WIP Title

Kexing Ying

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Contents

1 Introduction

1.1 Structure of this essay

2 The KLS and Thin-Shell Conjecture

While the KLS conjecture is often phrased as a isoperimetric problem, we will mainly consider the conjecture here as a problem regarding measures while providing some discussions regarding its equivalent formulations.

Mention we only work with Borel measures

2.1 Concentration

Definition 2.1 (Concentration, [Eld18])**.** Let μ be a measure on $\mathbb{R}^n$, then μ is said to be C-(inversely)-concentrated if for all 1-Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}_\mu[\phi] = \mathrm{Var}_{X\sim\mu}[\phi(X)] \leq C^2. \tag{1}$$

We denote the least possible such C by C^μ_{con}.

Heuristically, the concentration measures the relation between μ and the Euclidean metric by providing a numerical control for the variance of its norm. This is perhaps best illustrated by the following proposition.

Proposition 2.1. Let X be a $\mathbb{R}^n$-valued random variable. Then for all K-Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}[\phi(X)] \leq K^2 \mathrm{Var}[\|X\|^2].$$

Proof. WLOG. by subtracting its expectation from X, we may assume $\mathbb{E}[X] = 0$. Let X' be a i.i.d. copy of X on the same probability space. Then for all K-Lipschitz function ϕ, we have

$$\begin{aligned}
2\mathrm{Var}[\phi(X)] &= \mathrm{Var}[\phi(X) - \phi(X')] && \text{(i.i.d.)}\\
&= \mathbb{E}[(\phi(X) - \phi(X'))^2] - \mathbb{E}[\phi(X) - \phi(X')]^2 \\
&= \mathbb{E}[(\phi(X) - \phi(X'))^2] && \text{(identically distributed)}\\
&\leq K^2 \mathbb{E}[\|X - X'\|^2] && \text{(as } \phi \text{ is } K\text{-Lipschitz)}\\
&= K^2 \mathbb{E}[X^T X + X'^T X' - X^T X' - X'^T X] \\
&= 2K^2 \mathrm{Var}[\|X\|^2] - 2K^2 \mathrm{Cov}(X, X') = 2K^2 \mathrm{Var}[\|X\|^2]. && \text{(independence)}
\end{aligned}$$

implying $\mathrm{Var}[\phi(X)] \leq K^2 \mathrm{Var}[\|X\|^2]$ as claimed. ☐

With this proposition in mind, it is clear that for $\mathbb{R}$-valued random variables X, its law μ has concentration $C^\mu_{\mathrm{con}} = \mathrm{Var}[X]$. Furthermore, by considering the projection maps, it follows that the standard Gaussian measure on $\mathbb{R}^n$ is 1-concentrated.

We note that the definition we are presenting here is slightly non-standard. However, utilising a remarkable result due to Milman, we show that this definition is equivalent to the following definitions in a specific sense.

Definition 2.2 (Exponential concentration, [Mil18])**.** Given a measure μ on $\mathbb{R}^n$, we say μ has exponential concentration if there exists some $c, D > 0$ such that for all 1-Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$, $t > 0$, we have

$$\mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t) \leq c e^{-Dt}. \tag{2}$$

Fixing $c = 1$, we denote the largest possible D as $D^\mu_{\exp}$.

Definition 2.3 (First-moment concentration, [Mil18]). Again, given μ a measure on $\mathbb{R}^n$, we say μ has first-moment concentration if there exists some $D > 0$ such that for all 1-Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$, we have

$$\mathbb{E}_\mu[|\phi - \mathbb{E}_\mu[\phi]|] \leq \frac{1}{D}. \tag{3}$$

We denote the largest possible D by D^μ_{FM}.

It is clear that exponential concentration implies first-moment concentration. Indeed, if μ has exponential concentration with constant D (taking $c = 1$), then by the tail probability formula,

$$\mathbb{E}_\mu[|\phi - \mathbb{E}_\mu[\phi]|] = \int_0^\infty \mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t)\mathrm{d}t \leq \int_0^\infty e^{-Dt}\mathrm{d}t = \frac{1}{D}.$$

On the other hand, Milman showed that for log-concave measures (namely, measures of the form $\mathrm{d}\mu = \exp(-H)\mathrm{dLeb}^n$ for some convex function $H : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$) on $\mathbb{R}^n$, exponential concentration and first-moment concentration are equivalent in the following sense.

Theorem 1 (Milman, [Mil08]). For all log-concave measure μ on $\mathbb{R}^n$, μ has exponential concentration if and only if μ has first-moment concentration. Furthermore, $D^\mu_{\exp} \simeq D^\mu_{\mathrm{FM}}$ where we write $A \simeq B$ if there exists universal constants $C_1, C_2 > 0$ such that $C_1 A \leq B \leq C_2 A$.

With this theorem in mind, we establish the following correspondence.

