Non-Explosion for RDEs with Coefficients Having Unbounded Derivatives

Xue-Mei Li and Kexing Ying

EPFL, Institute of Mathematics. Email: kexing.ying@epfl.ch

Introduction

Fix $\mathbf{X} = (X, \mathbb{X})$ a rough path of $\alpha \in (\frac{1}{3}, 1]$ regularity (including Young regime) and b, σ (not necessarily bounded) vector fields, we consider the RDE

$$\mathrm{d}Y_t = b(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}\mathbf{X}_t.$$

Question: What sufficient conditions on b, σ ensure that the RDE does not explode?

Additive Noise: Spinning Out of Control

For $\sigma = 0$, we require b to have linear growth along the process (Osgood's criterion), i.e.

$$\left\langle \frac{x}{\|x\|}, b(x) \right\rangle \le f(\|x\|) \text{ for some } f \text{ s.t. } \int_{0+}^{\infty} \frac{1}{f(r)}\mathrm{d}r = \infty. \qquad (*)$$

This turns out to be insufficient even for the case of additive noise, i.e. σ is some constant matrix. Consider of the 2-dimensional SDE

$$\mathrm{d}x_t = \|x_t\|^5 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x_t \,\mathrm{d}t + \mathrm{d}W_t.$$

Easy to check for any initial condition, the SDE does not explode a.s.

Evolution of unit ball under same realization of the noise:

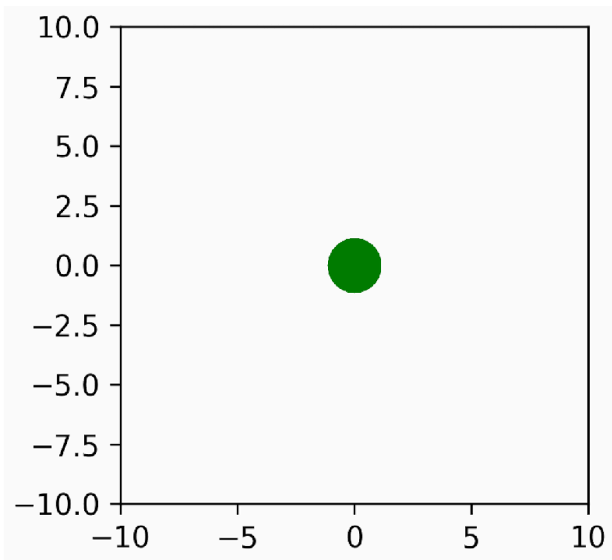


Figure 1: Simulation at $t = 0$.

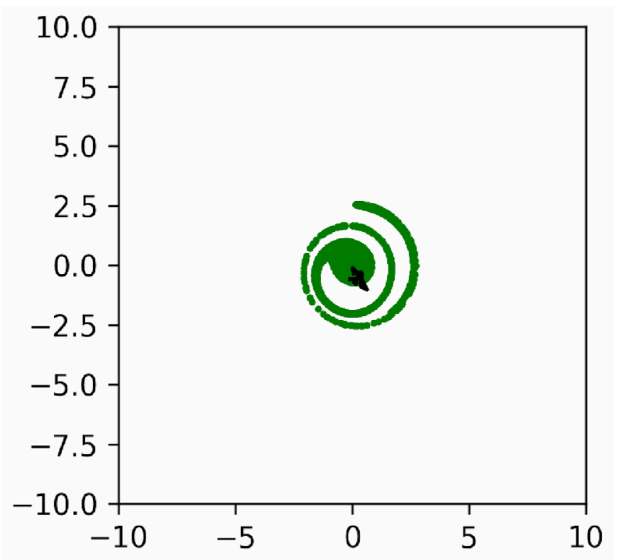


Figure 2: Simulation at $t = 1$.

