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**Fractional Brownian Motion and its
Application in Financial Mathematics**

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Abstract

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1 Introduction

2 Gaussian Process and Brownian Motion

In this section we start off the general concept of probability spaces and stochastic processes. Of this, a most important case we then describe, is Gaussian process. It brings us to introduce the Brownian Motion as a fine example.

2.1 Probability Space and Stochastic Process

DEFINITION 2.1. Let \mathcal{A} be a collection of subsets of a set Ω . \mathcal{A} is then a σ -Algebra on Ω if it satisfies the following conditions:

- (i) $\Omega \in \mathcal{A}$.
- (ii) For any set $F \in \mathcal{A}$, its complement $F^c \in \mathcal{A}$.
- (iii) If a series $\{F_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$, then $\cup_{n \in \mathbb{N}} F_n \in \mathcal{A}$.

DEFINITION 2.2. A mapping \mathcal{P} is said to be a *probability measure* from \mathcal{A} to $\mathcal{B}(\mathbb{R}^n)$, if $\mathcal{P}[\sum_{n=1}^{\infty} F_n] = \sum_{n=1}^{\infty} \mathcal{P}[F_n]$ for any $\{F_n\}_{n \in \mathbb{N}}$ disjoint in \mathcal{A} satisfying $\sum_{n=1}^{\infty} F_n \in \mathcal{A}$.

DEFINITION 2.3. A *probability space* is defined as a triple $(\Omega, \mathcal{A}, \mathcal{P})$ of a set Ω , a σ -Algebra \mathcal{A} of Ω and a measure \mathcal{P} from \mathcal{A} to $\mathcal{B}(\mathbb{R}^n)$.

The σ -Algebra generated of all open sets on \mathbb{R}^n is called the *Borel σ -Algebra* which we denote as usual by $\mathcal{B}(\mathbb{R}^n)$. Let μ be a probability measure on \mathbb{R}^n . Indeed, $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu)$ is a special case that probability space on \mathbb{R}^n . A function f mapping from $(\mathcal{D}, \mathcal{D}, \mu)$ into $(\mathcal{E}, \mathcal{E}, \nu)$ is *measurable* if its collection of the inverse image of \mathcal{E} is a subset of \mathcal{D} . A *random variable* is a \mathbb{R}^n -valued measurable function on some probability space. Let \mathcal{P} represent a probability measure, recall that in probability theory, for $B \in \mathcal{B}(\mathbb{R}^n)$ we call $\mathcal{P}[\{X \in B\}]$ the *distribution* of X . We write also $\mathcal{P}_X[\cdot]$ or $\mathcal{P}[X]$ for convenience of the notation above.

DEFINITION 2.4. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. A *n-dimensional stochastic process* (X_t) is a family of random variable such that $X_t(\omega) : \Omega \rightarrow \mathbb{R}^n, \forall t \in T$, where T denotes the set of Index of Time.

DEFINITION 2.5. A stochastic process $(X_t)_{t \in T}$ is said to be *stationary*, if the joint distribution

$$\mathcal{P}[X_{t_1}, \dots, X_{t_n}] = \mathcal{P}[X_{t_1+\tau}, \dots, X_{t_n+\tau}]$$

for t_1, \dots, t_n and $t_1 + \tau, \dots, t_n + \tau \in T$.

Remark that, definition 2.5 means the distribution of a stationary process is independent of a shift of time.

2.2 Definition of Gaussian Process

DEFINITION 2.6 (1-dimensional normal distribution). A \mathbb{R} -valued random variable X is said to be *standard normal distributed*, if its distribution can be described as

$$\mathcal{P}[X \leq x] = \int_{-\infty}^x (2\pi)^{-\frac{1}{2}} e^{-\frac{u^2}{2}} du$$

for $x \in \mathbb{R}$.

DEFINITION 2.7. A \mathbb{R} -valued random variable X is said to be *normal distributed* with a mean μ and a variance σ^2 , if

$$(X - \mu)/\sigma$$

is standard normal distributed.

We use a notation $X \sim Y$, which means X and Y have the same distribution. In similar way it is denoted by $X \sim (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} dx$, if it is standard normal distributed. In order to identifying the behaviour of a normal distributed random variable we recall the characteristic function in probability theory, see[1].

