PART III DIFFERENTIAL GEOMETRY

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Contents

Introduction

Preamble. These are notes for my version of the Cambridge Part III course *Differential Geometry*, as lectured in Michaelmas 2022. The course doesn't follow any particular book, but I will (very) occasionally refer to the following for proofs of technical results that we will take as black boxes:

- Liviu I. Nicolaescu, *Lectures on the geometry of manifolds.* 2nd edition. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007. https://www3.nd.edu/~lnicolae/Lectures.pdf
- John M. Lee, *Introduction to smooth manifolds.* 2nd edition. Graduate Texts in Mathematics, 218. Springer, New York, 2013.

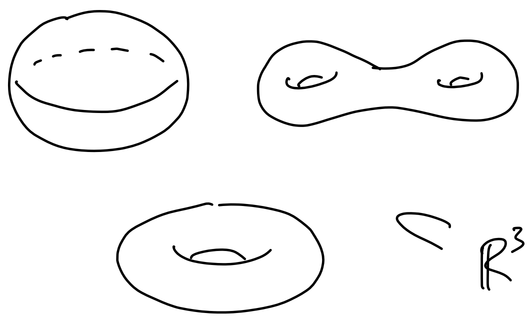
Such proofs are non-examinable.

I would be very glad to receive any comments or corrections at j.smith@dpmms.cam.ac.uk.

What is differential geometry? Differential geometry is the study of *smooth manifolds*: spaces which locally look like $\mathbb{R}^n$ in a smooth way. Here 'smooth' means we have a good notion of smooth (i.e. infinitely differentiable) function on the space, so we can use calculus.

There are two ways to define manifolds:

- *Embedded* manifolds are spaces that sit nicely inside $\mathbb{R}^N$ for some N, e.g.

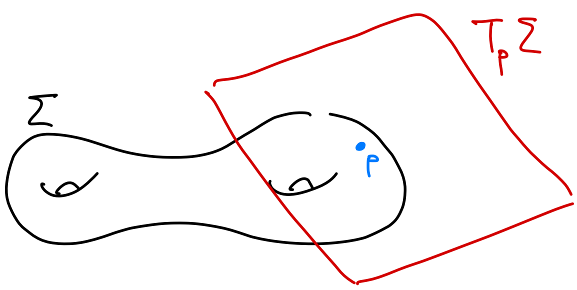


  or $\{y^2 = x^2 + 1\} \subset \mathbb{R}^2$ or $\mathrm{SO}(n) \subset \mathbb{R}^{n^2}$. This is an *extrinsic* picture.
- *Abstract* manifolds are (reasonable) topological spaces that come with 'local coordinates' about each point such that coordinate transformations are smooth. This is an *intrinsic* picture.

We'll use the abstract definition as it is cleaner and allows us to isolate intrinsic properties from those that require extra structure. But the two definitions turn out to be equivalent—every embedded manifold has local coordinates induced from $\mathbb{R}^N$, and every abstract manifold can be embedded in $\mathbb{R}^N$ for some $N \gg 0$ (Whitney embedding theorem).

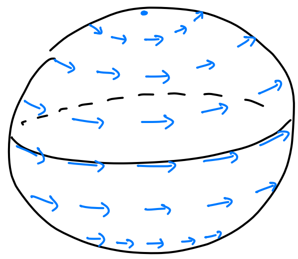
After defining manifolds we can set up many fundamental concepts:

- Tangent spaces, i.e. linear approximations to the manifold about each point. This is easy for embedded manifolds but less obvious for abstract ones.



- Smooth maps between manifolds and their derivatives, i.e. linear approximations.

- Vector fields and flows.



- Submanifolds. Embedded manifolds are just submanifolds of $\mathbb{R}^N$.

We can also equip manifolds with extra structure, e.g. a smooth group structure—these are *Lie groups*. The tangent space at the identity has the structure of a *Lie algebra*, and there is an *exponential map* from this tangent space to the Lie group. For $\mathrm{GL}(n, \mathbb{R})$, the tangent space at I is the space of $n \times n$ matrices, the Lie algebra structure is the matrix commutator $[A, B] = AB - BA$, and the exponential map is

$$A \mapsto e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots.$$

These have intrinsic geometric definitions that work for all Lie groups.

But some things are more subtle than you might expect...

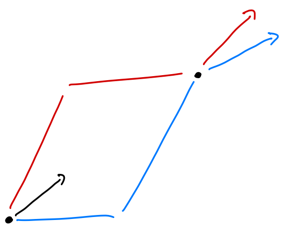
How do you differentiate a vector field? It's easy to differentiate a vector field on $\mathbb{R}^3$—just take partial derivatives of each component. But what about a vector field on (and tangent to) an embedded surface in $\mathbb{R}^3$?

- You can't differentiate in directions out of the surface. This is a flaw in the extrinsic picture.
- If you differentiate in a direction along the surface, the result need no longer be tangent to the surface. This is a flaw in the intrinsic picture.
- You could project the result orthogonally back onto the surface but a priori this depends on the embedding in $\mathbb{R}^3$.

So what kind of object should the derivative be? What extra structure on the surface do we need in order to define it? Can we talk about this intrinsically, i.e. without reference to an embedding in $\mathbb{R}^N$? What should it mean for the derivative to be zero?

Answering these questions leads to many of the concepts we will study:

- Tensors and differential forms—the right kind of objects for talking about derivatives.
- Connections—structures that allow you to compare vectors in nearby tangent spaces.
- Parallel transport—dragging a vector along a path in a way that is 'constant' with respect to a given connection.
- Curvature—the difference between parallel transport around two sides of an infinitesimal parallelogram and parallel transport around the other two sides.



An important class of examples are *Riemannian manifolds*: manifolds equipped with a smoothly varying inner product on each tangent space. Here there is a distinguished choice of connection, the *Levi-Civita* connection, and its curvature is the famous *Riemann tensor*.

More abstract examples. We'll actually spend a lot of time studying connections and curvature in the abstract setting of *vector bundles* and *principal bundles*. This will allow us to differentiate things more general than vector fields.

We'll soon meet *complex projective space*, $\mathbb{CP}^n$, which is the space of complex lines through the origin in $\mathbb{C}^{n+1}$. Associated to each point $p \in \mathbb{CP}^n$ is a 1-dimensional complex vector space E_p, namely the line in $\mathbb{C}^{n+1}$ represented by p. This family forms a *complex line bundle over* $\mathbb{CP}^n$, called the *tautological bundle*. A *section* of this bundle is a choice of point $s(p) \in E_p$ for each p, varying smoothly with p. This is our analogue of a vector field. Each *fibre* E_p is isomorphic to $\mathbb{C}$, but not canonically. Given points z and w in E_p it makes sense to talk about their absolute values (using the Euclidean norm on $\mathbb{C}^{n+1}$) and their relative phases, but not their absolute phases.

In physics, spacetime is a manifold X. A simple quantum particle is described by a wavefunction ψ, which is almost—but not quite—a map $X \to \mathbb{C}$. What matters about ψ is $|\psi|^2$ and its phase relative to other wavefunctions. So given any map $g : X \to \mathrm{U}(1)$ we can multiply all wavefunctions by g without changing any measurements. But to write down the equations of motion (the Schrödinger equation) for ψ we need to differentiate it. So what is ψ, and how can we differentiate it in a way that takes the phase ambiguity into account?

