{\int_F \kappa(x)|f(x)|^\gamma \cdot sgn(f(x))\mu(dx)}{\int_F |f(x)|^\gamma \mu(dx)}, \\
\theta &= \begin{cases} 0, \quad \text{if} \quad \gamma \in (0,1) \cup (1,2], \\ -\frac{2}{\pi} \int_F \kappa(x) f(x) \ln |f(x)| \mu(dx), \quad \text{if} \quad \gamma = 1, \end{cases} \\
\delta &= \left( \int_F |f(x)^\gamma| \mu(dx) \right)^{\frac{1}{\gamma}},
\end{aligned}$$

of the stable measure $\nu$.

**Proof.** See.[13], p.124, Proposition 3.4.1 . □

**Proposition 3.11.** The stable integral is linear, in fact,

$$J(c_1 f_1 + c_2 f_2) \overset{a.s.}{=} c_1 J(f_1) + c_2 J(f_2) \tag{3.7}$$

for any $f_1, f_2$ integrable with respect to some stable measure and real numbers $c_1, c_2$.

**Proof.** See [13], p.117, Property 3.2.3 . □

# 4 Fractional Brownian Motion

The fractional Brownian motion(FBM) was defined by Kolmogorov primitively. After that Mandelbrot and Van Ness has presented the work in detail. This section is concerned with the definition and some properties of it.

## 4.1 Definition of Fractional Brownian Motion

Mandelbrot and Van Ness [15] gave a integration representation of FBM.

**Definition 4.1.** Let $(U_H(t))_{t\in\mathbb{R}}$ be a $\mathbb{R}$-valued stochatstic process and $H$ be such that $0 < H < 1$. $(U_H(t))$ is said to be *fractional Brownian motion* if

$$U_H(t) - U_H(s) \quad = \quad \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{\mathbb{R}} \mathbb{1}_{\{t>u\}}\cdot(t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{s>u\}}(-u)^{H-\frac{1}{2}}\ dB_u\right) \tag{4.1}$$

for $t \geq s, t, s \in \mathbb{R}$. Where $(B_u)$ is defined as two-sides Brownian motion and the integral is in sense of stable integral as in previous section. $H$ is called *Hurst exponent* or *Hurst index* of FBM.

As usual, we set $U_H(0) = 0$, then equation (4.1) is equivalent to

$$U_H(t) \quad = \quad \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{\mathbb{R}} \mathbb{1}_{\{t>u\}}\cdot(t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u<0\}}(-u)^{H-\frac{1}{2}}\ dB_u\right). \tag{4.2}$$

**Lemma 4.2.** The equation (4.2) is well-defined, $U_H(t)$ has stable distribution and

$$U_H(t) \sim \Lambda(2, 0, 0, \frac{1}{\Gamma(H+\frac{1}{2})}(\int_{\mathbb{R}} |f(u)^2|\ \frac{du}{2})^{\frac{1}{2}}),$$

Where $f(x)$ is the integrand of integral in (4.2).

**Proof.** Firstly, $B_t$ is Gaussian and symmetric stable measure with zero mean and $\frac{|\cdot|}{2}$ is the control measure of it shown in Example 3.9.
Secondly, by $H = \frac{1}{2}$, $\int_{\mathbb{R}} f^2 \frac{du}{2} = \frac{1}{2}\int_0^{|t|} du = \frac{1}{2}|t| < \infty$. By $H \neq \frac{1}{2}$, we deal it with Taylor expansion. As $u$ goes to, $-\infty$ $f(u) = -(H-\frac{1}{2})(-u)^{H-\frac{3}{2}} + o(u)$. Where $o(u)$ tends to zero when $u$ near around $-\infty$, because $H - \frac{3}{2} < 0$ then $f(u)$ is square integrable around $-\infty$. As $u$ goes to $t$, $f(u) \propto \mathbb{1}_{\{t>u\}}(t-u)^{H-\frac{1}{2}}$. Hence, $f(u)$ is also square integrable around $u = t$. It is clear $f(u) = 0$ when $u$ at around $\infty$. Then it satisfies the condition $\int_{-\infty}^{\infty} f^2(u)\ \frac{du}{2} < \infty$.
Finally, in terms of Proposition 3.10, we get the claim. $\square$

Remark that, if we take $H = \frac{1}{2}$, choosing a restriction of the integrand on $\mathbb{R}_+$, $U_{\frac{1}{2}}(t) = \frac{1}{\Gamma(H+\frac{1}{2})}B_t$ is a Brownian motion.

**Lemma 4.3.** Let $(U_H(t))_t$ be a FBM. Then $U_H(t) \sim \mathcal{N}(0, \frac{1}{\Gamma(H+\frac{1}{2})^2}(\int_{\mathbb{R}} |f(u)^2|\, du))$.

**Proof.** In terms of Lemma (4.2), $\mathrm{E}[i\xi U_H(t)] = \exp\{-\xi^2 \frac{1}{2\Gamma(H+\frac{1}{2})^2}(\int_{\mathbb{R}} |f(u)^2|\, du)\}$. The rest is clear thanks to the form of characteristic function of a Gaussian random variable. □

Notice that we can also define $U_H(t)$ as Itó integral by 4.2, then is has the same mean and variance as defined by stable integral. Since $U_H(t)$ is Gaussian, both of two versions have the same distribution. Follwing properties remains true also by Ito integral version.

**Corollary 4.4.** $U_H(t) - U_H(s) \sim \mathcal{N}(0, \frac{1}{\Gamma(H+\frac{1}{2})^2}(\int_{\mathbb{R}} |f_t(u) - f_s(u)|^2\, du)$

**Proof.** This Corollary follows Proposition 3.7 and Lemma 4.3. □

**Theorem 4.5.** Let $(U_H(t))_t$ be a FBM. Then $U_H(t)$ has an expected value 0 and variance $\frac{1}{(\Gamma(H+\frac{1}{2}))^2} t^{2H}\, \mathrm{E}U_H^2(1)$ for any $t \in \mathbb{R}$.

**Proof.** It is clear that $U_H$ is Gaussian with zero mean due to Lemma 4.2. We suppose that $t \geq s \geq 0, c(H) = \frac{1}{(\Gamma(H+\frac{1}{2}))^2}$.

$$
\begin{aligned}
& \mathrm{E}[(U_H(t) - U_H(s))^2] \\
\overset{\text{Corollary 4.4}}{=} \quad & c(H)\mathrm{E}[\int_{\mathbb{R}} \left([\mathbb{1}_{\{t>u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{s>u\}} \cdot (s-u)^{H-\frac{1}{2}}\right)^2 du] \\
= \quad & c(H)\mathrm{E}[\int_{\mathbb{R}} \left(\mathbb{1}_{\{t-s>u\}} \cdot (t-s-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{0>u\}} \cdot (-u)^{H-\frac{1}{2}}\right)^2 du] \\
\overset{m=t-s}{=} \quad & c(H)\mathrm{E}[\int_{\mathbb{R}} \left(\mathbb{1}_{\{m>u\}} \cdot (m-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{0>u\}} \cdot (-u)^{H-\frac{1}{2}}\right)^2 du] \\
\overset{u=ml}{=} \quad & c(H)\mathrm{E}[\int_{\mathbb{R}} \left(\mathbb{1}_{\{m>ml\}} \cdot (m-ml)^{H-\frac{1}{2}} - \mathbb{1}_{\{0>ml\}} \cdot (-ml)^{H-\frac{1}{2}}\right)^2 m \cdot dl] \\
= \quad & c(H)\mathrm{E}[\int_{\mathbb{R}} \left(\mathbb{1}_{\{1>l\}} \cdot (1-l)^{H-\frac{1}{2}} - \mathbb{1}_{\{0>l\}} \cdot (-l)^{H-\frac{1}{2}}\right)^2 \cdot m^{2H-1} \cdot m\, dl] \\
= \quad & c(H) m^{2H} \mathrm{E}[U_H(1)^2] \\
= \quad & c(H)(t-s)^{2H} \mathrm{E}[U_H(1)^2] \qquad (4.3)
\end{aligned}
$$

Using the same calculation, we get

$$
\mathrm{E}[(U_H(t)^2] = c(H) t^{2H} \mathrm{E}[U_H(1)^2]. \qquad (4.4)
$$

(4.4) is variance of $U_H(t)$ due to $\mathrm{E}[(U_H(t)] = 0$. □

To normalize the variance, a definition of standard FBM is given.

**Definition 4.6.** A stochastic process $(U_H(t))_t$ is said to be a *standrad fractional Brownian motion*(sFBM) if

$$U_H(t) = \hat{c}(H) \int_{\mathbb{R}} \mathbb{1}_{\{t \geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u \leq 0\}} (-u)^{H-\frac{1}{2}} \, dB_u \,. \tag{4.5}$$

Where $\hat{c}(H) = \frac{1}{(\Gamma(H+\frac{1}{2})^2)\mathrm{E}[U_H(1)^2]}$.

We consider from now on sFBM as FBM.

**Theorem 4.7.** Let $(U_H(t))_t$ be a FBM. The covariance of $U_H(t)$ and $U_H(s)$ is $\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ for $t, s \in \mathbb{R}$.

**Proof.** Cf.[15], Theorem 5.3 .

$$\begin{aligned}
\mathrm{Cov}[U_H(t), U_H(s)] &= \mathrm{E}[U_H(t)U_H(s)] \\
&= \frac{1}{2}\left(\mathrm{E}[U_H(t)^2] + \mathrm{E}[U_H(s)^2] - \mathrm{E}[(U_H(t) - U_H(s))^2]\right) \\
&\overset{(4.4)}{=} \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})
\end{aligned} \tag{4.6}$$

□

**Theorem 4.8.** $(U_H(t))_t$ is Gaussian process.

**Proof.** We just need to prove that for any finite linear combination of $(U_H(t))_t$ is Gaussian. We take $t_1, \ldots, t_k \in \mathbb{R}, c_1, \ldots, c_k \in \mathbb{R}$ and the stable integral $J(f)$ is a linear functional with $\gamma = 2, \kappa = 0, \theta = 0, \delta = (\frac{1}{2}\int_{-\infty}^{\infty} f^2(u)\, du)^{\frac{1}{2}}$ due to Corollary 3.7. Suppose $f_1, \ldots, f_k$ are integrands of the form of stable integral of $U_H(t_1), \ldots, U_H(t_k)$.
Consider now, according to the Minkowski inequality,

$$\begin{aligned}
\int_{-\infty}^{\infty} (\sum_{j=1}^{k} c_j f_j)^2 \, du &\leq \sum_{j=1}^{k} \underbrace{\int_{-\infty}^{\infty} (c_j f_j)^2 \, du}_{<\infty} \\
&< \infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{j=1}^{k} c_j U_H(t_j) &= \sum_{j=1}^{k} c_j J(f_j) \\
&= J(\sum_{j=1}^{k} c_j f_j) \\
&\sim \Lambda(2, 0, 0, (\frac{1}{2}\int_{-\infty}^{\infty} (\sum_{j=1}^{k} c_j f_j)^2 \, du)^{\frac{1}{2}})
\end{aligned}$$

is Gaussian and the rest follows from Corollary 2.15. □

**Corollary 4.9.** Let $(U_H(t))_t$ be a FBM, then $(U_H(t))_t$ has stationary and H-self similar increments .

**Proof.** Assume that $s \geq u$. Because the joint distribution of $(U_H(s), U_H(u))^T$ is Gaussian, $(1, -1) \cdot (U_H(s), U_H(u))^T$ is Gaussian. From (4.3), $U_H(t_k + \tau) - U_H(s_k + \tau) \sim U_H(t_k) - U_H(s_k) \sim \mathcal{N}(0, (t_k - s_k)^{2H})$ for $k \in \{1 \ldots d\}$. Corresponding to (2.3),

$$\mathrm{E}[i \sum_{k=1}^{d} \xi_k (U_H(t_k + \tau) - U_H(s_k + \tau))] = \mathrm{E}[i \sum_{k=1}^{d} \xi_k (U_H(t_k) - U_H(s_k))]$$

due to $(U_H(t_{k+\tau} - s_{k+\tau}))_{k=1}^d, (U_H(t_k - s_k))_{k=1}^d$ have same expected vector and covariance matrix in their characteristic function. Hence, $(U_H(t))$ has stationary increments.
To show FBM has H-self similar increments, we have to prove
$(U_H(zt_1), U_H(zt_2), \ldots, U_H(zt_n))) \sim (z^H U_H(t_1), z^H U_H(t_2), \ldots, z^H U_H(t_n))$ for any $z > 0$. Obviously, the former and the latter of the term are Gaussian and $\mathrm{Var}[U_H(zt_i), U_H(zt_j)] = \mathrm{Var}[z^H U_H(t_i), z^H U_H(t_j)] = \frac{1}{2} z^{2H}(t_i^{2H} + t_j^{2H} - |t_i - t_j|^{2H})$. Thus they have the same expected vector and covariance matrix in their characteristic function. Using uniqueness theorem we have proved the claim. □

## 4.2 Regularity

**Theorem 4.10** (Kolmogorov Chentsov)**.** FBM has almost surely continuous sample path.

**Proof.** Cf.[15] Proposition 4.1 . Let $(U_H(t))_t$ be a FBM with Hurst index $H$. Fix $\alpha$ such that $1 < \alpha H$. Let us have a look at the expectation of $(U_H(t) - U_H(s))^\alpha$ using the same calculation in (4.3)

$$\begin{aligned} \mathrm{E}[(U_H(t) - U_H(s))^\alpha] &= |t - s|^{\alpha H} \cdot \underbrace{\mathrm{E}\left( \int_{\mathbb{R}} \mathbb{1}_{\{1>u\}} \cdot (1-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u<0\}} (-u)^{H-\frac{1}{2}} \, dB_u \right)^\alpha}_{c(\alpha,H)} \\ &= c(\alpha, H) \cdot |t - s|^{\alpha H}. \end{aligned} \tag{4.7}$$

We choose $\beta = \alpha H - 1$ and $\gamma \in (0, H - \frac{1}{\alpha})$ then the rest follows from Theorem 2.26 . □

Remark, $(U_H(t))_t$ is, in fact, $\gamma$-Hölder continuous with $\gamma < H$ almost surely.