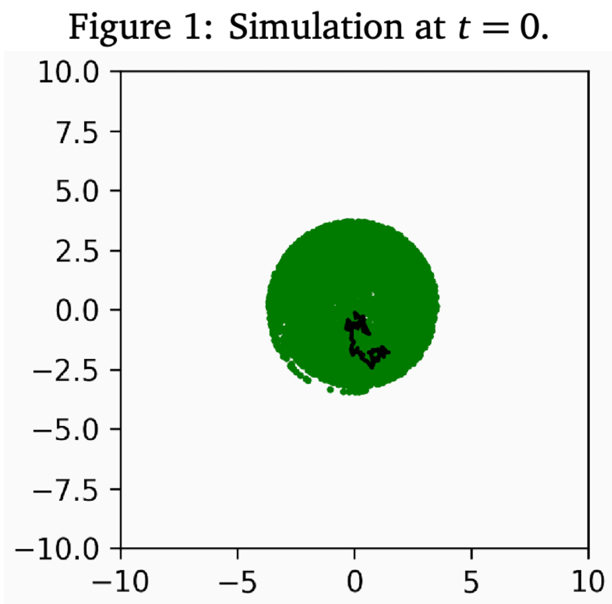


Figure 3: Simulation at $t = 3$.

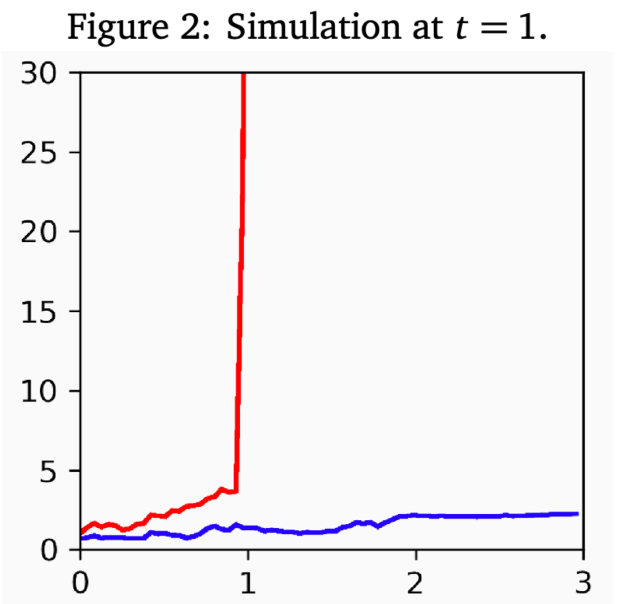


Figure 4: Norm of special points.

Denoting ϕ for the flow of this SDE, the Red line is $\sup_{\|x\|\le 1} \|\phi_t(x)\|$ and the Blue line is $\frac{1}{\pi}\int_{\|x\|\le 1} \|\phi_t(x)\|\mathrm{d}x$. Namely, for fixed realization, the SDE explodes for some initial condition! Thus, taking $\mathbf{X}$ the Itô lift of a Brownian motion provides a counter-example.

Explicit and *deterministic* construction of a counter-example also exists (c.f. Section 3.3 of [LY25]).

Additive Noise: Criterion

For ODE with additive noise $\mathrm{d}x_t = b(x_t)\mathrm{d}t + \mathrm{d}\xi_t$, we provide the following sufficient and *sharp* criterion for non-explosion

- $\xi \in C^\alpha$ for some $\alpha \in (0, 1)$ ($\alpha = 1$ can be dealt directly);
- Equation $(*)$ holds and moreover, for all $x \perp y$

$$\left| \left\langle \frac{y}{\|y\|}, b(x) \right\rangle \right| \le (1 + \|x\|) f(\|x\|)^{\beta}$$

for some $\beta \le 1 + \alpha$.

Namely, more regular noise $\implies$ more growth of b allowed.

Similar result hold replacing Hölder regularity by p-variation.

Rough Differential Equations

For $b = 0$, consider the RDE

$$\mathrm{d}\begin{pmatrix} Y_t^1 \\ Y_t^2 \end{pmatrix} = \begin{pmatrix} Y_t^1 \sin Y_t^2 \\ Y_t^1 \end{pmatrix} \mathrm{d}\mathbf{X}_t$$

with $\mathbf{X}_t = (0, t \otimes t)$ explodes $\implies$ linear growth condition is insufficient. Moreover, for $\mathbf{X}$ the Itô lift of a Brownian motion, σ bounded and smooth is insufficient for non-explosion (c.f. [LS11]).

Simple criterion when $b = 0$ [GZ25, Theorem 3.12]: RDE does not explode if $\|D\sigma\|_\infty, \|D^2\sigma\|_\infty < \infty$.