Answer. ψ is a section of a complex line bundle over X and to differentiate it we need a suitable connection on this bundle. It turns out that the connection corresponds precisely to the electromagnetic potential, and its curvature is the electromagnetic field strength! This is the basic idea of *gauge theory*, which is ubiquitous in physics and has become a powerful tool in geometry too.

Course plan. *[Not covered in lectures]*

- Manifolds, smooth maps, submanifolds, and transversality.
- Vector bundles and tensors.
- Differential forms, de Rham cohomology, integration, and Stokes's theorem.
- Connections on vector bundles, including parallel transport and curvature.
- Flows and Lie derivatives.
- Frobenius integrability and its significance for curvature.
- Lie groups and Lie algebras, and their actions.
- Principal bundles, and connections revisited.
- Introduction to Riemannian geometry.

Where next? *[Not covered in lectures]*

- Geometric topology: topology of smooth manifolds
  - Low-dimensional (3- and 4-dimensional) topology, e.g. generalised Poincaré conjecture
  - Knot theory
- Manifolds with extra structure
  - (Pseudo)Riemannian geometry, e.g. mathematical GR, exceptional holonomy
  - Symplectic geometry and topology
  - Complex geometry, Kähler geometry, closely related to algebraic geometry
- Applications in other fields
  - Theoretical physics
  - Dynamical systems
  - Statistics

1. Manifolds and smooth maps

1.1. **Manifolds.** A manifold is a space which locally looks like $\mathbb{R}^n$.

Definition 1.1.1. A *topological n-manifold* is a topological space X such that for every point p in X there exists an open neighbourhood U of p in X, an open set V in $\mathbb{R}^n$, and a homeomorphism $\varphi : U \xrightarrow{\sim} V$.

We also require X to be

- *Hausdorff*: given distinct points p_1 and p_2 in X there exist disjoint open neighbourhoods U_1 and U_2 of p_1 and p_2.
- *second-countable*: there exists a countable collection of open sets which form a basis for the topology, i.e. every open set is a union of sets in the collection.

Example 1.1.2. $\mathbb{R}^n$ is a topological n-manifold.

Remark 1.1.3. 'Hausdorff and second-countable' is important but is not restrictive in practice: for a space locally homeomorphic to $\mathbb{R}^n$ it's equivalent (via a non-trivial argument) to 'X is metrisable and has countably many components'.

Example 1.1.4. If X is a topological n-manifold, so is any open $W \subset X$: replace each $\varphi : U \xrightarrow{\sim} V$ with $\varphi|_{U\cap W} : U \cap W \xrightarrow{\sim} \varphi(U \cap W)$. Hausdorffness and second-countability are inherited from X.

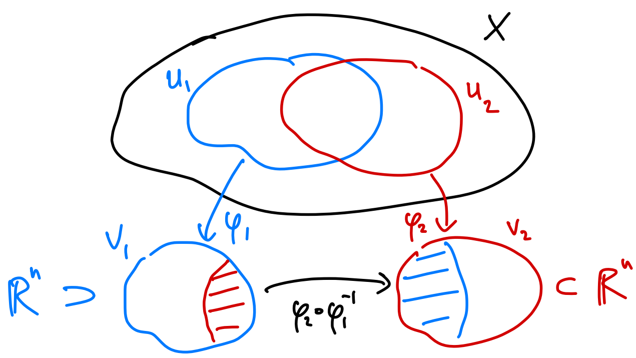
Terminology:

- φ is a *chart (about p)*.
- A collection of charts covering X is an *atlas*.
- U is a *coordinate patch*.
- If $x_1, \ldots, x_n$ are standard coordinates on $\mathbb{R}^n$ then

$$x_1 \circ \varphi, \ldots, x_n \circ \varphi$$

  are *local coordinates* ('on U' or 'about p'). Usually we'll just call them $x_1, \ldots, x_n$ or similar.
- The inverse of a chart is a *parametrisation*. (It's easier to remember which direction a parametrisation goes than a chart!)
- Given overlapping charts $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$, the corresponding local coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are related by the *transition map*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2).$$



  This expresses the y_i as functions of the x_i.

Definition 1.1.5. Given an atlas $\mathbb{A}$ and open $W \subset X$, a function $f : W \to \mathbb{R}$ is *smooth with respect to* $\mathbb{A}$ if $f \circ \varphi_\alpha^{-1}$ is smooth for all α, i.e. if all local coordinate expressions $f(x_1, \ldots, x_n)$ are smooth. An atlas is *smooth* if every transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth. (A map from an open subset of $\mathbb{R}^a$ to $\mathbb{R}^b$ is *smooth* if each component has all partial derivatives of all orders.)

Lemma 1.1.6. *If $\mathbb{A}$ is smooth then f is smooth with respect to $\mathbb{A}$ if and only if for all p in W there exists a chart φ_α about p such that $f \circ \varphi_\alpha^{-1}$ is smooth, i.e. if $f(x_1, \ldots, x_n)$ is smooth for* some *local coordinates $x_1, \ldots, x_n$ about p.*

Proof. The 'only if' direction is obvious. For the converse, take an arbitrary chart φ_α. We need to show that $f \circ \varphi_\alpha^{-1}$ is smooth near $\varphi_\alpha(p)$ for each $p \in W \cap U_\alpha$. For each such p, by assumption there exists a chart φ_β about p such that $f \circ \varphi_\beta^{-1}$ is smooth. Then near $\varphi_\alpha(p)$ we have

$$f \circ \varphi_\alpha^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}).$$

This is a composition of smooth functions, hence is smooth. □

Corollary 1.1.7. *Given a smooth atlas $\mathbb{A}$ all local coordinate functions are smooth with respect to the atlas.*

Proof. Each coordinate function is smooth in its own chart. Now use the previous lemma. □

We'll think of two smooth atlases as being the same if they have the same smooth functions.

Definition 1.1.8. Two smooth atlases are *smoothly equivalent* if and only if their union is smooth (this is an equivalence relation). A *smooth structure* on X is an equivalence class of smooth atlases under this relation. A *smooth n-manifold* is a topological n-manifold equipped with a choice of smooth structure.

Some people always work with the *maximal atlas*—the union of all atlases representing the smooth structure. But this obscures the fact that it's only the equivalence class that matters.

Lemma 1.1.9. *Smooth atlases $\mathbb{A}$ and $\mathbb{B}$ are smoothly equivalent if and only if they have the same smooth functions in the following sense: for any open set $W \subset X$ and any function $f : W \to \mathbb{R}$, f is smooth with respect to $\mathbb{A}$ if and only if it's smooth with respect to $\mathbb{B}$.*

Proof. Example Sheet 1. □

Definition 1.1.10. Given a smooth n-manifold X, a function $F : W \to \mathbb{R}$ is *smooth* if and only if it's smooth with respect to some (or, equivalently, all) smooth atlas(es) representing the smooth structure.