**Theorem 4.11.** The sample path of FBM is almost surely not differentiable.

**Proof.** Cf. [15] Proposition 4.2 . Fix $\omega \in \Omega$, we assume $c > 0, t_j \to s$.

$$\begin{aligned} & \mathcal{P}[\limsup_{t\to s} |\frac{U_H(t) - U_H(s)}{t-s}| > c] \\ = \quad & \mathcal{P}[\lim_{j\to\infty} \sup_{t_j \neq s} |\frac{U_H(t_j) - U_H(s)}{t_j - s}| > c] \end{aligned} \tag{4.8}$$

Since continuity of measures from above, then

$$\begin{aligned} (4.8) \quad = \quad & \lim_{j\to\infty} \mathcal{P}[\sup_{t_j\neq s} |\frac{U_H(t_j) - U_H(s)}{t_j - s}| > c] \\ \geq \quad & \lim_{j\to\infty} \mathcal{P}[|\frac{U_H(t_j) - U_H(s)}{t_j - s}| > c] \\ = \quad & \lim_{j\to\infty} \mathcal{P}[\frac{|(t_j - s)^H U_H(1)}{t_j - s}| > c] \\ = \quad & \lim_{j\to\infty} \mathcal{P}[|(t_j - s)^{H-1} U_H(1)| > c] \\ = \quad & \lim_{j\to\infty} \mathcal{P}[|U_H(1)| > \underbrace{|t_j - s|^{1-H}}_{\overset{j\to\infty}{\longrightarrow} 0} c] \\ \overset{j\to\infty}{\longrightarrow} \quad & 1 \end{aligned}$$

□

**Theorem 4.12.** Let $(U_H(k))_k$ be a FBM. The conditional expectation of $U_H(s)$ given $U_H(t) = x$ is

$$\frac{|\frac{s}{t}|^{2H} + 1 - |\frac{s}{t} - 1|^{2H}}{2} \cdot x$$

for all $s \geq t$.

**Proof.** Cf. [15] Theorem 5.3. Taking conditional expectation of $U_H(s)$ given $U_H(t)$,

$$\begin{aligned} & \mathrm{E}[U_H(s)|U_H(t)] \\ \overset{\text{Corollary 2.19}}{=} \quad & \mu_s + \rho_{s,t}(\frac{\sigma_s}{\sigma_t} U_H(t) - \mu_t) \\ = \quad & \rho_{s,t} \frac{\sigma_s}{\sigma_t} U_H(t) \\ = \quad & \frac{\rho_{s,t} \cdot \sigma_s \sigma_t \cdot U_H(t)}{\sigma_t^2} \\ = \quad & \frac{\mathrm{E}[U_H(s) U_H(t)]}{\mathrm{E}[U_H^2(t)]} \cdot U_H(t) \\ \overset{(4.6)}{=} \quad & \frac{s^{2H} + t^{2H} - |s-t|^{2H}}{2\mathrm{E}[U_H^2(t)]} \cdot U_H(t) \\ = \quad & \frac{s^{2H} + t^{2H} - |s-t|^{2H}}{2t^{2H}} \cdot U_H(t) \\ = \quad & \frac{|\frac{s}{t}|^{2H} + 1 - |\frac{s}{t} - 1|^{2H}}{2} \cdot U_H(t) \end{aligned}$$

□

## 4.3 Fractional Brownian Noise

**Definition 4.13.** Let $(U_H(t))_t$ be a FBM. The *fractional Brownian noise* is a sequence $(S_k)_k$ forms

$$S_H(k) \quad = \quad U_H(k+1) - U_H(k)$$

for $\tau \in \mathbb{R}$.

**Proposition 4.14.** Fractional Brownian noise is stationary and its autocovariance is

$$\varsigma_{S_H}(\tau) \quad = \quad \frac{1}{2}(|\tau+1|^{2H} - 2|\tau|^{2H} + |\tau-1|^{2H}) \tag{4.9}$$

for $\tau \in \mathbb{R}$.

**Proof.** Cf. [16], p.333, Proposition 7.2.9 .
The first part of the claim is clear due to FBM has stationary increments.
In terms of definition of fractional Brownian noise, fixing a $k$, we have

$$\begin{aligned}
& \varsigma_{S_H}(\tau) \\
= \quad & \mathrm{E}[S_H(k+\tau)S_H(k)] \\
= \quad & \mathrm{E}[(U_H(\tau+k+1) - U_H(\tau+k))(U_H(k+1) - U_H(k))] \\
= \quad & \mathrm{E}[U_H(\tau+k+1)U_H(k+1)] - \mathrm{E}[U_H(\tau+k+1)U_H(\tau)] \\
- \quad & \mathrm{E}[U_H(\tau+k)U_H(k+1)] + \mathrm{E}[U_H(\tau+k)U_H(k)] \\
\overset{(4.6)}{=} \quad & \frac{1}{2}((\tau+k+1)^{2H} + (k+1)^{2H} - |\tau|^{2H} - (\tau+k+1)^{2H} - k^{2H} + |\tau+1|^{2H} \\
- \quad & (\tau+k)^{2H} - (k+1)^{2H} + |\tau-1|^{2H} + (\tau+k)^{2H} + k^{2H} - |\tau|^{2H}) \\
= \quad & \frac{1}{2}(|1+\tau|^{2H} - 2|\tau|^{2H} + |1-\tau|^{2H})
\end{aligned}$$

for $\tau \in \mathbb{R}$. □

**Definition 4.15.** A stationary stochastic process $(X_t)_t$ is said to have *long memory* if its autocovariance $\varsigma_X(\tau)$ tend to 0 so slowly such that $\sum_{\tau=0}^{\infty} \varsigma_X(\tau)$ diverges.

**Lemma 4.16** (Cauchy Condensation test)**.** Let $(a_n)$ be a $\mathbb{R}$-valued positive non-increasing sequence. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

**Proof.** See [12], p.391, Theorem 13.13 . □

**Theorem 4.17.** The fractional Brownian noise with $H \in (\frac{1}{2}, 1)$ has long memory.

**Proof.** Cf. [16], p.335, Proposition 7.2.10 .
Without loss of generality, we suppose $\tau \in \mathbb{N}_0$ because $\varsigma_{S_H}(0) = 1$.

$$\begin{aligned} & \varsigma_{S_H}(\tau) \\ = & \frac{1}{2}\tau^{2H-2}\{\tau^2[(1+\frac{1}{\tau})^{2H} - 2 + (1-\frac{1}{\tau})^{2H}]\} \\ = & \frac{1}{2}\tau^{2H-2}\{\frac{(1+\frac{1}{\tau})^{2H}-1}{\frac{1}{\tau^2}} - \frac{1-(1-\frac{1}{\tau})^{2H}}{\frac{1}{\tau^2}}\} \end{aligned}$$

We deal with the former of the content in { } with L'Hôpital's rule as $\tau$ tend to infinity.

$$\begin{aligned} & \frac{(1+\frac{1}{\tau})^{2H}-1}{\frac{1}{\tau^2}} \\ = & \frac{2H(1+\frac{1}{\tau})^{2H-1}(-\frac{1}{\tau^2})}{-\frac{4}{\tau^3}} + \mathrm{o}(\tau) \\ = & \frac{H(1+\frac{1}{\tau})^{2H-1}}{\frac{1}{\tau}} + \mathrm{o}(\tau) \end{aligned}$$

We calculate the Latter of the content in { } in a similar way. Then

$$\begin{aligned} & \varsigma_{S_H}(\tau) \\ = & \tau^{2H-2}\frac{1}{2}\{\frac{H(1+\frac{1}{\tau})^{2H-1}}{\frac{1}{\tau}} - \frac{H(1-\frac{1}{\tau})^{2H-1}}{\frac{1}{\tau}}\} + \mathrm{o}(\tau) \\ \overset{\text{L'Hôpital}}{=} & \frac{1}{2}\tau^{2H-2}\{\frac{H(2H-1)(1+\frac{1}{\tau})^{2H-2}(-\frac{1}{\tau^2})}{-\frac{1}{\tau^2}} - \frac{H(2H-1)(1-\frac{1}{\tau})^{2H-2}\frac{1}{\tau^2}}{-\frac{1}{\tau^2}}\} + \mathrm{o}(\tau) \\ = & \frac{1}{2}\tau^{2H-2}\{H(2H-1)(1+\frac{1}{\tau})^{2H-2} + H(2H-1)(1-\frac{1}{\tau})^{2H-2}\} + \mathrm{o}(\tau) \\ = & \frac{1}{2}\tau^{2H-2}2H(2H-1) + \mathrm{o}(\tau) \\ = & H(2H-1)\tau^{2H-2} + \mathrm{o}(\tau) \end{aligned}$$

For $H \in (0, \frac{1}{2})$, $\sum\limits_{\tau=1}^{\infty} \tau^{2H-2}$ converges due to if we take the Cauchy condesation test for it

$$\begin{aligned} & \sum_{\tau=0}^{\infty} 2^{\tau}(2^{\tau})^{2H-2} \\ = & \sum_{\tau=0}^{\infty} 2^{\tau(2H-1)} \end{aligned}$$

This is geometric series if $2H - 1 < 0$, namely, $H \in (0, \frac{1}{2})$.
Otherwise, if $\frac{1}{2} < H < 1$, namely, $-1 < 2H - 2 < 0$, $\sum\limits_{\tau=1}^{\infty} \tau^{2H-2}$ diverges because it is greater than the harmolic series. $\sum\limits_{\tau=1}^{\infty} \varsigma(\tau)$ converges if and only if $\sum\limits_{\tau=1}^{\infty} \tau^{2H-2}$ converges. We have also $\lim\limits_{\tau\to\infty} \varsigma_{S_H}(\tau) = 0$, the claim is proved. □

**Corollary 4.18.** Let $S_H$ be fractional Brownian noise and $\varsigma_{S_H}(\cdot)$ be its autocovariance. Then $\sum_{\tau=0}^{\infty} \varsigma_{S_H}^2(\tau) < \infty$ if and only if $H < \frac{3}{4}$.

**Proof.** Cf. [16], p.72, Lemma 6.3. As by Theorem 4.17, we have $\varsigma_{S_H}^2(\tau) = H^2(2H-1)^2\tau^{4H-4} + o(\tau)$. The sum of it over the range of $\tau$ is finite if and only if, according to the same reason as in Theorem 4.17, $4H < 3$. That means $H < \frac{3}{4}$. □

## 4.4 FBM is not Semimartingale for $H \neq \frac{1}{2}$

Let us have a look at our Integration representation for FBM. In the case of FBM with Hurst index $\frac{1}{2}$, it must be an ordinary Brownian motion. Except for this, we'll show FBM is not a seimimartingal.

**Definition 4.19.** The *Hermite polynomials* forms as following

$$H_n(u) = (-1)^n e^{\frac{u^2}{2}} (\frac{\partial^n}{\partial u^n} e^{-\frac{u^2}{2}}), \tag{4.10}$$

for $u \in \mathbb{R}, n \in \mathbb{N}_0$.

**Proposition 4.20.** Let $(H_n)_{n\in\mathbb{N}}$ be a family of Hermite polynomials, $f, g : \mathbb{R} \to \mathbb{R}$ continuously differentiable. It has then following properties

(i) $H_{n+2}(u) = u \cdot H_{n+1}(u) - (n+1)H_n(u)$ and $H_{n+1}(u) = (n+1)H_n^{'}(u)$ for all $n \in \mathbb{N}_0, u \in \mathbb{R}$.