Proposition 2.2. For all measures μ on $\mathbb{R}^n$, we have

$$\text{Exponentially concentrated} \implies \text{Concentrated} \implies \text{First-moment concentrated}$$

and $D^\mu_{\exp} \leq \sqrt{2}(C^\mu_{\mathrm{con}})^{-1}$ and $(2C^\mu_{\mathrm{con}})^{-1} \leq D^\mu_{\mathrm{FM}}$. Hence, if μ is log-concave, $D^\mu_{\exp} \simeq D^\mu_{\mathrm{FM}} \simeq (C^\mu_{\mathrm{con}})^{-1}$.

Proof. Assume first that μ is C-concentrated. Then by the Chebyshev inequality, we have

$$\mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t) \leq \frac{1}{t^2}\mathrm{Var}_\mu[\phi] \leq \frac{C^2}{t^2},$$

for all 1-Lipschitz ϕ. Thus, by tail probability,

$$\begin{aligned}
\mathbb{E}_\mu[|\phi - \mathbb{E}_\mu[\phi]|] &= \int_0^\infty \mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t)\mathrm{d}t \\
&\leq \inf_{a>0}\left\{\int_0^a \mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t)\mathrm{d}t + C^2\int_a^\infty \frac{1}{t^2}\mathrm{d}t\right\} \\
&\leq \inf_{a>0}\left\{a + \frac{C^2}{a}\right\} = 2C,
\end{aligned}$$

implying μ is first-moment concentrated with respect to the constant $(2C)^{-1}$.

On the other hand, if μ is exponential concentration with some constant D, then again by the tail probability,

$$\mathrm{Var}_\mu[\phi] = \int_0^\infty \mu((\phi - \mathbb{E}_\mu[\phi])^2 \geq t)\mathrm{d}t \leq \int_0^\infty e^{-D\sqrt{t}}\mathrm{d}t = \frac{2}{D^2}$$

implying μ is $\sqrt{2}D^{-1}$-concentrated. □

2.2 Example: concentration of the Gaussian

2.3 The KLS and thin-shell conjecture

Informally, the KLS conjecture suggests that any log-concave measure on $\mathbb{R}^n$ admits the same concentration as that of the Gaussian measure. However, unlike the Gaussian, as the concentration of measures is not invariant under linear transformations, it is clear that the KLS conjecture would not hold without a suitable normalization. This leads us to the following formulation of the KLS conjecture.

Conjecture 1 (Kannan-Lovász-Simonovitz, [Eld18])**.** Denoting $\mathscr{M}^n_{\text{con}}$ the set of all log-concave probability measures μ on $\mathbb{R}^n$ satisfying $\text{Var}_\mu[T] \leq 1$ for all 1-Lipschitz linear maps $T : \mathbb{R}^n \to \mathbb{R}$, there exists a *universal* constant C such that for all $\mu \in \mathscr{M}^n_{\text{con}}$, μ is C-concentrated.

We remark that C is universal in the sense that it does not depend on any parameter and in particular is independent of the dimension n.

Conjecture 2 (Thin-shell, [Eld13])**.** Taking $\mathscr{M}^n_{\text{con}}$ as above, there exists a universal constant C such that for all $\mu \in \mathscr{M}^n_{\text{con}}$, we have

$$\sqrt{\text{Var}_\mu[\|\cdot\|]} \leq C.$$

As the norm function is 1-Lipschitz, it is *a priori* that the thin-shell conjecture is weaker than that of the KLS conjecture. On the other hand, as we shall describe in the next section, as a consequence of the stochastic localisation scheme, Eldan [Eld13] provides a reduction of the KLS conjecture to the thin-shell conjecture up to logarithmic factors.

Theorem 2 (Eldan, [Eld13])**.** Denoting $\mathscr{M}^n_{\text{con}}$ as above, we define

$$C^n_{\text{con}} := \inf\left\{C \mid \forall \mu \in \mathscr{M}^n_{\text{con}}, \, \mu \text{ is } C\text{-concentrated}\right\},$$

and

$$C^n_{\text{TS}} := \inf\left\{C \mid \forall \mu \in \mathscr{M}^n_{\text{con}}, \, \sqrt{\text{Var}_\mu[\|\cdot\|]} \leq C\right\} = \sup_{\mu \in \mathscr{M}^n_{\text{con}}} \sqrt{\text{Var}_\mu[\|\cdot\|]},$$

we have,

$$C^n_{\text{TS}} \leq C^n_{\text{con}} \leq C^n_{\text{TS}} \log n.$$

We remark that while the constants in theorem 2 depends on the dimension n, if $\sup_n C^n_{\text{TS}} \log n < \infty$, we can obtain the universal bound for the KLS conjecture by taking the constant $C = \sup_n C^n_{\text{con}} \leq \sup_n C^n_{\text{TS}} \log n < \infty$.