Example 1.1.11. (i) $\mathbb{R}^n$ is naturally an n-manifold via the atlas $\{\mathrm{id} : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n\}$.
(ii) If X is an n-manifold, then any open $W \subset X$ inherits the structure of an n-manifold, by restricting charts on X to W.
(iii) If X is an n-manifold and Y an m-manifold then $X \times Y$ is naturally an $(m+n)$-manifold, by equipping it with the product topology and the smooth structure induced by products of charts on X and Y.

Remark 1.1.12. (i) Being a topological n-manifold is a *property* of a topological space.
(ii) Being a smooth n-manifold is a property (being a topological n-manifold and admitting a smooth structure) *plus* a choice of smooth structure.
(iii) When $n \leq 3$, every topological n-manifold admits an essentially unique smooth structure.
(iv) For $n \geq 4$ a topological n-manifold may admit no smooth structure (e.g. the E_8 4-manifold) or many essentially different smooth structures (e.g. exotic 7-spheres, or exotic $\mathbb{R}^4$). But these results are hard.

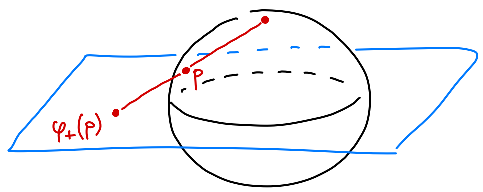
Definition 1.1.13. The integer n is the *dimension* of X, denoted $\dim X$.

Example 1.1.14. The *n-sphere*, S^n, is the smooth n-manifold whose underlying topological space is

$$\{\mathrm{y} = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} : \|\mathrm{y}\|^2 = 1\}$$

with the subspace topology, and whose smooth structure is defined by the following atlas. There are two charts $\varphi_\pm : U_\pm \xrightarrow{\sim} \mathbb{R}^n$, where $U_\pm = S^n \setminus \{(0, \ldots, 0, \pm 1)\}$ and $\varphi_\pm$ is stereographic projection

$$\varphi_\pm(y_1, \ldots, y_{n+1}) = \frac{1}{1 \mp y_{n+1}}(y_1, \ldots, y_n).$$



The local coordinates $\mathsf{x}^{\pm}$ associated to $\varphi_{\pm}$ satisfy $x_i^{\pm} = y_i/(1\mp y_{n+1})$. The *height function* $y_{n+1} : S^n \to \mathbb{R}$ is smooth, since it is given by

$$y_{n+1} = \pm\frac{\|\mathsf{x}^{\pm}\|^2 - 1}{\|\mathsf{x}^{\pm}\|^2 + 1} \quad \text{on} \quad U_{\pm}.$$

From now on, 'manifold' means 'smooth manifold'. All of our definitions and constructions with manifolds will depend only on their smooth structures, not the specific choices of atlas, and we won't keep saying this explicitly. When we talk about charts or local coordinates they will always be compatible with the smooth structures in the obvious sense: they belong to atlases representing the smooth structures. We are free to modify charts as long as we preserve this compatibility, e.g. by translating them or shrinking their domains.

1.2. **Manifolds from sets.** In practice we often want to define a smooth n-manifold structure on a set without explicitly describing the topology, just by covering it with pieces identified with open sets in $\mathbb{R}^n$. An important special case is when the set itself is defined by gluing together open sets in $\mathbb{R}^n$.

To formalise this, suppose we're given:

- A set X.
- A collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of subsets covering X.
- For each α an open set $V_\alpha \subset \mathbb{R}^n$ and a bijection $\varphi_\alpha : U_\alpha \to V_\alpha$.

Suppose also that for all α and β the set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α (equivalently, open in $\mathbb{R}^n$), and

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$$

is smooth.

Definition 1.2.1 (Non-standard)**.** Call this data a *smooth pseudo-atlas*, and each φ_α a *pseudo-chart*. Say two smooth pseudo-atlases are *equivalent* if their union is also a smooth pseudo-atlas.

Declare a subset $W \subset X$ to be open if and only if $\varphi_\alpha(W \cap U_\alpha)$ is open in V_α for all α.

Lemma 1.2.2. *(i) This defines a topology on X.*

(ii) Apart from the possible failure of 'Hausdorff and second-countable', the resulting topological space X is a topological n-manifold and the pseudo-atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\mathcal{A}}$ forms a smooth atlas.

(iii) Equivalent smooth pseudo-atlases define the same topology and smooth structure on X.

Proof. Easy check. □

Example 1.2.3. The *n-dimensional real projective space*, denoted $\mathbb{RP}^n$, is the space of lines (1-dimensional linear subspaces) in $\mathbb{R}^{n+1}$.

- Any non-zero x in $\mathbb{R}^{n+1}$ defines a point in $\mathbb{RP}^n$, namely $\mathbb{R}\mathsf{x}$.
- All lines arise in this way.
- Two points define the same line $\iff$ they differ by rescaling.

So we can label points of $\mathbb{RP}^n$ by the ratios $[x_0 : \cdots : x_n]$, called *homogeneous coordinates*. Explicitly $[x_0 : \cdots : x_n] = [y_0 : \cdots : y_n]$ if and only if there exists $\lambda \in \mathbb{R}^*$ such that $\mathsf{y} = \lambda\mathsf{x}$. We can remove the rescaling ambiguity by dividing through by one of the coordinates, as long as it's non-zero.

We thus define the following pseudo-charts. For $i = 0, \ldots, n$ let

$$U_i = \{[x_0 : \cdots : x_n] : x_i \neq 0\}$$

and define a bijection $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i([x_0 : \cdots : x_n]) = \frac{1}{x_i}(x_0, \ldots, \widehat{x}_i, \ldots, x_n).$$

These form a smooth pseudo-atlas and make $\mathbb{RP}^n$ into an n-manifold (proved on Example Sheet 1).

Similarly, the space of complex lines in $\mathbb{C}^{n+1}$, called *complex projective space* and denoted $\mathbb{CP}^n$, is covered by homogeneous coordinate pseudo-charts, now taking values in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. These form a smooth pseudo-atlas, making $\mathbb{CP}^n$ into a smooth $2n$-manifold.

Checking second-countability is usually easy: it's enough for X to be covered by countably many pseudo-charts. But there's no easy general method for checking Hausdorffness.

1.3. **Smooth maps.** Fix manifolds X and Y of dimensions n and m with smooth atlases

$$\{\varphi_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha\}_{\alpha \in \mathcal{A}} \quad \text{and} \quad \{\psi_\beta : S_\beta \to T_\beta\}_{\beta \in \mathcal{B}}.$$

Definition 1.3.1. A map $F : X \to Y$ is *smooth* if it's continuous and if for all α and β the map

$$\psi_\beta \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(F^{-1}(S_\beta)) \to T_\beta$$

is smooth as a map between open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$.

If $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are the corresponding local coordinates, then F makes the y_i into functions of the x_j and we are just asking that each y_i has all partial derivatives with respect to the x_j.

Remark 1.3.2. Continuity of F means $F^{-1}(S_\beta)$ is open, so the domain of

$$\psi_\beta \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(F^{-1}(S_\beta)) \to T_\beta$$

is open, so its smoothness makes sense.