(ii) Let $W, V$ be standard Gaussian distributed such that $(W, V)$ have a disjoint Gaussian distribution. Then

$$\int_\Omega H_j(W) \cdot H_k(V)\, \mathcal{P} = \begin{cases} j!\,(\mathrm{E}[WV])^j & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Let $W$ be standard Gaussian distributed, then

$$\frac{1}{j!}\int_\Omega H_j(W) H_k(W)\, \mathcal{P} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Remark that, $(iii)$ means the fact, $\{\frac{1}{\sqrt{j!}} \cdot H_j(x)\}_{j=0}^{\infty}$ is an orthonormal basis in $\mathcal{L}^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), e^{-\frac{x^2}{2}}\, dx)$.

**Proof.** See [16] p.3, Propostion 1.3. □

**Lemma 4.21.** Let $(U_H(t))_t$ be a FBM, $W$ standrad Gaussian variable, and $f:\mathbb{R}\to\mathbb{R}$, Borel-measurable function such that $\mathrm{E}[f^2(W)]<\infty$. Then,

$$\frac{1}{n}\sum_{j=1}^{n} f(U_H(j)-U_H(j-1)) \overset{\text{in } \mathcal{L}^2(\mathcal{P})}{\rightarrow} \mathrm{E}[f(W)],$$

as $n$ tend to $\infty$. In particular,

$$\sum_{j=1}^{n} |U_H(\frac{j}{n})-U_H(\frac{j-1}{n})|^{\beta} \overset{\text{in } \mathcal{L}^2(\mathcal{P})}{\rightarrow} \begin{cases} 0 & \text{if } \beta > \frac{1}{H} \\ \mathrm{E}[|W|^{\beta}] & \text{if } \beta = \frac{1}{H} \\ \infty & \text{if } \beta < \frac{1}{H} \end{cases} \tag{4.11}$$

as $n$ tend to $\infty$.

**Proof.** C.f. [16], p.17, Theorem 2.1 .
Firstly, because $\mathrm{E}[f^2(W)]<\infty$ one has $f\in\mathcal{L}^2(\mathbb{R},\mathscr{B}(\mathbb{R}),e^{-\frac{x^2}{2}}\,dx)$. In terms of Proposition 4.20(iii), taking expectation

$$\mathrm{E}[f(x)] = \mathrm{E}[\sum_{j=0}^{\infty}\frac{a_j H_j(x)}{\sqrt{j!}}],$$

for $x\in\mathbb{R}$. Notice $H_0(u)=1$ for $u\in\mathbb{R}$ due to (4.10). Setting $x=W$, Equating coefficients leads to $a_0=\mathrm{E}[f(W)]$. Moreover,

$$\begin{aligned}
& \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n} f(U_H(j)-U_H(j-1))-\mathrm{E}[f(W)]\}^2] \\
= \; & \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n}(f(U_H(j)-U_H(j-1))-\mathrm{E}[f(W)])\}^2] \\
= \; & \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n}(\sum_{k=0}^{\infty}\frac{a_k}{\sqrt{k!}}H_k(U_H(j)-U_H(j-1)))-\mathrm{E}[f(W)]\}^2] \\
= \; & \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n}(\sum_{k=1}^{\infty}\frac{a_k}{\sqrt{k!}}H_k(U_H(j)-U_H(j-1)))\}^2]
\end{aligned}$$

Consider now

$$\mathrm{E}[f^2(W)]<\infty,$$

which requires $\sum_{k=1}^{\infty}(a_k)^2 < \infty$. Then

$$
\begin{aligned}
& \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n} f(U_H(j) - U_H(j-1)) - \mathrm{E}[f(W)]\}^2] \\
= \quad & \frac{1}{n^2}\mathrm{E}[\sum_{k=1}^{\infty}\frac{a_k^2}{k!}(\sum_{j=1}^{n} H_k(U_H(j) - U_H(j-1)))^2] \\
= \quad & \frac{1}{n^2}\sum_{k=1}^{\infty}\frac{a_k^2}{k!}\sum_{j=1,m=1}^{n}\mathrm{E}[H_k(U_H(j) - U_H(j-1))H_k(U_H(m) - U_H(m-1)] \\
\overset{\text{Proposition 4.20(ii)}}{=} \quad & \frac{1}{n^2}\sum_{k=1}^{\infty} a_k^2 \sum_{j=1,m=1}^{n}(\mathrm{E}[(U_H(j) - U_H(j-1))(U_H(m) - U_H(m-1))])^k \\
= \quad & \frac{1}{n^2}\sum_{k=1}^{\infty} a_k^2 \sum_{j=1,m=1}^{n}(\mathrm{E}[S_H(j-1)S_H(m-1)])^k \\
= \quad & \frac{1}{n^2}\sum_{k=1}^{\infty} a_k^2 \sum_{j=1,m=1}^{n}(\varsigma_{S_H}(j-m))^k
\end{aligned}
$$

Notice that,

$$
\begin{aligned}
|\varsigma_{S_H}(k)| \quad & = \quad |\varsigma_{S_H}(|k|)| \\
& = \quad \mathrm{E}[(U_H(1) - U_H(0))(U_H(|x|+1) - U_H(|x|))] \\
& \overset{\text{Cauchy Schwartz}}{\leq} \quad \underbrace{\sqrt{\mathrm{E}[U_H(1)^2]}}_{=1} \cdot \underbrace{\sqrt{\mathrm{E}[U_H(|k|+1) - U_H(|k|)]^2}}_{=1} \\
& = \quad 1
\end{aligned}
$$

Consequently, $(\varsigma_{S_H}(j-m))^k \leq |\varsigma_{S_H}(j-m)|$. In fact,

$$
\begin{aligned}
& \mathrm{E}[\{\frac{1}{n}\sum_{j=1}^{n} f(U_H(j) - U_H(j-1)) - \mathrm{E}[f(W)]\}^2] \\
\leq \quad & \frac{1}{n^2}\underbrace{\sum_{k=1}^{\infty} a_k^2}_{=:\alpha<\infty}\sum_{j=1,m=1}^{n}|\varsigma_{S_H}(j-m)| \\
= \quad & \frac{\alpha}{n^2}\sum_{j=1,m=1}^{n}|\varsigma_{S_H}(j-m)| \\
= \quad & \frac{\alpha}{n^2}2\cdot\sum_{j=1}^{n}\sum_{m<j}|\varsigma_{S_H}(j-m)| \\
\leq \quad & \frac{\alpha}{n^2}2n\sum_{k=1}^{n-1}|\varsigma_{S_H}(k)| \\
= \quad & \frac{2\alpha}{n}\sum_{k=1}^{n-1}|\varsigma_{S_H}(k)|,
\end{aligned}
$$

As in the proof in Theorem 4.17,

$$\begin{aligned}
& \sum_{k=1}^{n-1} |\varsigma_{S_H}(k)| \\
\propto\ & H(2H-1)\sum_{k=1}^{n-1} j^{2H-2} \\
\propto\ & H(2H-1)n\cdot n^{2H-2} \\
\propto\ & n^{2H-1},
\end{aligned}$$

as n goes to infinity. Then

$$\frac{2\alpha}{n}\sum_{k=1}^{n-1} |\varsigma_{S_H}(k)| \propto n^{2H-2}$$

as n goes to infinity. This leads to, for $0 < H < 1$ as $n \to \infty$, $\mathrm{E}[\{\frac{1}{n}\sum\limits_{j=1}^{n} f(U_H(j) - U_H(j-1)) - \mathrm{E}[f(W)]\}^2] \to 0$ due to $n^{2H-2} \to 0$. I.e., $\frac{1}{n}\sum\limits_{j=1}^{n} f(U_H(j) - U_H(j-1)) \overset{\text{in}\mathcal{L}^2}{\to} \mathrm{E}[f(W)]$.

Secondly, we apply previous result for (4.11), In fact,

$$\begin{aligned}
& \sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\beta \\
= \ & \frac{1}{n^{\beta H}}\sum_{j}^{n} |U_H(j) - U_H(j-1)|^\beta \\
= \ & \frac{1}{n^{\beta H-1}}\frac{1}{n}\sum_{j}^{n} |U_H(j) - U_H(j-1)|^\beta \\
\overset{\text{in L}^2}{\longrightarrow}\ & n^{1-\beta H}\mathrm{E}[|W|^\beta]
\end{aligned}$$

Due to $\mathrm{E}[|W|^\beta] < \infty$, (4.11) holds as well as $n \to \infty$. □

**Theorem 4.22.** FBM is not a semimartingale for $H \neq \frac{1}{2}$.

**Proof.** Without loss of generality, we set the Time index $T = [0,1]$. Choosing $\beta = 2$ in (4.11), we suppose $U_H(t)$ were a semimartingale.
Case $H < \frac{1}{2}$. Then $\sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^2 \to \infty$ contradicts that semimartingale has finite quadratic variation.
Case $H > \frac{1}{2}$. $\sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^2 \to 0$. On the one hand, according to Doob-Meyer decomposition, $U_H(t) = M(t)+A(t)$, where $M(t)$ is a local martingal and $A(t)$ is local finite variation process. Hence $A(t)$ has quadratic variation zero and we have $0 = [U_H, U_H] = [M, M]$, where $[\cdot,\cdot]$ is denoted for quadratic variation. Consequently, $M(t)$ is zero process

due to Cauchy Schwarz inequality. In other words, $U_H(t) = A(t)$ which has finite variation. On the ohter Hand, choosing $1 < \gamma < \frac{1}{H}$, then $\sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\gamma \to \infty$. Precisily,

$$
\begin{aligned}
&\infty \leftarrow \sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^\gamma \\
&\leq \underbrace{\sup_{1\leq j\leq n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|^{\gamma-1}}_{\overset{(\gamma-1)-\text{Hölder}}{\rightarrow} 0} \cdot \sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})|,
\end{aligned}
$$

this leads to $\sum_{j=1}^{n} |U_H(\frac{j}{n}) - U_H(\frac{j-1}{n})| \to \infty$ contradicts mentioned above that $U_H(t)$ has finite variation.

Given all that, FBM is not a semimartingal for $H \neq \frac{1}{2}$. $\square$

# 5 Fractional Ornstein-Uhlenbeck Process

In this section we turn our attention on the fractional Ornstein-Uhlenbeck process (for short, we denote FOU).

## 5.1 Fractional Ornstein-Uhlenbeck Process

Consider the following stochastic dynamics

$$dX_t = -aX_t \, dt + \gamma \, dU_H(t), \tag{5.1}$$

where $(X_t)_{t\geq 0}$ is a stochastic process, $a, \gamma \in \mathbb{R}_+$ and $(U_H(t))_{t\geq 0}$ a FBM with Hurst exponent $H$. In fact, given an initial condition $X_0(\omega) = b(\omega)$, then in the theory of SDE, (5.1) is understood as

$$X_t(\omega) = b(\omega) - a\int_0^t X_u(\omega) \, du + \gamma U_H(t)(\omega) \tag{5.2}$$

for $t \geq 0$.

Cheridito et al.[4] shows the following Lemma with Hölder continuity of $U_H(t)$.

**Lemma 5.1.** Let $U_H(t)$ be a FBM, $a \in \mathbb{R}_+$ and $u, d \in \mathbb{R}$ such that $d \leq s$. Then there exist a Riemann-Stieltjes integral such that

$$\int_d^s e^{au} \, dU_H(u) = e^{as}U_H(s) - e^{ad}U_H(d) - a\int_d^s U_H(u)e^{au} \, du \, . \tag{5.3}$$

**Proof.** See. [4], p.11, Proposition A.1 . □

**Lemma 5.2.** Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), a > 0$ and $-\infty \leq m < n \leq j < k < \infty$. Then

$$\begin{aligned} & \mathrm{E}[\int_m^n e^{au} \, dU_H(u) \int_j^k e^{as} \, dU_H(s)] \\ = \; & H(2H-1)\int_m^n e^{au}\left(\int_j^k e^{as}(s-a)^{2H-2} \, ds\right) du \, . \end{aligned}$$

**Proof.** See. [4], p.5, Lemma2.1 . □

**Theorem 5.3.** $\hat{X}_t^{b,H} := e^{-at}\left(b + \gamma\int_0^t e^{au} \, dU_H(u)\right)$ is the solution that solves (5.2) for $u \geq 0$.