PROPOSITION 2.8. Let X be a \mathbb{R} -valued standard normal distributed random variable. The characteristic function of X

$$\Psi_X(\xi) := \int_{\mathbb{R}} e^{ix\xi} \mathcal{P}[X \in dx] = e^{-\frac{\xi^2}{2}} \quad (2.1)$$

for $\xi \in \mathbb{R}$.

Proof. According the definition of characteristic function

$$\Psi_X(\xi) = \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} dx,$$

take differentiating both sides of the equation by ξ , then

$$\begin{aligned} \Psi'_X(\xi) &= \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} ix dx \\ &= (-i) \cdot \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} \left(\frac{d}{dx} e^{-\frac{x^2}{2}} \right) e^{ix\xi} dx \\ &\stackrel{\text{part.int.}}{=} - \int_{\mathbb{R}} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} e^{ix\xi} \xi dx \\ &= -\xi \Psi_X(\xi). \end{aligned}$$

Obviously, $\Psi(\xi) = \Psi(0)e^{-\frac{\xi^2}{2}}$ is the solution of the partial differential equation above, and $\Psi(0)$ is equal to 1. □

2.2 Definition of Gaussian Process

In particular, the characteristic function of a normal distributed random variable with a mean μ and a variance σ^2 , which denoted by $\Psi_{X_{\mu, \sigma^2}}(\xi)$, is $e^{i\mu\xi - \frac{1}{2}(\sigma\xi)^2}$. To achieve this result, we just need to substitute x by $(x - \mu)/\sigma$ in the calculation before.

DEFINITION 2.9. Let X be a \mathbb{R}^n -valued random variable. X is said to be *normal distributed*, if for any $d \in \mathbb{R}^n$ such that $d^T X$ is normal distributed on \mathbb{R} .

PROPOSITION 2.10. Let X be a \mathbb{R}^n -valued normal distributed. Then there exist $m \in \mathbb{R}^n$ and a positive definite symmetric matrix $\Sigma \in \mathbb{R}^{n \times n}$ such that,

$$\mathbb{E} e^{i\xi^T X} = e^{i\xi^T m - \frac{1}{2}\xi^T \Sigma \xi} \quad (2.2)$$

For $\xi \in \mathbb{R}^n$. Furthermore, the density function of X is

$$(2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-m)^T \Sigma^{-1}(x-m)} dx. \quad (2.3)$$

Remark, the equation (2.2) can also be as definition of characteristic function of a n -dimensional normal distributed random variable.

Proof. Since X normal distributed on \mathbb{R}^n , then $\xi^T X$ is normal distributed on \mathbb{R} . Due to the proposition 2.8 there is

$$\begin{aligned} \mathbb{E} e^{i\xi^T X} &= \mathbb{E} e^{i \cdot 1 \cdot \xi^T X} \\ &= e^{i\mathbb{E}[\xi^T X] - \frac{1}{2}\text{Var}[\xi^T X]} \\ &= e^{i\xi^T \mathbb{E}[X] - \frac{1}{2}\xi^T \text{Var}[X]\xi}. \end{aligned}$$

According the uniqueness theorem of characteristic function (Satz 23.4 in [1]), then we can deduce the density function of the equation (2.3). \square

A normal distributed normal random variable can be characterized by its mean and variance respectively mean vector and covariance vector because of the characteristic function.

DEFINITION 2.11. Let $(X_t)_{t \in T}$ be a \mathbb{R}^n -valued stochastic process. (X_t) is said to be a *gaussian process* if

$$c_1^T X_{t_1} + \cdots + c_n^T X_{t_n}$$

has a normal distribution for any $c_1 \cdots c_n \in \mathbb{R}^n$, $t_1 \dots t_n \in T$ and $n \in \mathbb{N}$.

The definition immediately shows for every X_t in gaussian process has a distribution.

3 Fractional Brownian Motion

4 Fractional Ornstein Uhlenbeck Process Model

5 Application in Financial Mathematics

6 Conclusion

References

- [1] BAUER, H. (2002). Wahrscheinlichkeitstheorie(5th. durchges. und verb. Aufl.). Berlin:
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