Example 1.3.3.
- (i) Identity maps, constant maps, and the projections $X \times Y \to X, Y$ are smooth.
- (ii) The inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.
- (iii) A map $f : X \to \mathbb{R}$ is smooth iff it's a smooth function in the sense of Section 1.1.
- (iv) A map from an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^m$ is smooth if and only if it's smooth in the usual multi-variable calculus sense.

We have the following basic properties.

Lemma 1.3.4.
- *(i) Smoothness is* local in the source, *meaning it can be checked locally about each $p \in X$.*
- *(ii) A composition of smooth maps is smooth.* □

Example 1.3.5. Viewing $\mathbb{C}^{n+1}$ as $\mathbb{R}^{2(n+1)}$, we can think of S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$. Sending a point on this sphere to the complex line through that point gives a map

$$H : S^{2n+1} \to \mathbb{CP}^n$$

called the *Hopf map*. This is smooth (see Example Sheet 1).

Definition 1.3.6. A *diffeomorphism* from one manifold to another is a smooth map with a smooth two-sided inverse. Two manifolds are *diffeomorphic*, written $\cong$, if there exists a diffeomorphism between them. This is obviously an equivalence relation.

Example 1.3.7. $\mathbb{CP}^1$ is diffeomorphic to S^2 (Example Sheet 1), so it makes sense to call $\mathbb{CP}^1$ the *Riemann sphere* and talk about the Hopf map $S^3 \to S^2$.

Lemma 1.3.8. *If X and Y are diffeomorphic non-empty manifolds then $n = m$ (i.e. $\dim X = \dim Y$).*

Proof. Fix a diffeomorphism $F : X \to Y$ and a point p in X. Take charts $\varphi : U \xrightarrow{\sim} V$ on X about p and $\psi : S \xrightarrow{\sim} T$ on Y about $F(p)$. By shrinking U, V, S, and T if necessary, WLOG $F(U) = S$.

Then $G := \psi \circ F \circ \varphi^{-1}$ and $H := \varphi \circ F^{-1} \circ \psi^{-1}$ are mutually inverse smooth maps between open subsets $V \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$. So from multi-variable calculus the derivatives $D_{\varphi(p)}G$ and $D_{\psi(F(p))}H$ are mutually inverse linear maps between $\mathbb{R}^n$ and $\mathbb{R}^m$, so $n = m$. □

1.4. **Tangent spaces.** The tangent space parametrises infinitesimal directions in a manifold. The idea is to represent these directions by curves tangent to them.

Fix an n-manifold X and a point p in X.

Definition 1.4.1. A *curve based at* p is a smooth map $\gamma : I \to X$ from an open neighbourhood I of 0 in $\mathbb{R}$, satisfying $\gamma(0) = p$. We say that curves $\gamma_1 : I_1 \to X$ and $\gamma_2 : I_2 \to X$ *agree to first order* if there exists a chart $\varphi : U \xrightarrow{\sim} V$ about p such that

$$(\varphi \circ \gamma_1)^{\bullet}(0) = (\varphi \circ \gamma_2)^{\bullet}(0) \tag{1.4.1}$$

as vectors in $\mathbb{R}^n$.

Lemma 1.4.2. *If* (1.4.1) *holds for some chart* φ *about* p *then it holds for all such charts.*

Proof. Given a chart φ about p, let

$$\pi_p^{\varphi} : \{\text{curves based at } p\} \to \mathbb{R}^n$$

denote the map

$$\gamma \mapsto (\varphi \circ \gamma)^{\bullet}(0).$$

Now suppose φ_1 and φ_2 are two charts about p. We want to show that if two curves have the same image under $\pi_p^{\varphi_1}$ then they also have the same image under $\pi_p^{\varphi_2}$. This holds because the chain rule gives $\pi_p^{\varphi_2} = A \circ \pi_p^{\varphi_1}$, where $A = D_{\varphi_1(p)}(\varphi_2 \circ \varphi_1^{-1})$. □

Corollary 1.4.3. *Agreement to first order is an equivalence relation on curves based at* p. □

Definition 1.4.4. The *tangent space to* X *at* p, denoted T_pX, is the set of curves based at p modulo agreement to first order. Elements are called *tangent vectors at* p. We write $[\gamma]$ for the tangent vector represented by a curve γ. Intuitively this is the infinitesimal direction in which γ passes through p.

Proposition 1.4.5. T_pX *naturally carries the structure of an* n*-dimensional real vector space, in such a way that each* π_p^{φ} *is a linear isomorphism.*

Proof. For each chart φ about p the map π_p^{φ} embeds T_pX into $\mathbb{R}^n$. We claim that each π_p^{φ} is in fact surjective, so induces a bijection $T_pX \to \mathbb{R}^n$. For different choices of chart these bijections differ by a linear automorphism of $\mathbb{R}^n$ (the map A above), which proves the proposition.

It remains to show π_p^{φ} is surjective. Given a vector $\mathsf{v} \in \mathbb{R}^n$, define a curve γ based at p by

$$\gamma(t) = \varphi^{-1}(\varphi(p) + t\mathsf{v}),$$

for all t in a small open neighbourhood of 0 in $\mathbb{R}$. By construction this satisfies $\pi_p^{\varphi}(\gamma) = \mathsf{v}$. □

Definition 1.4.6. If $x_1, \dots, x_n$ are the local coordinates associated to the chart φ then we denote by $\partial/\partial x_i$ the tangent vector $(\pi_p^{\varphi})^{-1}(\mathsf{e}_i)$, where e_i is the ith standard basis vector. (Here $(\pi_p^{\varphi})^{-1}$ really means $(\overline{\pi}_p^{\varphi})^{-1}$, where $\overline{\pi}_p^{\varphi}$ is the isomorphism $T_pX \to \mathbb{R}^n$ induced by π_p^{φ}.) By construction, the $\partial/\partial x_i$ form a basis for T_pX.

We may abbreviate $\partial/\partial x_i$ to ∂_{x_i} or even ∂_i if the chart is clear. Intuitively it is the infinitesimal direction obtained by running at unit speed along the x_i-axis, i.e. the curve along which all other x_j are constant. The notation $\partial/\partial x_i$ is just that: a piece of notation. We shall see shortly that it is justified by the fact that these tangent vectors can be interpreted as the obvious differential operators, and that they transform in the way the notation suggests.

Note also that ∂_{x_i} may denote a tangent vector at any point in the domain of the chart φ, and we will either be thinking of it as this whole family of vectors (a simple example of a *vector field*) or we will specify at which specific point p we are looking.

Warning! Each vector ∂_{x_i} depends on *all* of the local coordinates $x_1, \dots, x_n$, not just on x_i itself. Said another way, if $y_1, \dots, y_n$ are local coordinates associated to another chart about p, and if $y_i = x_i$ for some i, then it does not automatically follow that $\partial_{x_i} = \partial_{y_i}$.

The correct transformation between different coordinate systems is the following.