**Proof.** Cf. [4], p.11, Proposition A.1 . We define

$$Y(t) := \int_0^t X_u \, du,$$

for $t \geq 0$. Rewirte (5.2) with $Y_t$ and $Y(0) = 0$,

$$Y'(t) = b - aY(t) + \gamma U_H(t)$$

And the solution of that linear differential equation with $Y(0) = 0$ is

$$Y(t) = e^{-at} \int_0^t e^{au} (b + \gamma U_H(u)) \, du,$$

in terms of Definition above, using (5.3)

$$\begin{aligned}
& X(t) = Y'(t) \\
= \quad & -ae^{-at} \int_0^t e^{au}(b + \gamma U_H(u)) \, du + e^{-at} e^{at} (b + \gamma U_H(t)) \\
= \quad & -ae^{-at} \int_0^t e^{au}(b + \gamma U_H(u)) \, du + b + \gamma U_H(t) \\
= \quad & e^{-at} \left( \underbrace{-a \int_0^t e^{au} \gamma U_H(u) \, du + e^{at} \gamma U_H(t)}_{\gamma \int_0^t e^{au} \, dU_H(u)} - a \int_0^t e^{au} du \cdot b \right) + b \\
= \quad & e^{-at} \left( \gamma \int_0^t e^{au} \, dU_H(u) - e^{au}|_{u=0}^{u=t} \cdot b \right) + b \\
= \quad & e^{-at} \left( \gamma \int_0^t e^{au} \, dU_H(u) + b \right)
\end{aligned}$$

□

In order to have a stationary solution, we assume that the initial value is normally distributed that $\hat{X}_{H,t} := \hat{X}_t^{\gamma \int_{-\infty}^0 e^{au} \, dU_H(u), H} := e^{-at} \left( \gamma \int_{-\infty}^t e^{au} \, dU_H(u) \right)$.

**Theorem 5.4.** $(\hat{X}_{H,t})_{t \geq 0}$ is centered Gaussian and stationary.

**Proof.** For the sake of simplicity, we let $\hat{X}_{H,t} = \int_{-\infty}^t e^{au} \, dU_H(u)$. Fix $\epsilon^j > 0, H \in (0,1)$ , then there exists $\{u_0^j < u_1^j < \cdots < u_{k_j}^j \leq t_j\}$ such that

$$\begin{aligned}
& |\hat{X}_{H,t_j} - \sum_{l=0}^{k_j - 1} e^{au_l^j} (U_H(u_{l+1}^j) - U_H(u_l^j))| \\
= \quad & |\int_{-\infty}^{t_j} e^{au} \, dU_H(u) - \sum_{l=0}^{k_j - 1} e^{au_l^j} (U_H(u_{l+1}^j) - U_H(u_l^j))| \\
< \quad & \epsilon^j.
\end{aligned}$$

for $0 \leq j \leq d$.
In the one hand, we calculate the characterstic function of $(\hat{X}_{H,t_1}, \ldots, \hat{X}_{H,t_d})$ approximately with respect to $\epsilon^1, \ldots, \epsilon^d$.

$$\sum_{j=1}^{d} \xi_j X\hat{_{H,t_j}}_{t_j} \approx \sum_{j=1}^{d} \xi_j \left( \sum_{l=0}^{k_j-1} e^{au_l^j} (U_H(u_{l+1}^j) - U_H(u_l^j)) \right)$$

Notice that since $(U_H(t))$ is centered Gaussian process, any infinite linear combination of its instances is centered Gaussian again. In other words,

$$\sum_{j=1}^{d} \xi_j \left( \sum_{l=0}^{k_j-1} e^{au_l^j} (U_H(u_{l+1}^j) - U_H(u_l^j)) \right)$$

is centered Gaussian. Passing $\epsilon^1, \ldots, \epsilon^d$ to zero, it implies immediately $(\hat{X}_{t_1}, \ldots, \hat{X}_{t_d})$ is centered Gaussian process due to Corolary 2.15.
On the other hand,

$$\mathrm{E}[\exp\{i \sum_{j=1}^{d} \xi_j \hat{X}_{H,t_j}\}] \approx \mathrm{E}[\exp\{i \sum_{j=1}^{d} \xi_j \left( \sum_{l=0}^{k_j-1} e^{au_l^j} (U_H(u_{l+1}^j) - U_H(u_l^j)) \right)\}]$$

Since $\{U_H(t)\}$ has stationary increments, then

$$\begin{aligned} & \mathrm{E}[\exp\{i \sum_{j=1}^{d} \xi_j \left( \sum_{l=0}^{k_j-1} e^{au_l^j} (U_H(u_{l+1}^j) - U_H(u_l^j)) \right)\}] \\ = \; & \mathrm{E}[\exp\{i \sum_{j=1}^{d} \xi_j \left( \sum_{l=0}^{k_j-1} e^{au_l^j} (U_H(u_{l+1}^j + \tau) - U_H(u_l^j + \tau)) \right)\}] \end{aligned}$$

which converges if $\epsilon$'s tend to zero and must be equal $\mathrm{E}[\exp\{i \sum_{j=1}^{d} \xi_j \hat{X}_{H,t_j+\tau}\}]$, i.e., $(\hat{X}_{H,t})$ is stationary process. $\square$

**Theorem 5.5.** Let $H$ be that $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then

$$\varsigma_{\hat{X}_H}(\tau) = \frac{1}{2}\gamma^2 \sum_{k=1}^{N} a^{-2k} \left( \prod_{j=0}^{2k-1} (2H - j) \right) \tau^{2(H-k)} + \mathrm{o}(\tau)$$

for $N \in \mathbb{N}_0, \tau \in \mathbb{R}$, $\gamma, a \in \mathbb{R}_+$ as in (5.2).

**Proof.** See [4], p.7, Theorem 2.3 . $\square$

**Corollary 5.6.** $(\hat{X}_{H,t})_{t\geq 0}$ has long memory for $H \in (\frac{1}{2}, 1)$.

**Proof.** Consider the autocovariance of $(\hat{X}_{H,t})$, with a given function $c(a, \gamma, N, H)$,

$$\varsigma_{\hat{X}_H}(\tau) = \sum_{k=1}^{N} c(a, \gamma, N, H)\tau^{2H-2k} + \mathrm{o}(\tau)$$

is obviously tend to zero as $\tau$ goes to infinity. And To check convergence of $\sum_{\tau=1}^{\infty} \varsigma_{\hat{X}_H}(\tau)$, we only need to check the term of $k = 1$ in the summation, because in the case $k > 1$, $\tau^{2H-2k} < \tau^{2H-2}$ for $\tau \in \mathbb{N}_+$. We deal with it in same way as in Theorem 4.17(use Cauchy condesation test). That means, if $2H < 1$, namely $H < \frac{1}{2}, \sum_{\tau=1}^{\infty} \varsigma_{X_H}(\tau)$ converges. For $H \in (\frac{1}{2}, 1)$, it diverges. Thus $(\varsigma_{\hat{X}_H}(t))_{t\geq 0}$ has long memory for $H \in (\frac{1}{2}, 1)$. □

# 6 Applications in Financial Mathematics

## 6.1 Fractional Black-Scholes Model

In this subsection, we'll introduce FBM to the Black-Scholes model. To be specific, our finance market is modeled with two stochastic processes that a process of a riskless asset $(A_t)_t$ and a process of price of a stock $(S_t)_t$. The stock is assumed that it pays no dividends. Setting initial values $A_0 = 1, S_0 = 1$, we give our fractional Black-Scholes model as follows

$$\begin{aligned} A_t &= \exp(rt) \\ S_t &= \exp(rt + \mu(t) + \sigma U_H(t)), t \in [0, T], \end{aligned} \tag{6.1}$$

where $r \in \mathbb{R}, \sigma \in \mathbb{R}_+, \sup_{t\in[0,T]} \mu(t) < \infty$. Through out this section we denote by $(\mathcal{F}_t^X)_t$ the filtration of a stochatic process $(X_t)_t$ and give definitions

**Definition 6.1.** A $\mathbb{R}^2$-valued stochastic process $(\xi_t^0, \xi_t^1)_{t\in[0,T]}$ is said to be a *strategy* to (6.1) , if $\xi_t^0 \in \mathcal{F}_j^A$ and $\xi_t^1 \in \mathcal{F}_j^S$, for $0 \le j \le t$.

**Definition 6.2.** A stochastic process $(V_t)_{t\in[0,T]}$ is said to be *value process* with respect to strategy $(\xi_t^0, \xi_t^1)$, if

$$V_t = \xi_t^0 A_t + \xi_t^1 S_t$$

for $t \in [0, T]$.

**Definition 6.3.** A stochastic process $(\tilde{V}_t)_{t\in[0,T]}$ is said to be *discounted value process* of a value process $(V_t)_{t\in[0,T]}$ with respect to $(\xi_t^0, \xi_t^1)$, if

$$\tilde{V}_t = \frac{V_t}{A_t}$$

for $t \in [0, T]$.

Obviously, $\tilde{V}_t = \xi^0 + \xi^1 \tilde{S}_t$ with $\tilde{S}_t = \exp(\mu(t) + \sigma U_H(t))$.

**Definition 6.4.** A strategy $(\xi_t^0, \xi_t^1)_{t\in T}$ is said to be *self-financing*, if

$$V_T = V_0 + \sum_{j=1}^{m} \xi_{s_j}^0 (A_{s_j} - A_{s_{j-1}}) + \xi_{s_j}^1 (S_{s_j} - S_{s_{j-1}}). \tag{6.2}$$

for $0 = s_0 \le s_1 \le \cdots \le s_m = T$.

**Definition 6.5.** A self-financing strategy $(\xi_t^0, \xi_t^1)_{t\in T}$ is said to be have *arbitrage*, if its discounted process satisfies following conditions

(i) $\mathcal{P}[\tilde{V}_T - \tilde{V}_0] = 1$.

(ii) $\mathcal{P}[\tilde{V}_T > 0] > 0$.

Cheridito proved that in reality, if there exist a minimal amount of time between two successive transactions, the market(6.1) is arbitagefree. In following we complete the proof in [3] of the claim.

**Lemma 6.6.** Let $(X_t)_{t\geq 0}$ be a stochastic process continueous in $t$. If $(X_t)$ is a modification of the process

$$\left(\int_0^t (t-u)^{H-\frac{1}{2}}\, dB_u\right)_{t\geq 0}$$

for $(B_t)_{t\geq 0}$ a Brownian motion and $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, then

$$\mathcal{P}[\sup_{t\in[a,b]} X_t \leq -c] > 0$$

for $c \geq 0$ and $0 < a \leq b$.

**Proof.** See [3], p.15, Lemma 4.2 . □

**Theorem 6.7.** Let $(S_t)_{t\in[0,T]}$ be a stochastic process such that

$$\tilde{S}_t = \exp\left(\mu(t) + \sigma U_H(t)\right), \tag{6.3}$$

where $\mu, \sigma$ are as in (6.1), $U_H(t)$ is a FBM. If there exist

$$\xi_t^1 = f_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{n-1} f_k \mathbb{1}_{(\tau_k, \tau_{k+1}]}(t)$$

where $t \in [0,T]$, $\{f_k\}_0^{n-1}$ is family of $(\mathcal{F}_k^{U_H})_k$-measurable function, $0 = \tau_1 < \cdots < \tau_n = T$ are stopping times with respect to $(\mathcal{F}_k^{U_H})_k$ respectively, with $\tau_{k+1} - \tau_k \geq m$ for some $m > 0$ and $\mathcal{P}[f_k \neq 0] > 0$ for $k \in \{0, \ldots, n-1\}$, then

$$\mathcal{P}[(\xi^1 \cdot \tilde{S})_T < 0] > 0.$$

**Proof.** Cf.[3], p.18, Theorem 4.3 . $\xi^1$ is predictable. Assume $\mathcal{P}[(\xi^1 \cdot \tilde{S})_T < 0] = 0$, then there exist

$$l = \min\{j : \mathcal{P}[f_j \neq 0] > 0, \quad \mathcal{P}[\sum_{k=1}^{j} f_k (e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) \geq 0] = 1\}$$

This leads to

$$\mathcal{P}\left[\left(\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}\right) \leq 0\right] = 1$$

Ignoring constant term, we define

$$U_H(t)(\omega) = \int_{\mathbb{R}} \mathbb{1}_{\{t\geq u\}} \cdot (t-u)^{H-\frac{1}{2}} - \mathbb{1}_{\{u\leq 0\}}(-u)^{H-\frac{1}{2}} \; d\omega(u)\,.$$

where $\omega(u) := B_u(\omega)$ for all $\omega \in \Omega^w$. We give the filtration $(\mathcal{F}_t^{\Omega^w})$ denoting by

$$\mathcal{F}_t^{\Omega^w} := \sigma(\{\{w \in \Omega^w : \omega(u) \in \mathbb{R}\} : -\infty < u \leq t, t \in \mathbb{R}\}).$$

Then $\tau_k$ is also stopping time of $\mathcal{F}_t^{\Omega^w}$ due to

$$\mathcal{F}_t^{U_H} \subset \mathcal{F}_t^{\Omega^w}, t \in \mathbb{R}$$

For $\omega \in \Omega^w$, we split it into two parts as follows

$$\begin{aligned} \psi_\omega(u) &:= \omega(u)\mathbb{1}_{(-\infty,\tau_l(\omega)]}, u \in \mathbb{R} \\ \phi_\omega(u) &:= \omega(\tau_l(\omega)+u) - \omega(\tau_l(\omega)), u \geq 0. \end{aligned}$$

Corresponding to each part, we define

$$\begin{aligned} \Omega^1 &:= \{\psi_\omega \in \mathcal{C}(\mathbb{R}) : \omega \in \Omega^w\} \\ \Omega^2 &:= \{\phi_\omega \in \mathcal{C}([0,\infty)) : \omega \in \Omega^w\} \end{aligned}$$

And for the smallst $\sigma$-algebra of all subsets, respectively, of $\Omega^1, \Omega^2$ denoted by $\mathscr{B}^1, \mathscr{B}^2$. Notice that

$$\begin{aligned} \{\tau_l \leq t\} \cap \{\psi_\omega \in \Omega^w\} &= \{\{\omega \in \Omega^w : \omega(u) \in \mathbb{R}\} : -\infty < u \leq t\} \\ &\in \mathcal{F}_t^{\Omega^w}, \end{aligned}$$

therefore is $\psi_\omega$ a $\mathcal{F}_{\tau_l}^{\Omega^w}$- measurable mapping. Moreover, since the strong Markovian property of Brownian motion, $\phi_\omega$ is independent of $\mathcal{F}_{\tau_l}^{\Omega^w}$ and it must be a Brownian motion.