2.3.1 Equivalent formulation of the KLS conjecture

While the formulation of the KLS conjecture above is very useful, as mentioned previously, the KLS conjecture has several equivalent formulations; we will quickly present them here. However, before stating these formulations, let us quickly complete theorem 1 with two additional equivalences.

Definition 2.4 (Cheeger's inequality, [Mil08])**.** Given a measure μ on $\mathbb{R}^n$, we recall the Minkowski's boundary measure defined by

$$\mu^+(A) := \liminf_{\epsilon \downarrow 0} \frac{\mu(A_\epsilon) - \mu(A)}{\epsilon}$$

where $A_\epsilon := \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, A) \leq \epsilon\}$ is the ϵ-thickening of some Borel set A. Then, we say μ satisfy Cheeger's inequality if there exists some D such that for all A satisfying $\mu(A) \leq \frac{1}{2}$,

This definition is from Eldan not Milman, can we show equivalence?

$$\mu(A) \leq D\mu^+(A).$$

We denote the largest such D by D_{C}^{μ}.

Theorem 3 (Milman, [Mil08])**.** For all log-concave measure μ on $\mathbb{R}^n$, the following are equivalent

- μ has exponential concentration with constant $D_{\exp}^{\mu}$.
- μ has first-moment concentration with constant D_{FM}^{μ}.
- μ satisfy the Cheeger's inequality with constant D_{C}^{μ}.
- μ satisfy the Poincaré inequality: there exists some $D > 0$ such that for all differentiable $\phi : \mathbb{R}^n \to \mathbb{R}$ satisfying $\int \phi \,\mathrm{d}\mu = 0$, we have

  $$D \cdot \mathrm{Var}_\mu[\phi] \leq \int \|\nabla \phi\|^2 \mathrm{d}\mu.$$

  We denote the largest such D by D_{P}^{μ}.

Furthermore, $D_{\exp}^{\mu} \simeq D_{\mathrm{FM}}^{\mu} \simeq D_{\mathrm{P}}^{\mu} \simeq D_{\mathrm{C}}^{\mu}$.

With this theorem and proposition 2.2 in mind, it is clear that the KLS conjecture can be instead formulated with any of these inequalities instead.

Conjecture 3 (KLS, [Eld13])**.** Denoting $\mathscr{M}_{\mathrm{iso}}^n$ the set of all log-concave probability measures μ on $\mathbb{R}^n$ satisfying $\int x\mu(\mathrm{d}x) = 0$ (namely μ is isotropic), there exists a *universal* constant D such that for all $\mu \in \mathscr{M}_{\mathrm{iso}}^n$, μ satisfy the Cheeger's inequality with constant D.

Why are the normalisations equivalent?

Similarly, we may also reformulate the KLS conjecture using the constant provided by the Poincaré inequality.

Conjecture 4 (KLS, [Eld13])**.** Denoting $\mathscr{M}_{\mathrm{iso}}^n$ as above, there exist a *universal* constant D such that for all $\mu \in \mathscr{M}_{\mathrm{iso}}$, μ satisfy the Poincaré inequality with constant D.

By observing that $\mathrm{Var}_\mu[\phi] = \int \phi^2 \mathrm{d}\mu$ for all differentiable ϕ satisfying $\int \phi \,\mathrm{d}\mu = 0$, we remark that the Poincaré inequality can be alternatively written as

$$\frac{\int \phi^2 \mathrm{d}\mu}{\int \|\nabla \phi\|^2 \mathrm{d}\mu} \leq D^{-1}.$$

3 The Stochastic Localisation Scheme

We will in this section provide a description of the stochastic localisation scheme introduced by Eldan [Eld13] and describe its application in reducing (up to a logarithmic factor) the KLS conjecture into the thin-shell conjecture, i.e. we will describe a proof of 2.

3.1 Construction and basic properties

Definition 3.1 (Barycenter)**.** Given a (probability) measure μ on $\mathbb{R}^n$, we define its barycenter with respect to the function $F : \mathbb{R}^n \to \mathbb{R}$ to be

$$\bar{\mu}(F) := \int_{\mathbb{R}^n} xF(x)\mu(\mathrm{d}x).$$

In the case that $F = \mathrm{id}$, we simply write $\bar{\mu} = \bar{\mu}(F) = \mathbb{E}_{X\sim\mu}[X]$.