Lemma 1.4.7. *For each i we have*

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}.$$

Proof. Let φ_1 and φ_2 be the charts defining x and y. Following our earlier notation we have $\pi_p^{\varphi_2} = A \circ \pi_p^{\varphi_1}$, so for each i we get

$$\partial_{y_i} = (\pi_p^{\varphi_2})^{-1}(\mathsf{e}_i) = (\pi_p^{\varphi_1})^{-1}(A^{-1}\mathsf{e}_i). \tag{1.4.2}$$

The map A^{-1} is the derivative of $\varphi_1 \circ \varphi_2^{-1}$, which expresses the x_j in terms of the y_i, so

$$A^{-1}\mathsf{e}_i = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} \mathsf{e}_j.$$

Plugging into (1.4.2) and using linearity of $(\pi_p^{\varphi_1})^{-1}$ then gives

$$\frac{\partial}{\partial y_i} = (\pi_p^{\varphi_1})^{-1}(A^{-1}\mathsf{e}_i) = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i}(\pi_p^{\varphi_1})^{-1}\mathsf{e}_j = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i}\frac{\partial}{\partial x_j}.$$ □

If a vector $[\gamma]$ is given by $\sum_i a_i \partial_{x_i}$ then we have

$$(\varphi \circ \gamma)^{\bullet}(0) = \pi_p^{\varphi}(\gamma) = \sum_{i=1}^{n} a_i \mathsf{e}_i.$$

Equating components of e_i we obtain

$$(x_i \circ \gamma)^{\bullet}(0) = a_i,$$

so the coefficients of the ∂_{x_i} are the derivatives of the x_i along γ.

1.5. **Derivatives.** Fix manifolds X and Y of dimensions n and m, and a smooth map $F : X \to Y$.

Definition 1.5.1. The *derivative of F at p* is the map $D_pF : T_pX \to T_{F(p)}Y$ given by $[\gamma] \mapsto [F \circ \gamma]$. We sometimes denote D_pF by F_*, the *pushforward* by F on tangent vectors.

Lemma 1.5.2. *The map D_pF is well-defined and linear.*

Proof. Fix charts φ and ψ about p and $F(p)$. Then for any curve γ based at p we have

$$\pi_{F(p)}^{\psi}(F \circ \gamma) = (\psi \circ F \circ \gamma)^{\bullet}(0) = \big((\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ \gamma)\big)^{\bullet}(0) = T\pi_p^{\varphi}(\gamma),$$

where T is the derivative of $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$. So if $[\gamma_1] = [\gamma_2]$ then $[F \circ \gamma_1] = [F \circ \gamma_2]$, and we have

$$\begin{tikzcd}
T_pX \arrow[r, "D_pF"] \arrow[d, "\pi_p^{\varphi}"', "\wr"] & T_{F(p)}Y \arrow[d, "\pi_{F(p)}^{\psi}", "\wr"'] \\
\mathbb{R}^{\dim X} \arrow[r, "T"] & \mathbb{R}^{\dim Y}
\end{tikzcd}$$

so D_pF is linear. □

If x and y are the local coordinates associated to φ and ψ then T is precisely the matrix representing D_pF with respect to the ∂_{x_i} and ∂_{y_j}. But T is the derivative of $\psi \circ F \circ \varphi^{-1}$, which expresses the y_j as functions of the x_i via F. So

$$D_pF(\partial_{x_i}) = \sum_j \left.\frac{\partial y_j}{\partial x_i}\right|_p \partial_{y_j}.$$

Remark 1.5.3. (i) This shows that the new notion of derivative coincides with the multi-variable calculus version when X and Y are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$.

(ii) If $f : X \to \mathbb{R}$ is a smooth function and $x_1, \ldots, x_n$ are local coordinates about p then
$$D_p f(\partial_{x_i}) = \frac{\partial f}{\partial x_i}.$$
(iii) For a curve γ based at p we can write $[\gamma]$ as $D_0\gamma(\partial_t)$, where t is the standard coordinate on the domain of γ.

With our new definition, the chain rule is tautological.

Proposition 1.5.4 (The chain rule)**.** *If $F : X \to Y$ and $G : Y \to Z$ are smooth maps between manifolds then $G \circ F$ is also smooth and for all p in X we have $D_p(G \circ F) = D_{F(p)}G \circ D_p F$.*

Proof. For all $[\gamma]$ in T_pX we have
$$D_p(G \circ F)([\gamma]) = [(G \circ F) \circ \gamma] = [G \circ (F \circ \gamma)] = D_{F(p)}G \circ D_pF([\gamma]). \qquad \square$$

1.6. **Immersions, submersions, and local diffeomorphisms.** Fix manifolds X and Y of dimensions n and m as above, and a smooth map $F : X \to Y$.

Definition 1.6.1. F is an *immersion/submersion/local diffeomorphism* (at p) if DF is respectively injective/surjective/an isomorphism everywhere (respectively at p). The points p at which F is a submersion are called *regular points* of F. The non-regular points are called *critical points* of F. A point q in Y is a *regular value* of F if every p in $F^{-1}(q)$ is a regular point. Otherwise q is a *critical value* of F.

The name 'local diffeomorphism' is justified by the following.

Lemma 1.6.2. *If D_pF is an isomorphism then there exists an open neighbourhood U of p in X and an open neighbourhood S of $F(p)$ in Y such that $F|_U$ is a diffeomorphism $U \to S$.*

Proof. Pick charts $\varphi : U \xrightarrow{\sim} V$ and $\psi : S \xrightarrow{\sim} T$ about p and $F(p)$ respectively. By shrinking the first chart if necessary we may assume that $F(U) \subset S$. Now apply the inverse function theorem to
$$G := \psi \circ F \circ \varphi^{-1} : V \to T.$$
We obtain open subsets $V' \subset V$ and $T' \subset T$ such that $G|_{V'}$ is a diffeomorphism $V' \to T'$. Replace U with $\varphi^{-1}(V')$ and S with $\psi^{-1}(T')$ to get the result. $\square$

Notice that in the proof of Lemma 1.6.2 we could take $\psi \circ F$ and ψ (or φ and $\varphi \circ F|_U^{-1}$) as charts on U and V respectively. In the induced local coordinates F is then given by the identity map. There are also nice local forms for immersions and submersions.

Lemma 1.6.3. *Suppose F is an immersion at p and $x_1, \ldots, x_n$ are local coordinates about p. Then there exist local coordinates $y_1, \ldots, y_m$ about $F(p)$ such that, in terms of x and y, F is given on a neighbourhood of p by the inclusion*
$$\mathbb{R}^n = \mathbb{R}^n \oplus 0 \hookrightarrow \mathbb{R}^n \oplus \mathbb{R}^{m-n} = \mathbb{R}^m.$$
In other words $\mathsf{y} \circ F = (x_1, \ldots, x_n, 0, \ldots, 0)$.

Similarly, if F is a submersion at p then given local coordinates y about $F(p)$ there exist local coordinates x about p in which F is given by the projection
$$\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m} \twoheadrightarrow \mathbb{R}^m,$$
i.e. $\mathsf{y} \circ F = (x_1, \ldots, x_m)$.

Proof. The submersion case is on Example Sheet 1, and the immersion case is similar. $\square$

1.7. **Submanifolds.** Fix an n-manifold X.

Definition 1.7.1. A subset $Z \subset X$ is a *submanifold (of codimension k)* if for all p in Z there exists an open neighbourhood U of p in X, and local coordinates $x_1, \dots, x_n$ defined on U, such that $Z \cap U$ is given by $x_1 = \cdots = x_k = 0$. It is a *properly embedded submanifold* if this holds for all p in X.