Plugging $\psi_\omega, \phi_\omega$ into $\Omega$, we calculate the value process

$$\begin{aligned}
& \left(\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) + f_l(e^{U_H(\tau_l+m)} - e^{U_H(\tau_l)})\right)(\omega) \\
= \ & \underbrace{\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)})(\omega)}_{:=J^1} + f_l\left(e^{U_H(\tau_l)}\left((e^{U_H(\tau_l+m)} - e^{U_H(\tau_l)}) - 1\right)\right)(\omega) \\
= \ & J^1(\omega) + f_l(\exp\{\int_{\mathbb{R}} \mathbb{1}_{\{\tau_l \geq u\}}(\tau_l(\omega) + m - u)^{H-\frac{1}{2}} - \mathbb{1}_{\{\tau_l \geq u\}}(\tau_l(\omega) - u)^{H-\frac{1}{2}}\, d\omega(u) \\
+ \ & \int_{\mathbb{R}} \mathbb{1}_{(\tau_l, \tau_l+m]}(\tau_l + m - u)^{H-\frac{1}{2}}\, d\omega(u)\}) \\
= \ & J^1(\psi_\omega) + \underbrace{f_l\left(\exp\{\int_{-\infty}^{\tau_l(\psi_\omega)} (\tau_l(\psi_\omega) + h + u)^{H-\frac{1}{2}} - (\tau_l(\psi_\omega) - u)^{H-\frac{1}{2}}\, d\psi_\omega(u)\}\right)}_{=:J^2(\psi_\omega, m)} \\
\cdot \ & \underbrace{\left(\exp\{\int_0^m (m-u)^{H-\frac{1}{2}}\, d\phi_\omega(u)\} - 1\right)}_{=:J^3(\phi_\omega, m)} \\
= \ & J(\psi_\omega, \phi_\omega, m)
\end{aligned}$$

where $J$ is defined as

$$J(\psi, \phi, t) := J^1(\psi) + J^2(\psi, t) \quad \cdot \quad J^3(\phi, t)$$

for $\psi \in \Omega^1, \phi \in \Omega^2$.

Indeed, fixing $\psi \in \Omega^1, \phi \in \Omega^2$, $J(\psi, \phi, \cdot)$ has continueous path on $(\Omega^1 \times \Omega^2, \mathscr{B}^1 \otimes \mathscr{B}^2)$, then we can define a $\mathscr{B}^1 \otimes \mathscr{B}^2$- measurable set

$$E := \{(\psi, \phi) : \sup_{m \leq t \leq T} J(\psi, \phi, t) < 0\}$$

$$\begin{aligned}
\mathrm{E}[\mathbb{1}_E(\omega_1, \phi_\omega) | \psi_\omega = \omega_1] &= \mathcal{P}^\omega[\sup_{m \leq t \leq T} J(\omega_1, \phi_\omega, t) < 0] \\
&\geq \mathcal{P}^w[J^1(\omega_1) + \sup_{m \leq t \leq T} J^2(\omega_1, t) \cdot \sup_{m \leq t \leq T} J^3(\phi_\omega, t) < 0]
\end{aligned}$$

It is clear $J^2(\omega_1)$ is bounded for a fixed $\omega_1$ on the compact set and $J^1(\omega_1)$ also. We set $J^1(\omega_1) = c_1$, $\sup_{m \leq t \leq T} J^2(\omega_1, t) = c_2$. Thanks to Lemma 6.6, there exist $D \subset \Omega^w$ such that $\mathcal{P}^w[D] > 0$ and

$$\int_0^m (m-u)^{H-\frac{1}{2}}\, d\phi_\omega(u) < c_3$$

for $\omega \in D$. where $c_3$ could be small enough that $c_2 \cdot e^{c_3 - 1} < 0$. Under assumption at beginning, $c_1 \leq 0$ almost surely. All of this leads to

$$\begin{aligned} & \mathcal{P}^w[\mathrm{E}[\mathbb{1}_E(\omega_1, \phi_\omega)|\psi_\omega = \omega_1]] \\ = \; & \mathcal{P}^w[J^1(\omega_1) + \sup_{m \leq t \leq T} J^2(\omega_1, t) \cdot \sup_{m \leq t \leq T} J^3(\phi_\omega, t)] \\ > \; & 0 \end{aligned}$$

for $\omega \in \Omega^w$. Then

$$\begin{aligned} & \mathcal{P}^w[\sum_{k=1}^{l} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) < 0] \\ \geq \; & \mathcal{P}^w[\sum_{k=1}^{l-1} f_k(e^{U_H(\tau_{k+1})} - e^{U_H(\tau_k)}) + \sup_{m \leq t \leq T} f_l(e^{U_H(\tau_l + t)} - e^{U_H(\tau_l)}) < 0] \\ = \; & \mathrm{E}[\mathrm{E}[\mathbb{1}_E(\omega_1, \phi_\omega)|\psi_\omega = \omega_1]] > 0 \end{aligned}$$

this contradicts our assumption. It must be that $\mathcal{P}[(\xi^1 \cdot S)_T < 0] > 0$. □

**Corollary 6.8.** Let strategy $(\xi^0, \xi^1)$ be such that, $\xi^1$ is given as in Theorem 6.7. Then the strategy has no arbitrage in our finance market.

**Proof.** Assume $(\xi^0, \xi^1)$ is a self-financing strategy. In terms of Definition 6.4,

$$\begin{aligned} \tilde{V}_T - \tilde{V}_0 &= \sum_{k=1}^{n} \frac{(\xi_k^0 A_k + \xi_k^1 S_k)}{A_k} - \frac{(\xi_k^0 A_{k-1} + \xi_k^1 S_{k-1})}{A_{k-1}} \\ &= \sum_{k=1}^{n} \xi_k^1 (\tilde{S}_k - \tilde{S}_{k-1}) \end{aligned}$$

It follows then from Theorem 6.7, $\mathcal{P}[(\tilde{V}_T - \tilde{V}_0) < 0] > 0$, since and therefore there is no arbitrage in this sense. □

## 6.2 Fractional Stochastic Volatility Model

In framework of the Black-Scholes, a risky asset price is modelded as follows

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t \tag{6.4}$$

for $r, \sigma \in \mathbb{R}_+$. In the simplest case, the volatility is assumed as a constant or a deterministic function of time and underlying price of the asset. Such models, however, generate unrealistic volatility dynamics. For solve this problem, in such Hull-White, Heston or SABR models, the volatility also modelded as a stochastic process(e.g. semimartingal).

In our paper, the log-volatility is assumed to obey fractional Ornstein-Uhlenbeck process. That means

$$dX_t = -aX_t\,dt + \gamma\,dU_H(t) \tag{6.5}$$

where $a, \gamma \in \mathbb{R}_+$. By the proceeding section, we have a stationary solution

$$\hat{X}_{H,t} = e^{-at}\int_{-\infty}^{t} e^{au}\,dU_H(u) \tag{6.6}$$

for an appropriate initial condition. $\hat{\sigma}_{H,t} = \exp\{\hat{X}_{H,t}\}$ has following property.

**Proposition 6.9.** Let $\hat{X}_{H,t}$ be such that as in (6.6) and $\hat{\sigma}_{H,t} = \exp\{\hat{X}_{H,t}\}$, then $(\hat{\sigma}_{H,t})$ is weak stationary and has long memory for $H \in (\frac{1}{2}, 1)$.

**Proof.** We start our proof by definition of the autocovariance of $\hat{\sigma}_{H,t}$

$$\begin{aligned}
&\varsigma_{\hat{\sigma}_H}(\tau) \\
=\ & \mathrm{Cov}[\hat{\sigma}_{H,t}, \hat{\sigma}_{H,t+\tau}] \\
=\ & \mathrm{E}[\hat{\sigma}_{H,t}\hat{\sigma}_{H,t+\tau}] - \mathrm{E}[\hat{\sigma}_{H,t}]\mathrm{E}[\hat{\sigma}_{H,t+\tau}] \\
=\ & \mathrm{E}[\exp(\hat{X}_{H,t} + \hat{X}_{H,t+\tau})] - \mathrm{E}[\exp(\hat{X}_{H,t})]\mathrm{E}[\exp(\hat{X}_{H,t+\tau})]
\end{aligned}$$

Since $\hat{X}_{H,t}, \hat{X}_{H,t+\tau}$ are centred Gaussian, we apply (2.7) for it

$$\begin{aligned}
&\varsigma_{\hat{\sigma}_H}(\tau) \\
=\ & \exp(\frac{1}{2}\mathrm{Var}[\hat{X}_{H,t} + \hat{X}_{H,t+\tau}]) - \exp(\frac{1}{2}\mathrm{Var}[\hat{X}_{H,t}])\exp(\frac{1}{2}\mathrm{Var}[\hat{X}_{H,t+\tau}])
\end{aligned}$$

Since $(\hat{X}_{H,t})_t$ is stationary, we have

$$\begin{aligned}
&\varsigma_{\hat{\sigma}_H}(\tau) \\
=\ & \exp(\frac{1}{2}\mathrm{Var}[\hat{X}_{H,t} + \hat{X}_{H,t+\tau}]) - \exp(\mathrm{Var}[\hat{X}_{H,t}]) \\
=\ & \exp(\frac{1}{2}(\mathrm{Var}[\hat{X}_{H,t}] + \mathrm{Var}[\hat{X}_{H,t+\tau}] + 2\mathrm{E}[\hat{X}_{H,t}\hat{X}_{H,t+\tau}])) - \exp(\mathrm{Var}[\hat{X}_{H,t}]) \\
=\ & \exp(\mathrm{Var}[\hat{X}_{H,t}] + \mathrm{Cov}[\hat{X}_{H,t}, \hat{X}_{H,t+\tau}]) - \exp(\mathrm{Var}[\hat{X}_{H,t}])
\end{aligned}$$

The term $\mathrm{Var}[\hat{X}_{H,t}]$ is independent of $\tau$. We define $\mathrm{Var}[\hat{X}_{H,t}] = C$. Then

$$\begin{aligned}
&\varsigma_{\hat{\sigma}_H}(\tau) \\
=\ & \exp(C)\exp(\varsigma_{\hat{X}_H}(\tau)) - \exp(C) \\
=\ & \exp(C)(\exp(\varsigma_{\hat{X}_H}(\tau)) - 1) \\
=\ & \kappa(\exp(\varsigma_{\hat{X}_H}(\tau)) - 1)
\end{aligned}$$

for some $\kappa$. Obviously, $\mathbb{E}[\hat{\sigma}_H(t)] = 1$. With $\varsigma_{\hat{\sigma}_H}(0) = \kappa(\exp(\varsigma_{\hat{X}_H}(0)) - 1)$, the first claim is proved.