Unbounded derivatives

We provide a sufficient condition for non-explosion when $b \ne 0$ and σ has unbounded derivatives (stated here for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$):

- b satisfies Equation $(*)$ and $\|b(x)\| \le f(\|x\|)^{1+\kappa\alpha}$;
- and $\|D^n\sigma(x)\| \le f(\|x\|)^{(1-n\kappa)\alpha-}$ for $n = 0, 1, 2$

for some $\kappa \in [0, \frac{1}{2})$.

Decaying second derivatives (WIP)

If σ has decaying second derivative, we can achieve a non-explosion criterion with better growth condition while still allowing unbounded first order derivative (here we take $b = 0$ and $\alpha \in (\frac{1}{3}, \frac{1}{2}]$):

- $\|D\sigma(x)\| \lesssim (\log(1 + \|x\|))^\alpha$;
- $\|D^n\sigma(x)\| \lesssim (1 + \|x\|)^{(1-n)\gamma}$ for $n = 0, 2$.

for some $\gamma \in [\alpha, 2\alpha)$.

Going further downwards (WIP)

For $\alpha < \frac{1}{3}$, same argument holds interpreting the integral as a branched rough integral. This allows arbitrarily close to linear growth of by dropping down a level of regularity, e.g. treating a $\frac{2}{5}$-Hölder rough path as a $\frac{1}{3} - \epsilon$ branched rough path via its canonical lift.

Proof Sketch

Treating the rough integral $\eta_t := \int_0^t Y_s \mathrm{d}\mathbf{X}_s$ as the noise part of an ODE with additive noise, we can apply the previous criterion if we can estimate its regularity on a partition of the time interval.

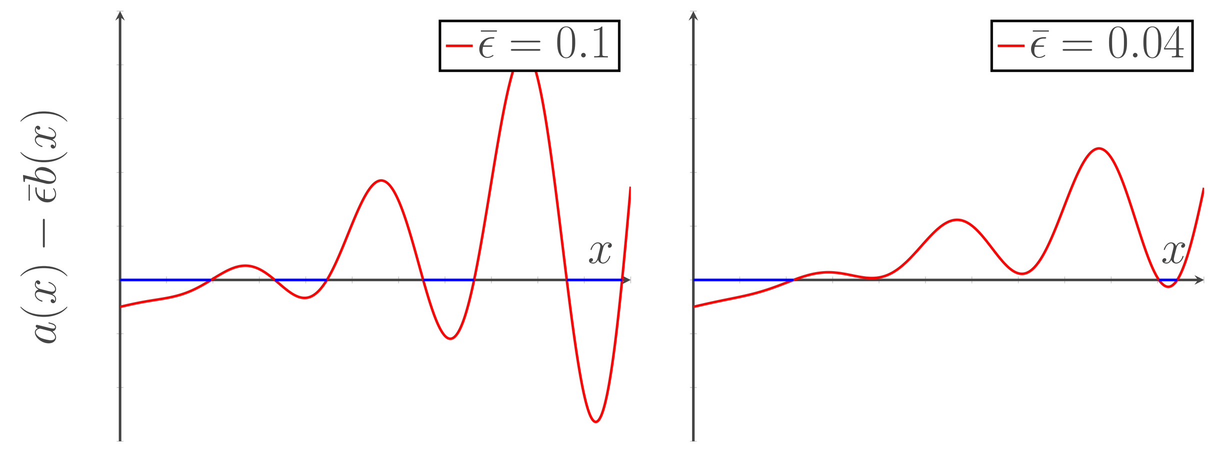
For $\alpha \in (\frac{1}{2}, 1]$, apply sewing lemma to obtain the desired estimate.

For $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, estimate from sewing lemma gives a non-linear inequality and naïve estimate no longer works.

Key lemma: *uniform* estimate for class of functions f satisfying a (non-linear) inequality

$$a(f(\epsilon)) \le \epsilon b(f(\epsilon)) \text{ for all } \epsilon > 0.$$

Idea: Cannot live in any region other than the first one as all others vanish in the limit $\epsilon \to 0$.



In our case ϵ is the interval size and f is the Hölder norm.

References

[GZ25] Massimiliano Gubinelli and Lorenzo Zambotti. Twelve lectures on rough paths, 2025.

[LS11] Xue-Mei Li and Michael Scheutzow. Lack of strong completeness for stochastic flows. 2011.

[LY25] Xue-Mei Li and Kexing Ying. Strong completeness of sdes and non-explosion for rdes with coefficients having unbounded derivatives, 2025.