For instance, the set $Z = \{0\} \times \mathbb{R} \subset X = \mathbb{R}^2$ is a properly embedded submanifold, using the standard coordinates. The set $Z' = \{0\} \times \mathbb{R}^*$ is a submanifold but not properly embedded.

Fix a submanifold $Z \subset X$ of codimension k.

- Z carries a subspace topology, which is Hausdorff and second-countable because X is.
- For each point p in Z there exist local coordinates x about p such that $Z = \{x_1 = \cdots = x_k = 0\}$. Then $x_{k+1}, \dots, x_n$ form local coordinates on a neighbourhood of p in Z, i.e. they define a chart on Z about p.
- The transition functions between different charts constructed in this way are smooth, because the original transition functions on X were smooth. So doing this for all p, U and x, we obtain a smooth atlas on Z.

Equivalent atlases on X induce equivalent atlases on Z so we get the following.

Proposition 1.7.2. *A codimension-k submanifold $Z \subset X$ is naturally an $(n-k)$-manifold. The inclusion $\iota : Z \to X$ is a smooth immersion and a homeomorphism onto its image. Composition with ι gives a bijection*

$$\{\textit{smooth maps to } Z\} \xrightarrow[\iota\circ -]{\sim} \{\textit{smooth maps to } X \textit{ with image contained in } Z\}. \qquad \square$$

Definition 1.7.3. An *embedding* of Y into X is a smooth immersion $F : Y \to X$ that is a homeomorphism onto its image.

Lemma 1.7.4. *The image of an embedding $F : Y \to X$ is a submanifold Z of X, and F induces a diffeomorphism from Y to Z.* $\square$

E.g. the inclusion $S^n \to \mathbb{R}^{n+1}$ is an embedding, so the manifold structure we defined on S^n coincides with the structure it inherits as a submanifold of $\mathbb{R}^{n+1}$.

Finding nice local coordinates about each point of $Z \subset X$ is fiddly. But there is a much easier way to check that Z is a submanifold.

Proposition 1.7.5. *Suppose $F : X \to Y$ is a smooth map and $q \in Y$ is a regular value (F is a submersion at p, i.e. D_pF is surjective, for all p in $F^{-1}(q)$). Then $F^{-1}(q)$ is a submanifold of X of codimension* $\dim Y$.

Proof. For each $p \in F^{-1}(q)$ we know that F is a submersion at p, so there exist local coordinates x about p and y about q such that

$$\mathsf{y} \circ F = (x_1, \dots, x_m).$$

If we translate the local coordinates so that $\mathsf{y}(q) = 0$, then on the domain of x we have that $F^{-1}(q)$ is given by $x_1 = \cdots = x_m = 0$. $\square$

E.g. Define $F : \mathbb{R}^{n+1} \to \mathbb{R}$ by $F(x) = \|x\|^2$. The only critical point is 0 in $\mathbb{R}^{n+1}$, so the only critical value is 0. In particular, $S^n = F^{-1}(1)$ is a submanifold of $\mathbb{R}^{n+1}$.

In order to use this criterion to produce submanifolds, we need regular values to be plentiful. Fortunately, this is the case.

Theorem 1.7.6 (Sard's theorem)**.** *The set of critical values has measure 0 in Y. More precisely, given any chart $\psi : S \xrightarrow{\sim} T$ on Y, the set*

$$\psi(S \cap \{\textit{critical values of } F\}) \subset T \subset \mathbb{R}^m$$

has measure 0 with respect to the Lebesgue measure on $\mathbb{R}^m$.

Proof. See Lee's *Introduction to smooth manifolds* (Theorem 6.10 in the second edition), or Nicolaescu's *Lectures on the geometry of manifolds* (Theorem 2.1.18 in the September 9, 2018 version). □

Corollary 1.7.7. *The regular values of F are dense in Y. In particular, F has at least one regular value (assuming Y is non-empty!).* □

Warning! Regular *points* need not exist.

If two submanifolds intersect nicely then their intersection is also a submanifold.

Definition 1.7.8. Submanifolds Y and Z in X are *transverse* if for all $p \in Y \cap Z$ we have

$$T_pY + T_pZ = T_pX.$$

(This is equivalent to the annihilators of T_pY and T_pZ intersecting trivially in $(T_pX)^\vee$.)

Lemma 1.7.9. *If Y and Z are transverse then $Y \cap Z$ is a submanifold of X.*

Proof. Given $p \in Y \cap Z$, pick local coordinates $y_1, \dots, y_n$ and $z_1, \dots, z_n$ on X about p in which Y is $\{y_1 = \cdots = y_k = 0\}$ and Z is $\{z_1 = \cdots = z_l = 0\}$. Now pick a small open neighbourhood U of p, on which both sets of coordinates are defined, and consider the map $F : U \to \mathbb{R}^{k+l}$ defined by $(y_1, \dots, y_k, z_1, \dots, z_l)$. Transversality tells us that the projection

$$T_pX \to (T_pX/T_pY) \oplus (T_pX/T_pZ)$$

is surjective, so F is a submersion at p. There thus exist coordinates $x_1, \dots, x_n$ about p with $x_1 = y_1, \dots, x_k = y_k$ and $x_{k+1} = z_1, \dots, x_{k+l} = z_l$. Then $Y \cap Z$ is locally given by $\{x_1 = \cdots = x_{j+k} = 0\}$. □

2. Vector bundles and tensors

2.1. **Vector bundles.** A vector bundle is a family of vector spaces parametrised by a manifold.

Definition 2.1.1. A *vector bundle of rank k* over a manifold B is a manifold E equipped with:

- A smooth map $\pi : E \to B$.
- An open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of B.
- For each α a diffeomorphism

  $$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k,$$

  such that:
  - $\mathrm{pr}_1 \circ \Phi_\alpha = \pi$.
  - On overlaps $U_\alpha \cap U_\beta$ the map $\Phi_\beta \circ \Phi_\alpha^{-1}$ has the form

    $$\begin{aligned}(U_\alpha \cap U_\beta) \times \mathbb{R}^k &\to (U_\alpha \cap U_\beta) \times \mathbb{R}^k \\ (b, \mathrm{x}) &\mapsto (b, g_{\beta\alpha}(b)(\mathrm{x})),\end{aligned}$$

    for some (necessarily smooth) map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$.

E is the *total space*, B is the *base*, π is the *projection*, the Φ_α are *local trivialisations*, and the maps $g_{\beta\alpha}$ are *transition functions*. We denote the bundle by $\pi : E \to B$, and the *fibre* $\pi^{-1}(b)$ by E_b.

Remark 2.1.2. Each fibre E_b has a canonical vector space structure, defined by picking U_α containing p and using $\Phi_\alpha : E_b \to \{b\} \times \mathbb{R}^k$. (If we chose β instead of α then the resulting identification of E_b with $\mathbb{R}_k$ would differ from the previous one by the linear automorphism $g_{\beta\alpha}(b)$ of $\mathbb{R}^k$, so the vector space structure is independent of this choice.)

Complex vector bundles can be defined analogously.