In the one hand, as we known, $\varsigma_{\hat{X}_H}(\tau)$ vanishes, for $H \in (\frac{1}{2}, 1)$, as $\tau$ goes to infinity. I.e.,

$$\lim_{\tau\to\infty} \kappa(\exp(\varsigma_{\hat{X}_H}(\tau)) - 1) = 0.$$

On the other hand, consider in Theorem 5.5,

$$\varsigma_{\hat{X}_H}(\tau)) = \mu\tau^{2H-2} + \mathrm{o}(\tau)$$

for some $\mu$. And $\varsigma_{\hat{X}_H}(\tau)$ is the equivalence infinitesimal of $\exp(\varsigma_{\hat{X}_H}(\tau)) - 1$. Hence,

$$\begin{aligned} & \lim_{\tau\to\infty} \frac{\varsigma_{\hat{\sigma}_H}(\tau) - \kappa\mu\tau^{2H-2}}{\tau} \\ = \; & \lim_{\tau\to\infty} \frac{\kappa(\exp(\varsigma_{\hat{X}_H}(\tau)) - 1) - \kappa\mu\tau^{2H-2}}{\tau} \\ = \; & \lim_{\tau\to\infty} \frac{\kappa(\varsigma_{\hat{X}_H}(\tau) - \mu\tau^{2H-2})}{\tau} \\ = \; & 0 \end{aligned}$$

This implies $\varsigma_{\hat{\sigma}_H}(\tau) = \kappa\mu\tau^{2H-2} + \mathrm{o}(\tau)$. For the same reason as in Theorem 4.17, the long memory property requires $1 < 2H$, namely, $H \in (\frac{1}{2}, 1)$. □

The Long memory property may explain why in stock market, large upheavals tend to be followed by large upheavals and small upheavals happend after by small upheavals. In order to model long memory volatility process, Comte and Renault are forced to set $H \in (\frac{1}{2}, 1)$ in (6.6) named fractional volatility stochastic(FSV), see[7].

## 6.3 Rough Fractional Stochastic Volatility Model

In order to generate a disirable volatility dynamics, Gatheral et al.(2014) take the implied volatility $\sigma^{BS}(m, \tau)$ into account. Where $m$ is the log-moneyness and $\tau$ is time to expiration date. The implied volatility refer to the value of volatility required in the Black-Scholes model such that the pricing is coincide with the asset price observed. Graphing implied Volatility as a function of moneyness and time to expiriation seems to be a U-shape which is so-called volatility smile. In particular, the term structure of volatility skew of at-the-money of stylized data

$$\kappa(\tau) = |\frac{\partial}{\partial m}\sigma^{BS}(m, \tau)|_{m=0}|,$$

acts as a power law with exponent around $-\frac{1}{2}$, which is explained for example in [10]. In the one hand, in the FSV model with $H \in (\frac{1}{2}, 1)$, the volatility smile is depressed by arising $\tau$, see [8], p.350, Eq. (4.7). On the other hand, in [9], the volatility is driven by FBM with Hurst exponent $H$ in Fukasawa's model whose volatility skew of at-the-money has a form $\kappa(\tau) \sim \tau^{H-\frac{1}{2}}$ as $\tau$ goes to zero. This requires that $H$ is near zero to match the power law decay of $\kappa(\tau)$. All of this suggest us, we should apply a stochastic volatility model which is driven by FBM with Hurst exponent $H \in (0, \frac{1}{2})$.
Replaced by $H \in (0, \frac{1}{2})$ in (6.5), a fractional stochastic volatility model (RFSV) is

$$dX_t = -aX_t\,dt + \gamma\,dU_H(t), \tag{6.7}$$

for $a, \gamma \in \mathbb{R}_+$ and a stationary solution

$$\hat{X}_{H,t} = \mu + e^{-at}\gamma \int_{-\infty}^{t} e^{au}\,dU_H(u)\,. \tag{6.8}$$

Consider a quantity defined on $[0, T]$ with mesh $\tau$

$$s(\tau, \sigma) = \frac{1}{N}\sum_{k=1}^{N} |\log(\sigma_{k\tau}) - \log(\sigma_{(k-1)\tau})|^2 \tag{6.9}$$

Where $N = \lfloor T/\tau \rfloor$. Due to the volatilities are not observable, we could take spot volatility values instead of them. In [3], the daily spot variances by daily realized variance estimates $\tilde{\sigma}$ are used. A plotting of $\log(s(\tau, \tilde{\sigma}))$ against $\log(\tau, \tilde{\sigma})$ looks then as straight line, i.e.,

$$s(\tau, \tilde{\sigma}) = k\tau^z \tag{6.10}$$

There is a linear relationship between $log(s(\tau))$ and $\tau$ and the RFSV model does match the observed phenomenon.Insteading $\log(\sigma(t))$ by $\hat{X}_{H,t}$

$$s(\tau, \hat{X}_H) \quad = \quad \frac{1}{N}\sum_{k=1}^{N} |\hat{X}_{H,k\tau} - \hat{X}_{H,(k-1)\tau}|^2$$

Since $(\hat{X}_{H,t})$ is stationary, we could apply weak law of large number, as $N$ goes to infinity,

$$\begin{aligned} s(\tau, \hat{X}_H) &\overset{\tau\downarrow 0}{\to} \mathrm{E}[|\hat{X}_{H,t+\tau} - \hat{X}_{H,t}|^2] \\ &= 2\mathrm{Var}[\hat{X}_{H,t}] - 2\mathrm{Cov}[\hat{X}_{H,t}, \hat{X}_{H,t+\tau}] \end{aligned}$$

**Theorem 6.10.** Let $(\hat{X}_{H,t,a})$ be as in (6.8) driven by a FBM $(U_H(t))$ with $H \in (0, \frac{1}{2})$, then

$$\mathrm{E}[\sup_{t\in[0,T]} |\hat{X}_{H,t+\tau,a} - \gamma U_H(t)|] \to 0$$

as $a \to 0$ for $T > 0$.

**Proof.** Cf. [10], p.15, Proposition 3.1 . □

The theorem shows if $a$ is small enough, $(\hat{X}_{H,t})_t$ behaves essentially as FBM at a compact time scale.

**THEOREM 6.11.** Let $(\hat{X}_{H,t,a})$ the solution by (6.8) with $H \in (0, \frac{1}{2})$, then

$$\mathrm{Var}[\hat{X}_{H,t,a}] - \mathrm{Cov}[\hat{X}_{H,t,a}, \hat{X}_{H,t+\tau,a}] \to \frac{1}{2}\gamma^2\tau^{2H} \tag{6.11}$$

as $a$ goes to zero, for $t > 0, \tau > 0$.

**Proof.** Cf. [10], p.16, Corollary 3.2 . □

(6.11) shows, the choise of $H \in (0, \frac{1}{2})$ enable us to model the log-volatility process has a form of (6.10). It may turn out to be RFSV is more reasonable than FSV for this empirical result.

## 6.4 Fractional Calculus

The fractional integral could be derived from the repeated integral which is approached by Riemann-Liouville integral.

$$\begin{aligned}
&\int_0^s \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{n-1}} f(s_n) ds_n \cdots ds_2 ds_1 \\
= \; &\frac{1}{(n-1)!} \int_0^s (s-u)^{n-1} f(u)\, du
\end{aligned}$$

for $n \in \mathbb{N}_+$. Where $n$ is said to be *the order* of the fractional integral. We extend it with the order $\alpha \in \mathbb{R}_+$.

**DEFINITION 6.12.** Let $f$ is locally integrable function. Then we define *fractional integral of order* $\alpha$ as follows

$$I^\alpha f(s) := \frac{1}{\Gamma(\alpha)} \int_0^s (s-u)^{\alpha-1} f(u)\, du \tag{6.12}$$

for $\alpha \in \mathbb{R}_+$.

Remark, $I^\alpha I^\beta = I^\beta I^\alpha = I^{\alpha+\beta}$. Let $\Phi_\alpha(s) := \frac{s^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha \in \mathbb{R}_+$, then

$$I^\alpha f(s) = \Phi_\alpha(s) * f(s) \tag{6.13}$$

We give the definition of fractional derivative.

**DEFINITION 6.13.** Let $f$ is locally integrable function. Then we define *fractional derivative of order* $\alpha$ as follows

$$D^\alpha f(s) := \begin{cases} \frac{d^n}{ds^n}\left[\frac{1}{\Gamma(l-\alpha)} \int_0^s \frac{f(u)}{(s-u)^{\alpha+1-n}}\, du\right] & \text{if } n-1 < \alpha < n, \\ \frac{d^n}{ds^n} f(s) & \text{if } \alpha = n \end{cases} \tag{6.14}$$

### 6.4.1 Discretization Fractional Process

**Definition 6.14.** The *fractional process of order $\alpha$* is defined as

$$X(t) = \int_0^t g(t-u)\,dB_u \tag{6.15}$$

where

$$\begin{aligned} g(s) &= \Phi_{\alpha+1}(s)h(s) \\ &= \frac{s^\alpha h(s)}{\Gamma(1+\alpha)} \end{aligned}$$

with $h \in C^1([0,T])$, $|\alpha| < \frac{1}{2}$.

In [5], Comte gave a truncated representation FBM ($U_{\alpha,t}$).

$$U_{\alpha,t} = \int_0^t \frac{(t-u)^\alpha}{\Gamma(1+\alpha)}\,dB_u, \tag{6.16}$$

Where $|\alpha| < \frac{1}{2}, (B_u)$ Brownian motion. Notice that $\alpha$ is given by $H - \frac{1}{2}$ as in previous representation and the long memory property remains true.

**Proposition 6.15.** Let $X(t)$ be a fractional process of order $\alpha$ as by (6.15). Then

$$X(t) = \int_0^t c(t-u)\,dU_{\alpha,u} \tag{6.17}$$

$$:= \frac{d}{dt}\left(\int_0^t c(t-u)U_{\alpha,u}\,du\right) \tag{6.18}$$

where $c \in C([0,T])$, $c$ and $g$ are functions related by

$$c(s) = \frac{d}{ds}\left(\int_0^s \frac{(s-u)^{-\alpha}u^\alpha h(u)\,du}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\right). \tag{6.19}$$

$$u^\alpha h(u) = \frac{d}{du}\left(\int_0^u c(s)(u-s)^\alpha\,ds\right). \tag{6.20}$$

**Proof.** See [6], pp.106-108, Lemma 1, Proposition 1. □

**Definition 6.16.** The *fractional derivation of order $\alpha$* of (6.15) is defined as

$$\begin{aligned} &X^{(\alpha)}(t) \\ := &\int_0^t \frac{(t-u)^{-\alpha}}{\Gamma(1-\alpha)}\,dX_t \\ := &\frac{d}{dt}\int_0^t \frac{(t-u)^{-\alpha}}{\Gamma(1-\alpha)}X_t\,du \\ = &D^\alpha X(t) \end{aligned} \tag{6.21}$$

**Proposition 6.17.** Let $|\alpha| < \frac{1}{2}$, if $h(0) \neq 0$ and $h$ is twice continuously differentiable, then

$$X^{(\alpha)}(t) = \int_0^t c(t-s)\, dB_s\,.$$

where $c$ and $h$ are one-to-one related by (6.19) and (6.20).

**Proof.** See [6], p.111, Proposition 4. □

**Theorem 6.18.**

$$X(t) = \int_0^t \frac{(t-u)^\alpha}{\Gamma(1+\alpha)}\, dX_u^{(\alpha)} \tag{6.22}$$

**Proof.** Since all the integrands is nonnegative, we could apply Fubini theorem.

$$\begin{aligned}
& \int_0^t \frac{(t-u)^\alpha}{\Gamma(1+\alpha)}\, dX_u^{(\alpha)} \\
= & \frac{d}{dt}\int_0^t \frac{(t-u)^\alpha}{\Gamma(1+\alpha)}\left(\int_0^u \frac{(u-s)^{-\alpha}}{\Gamma(1-\alpha)}\, dX_s\right) du \\
= & \frac{d}{dt}\int_0^t \left(\int_s^t \frac{(t-u)^\alpha (u-s)^{-\alpha}}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\, du\right) dX_s
\end{aligned}$$

$$\begin{aligned}
& \int_s^t \frac{(t-u)^\alpha (u-s)^{-\alpha}}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\, du \\
= & \int_0^1 \frac{(t-s-(t-s)v)^\alpha (t-s)^{-\alpha}}{\Gamma(1+\alpha)\Gamma(1-\alpha)}(t-s)\, dv \\
= & \frac{(t-s)}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\int_0^1 (1-v)^\alpha v^{-\alpha}\, dv
\end{aligned}$$

Note that the beta function

$$\begin{aligned}
B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \\
&= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\end{aligned}$$

for $x, y \in \mathbb{R}_+$. Then,

$$\begin{aligned}
& \int_s^t \frac{(t-u)^\alpha (u-s)^{-\alpha}}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\, du \\
= & \frac{(t-s)}{\Gamma(1+\alpha)\Gamma(1-\alpha)} B(1+\alpha, 1-\alpha) \\
= & \frac{(t-s)}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \frac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{\Gamma(2)} \\
= & t-s
\end{aligned}$$