Given the above definition, we now define the following construction central to the stochastic localisation scheme. Let $(W_t)_{t\geq 0}$ be a standard Wiener process in $\mathbb{R}^n$, we define the random functions $(F_t)_{t\geq 0}$ to be the solution of the following infinite system of SDEs:

$$F_0 = 1, \mathrm{d}F_t(x) = \langle x - \bar{\mu}(F_t), \mathrm{d}W_t\rangle F_t(x), \tag{4}$$

for all $x \in \mathbb{R}^n$. We shall from this point forward denote the random variables $a_t := \bar{\mu}(F_t)$. By applying Itô's formula, we make the following useful observation: for all $x \in \mathbb{R}^n$,

Existence and uniqueness of F.

$$\mathrm{d}\log F_t(x) = \frac{\mathrm{d}F_t(x)}{F_t(x)} - \frac{\mathrm{d}[F(x)]_t}{2F_t(x)^2} = \langle x - a_t, \mathrm{d}W_t\rangle - \frac{1}{2}\|x - a_t\|^2\mathrm{d}t \tag{5}$$

where the second equality follows by the construction of F. Hence, as $\log F_0(x) = 0$, we observe

$$\begin{aligned}
\log F_t(x) &= \int_0^t \langle x - a_s, \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^t \|x - a_s\|^2\mathrm{d}s \\
&= \left(\langle x, W_t\rangle - \int_0^t \langle a_s, \mathrm{d}W_s\rangle\right) - \left(\frac{t}{2}\|x\|^2 + \frac{1}{2}\int_0^t \|a_s\|^2\mathrm{d}s - \int_0^t \langle x, a_s\rangle\mathrm{d}s\right) \\
&= -\left(\int_0^t \langle a_s, \mathrm{d}W_s\rangle + \frac{1}{2}\|a_s\|^2\mathrm{d}s\right) + \langle x, a_t + W_t\rangle - \frac{t}{2}\|x\|^2.
\end{aligned}$$

Thus, taking $\mathrm{d}z_t := \langle a_t, \mathrm{d}W_t\rangle + \frac{1}{2}\|a_t\|^2\mathrm{d}t$ and $v_t := a_t + W_t$, we observe $F_t(x)$ is of the form

$$F_t(x) = e^{z_t + \langle x, v_t\rangle - \frac{t}{2}\|x\|^2}, \tag{6}$$

for given Itô processes $(z_t), (v_t)$. With this formulation of $F_t(x)$ in mind, it follows F_t is non-negative, and so, we may define the measure-valued random variable μ_t such that $\mu_t = F_t\mu$, i.e. they have Radon-Nikodym derivative $\mathrm{d}\mu_t/\mathrm{d}\mu = F_t$.

Is $e^{z_t + \langle x, v_t\rangle}$ itself a Itô process?

Proposition 3.1. For all $t \geq 0$, μ_t is a probability measure almost everywhere (a.e.), i.e. $\mathbb{P}(\mu_t(\mathbb{R}^n) = 1) = 1$.

Proof. As μ is a probability measure, it suffices to show $\partial_t \mu_t(\mathbb{R}^n) = 0$. To prove this, we first consider the discrete stochastic integral on the lattice $\Lambda = \mathbb{Z}^n \mathrm{d}t$ for some $\mathrm{d}t > 0$. Then, constructing $\mu_t^{\mathrm{d}t}$ on Λ via the same process as μ_t, for all $t = k\mathrm{d}t \in \Lambda$,

$$\begin{aligned}\partial_t \mu_t^{\mathrm{d}t}(\mathbb{R}^n) &= \int_{\mathbb{R}^n} \mathrm{d}F_t(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}^n} \langle x - a_t, \mathrm{d}W_t\rangle F_t(x)\mu(\mathrm{d}x) \\ &= \left\langle \int_{\mathbb{R}^n} (x - a_t)F_t(x)\mu(\mathrm{d}x), \mathrm{d}W_t \right\rangle = \langle a_t - a_t\mu_t(\mathbb{R}^n), \mathrm{d}W_t\rangle = 0\end{aligned}$$

by induction on k. However, by the very construction of the stochastic integral, the densities of $\mu_t^{\mathrm{d}t}$: $F_t^{\mathrm{d}t}$ converges a.e. to F_t as $\mathrm{d}t \to 0$ implying $\mu_t^{\mathrm{d}t} \to \mu_t$ weakly and so, $\mu_t^{\mathrm{d}t}(\mathbb{R}^n) \to \mu_t(\mathbb{R}^n)$ resulting in $\mu_t(\mathbb{R}^n) = 1$ as required. □

We would also like to study the limiting behavior of (μ_t) as $t \to \infty$. To achieve this, we will consider the covariance matrices

$$A_t := \mathrm{Cov}[\mu_t] = \int (x - a_t) \otimes (x - a_t)\mu_t(\mathrm{d}x), \tag{7}$$

where $\otimes$ denotes the Kronecker product. In particular, we will show $(A_t)_{ij} \to 0$ for all $i, j \in \{1, \cdots, n\}$ as $t \to \infty$ allowing us to conclude (μ_t) converges weakly to some Dirac measure. Indeed, this is a direct consequence of the following lemma.