Example 2.1.3. (i) For any B and k the *trivial vector bundle* (of rank k over B) has $E = B \times \mathbb{R}^k$, with obvious projection $\pi : E \to B$ and *global* trivialisation $\Phi : E \to B \times \mathbb{R}^k$. We denote it by $\underline{\mathbb{R}}^k$ if B is clear.

(ii) The *tautological bundle* over $\mathbb{RP}^n$ is the line bundle (i.e. rank 1 vector bundle) defined by:
- $E = \{(p, \mathsf{v}) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : \mathsf{v} \text{ lies on the line } p\}$, which is a submanifold of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$.
- $\pi : E \to \mathbb{RP}^n$ given by $(p, \mathsf{v}) \mapsto p$.
- Open cover $\{U_0, \dots, U_n\}$, where $U_i = \{[x_0 : \cdots : x_n] : x_i \neq 0\}$.
- $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}$ given by
$$([x_0 : \cdots : x_n], \lambda(x_0, \dots, x_n)) \mapsto ([x_0 : \cdots : x_n], \lambda x_i).$$
- Then $\Phi_j \circ \Phi_i^{-1}([x_0 : \cdots : x_n], t) = ([x_0 : \cdots : x_n], x_j t/x_i)$ has the required form, with $g_{ji} = x_j/x_i \in \mathbb{R}^* = \mathrm{GL}(1, \mathbb{R})$.

(iii) The analogous tautological complex line bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$ over $\mathbb{CP}^n$.

(iv) The *tangent bundle* TX of an n-manifold X is a rank-n vector bundle over X, defined by:
- Total space $TX \coloneqq \coprod_{p \in X} T_pX$. Given local coordinates $x_1, \dots, x_n$ on an open set $U \subset X$, write $a_1, \dots, a_n$ for the components of a tangent vector with respect to $\partial_{x_1}, \dots, \partial_{x_n}$. This gives coordinates
$$(x_1, \dots, x_n, a_1, \dots, a_n) : \coprod_{p \in U} T_pU \to \mathbb{R}^{2n}.$$
Doing this for all coordinate patches U on X defines a smooth pseudo-atlas on TX, making it a manifold.
- Projection $\pi : (p, \mathsf{v}) \in T_pX \mapsto p \in X$.
- Open cover by coordinate patches, and trivialisations defined by
$$\Big(p, \sum_i a_i \partial_{x_i}\Big) \mapsto (p, (a_1, \dots, a_n)).$$

Really each local trivialisation is like a chart, and each collection of trivialisations that covers B is like an atlas. We should then define equivalence of such collections in terms of their union also forming a valid collection, and ask that E is equipped with an *equivalence class* of collection, rather than a specific one. All definitions and constructions should be independent of the choice of representative of the equivalence class. We'll not dwell on these issues.

Definition 2.1.4. A *section* of a vector bundle $\pi : E \to B$ over an open set $U \subset B$ is a smooth map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_B$. A *global section* is a section defined over $U = B$.

Example 2.1.5. (i) The *zero section* is the section which is 0 in each fibre.
(ii) A *vector field* on X is a section of TX.

Definition 2.1.6. Given a smooth map $F : B_1 \to B_2$, and vector bundles $\pi_i : E_i \to B_i$, a *morphism of vector bundles $E_1 \to E_2$ covering F* is a smooth map $G : E_1 \to E_2$ such that:
- $\pi_2 \circ G = F \circ \pi_1$.
- For each b the map $(E_1)_b \to (E_2)_{F(b)}$ induced by G is linear.

An isomorphism of vector bundles over B is a morphism covering id_B with a two-sided inverse. A vector bundle is *trivial* if it's isomorphic to a trivial bundle.

Example 2.1.7. If we think of S^1 as $\{e^{i\theta} : \theta \in \mathbb{R}\} \subset \mathbb{C}$ then although the local coordinate θ is multivalued if we try to define it globally, the vector ∂_θ is well-defined at every point. The map
$$(p, a) \in S^1 \times \mathbb{R} \mapsto (p, a\, \partial_\theta) \in TS^1$$
is then an isomorphism of vector bundles over S^1, so TS^1 is trivial. In contrast, TS^n is non-trivial for n even (Sheet 2).

Remark 2.1.8. A global section s of $\pi : E \to B$ is the same thing as a morphism $G : \underline{\mathbb{R}} \to E$ covering id_B. The section s is sent to the morphism G given by $(b \in B, t \in \mathbb{R}) \mapsto ts(b)$, where the multiplication by t uses the vector space structure on each fibre. Conversely, the morphism G is sent to the section $b \mapsto G(b, 1)$. More generally, a collection of k sections corresponds to a morphism $\underline{\mathbb{R}}^k \to E$. This is an isomorphism iff the sections form a basis in each fibre, so E is trivial iff a collection of sections forming a fibrewise basis exists.

Definition 2.1.9. A *subbundle* of $\pi : E \to B$ of rank l is a subset $F \subset E$ such that for each $b \in B$ there exists a trivialisation of E about b under which F corresponds to $\mathbb{R}^l = \mathbb{R}^l \oplus 0 \subset \mathbb{R}^k$ in each fibre. It is automatically a submanifold of E and inherits a vector bundle structure. Given a subbundle $F \subset E$ of rank l, the fibrewise quotient E/F forms a vector bundle of rank $k - l$. There are morphisms $F \to E \to E/F$ covering id_B.

For example, since each fibre of the tautological bundle E over $\mathbb{RP}^n$ is naturally a subspace of $\mathbb{R}^{n+1}$, E is a subbundle of $\underline{\mathbb{R}}^{n+1}$. The *Euler sequence* identifies the quotient $\underline{\mathbb{R}}^{n+1}/E$ as $T\mathbb{RP}^n(-1)$, where the (-1) amounts to 'tensor with E'.

2.2. **Constructing vector bundles by gluing.** We can reconstruct a vector bundle just from the base and the transition functions.

Precisely, to define a vector bundle over a manifold B it's enough to give the open cover $\{U_\alpha\}$, and for each α, β a smooth map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that

(i) For all α the map $g_{\alpha\alpha}$ is constant, equal to $\mathrm{id}_{\mathbb{R}^k}$. (In our setup, with the $g_{\beta\alpha}$ mapping to $\mathrm{GL}(k, \mathbb{R})$, this condition follows from (ii) with $\beta = \alpha$. If, however, we had simply asked the $g_{\beta\alpha}$ to map to $k \times k$ matrices, then we would have to impose condition (i) separately.)

(ii) For all α, β and γ the *cocycle condition*

$$g_{\gamma\alpha} = g_{\gamma\beta} g_{\beta\alpha}$$

holds over $U_\alpha \cap U_\beta \cap U_\gamma$.

(iii) ($g_{\beta\alpha} = g_{\alpha\beta}^{-1}$ but this follows from (i) and (ii) by taking $\gamma = \alpha$.)

The total space E can then be defined by

$$\coprod_\alpha (U_\alpha \times \mathbb{R}^k) \Big/ (b \in U_\alpha, \mathrm{x}) \sim (b \in U_\beta, g_{\beta\alpha}(b)(\mathrm{x})),$$

with π the obvious map $E \to B$. The manifold structure is defined by tautological pseudo-charts

$$(U_\alpha \times \mathbb{R}^k) \subset E \to U_\alpha \times \mathbb{R}^k,$$

which also give the local trivialisations.