Plugging it back,

$$\begin{aligned}
& \int_0^t \frac{(t-u)^\alpha}{\Gamma(1+\alpha)}\, dX_u^{(\alpha)} \\
= \; & \frac{d}{dt}\int_0^t t-s\, dX_s \\
= \; & \frac{d^2}{dt^2}\int_0^t (t-s)X_s\, ds \\
= \; & \frac{d^2}{dt^2}\int_0^t X_s \int_s^t du\, ds \\
= \; & \frac{d^2}{dt^2}\int_0^t \int_s^t X_s\, du\, ds \\
= \; & \frac{d^2}{dt^2}\int_0^t \int_0^u X_s\, ds\, du \\
= \; & X_t
\end{aligned}$$

□

**Example 6.19.** The truncated FOU process driven by $U_{\alpha,t}$ is

$$X_{\alpha,t} = \gamma \int_0^t e^{-a(t-u)}\, dU_{\alpha,u} \tag{6.23}$$

It is a form of (6.15) with

$$c(u) \;=\; \gamma e^{-au}$$

In terms of (6.20),

$$\begin{aligned}
& g(u) \\
= \; & \frac{\frac{d}{du}\int_0^u c(s)(u-s)^\alpha\, ds}{\Gamma(1+\alpha)} \\
= \; & \frac{\gamma}{\Gamma(1+\alpha)}\frac{d}{du}\int_0^u e^{-as}(u-s)^\alpha\, ds
\end{aligned}$$

Using partial integration,

$$\begin{aligned}
& g(u) \\
= \; & \frac{\gamma}{\Gamma(1+\alpha)}\left(\left(\frac{d}{du}(-\frac{1}{1+\alpha}e^{-as}(u-s)^{1+\alpha}|_{s=0}^{u})\right) - \left(\frac{d}{du}(\int_0^u -ae^{-as}(u-s)^\alpha)\, ds\right)\right) \\
= \; & \frac{\gamma}{\Gamma(1+\alpha)}\left(\left(\frac{d}{du}(\frac{u^{1+\alpha}}{1+\alpha})\right) - ae^{-au}\left(\int_0^u e^{-as}s^\alpha\, ds\right)\right) \\
= \; & \frac{\gamma}{\Gamma(1+\alpha)}(u^\alpha - ae^{-au}\int_0^u e^{-as}s^\alpha\, ds)
\end{aligned}$$

then $g = \gamma(1 + \frac{ae^{-au}\int_0^u e^{-as}s^\alpha\, ds}{u^\alpha})$ and therefore $g(0) \neq 0$ and it is twice continueously differable. I.e., its fractional derivative of order $\alpha$ is

$$\begin{aligned} X^{(\alpha)}(t) &= \int_0^t c(t-u)\, dB_u \\ &= \gamma \int_0^t e^{-a(t-u)}\, dB_u \end{aligned}$$

and

$$X_{\alpha,t} = \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)}\, dX^{(\alpha)}(s)\,.$$

To approximate $X_{\alpha,t}$ on a discrete time scale, according to (6.22), Comte had defined a quantity

$$\hat{X}_{\alpha,n}(t) = \sum_{k=1}^{\lfloor nt \rfloor} \frac{(t - \frac{k-1}{n})^\alpha}{\Gamma(1+\alpha)} \left( X^{(\alpha)}_{\frac{k}{n}} - X^{(\alpha)}_{\frac{k-1}{n}} \right) \tag{6.24}$$

**Theorem 6.20.** $\hat{X}_{\alpha,n}(t) \to X_{\alpha,n}(t)$ in distribution as $n$ tend to infinity.

**Proof.** See. [7], p.298, Proposition 3.1 . □

Since $\hat{X}_{\alpha,n}(t)$ is asymtotically stationary and Gaussian, an AR(1) approach of $X^{(\alpha)}$ could be given,

$$(1 - c_n L_n) X^{(\alpha)}_{\frac{k}{n}} = \epsilon(\frac{k}{n}) \tag{6.25}$$

where $\epsilon(\cdot) \sim \mathcal{N}(0, \mathrm{Var}[X^{(\alpha)}_{\cdot}]), L_n$ represents the Lag operator such that $L_n X^{(\alpha)}_{\frac{k}{n}} = X^{(\alpha)}_{\frac{k-1}{n}}$ and $c_n$ are normalised cofficients corresponding to (6.24). Moreover, in the one hand,

$$\begin{aligned} \hat{X}_{\alpha,n}(\frac{k}{n}) &= \sum_{l=1}^{k} \frac{(\frac{k}{n} - \frac{l-1}{n})^\alpha}{\Gamma(1+\alpha)} \left( X^{(\alpha)}_{\frac{l}{n}} - X^{(\alpha)}_{\frac{l-1}{n}} \right) \\ &= \sum_{l=1}^{k} \frac{(k-l+1)^\alpha}{\Gamma(1+\alpha)n^\alpha} \left( X^{(\alpha)}_{\frac{l}{n}} - X^{(\alpha)}_{\frac{l-1}{n}} \right) \\ &\overset{m=k-l}{=} \sum_{m=0}^{k-1} \frac{(m+1)^\alpha}{\Gamma(1+\alpha)n^\alpha} \left( X^{(\alpha)}_{\frac{k-m}{n}} - X^{(\alpha)}_{\frac{k-m-1}{n}} \right) \\ &= \left( \sum_{m=0}^{k-1} \frac{(m+1)^\alpha - m^\alpha}{\Gamma(1+\alpha)n^\alpha} \cdot L_n^m \right) X^{(\alpha)}_{\frac{k}{n}} \\ &\overset{(6.25)}{=} \left( \sum_{m=0}^{k-1} \frac{(m+1)^\alpha - m^\alpha}{\Gamma(1+\alpha)n^\alpha} \cdot L_n^m \right) (1 - c_n L_n)^{-1} \epsilon(\frac{k}{n}). \end{aligned} \tag{6.26}$$

We rewrite this then

$$(1 - c_n L_n) \left( \sum_{m=0}^{k-1} \frac{(m+1)^\alpha - m^\alpha}{\Gamma(1+\alpha) n^\alpha} \cdot L_n^m \right)^{-1} \hat{X}_{\alpha,n}(\frac{k}{n}) = \epsilon(\frac{k}{n}).$$

On the other hand, if we take consideration with the $ARFIMA(1, \alpha, 0)$, insteading of the content with $\{\}^{-1}$ by the $\alpha$-integrated term $(1 - L_n)^\alpha$. Then we have

$$(1 - c_n L_n)(1 - L_n)^\alpha \hat{X}_{\alpha,n}(\frac{k}{n}),$$

which is in general not Gaussian. Furthermore, $(1 - L_n)^\alpha \hat{X}_{\alpha,n}$ is in general not a $AR(1)$. In the other words, the ARFIMA$(1, \alpha, 0)$ may not fit such a high-frequency data that is comfortable with the fractional stochastic volatility model driven by FBM.

for $n \in \mathbb{Z}$ and $\beta \in \mathbb{R}$.

## 6.5 Weighted Fractional Stochastic Volatility Model

**Definition 6.21.** A *mixed fractional Brownian motion* is defined as follows

$$M_{\alpha,\beta,H_1,H_2}(t) = \alpha U_{H_1}(t) + \beta U_{H_2}(t) \tag{6.27}$$

for $t \in \mathbb{R}$, where $\alpha, \beta$ are real numbers and $U_{H_1}, U_{H_2}$ are two independent FBM's with Hurst exponents $H_1 \in (0, \frac{1}{2}), H_2 \in (\frac{1}{2}, 1)$ respectively.

**Proposition 6.22.** The mixed fractional Brownian motion $M_{\alpha,\beta,H_1,H_2}(t)_{t\in\mathbb{R}}$ has following properties

(i) $M_{\alpha,\beta,H_1,H_2}(0) = 0$ and $(M_{\alpha,\beta,H_1,H_2}(t))_t$ is a centered Gaussian process.

(ii) $\mathrm{Cov}[M_{\alpha,\beta,H_1,H_2}(t), M_{\alpha,\beta,H_1,H_2}(s)] = \alpha^2 \mathrm{Cov}[U_{H_1}(t), U_{H_1}(s)] + \beta^2 \mathrm{Cov}[U_{H_2}(t), U_{H_2}(s)] = \frac{1}{2} \left( \alpha^2 (t^{2H_1} + s^{2H_1} + |t-s|^{2H_1}) + \beta^2 (t^{2H_2} + s^{2H_2} + |t-s|^{2H_2}) \right).$

(iii) $M_{\alpha,\beta,H_1,H_2}(qt) \sim M_{\alpha q^{H_1}, \beta q^{H_2}, H_1, H_2}(t)$, for $q \in \mathbb{R}$.

**Proof.** (i): $M_{\alpha,\beta,H_1,H_2}(t) = \alpha U_{H_1}(t) + \beta U_{H_2}(t) = \alpha \cdot 0 + \beta \cdot 0 = 0$. To prove it is a Gaussian processes, let us have a look at it's characteristic function

$$\begin{aligned} & \mathrm{E}[\exp(i\xi \sum_{k=1}^{d} c_k M_{\alpha,\beta,H_1,H_2}(k))] \\ = \; & \mathrm{E}[\exp(i\xi \sum_{k=1}^{d} c_k (\alpha U_{H_1}(k) + \beta U_{H_2}(k)))] \end{aligned}$$

Since $U_{H_1}(t), U_{H_2}(t)$ are independent and $(U_{H_1}(t))_t, (U_{H_2}(t))_t$ are centered Gaussian process, $\sum_{k=1}^{d} c_k(\alpha U_{H_1}(k) + \beta U_{H_2}(k))$ is centered Gaussian and therefore $(M_{\alpha,\beta,H_1,H_2}(t))_t$ is

centered Gaussian process.
(ii): Using independence of $U_{H_1}(t)$ and $U_{H_2}(t)$, we have

$$\begin{aligned}
& \mathrm{Cov}[M_{\alpha,\beta,H_1,H_2}(t), M_{\alpha,\beta,H_1,H_2}(s)] \\
= \; & \mathrm{Cov}[\alpha U_{H_1}(t) + \beta U_{H_2}(t), \alpha U_{H_1}(t) + \beta U_{H_2}(s)] \\
= \; & \mathrm{E}[(\alpha U_{H_1}(t) + \beta U_{H_2}(t))(\alpha U_{H_1}(t) + \beta U_{H_2}(s))] \\
= \; & \mathrm{E}[\alpha^2 U_{H_1}(t) U_{H_1}(s)] + \underbrace{\mathrm{E}[\alpha\beta U_{H_2}(t) U_{H_1}(s)]}_{=0} + \underbrace{\mathrm{E}[\alpha\beta U_{H_1}(t) U_{H_2}(s)]}_{=0} + \mathrm{E}[\beta^2 U_{H_2}(t) U_{H_2}(s)] \\
= \; & \alpha^2 \mathrm{Cov}[U_{H_1}(t), U_{H_1}(s)] + \beta^2 \mathrm{Cov}[U_{H_2}(t), U_{H_2}(s)].
\end{aligned}$$

And the rest is clear.
(iii):

$$\begin{aligned}
M_{\alpha,\beta,H_1,H_2}(qt) &= \alpha U_{H_1}(qt) + \beta U_{H_2}(qt) \\
&\sim \alpha q^{H_1} U_{H_1} + \beta q^{H_2} U_{H_2} \\
&= M_{\alpha q^{H_1}, \beta q^{H_2}, H_1, H_2}(t)
\end{aligned}$$

□

We could give our stochastic volatility model diven by the mixed FBM. Given all parameter as in the assumption as before, we have

$$dX_{\alpha,\beta,H_1,H_2}(t) = -a X_{\alpha,\beta,H_1,H_2}(t)\, dt + \gamma\, dM_{\alpha,\beta,H_1,H_2}(t) \tag{6.28}$$

for $t \geq 0$ and where $a > 0, \gamma > 0$.

**Proposition 6.23.** For an appropriate initial condition of (6.28), there exist a solution $\hat{X}_t$ satisfies follwing properties

(i) $(\hat{X}_t)_{t\geq 0}$ is a centered Gaussian stationary process.
(ii) $\hat{X}_t$ has long memory.

**Proof.** In terms of (6.28), then

$$X_{\alpha,\beta,H_1,H_2}(t) = X_{\alpha,\beta,H_1,H_2}(0) - a \int_0^t X_u \, du + \gamma M_{\alpha,\beta,H_1,H_2}(t).$$

Recall by (5.3), the integral in sense of Riemann-Stieljet of $\int_0^t e^{au}\, dM_{\alpha,\beta,H_1,H_2}$ is well-defined because $\int_0^t e^{au}\, dU_{H_i}$ is well-defined for $i \in \{1, 2\}$.