Lemma 3.2 (Brascamp-Lieb inequality, [BL76])**.** Given $V : \mathbb{R}^n \to \mathbb{R}$ convex and $K > 0$, if ν is an isotropic probability measure on $\mathbb{R}^n$ of the form

$$\mathrm{d}\nu = Ze^{-V(x) - \frac{1}{2K}\|x\|^2}\mathrm{dLeb}^n$$

with Z being the normalization constant, then ν satisfy the Poincaré inequality, i.e. for all differentiable ϕ,

$$K\mathrm{Var}_\nu[\phi] \leq \int \|\nabla\phi\|^2 \mathrm{d}\nu.$$

Proof of lemma 3.2. TODO. Maybe go in appendix? □

With this lemma in mind, by taking $\nu = \mu_t$ using equation 6 and defining $\pi_i(x) := x_i$, we have by the Cauchy-Schwarz inequality

$$(A_t)_{ij} \leq \sqrt{\mathrm{Var}_{\mu_t}[\pi_i]}\sqrt{\mathrm{Var}_{\mu_t}[\pi_j]} \leq \max_{k=1,\cdots,n} \frac{1}{t} \int \|\nabla\pi_k\|^2 \mathrm{d}\mu_t$$

Again, using equation 6, we note that any realizations of $(F_t(x))$ is eventually decreasing in t for all $x \neq 0$, implying

$$\sup_{t>0} \max_{k=1,\cdots,n} \int \|\nabla\pi_k\|^2 \mathrm{d}\mu_t = \sup_{t>0} \max_{k=1,\cdots,n} \int x_k^2 \mathrm{d}\mu_t < \infty.$$

Thus, by taking $t \to \infty$ we have $(A_t)_{ij} \to 0$ for all $i, j \in \{1, \cdots, n\}$ as claimed and we have the following corollary.

Corollary 3.3. (μ_t) converges weakly to some Dirac measure almost everywhere. We denote this limiting (random) Dirac measure by δ_{a_∞} where a_∞ is some $\mathbb{R}^n$-valued random variable.

Since convergence implies relatively compact, applying the Dunford-Pettis theorem it follows that any realizations of (F_t) is uniformly integrable. Thus, we can make the following deductions about a_∞.

Corollary 3.4. The massive point a_∞ of the limiting Dirac measure is the limit of a_t as $t \to \infty$ and has law μ.

Proof. Since (F_t) is uniformly integrable we have convergence of means almost everywhere, namely

$$a_t = \int x\mu_t(\mathrm{d}x) \xrightarrow{\text{a.e.}} \int x\delta_{a_\infty}(\mathrm{d}x) =: a_\infty \text{ as } t \to \infty$$

implying that a_t converges a.e. to a_∞ as $t \to \infty$ as required.
Furthermore, taking $\phi : \mathbb{R}^n \to \mathbb{R}$ to be any bounded continuous function, we have

$$\int \phi(x)\mu(\mathrm{d}x) = \lim_{t\to\infty} \int \phi(x)\mu_t(\mathrm{d}x).$$

Then, taking expectation on both sides, we obtain

$$\begin{aligned}
\int \phi(x)\mu(\mathrm{d}x) &= \mathbb{E}\left[\lim_{t\to\infty}\int \phi(x)\mu_t(\mathrm{d}x)\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\int \phi(x)\mu_t(\mathrm{d}x)\right] && \text{(Dominated convergence)} \\
&= \lim_{t\to\infty} \mathbb{E}[\phi(a_t)] && \text{(LOTUS. theorem)} \\
&= \mathbb{E}[\phi(a_\infty)]. && \text{(Dominated convergence \& continuity of } \phi)
\end{aligned}$$

Thus, $\mathbb{E}_\mu[\phi] = \mathbb{E}[\phi(a_\infty)]$ for all bounded continuous ϕ implying $a_\infty \sim \mu$. ☐

Proposition 3.5. For all $x \in \mathbb{R}^n$, $(F_t(x))_{t\geq 0}$ is a martingale. Furthermore, for any continuous $\phi : \mathbb{R}^n \to \mathbb{R}$, defining the process $M_t := \int \phi \mathrm{d}\mu_t$, $(M_t)_{t\geq 0}$ is also a martingale.

Proof. By its very construction, $(F_t(x))$ is a martingale by observing equation 4 has no drift term.
Now, for all $s \leq t$ we have by the conditional Fubini's theorem,

$$\begin{aligned}
\mathbb{E}[M_t \mid \mathscr{F}_s] &= \mathbb{E}\left[\int \phi(x)F_t(x)\mu(\mathrm{d}x)\middle|\mathscr{F}_s\right] \\
&= \int \phi(x)\mathbb{E}[F_t(x) \mid \mathscr{F}_s]\mu(\mathrm{d}x) = \int \phi(x)F_s(x)\mu(\mathrm{d}x) = M_s
\end{aligned}$$

implying (M_t) is also a martingale. ☐

Using the same proof as corollary 3.4, we observe

$$M_t \xrightarrow{\text{a.e.}} M_\infty \sim \phi_*\mu \tag{8}$$

where $\phi_*\mu$ denotes the push-forward measure of μ along ϕ.