As in Theorem 5.3, we have the solution

$$\begin{aligned} & X_t \\ = \; & e^{-at}\left(\gamma \int_0^t e^{au}\, dM_{\alpha,\beta,H_1,H_2} + X_0\right) \\ = \; & e^{-at}\left(\gamma \int_0^t e^{au}\, d(\alpha U_{H_1} + \beta U_{H_2}) + X_0\right) \\ = \; & e^{-at}\left(\alpha\gamma \int_0^t e^{au}\, dU_{H_1} + \beta\gamma \int_0^t e^{au}\, dU_{H_2} + X_0\right) \end{aligned}$$

Given $X_0$ so that

$$X_{\alpha,\beta,H_1,H_2}(t) = \alpha \underbrace{\gamma e^{-at} \int_{-\infty}^t e^{au}\, dU_{H_1}}_{:=J_{H_1}(t)} + \beta \underbrace{\gamma e^{-at} \int_{-\infty}^t e^{au}\, dU_{H_2}}_{:=J_{H_2}(t)} . \tag{6.29}$$

Notice $(J_{H_1}(t)), (J_{H_2}(t))$ are stationary fractional Ornstein-Uhlenbeck process. Since $J_{H_1}(t), J_{H_2}(t)$ defined as integral of $U_{H_1}, U_{H_2}$ of Riemann-Stieljes sense, they are therefore independent. Hence $\sum_{k=1}^{d} c_k(\alpha J_{H_1}(k) + \beta J_{H_2}(k))$ are centered Gaussian.
For (i):

$$\begin{aligned} & \mathrm{E}[\exp(i\xi \sum_{k=1}^{d} c_k X_{\alpha,\beta,H_1,H_2}(k))] \\ = \; & \mathrm{E}[\exp(i\xi \sum_{k=1}^{d} c_k(\alpha J_{H_1}(k) + \beta J_{H_2}(k)))] \\ = \; & \mathrm{E}[\exp(i\xi \sum_{k=1}^{d} c_k(\alpha J_{H_1}(k+s) + \beta J_{H_2}(k+s)))] \end{aligned}$$

for a fixed $s$. This shows $(X_{\alpha,\beta,H_1,H_2}(t))$ is centered Gaussian and stationary.
For (ii):

$$\begin{aligned} & \varsigma(\tau) \\ = \; & \mathrm{Cov}[X_{\alpha,\beta,H_1,H_2}(0), X_{\alpha,\beta,H_1,H_2}(\tau)] \\ = \; & \mathrm{E}[X_{\alpha,\beta,H_1,H_2}(0) X_{\alpha,\beta,H_1,H_2}(\tau)] \\ = \; & \mathrm{E}[(\alpha J_{H_1}(0) + \beta J_{H_2}(0))(\alpha J_{H_1}(\tau) + \beta J_{H_2}(\tau)] \\ = \; & \alpha^2 \mathrm{E}[J_{H_1}(0) J_{H_1}(\tau)] + \alpha\beta \mathrm{E}[J_{H_1}(0) J_{H_2}(\tau)] + \alpha\beta \mathrm{E}[J_{H_2}(0) J_{H_1}(\tau)] + \beta^2 \mathrm{E}[J_{H_2}(0) J_{H_2}(\tau)]. \end{aligned}$$

they are centered Gaussian, using independece again, we have

$$\begin{aligned}
& \varsigma(\tau) \\
= \quad & \alpha^2 \mathrm{E}[J_{H_1}(0)J_{H_1}(\tau)] + \alpha\beta \mathrm{E}[J_{H_1}(0)]\mathrm{E}[J_{H_2}(\tau)] \\
+ \quad & \alpha\beta \mathrm{E}[J_{H_2}(0)]\mathrm{E}[J_{H_1}(\tau)] + \beta^2 \mathrm{E}[J_{H_2}(0)J_{H_2}(\tau)] \\
= \quad & \alpha^2 \mathrm{E}[J_{H_1}(0)J_{H_1}(\tau)] + \beta^2 \mathrm{E}[J_{H_2}(0)J_{H_2}(\tau)] \\
\overset{\text{Theorem 5.5}}{=} \quad & \frac{1}{2}(\alpha\gamma)^2 \sum_{k=1}^{N} a^{-2k} \left( \prod_{j=0}^{2k-1} (2H_1 - j) \right) \tau^{2(H_1-k)} + \mathrm{o}(\tau) \\
+ \quad & \frac{1}{2}(\beta\gamma)^2 \sum_{k=1}^{N} a^{-2k} \left( \prod_{j=0}^{2k-2} (2H_2 - j) \right) \tau^{2(H_2-k)} + \mathrm{o}(\tau).
\end{aligned}$$

□

As by Corollary 5.6, the summation over $\tau$ of the latter of the last line of the equation diverges, when $H_2 \in (\frac{1}{2}, 1)$. And not only the former but also the latter tend to zero as $\tau$ goes to zero. Thus, $(\hat{X}_t)$ has long memory property.

**Definition 6.24.** We add a restriction that $\alpha^2 + \beta^2 = 1$ and let $a$ so small such that it near zero to (6.28), then we get our *weighted fractional Brownian motion*.

**Proposition 6.25.** Let $M_{\alpha,\beta,H_1,H_2}$ be a weighted fractional brownian motion with respect to $U_{H_1}$ and $U_{H_2}$, $T, \tau > 0$, $a, \gamma$ are defined by (6.28), $J_{H_1}, J_{H_2}$ are defined by (6.29), $\phi = H_1 - \frac{1}{2}, \psi = H_2 - \frac{1}{2}$. Then

(i) $\mathrm{E}[\sup_{t\in[0,T]} |M_{\alpha,\beta,H_1,H_2}(t) - U_{H_1}(t)|] \to 0$ as $a \to 0, \alpha \to 1$.

(ii) $\mathrm{E}[|M_{\alpha,\beta,H_1,H_2}(t+\tau) - M_{\alpha,\beta,H_1,H_2}(t)|] \to \frac{1}{2}\gamma^2\tau^{2H}$ as $a \to 0, \alpha \to 1$.

(iii) Let

$$\begin{aligned}
& \hat{X}_{\alpha,\beta,H_1,H_2}(t) && (6.30) \\
= \quad & \alpha \sum_{k=1}^{\lfloor nt \rfloor} \frac{(t - \frac{k-1}{n})^{\phi}}{\Gamma(1+\phi)} \left( J_{H_1}^{(\phi)}(\frac{k}{n}) - J_{H_1}^{(\phi)}(\frac{k-1}{n}) \right) && (6.31) \\
+ \quad & \beta \sum_{k=1}^{\lfloor nt \rfloor} \frac{(t - \frac{k-1}{n})^{\psi}}{\Gamma(1+\psi)} \left( J_{H_2}^{(\psi)}(\frac{k}{n}) - J_{H_2}^{(\psi)}(\frac{k-1}{n}) \right) && (6.32)
\end{aligned}$$

then,

$$\hat{X}_{\alpha,\beta,H_1,H_2}(t) \to X_{\alpha,\beta,H_1,H_2}(t)$$

in distribution.

**Proof.** (i):

$$
\begin{aligned}
& \mathrm{E}[\sup_{t\in[0,T]} |M_{\alpha,\beta,H_1,H_2}(t) - U_{H_1}|] \\
= \; & \mathrm{E}[\sup_{t\in[0,T]} |M_{\alpha,\beta,H_1,H_2}(t) - \alpha J_{H_1}(t) + \alpha J_{H_1}(t) - U_{H_1}(t)|] \\
\leq \; & \mathrm{E}[\sup_{t\in[0,T]} \underbrace{|M_{\alpha,\beta,H_1,H_2}(t) - \alpha J_{H_1}(t)|]}_{\stackrel{\alpha\uparrow 1}{\to} 0} + \mathrm{E}[\sup_{t\in[0,T]} |\underbrace{\alpha J_{H_1}(t)}_{\stackrel{\alpha\uparrow 1}{\to} J_{H_1}(t)} - U_{H_1}(t)|] \\
\stackrel{\alpha\uparrow 1}{\to} \; & \mathrm{E}[\sup_{t\in[0,T]} |J_{H_1}(t) - U_{H_1}(t)|] \\
\stackrel{a\downarrow 0}{\to} \; & 0
\end{aligned}
$$

(ii):

$$
\begin{aligned}
& \mathrm{E}[|M_{\alpha,\beta,H_1,H_2}(t+\tau) - M_{\alpha,\beta,H_1,H_2}(t)|^2] \\
= \; & 2\mathrm{Var}[M_{\alpha,\beta,H_1,H_2}(t)] - 2\mathrm{Cov}[M_{\alpha,\beta,H_1,H_2}(t), M_{\alpha,\beta,H_1,H_2}(t+\tau)] \\
= \; & 2(\alpha^2\mathrm{Var}[J_{H_1}(t)] + \beta^2\mathrm{Var}[J_{H_2}(t)] \\
- \; & \alpha^2\mathrm{Cov}[J_{H_1}(t), J_{H_1}(t+\tau)] - \beta^2\mathrm{Cov}[J_{H_2}(t), J_{H_2}(t+\tau)]) \\
= \; & 2(\alpha^2\mathrm{Var}[J_{H_1}(t)] - \alpha^2\mathrm{Cov}[J_{H_1}(t), J_{H_1}(t+\tau)] \\
+ \; & \beta^2\mathrm{Var}[J_{H_2}(t)] - \beta^2\mathrm{Cov}[J_{H_2}(t), J_{H_2}(t+\tau)]) \\
\to \; & \alpha^2(\frac{1}{2}\gamma^2\tau^{2H_1}) + \beta^2\varsigma_{\hat{J}_{H_2}}(\tau)
\end{aligned}
$$

(iii): Suppose $\{Y_t\}_t, \{Z_t\}_t$ are two families of random variables with $Y_t \to Y, Z_t \to Z$ in distribution. $Y_t, Z_t$ are independent for each $t$. Then $Y_t + Z_t \to Y + Z$, because, using continuity theorem of characteristic function

$$
\begin{aligned}
\mathrm{E}[\exp i\xi(Y_t + Z_t)] &= \mathrm{E}[\exp i\xi Y_t]\mathrm{E}[\exp i\xi Z_t] \\
&\to \mathrm{E}[\exp i\xi Y]\mathrm{E}[\exp i\xi Z] \\
&= \mathrm{E}[\exp i\xi(Y+Z)].
\end{aligned}
$$

Consider

$$\alpha \sum_{k=1}^{\lfloor nt \rfloor} \frac{(t - \frac{k-1}{n})^{\phi}}{\Gamma(1+\phi)} \left( J_{H_1}^{(\phi)}(\frac{k}{n}) - J_{H_1}^{(\phi)}(\frac{k-1}{n}) \right)$$

and

$$\beta \sum_{k=1}^{\lfloor nt \rfloor} \frac{(t - \frac{k-1}{n})^{\psi}}{\Gamma(1+\psi)} \left( J_{H_2}^{(\psi)}(\frac{k}{n}) - J_{H_2}^{(\psi)}(\frac{k-1}{n}) \right)$$

are independent. According Theorem 6.20,

$$\alpha \sum_{k=1}^{\lfloor nt \rfloor} \frac{(t - \frac{k-1}{n})^{\phi}}{\Gamma(1+\phi)} \left( J_{H_1}^{(\phi)}(\frac{k}{n}) - J_{H_1}^{(\phi)}(\frac{k-1}{n}) \right) \to \alpha J_{H_1}(t)$$

and

$$\beta \sum_{k=1}^{\lfloor nt \rfloor} \frac{(t-\frac{k-1}{n})^{\psi}}{\Gamma(1+\psi)} \left( J_{H_2}^{(\psi)}(\frac{k}{n}) - J_{H_2}^{(\psi)}(\frac{k-1}{n}) \right) \to \beta J_{H_2}(t)$$

in distribution. Then

$$\begin{array}{rcl} \hat{X}_{\alpha,\beta,H_1,H_2}(t) & \overset{\text{Cramer Slutsky}}{\to} & \alpha J_{H_1}(t) + \beta J_{H_2}(t) \\ & = & X_{\alpha,\beta,H_1,H_2}(t) \end{array}$$

□

## 6.6 Discussion

As well known, we deal with the volatility as fractional stochastic volatility model. A high-frequency data may have required to take a great $H$. In FSV, we choose $H > \frac{1}{2}$ and that will make sure the solution of log-volatility SDE has long memory. In contrast, although it could not exhibit the long memory by a $H < \frac{1}{2}$, RFSV demonstrate a more reasonable smoothness of volatility. In another words, RFSV ensures $H$ to match the slop of the plotting of $log(s(\tau))$ agains $log(\tau)$, see (6.9).

The weighted-FSV model inherit long memory of FSV. With the manipulatable factor, one can achieve a result of smoothness of volatility near it by RFSV. As by RFSV, as $a$ goes to zero, the $log(\sigma)$ is modeled to act as FBM at any compact time scale.

Both in FSV and RFSV, there is a discretization motived by fractional derivation. An AR(1) approach can be applied in them. But

The future work is maybe the question, whether a dirsable discretization could have found which can compare with the usual models like ARFIMA.

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