3.2 Reduction of KLS to thin-shell

In this section we will present a proof of theorem 2 as described by [LV16] and reformulated in terms of concentration in [Eld18]. We recall the goal of theorem 2 is to control $\mathrm{Var}_\mu[\phi]$ by a logarithmic factor of $\mathrm{Var}_\mu[\|\cdot\|]$. As translating the barycenter of μ does not affect its variance, we may assume μ has its barycenter $\overline{\mu}$ at the origin.

Fix $\phi : \mathbb{R}^n \to \mathbb{R}$ some 1-Lipschitz function and let (M_t) be the martingale as described above and in particular we recall equation 3.4 and so, $\mathrm{Var}_\mu[\phi] = \mathrm{Var}[M_\infty]$ where $M_t \xrightarrow{\text{a.e.}} M_\infty$. Hence, for all $t > 0$, by the martingale property we have

$$\begin{aligned}\mathrm{Var}[M_t] + \mathbb{E}[\mathrm{Var}[M_\infty \mid \mathscr{F}_t]] &= (\mathbb{E}[M_t^2] - \mathbb{E}[M_t]^2) + \mathbb{E}\left[\mathbb{E}[M_\infty^2 \mid \mathscr{F}_t] - \mathbb{E}[M_\infty \mid \mathscr{F}_t]^2\right] \\ &= \mathbb{E}[M_t^2] + (\mathbb{E}[\mathbb{E}[M_\infty^2 \mid \mathscr{F}_t]] - \mathbb{E}[M_t^2]) \\ &= \mathbb{E}[M_\infty^2] = \mathrm{Var}[M_\infty],\end{aligned}$$

where the second equality follows as $\mathbb{E}[M_t] = \mathbb{E}[M_\infty] = \mathbb{E}_\mu[\phi] = 0$ as a linear map

On the other hand, as (M_t) is a martingale, $M_t^2 - [M]_t$ is also a martingale implying $\mathbb{E}[M_t^2] = \mathbb{E}[M]_t$ and so $\mathrm{Var}[M_t] = \mathbb{E}[M_t^2] - \mathbb{E}[M_t]^2 = \mathbb{E}[M]_t - \overline{\mu}^2 = \mathbb{E}[M]_t$. Hence, combining this with the above, we obtain the bound

$$\mathrm{Var}_\mu[\phi] = \mathrm{Var}[M_\infty] = \mathrm{Var}[M]_t + \mathbb{E}[\mathrm{Var}[M_\infty \mid \mathscr{F}_t]] = \mathbb{E}[M]_t + \mathbb{E}[\mathrm{Var}_{\mu_t}[\phi]]. \tag{9}$$

Now, observing that ϕ is 1-Lipschitz implies $\|\nabla\phi\|^2 \leq 1$, we have by lemma 3.2 the bound $\mathrm{Var}_{\mu_t}[\phi] \leq t^{-1}$. Thus, the second term $\mathbb{E}[\mathrm{Var}_{\mu_t}[\phi]]$ is bounded by t^{-1}. With this in mind, we see that the reduction of KLS to thin-shell relies on bounding $\mathbb{E}[M]_t$ which we will dedicate the remainder of this section to.

3.2.1 Differential of the quadratic variation

To bound the term $\mathbb{E}[M]_t$ we will compute its differential and bound it sufficiently such that we reobtain a bound for $[M]_t$ after integration. In particular, we will show $\mathrm{d}[M]_t$ is bounded by a quantity concerning A_t. This should not be at all surprising as both $\mathrm{d}[M]_t$ and A_t describes the variation of M_t in a infinitesimal time neighborhood of t.

We compute

$$\begin{aligned}\mathrm{d}M_t = d\int \phi(x)F_t(x)\mu(\mathrm{d}x) &= \int \phi(x)\langle x - a_t, \mathrm{d}W_t\rangle \mu_t(\mathrm{d}x) \\ &= \left\langle \int \phi(x)(x-a_t)\mu_t(\mathrm{d}x), \mathrm{d}W_t \right\rangle\end{aligned}$$

and so, by considering the component-wise quadratic variation, we have

$$\mathrm{d}[M]_t = \left\| \int \phi(x)(x-a_t)\mu_t(\mathrm{d}x) \right\|^2 \mathrm{d}t. \tag{10}$$

Then, denoting α the vector $\int \phi(x)(x-a_t)\mu_t(\mathrm{d}x)$ normalized to have norm 1, so

$$\left\langle \alpha, \int \phi(x)(x-a_t)\mu_t(\mathrm{d}x) \right\rangle = \left\| \int \phi(x)(x-a_t)\mu_t(\mathrm{d}x) \right\|$$

we observe,

$$
\begin{aligned}
\mathrm{d}[M]_t &= \left\langle \alpha, \int \phi(x)(x-a_t)\mu_t(\mathrm{d}x) \right\rangle^2 = \left\langle \alpha, \int (\phi(x)-a_t)(x-a_t)\mu_t(\mathrm{d}x) \right\rangle^2 \\
&= \left(\int (\phi(x)-a_t)\langle \alpha, x-a_t \rangle \mu_t(\mathrm{d}x) \right)^2 \\
&\leq \left(\int (\phi(x)-a_t)^2 \mu_t(\mathrm{d}x) \right) \left(\int \langle \alpha, x-a_t \rangle^2 \mu_t(\mathrm{d}x) \right) \\
&= \mathrm{Var}_{\mu_t}[\phi] \int \alpha^T (x-a_t)^{\otimes 2} \alpha \mu_t(\mathrm{d}x) = \mathrm{Var}_{\mu_t}[\phi] \alpha^T A_t \alpha \\
&\leq \mathrm{Var}_{\mu_t}[\phi] \|A_t\|_{\mathrm{op}}.
\end{aligned}
\tag{11}
$$

where the inequality follows by the Cauchy-Schwarz inequality and $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm. Thus, as we know $\mathrm{Var}_{\mu_t}[\phi] \leq t^{-1}$, the problem is now reduced to that of bounding $\|A_t\|_{\mathrm{op}}$.

3.2.2 Analysis of the covariance matrix

As demonstrated in section 3.1, we know the limiting behavior of the covariance matrices, namely $A_t \to 0$ point-wise as $t \to \infty$. This was important for us to establish the existence of the limit of (a_t) and (M_t). However, as shown above, we now require some quantitative bounds for A_t. For this purpose, we first compute some useful properties of A_t.

Observing

$$
\int \mathrm{d}F_t(x)\mu(\mathrm{d}x) = \int \langle x-a_t, \mathrm{d}W_t \rangle \mu_t(\mathrm{d}x) = \left\langle \int x\mu_t(\mathrm{d}x) - a_t, \mathrm{d}W_t \right\rangle = 0,
$$

we have

$$
\begin{aligned}
\mathrm{d}a_t &= \mathrm{d}\int xF_t(x)\mu(\mathrm{d}x) = \int x\mathrm{d}F_t(x)\mu(\mathrm{d}x) = \int (x-a_t)\mathrm{d}F_t(x)\mu(\mathrm{d}x) \\
&= \int (x-a_t)\langle x-a_t, \mathrm{d}W_t \rangle F_t(x)\mu(\mathrm{d}x) = \int (x-a_t)^{\otimes 2}\mathrm{d}W_t\mu_t(\mathrm{d}x) = A_t\mathrm{d}W_t
\end{aligned}
\tag{12}
$$

where the second to last equality used the fact that $v\langle v, w \rangle = v^{\otimes 2}w$ for any appropriate v, w. Similarly, computing using Itô's formula, we have

$$
\begin{aligned}
\mathrm{d}A_t &= \mathrm{d}\int (x-a_t)^{\otimes 2}F_t(x)\mu(\mathrm{d}x) \\
&= \int (x-a_t)^{\otimes 2}\mathrm{d}F_t(x) + F_t(x)\,\mathrm{d}(x-a_t)^{\otimes 2} \\
&\qquad - 2(x-a_t) \otimes \mathrm{d}[a_t, F_t(x)]_t + F_t(x)\mathrm{d}[a_t]_t\mu(\mathrm{d}x).
\end{aligned}
\tag{13}
$$

The second term vanishes as

$$
\int F_t(x)\,\mathrm{d}(x-a_t)^{\otimes 2}\,\mu(\mathrm{d}x) = -2\mathrm{d}a_t \otimes \overbrace{\int (x-a_t)\mu_t(\mathrm{d}x)}^{=0} = 0.
$$

Also, by equation 12, $\mathrm{d}a_t = A_t \mathrm{d}W_t$ implying $\mathrm{d}[a_t]_t = A_t^2 \mathrm{d}t$. Finally, as both (a_t) and $(F_t(x))$ are martingales, $\mathrm{d}[a_t, F_t(x)]_t = F_t(x) A_t x \mathrm{d}t$ and the third term becomes

$$\begin{aligned}
-2\int (x-a_t)\otimes \mathrm{d}[a_t, F_t(x)]\mu_t(\mathrm{d}x) &= -2A_t\left(\int (x-a_t)\otimes x\mu_t(\mathrm{d}x)\right)\mathrm{d}t \\
&= -2A_t\left(\overbrace{\int (x-a_t)^{\otimes 2}\mu_t(\mathrm{d}x)}^{A_t} + \overbrace{\int (x-a_t)\mu_t(\mathrm{d}x)\otimes a_t}^{=0}\right)\mathrm{d}t \\
&= -2A_t^2\mathrm{d}t.
\end{aligned}$$

Hence, combining these and equation 4 together in 13, we have

$$\mathrm{d}A_t = \int (x-a_t)^{\otimes 2}\langle x-a_t, \mathrm{d}W_t\rangle \mu_t(\mathrm{d}x) - A_t^2\mathrm{d}t \tag{14}$$

With this computation in mind, we now proceed to bound the operator norm of A_t. In particular, by recalling that $\|A_t\|_{\mathrm{op}} = \max_{i=1,\cdots,n}\lambda_i(t) = \|(\lambda_i(t))_{i=1}^n\|_\infty$ where $\lambda_i(t)$ denotes the distinct eigenvalues of A_t, it suffices to find a bound for the potential

$$\Phi_t^\alpha = \sum_{i=1}^n |\lambda_i(t)|^\alpha = \|(\lambda_i(t))_{i=1}^n\|_\alpha^\alpha \tag{15}$$

for some $\alpha > 0$. Furthermore, as A_t is positive semi-definite, $\lambda_i(t) \geq 0$ for all $i = 1,\cdots,n$ and thus we have $\Phi_t^\alpha = \sum_{i=1}^n \lambda_i(t)^\alpha$. Again, to proceed, we will attempt to compute $\mathrm{d}\Phi_t^\alpha$ at some $t = t_0 > 0$ utilizing the following lemma.

Lemma 3.6. If $A = [a_{ij}]$ is a diagonal matrix with distinct eigenvalues $\lambda_i, \cdots, \lambda_n$, then for all $i, j, k, l, m \in 1, \cdots n$, we have

- $\frac{\partial \lambda_i}{\partial a_{jk}} = \delta_{ij}\delta_{ik}$;
- whenever $i \neq j$, $\frac{\partial^2 \lambda_i}{\partial a_{ij}^2} = 2(\lambda_i - \lambda_j)^{-1}$;
- and for $j \neq l, k \neq m$ or $i \neq j$ and $i \neq k$, $\frac{\partial^2 \lambda_i}{\partial a_{jk}\partial a_{lm}} = 0$,

where δ_{ij} denotes the Kronecker delta function.

As this lemma requires the matrix to be diagonal, denoting $e_1, \cdots, e_n$ as the normalized eigenbasis of A_{t_0} (they are in fact orthonormal as A_{t_0} is positive semi-definite), we will consider A_t with respect to this basis by considering the entries

$$a_{ij}(t) := \langle e_i, A_t e_j\rangle.$$

Using equation 14, we compute

$$\begin{aligned}
\mathrm{d}a_{ij}(t) &= \left\langle e_i, \left(\int (x-a_t)^{\otimes 2}\langle x-a_t, \mathrm{d}W_t\rangle\mu_t(\mathrm{d}x)\right)e_j\right\rangle - \langle e_i, A_t^2 e_j\rangle \mathrm{d}t \\
&= \left\langle \int \langle e_i, (x-a_t)^{\otimes 2}e_j\rangle (x-a_t)\mu_t(\mathrm{d}x), \mathrm{d}W_t\right\rangle - \langle e_i, A_t^2 e_j\rangle \mathrm{d}t.
\end{aligned} \tag{16}$$

Since by construction, A_{t_0} is diagonal with respect to the eigenbasis $(e_i)_{i=1}^n$, we have $\langle e_i, A_{t_0}^2 e_j\rangle = \langle e_i, A_{t_0} e_j\rangle^2 \geq 0$ and thus the term $-\langle e_i, A_t^2 e_j\rangle \mathrm{d}t$ contributes negatively around t_0. Hence, as we are interested in bounding $\mathrm{d}a_{ij}(t)$ from above, we may ignore the second term. With this in mind, denoting

$$\xi_{ij} = \int \langle e_i, (x-a_t)^{\otimes 2} e_j\rangle (x-a_t)\mu_t(\mathrm{d}x),$$

we have $\mathrm{d}a_{ij}(t) \leq \langle \xi_{ij}, \mathrm{d}W_t\rangle$. Thus, combining this with lemma 3.6, we have by Itô's formula

$$\begin{aligned} \mathrm{d}\lambda_i(t) &= \sum_{j,k=1}^n \frac{\partial \lambda_i}{\partial a_{jk}} \mathrm{d}a_{jk}(t) + \frac{1}{2}\sum_{j,k=1}^n \sum_{l,m=1}^n \frac{\partial^2 \lambda_i}{\partial a_{jk}\partial a_{lm}} \mathrm{d}[a_{jk}, a_{lm}]_t \\ &\leq \langle \xi_{ii}, \mathrm{d}W_t\rangle + \sum_{j\neq i} \frac{\mathrm{d}[a_{ij}]_t}{\lambda_i - \lambda_j} \leq \langle \xi_{ii}, \mathrm{d}W_t\rangle + \sum_{j\neq i} \frac{\|\xi_{ij}\|^2}{\lambda_i - \lambda_j}. \end{aligned} \tag{17}$$

at $t = t_0$.

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