Example 2.2.1. For each $r \in \mathbb{Z}$ we can define a line bundle over $\mathbb{RP}^n$ by taking the open cover $\{U_i\}$ as above, and setting $g_{ji} = (x_j/x_i)^{-r}$. This is denoted by $\mathcal{O}_{\mathbb{RP}^n}(r)$. We similarly have a complex line bundle $\mathcal{O}_{\mathbb{CP}^n}(r)$ over $\mathbb{CP}^n$.

Proposition 2.2.2. *If $E \to B$ is a rank k vector bundle that can be trivialised over an open cover $\{U_\alpha\}$ with transition functions $g_{\beta\alpha}$, then its transition functions satisfy (i) and (ii) and E is isomorphic to the bundle constructed above by gluing.*

Proof. Since the transition functions are defined via $\Phi_\beta \circ \Phi_\alpha^{-1}$, it's clear that they satisfy (i) and (ii). The trivialisations and their inverses define a diffeomorphism

$$E \cong \coprod_\alpha (U_\alpha \times \mathbb{R}^k) \Big/ \sim$$

covering id_B, which is manifestly linear on fibres, so is a bundle isomorphism. □

Corollary 2.2.3. *Two bundles over B are isomorphic if and only if they can be trivialised over the same cover with the same transition functions.*

Proof. If they can both be trivialised over the same cover with the same transition functions then they are both isomorphic to the bundle constructed above by gluing. Conversely, if they're isomorphic then the isomorphism can be used to carry trivialisations of one to trivialisations of the other. □

E.g. the tautological bundle over $\mathbb{RP}^n$ is isomorphic to $\mathcal{O}_{\mathbb{RP}^n}(-1)$.

Example 2.2.4. Define the *Möbius bundle* $M \to \mathbb{RP}^1$ to be the line bundle trivialised over U_0 and U_1 with transition function $g_{10} = \operatorname{sign}(x_1/x_0)$. We claim it's isomorphic to $\mathcal{O}_{\mathbb{RP}^1}(-1)$, which has transition function x_1/x_0. To prove this, consider rescaling our M trivialisations $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}$ by maps $\psi_i : U_i \to \mathbb{R}^*$, i.e. replacing Φ_i with

$$(p, \mathsf{v}) \in \pi^{-1}(U_i) \mapsto (p, \psi_i(p) \operatorname{pr}_2 \circ \Phi_i(p, \mathsf{v})) \in U_i \times \mathbb{R}.$$

This changes the transition function to $(\psi_1/\psi_0) \operatorname{sign}(x_1/x_0)$, so it suffices to find ψ_0 and ψ_1 such that $\psi_1/\psi_0 = |x_1|/|x_0|$. The following works:

$$\psi_0 = \sqrt{\frac{x_0^2}{x_0^2 + x_1^2}} \quad \text{and} \quad \psi_1 = \sqrt{\frac{x_1^2}{x_0^2 + x_1^2}}.$$

Definition 2.2.5. If $\pi : E \to B$ is a vector bundle and $F : B' \to B$ a smooth map, then the *pullback bundle* F^*E is the bundle over B' with total space

$$\coprod_{b \in B'} E_{F(b)},$$

and the following bundle structure. If E is trivialised over $\{U_\alpha\}$ with transition functions $g_{\beta\alpha}$, then F^*E is trivialised over $\{F^{-1}U_\alpha\}$ with transition functions $g_{\beta\alpha} \circ F$.

E.g. Consider the Hopf map $H : S^{2n+1} \to \mathbb{CP}^n$. The complex line bundle $H^*\mathcal{O}_{\mathbb{CP}^n}(-1) \to S^{2n+1}$ is trivial, since it has a nowhere-zero global section

$$p \in S^{2n+1} \mapsto p \in (\text{line through } p) = (\text{fibre of } \mathcal{O}_{\mathbb{CP}^n}(-1) \text{ over } H(p)).$$

Definition 2.2.6. The *dual bundle*, $E^\vee$, is the vector bundle over B with total space

$$\coprod_{b \in B} (E_b)^\vee.$$

It's trivialised over $\{U_\alpha\}$ with transition functions $(g_{\beta\alpha}^\vee)^{-1}$ (cf. dual representation).

If E is trivialised over $U \subset B$ by local sections $s_1, \ldots, s_r$ that form a fibrewise basis then the fibrewise dual basis gives smooth sections $\sigma_1, \ldots, \sigma_r$ of $E^\vee$ that trivialise it over U.

2.3. **The cotangent bundle.** Fix an n-manifold X.

Definition 2.3.1. The dual of $TX \to X$ is the *cotangent bundle*, denoted T^*X, with fibre T_p^*X the *cotangent space* at p.

We defined T_pX in terms of curves through p, i.e. smooth maps $\mathbb{R} \to X$ sending 0 to p. We can dualise this picture to get a geometric interpretation for T_p^*X. Consider

$$\{\text{functions about } p\} = \{(U, f) : U \text{ an open neighbourhood of } p,\ f : U \to \mathbb{R} \text{ smooth}\}.$$

There is an obvious notion of agreement to first order on this set, i.e. $f_1 \sim f_2$ if $D_p f_1 = D_p f_2$.

Proposition 2.3.2. *There is a canonical isomorphism*

$$\{\textit{functions about } p\}/\textit{agreement to first order} \to T_p^*X.$$

Proof. There is a pairing between functions f and curves γ, given by

$$(\gamma, f) \mapsto (f \circ \gamma)^\bullet(0).$$

This induces a linear map

$$\{\text{functions about } p\} \to T_p^*X \quad \text{given by} \quad f \mapsto ([\gamma] \mapsto (f \circ \gamma)^\bullet(0)),$$

given in coordinates by

$$f \mapsto \left(\sum_{i=1}^{n} a_i \partial_{x_i} \mapsto \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} \right). \tag{2.3.1}$$

This is surjective since the functions x_i are sent to the dual basis to the ∂_{x_i}, and two functions have the same image if and only if they agree to first order at p. □

Given a smooth function $f : X \to \mathbb{R}$, by the above result it defines a class in each $T_p^* X$.

Lemma 2.3.3. *This defines a (smooth) section of T^*X, which we denote by $\mathrm{d}f$.*

Proof. We saw in the proof above that $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ are fibrewise dual to $\partial_{x_1}, \ldots, \partial_{x_n}$. They are therefore smooth. Then from (2.3.1) we get

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i,$$

which is smooth. □

Definition 2.3.4. A section of T^*X is called a 1-*form*. The 1-form $\mathrm{d}f$ is the *differential* of f. By construction, the value of $\mathrm{d}f$ applied to a tangent vector v is the derivative of f in the direction v.

Un-warning! Although each ∂_{x_i} depends on the whole set $x_1, \ldots, x_n$ of local coordinates, each $\mathrm{d}x_i$ manifestly only depends on x_i. In this sense the $\mathrm{d}x_i$ are more fundamental.

Definition 2.3.5. Given a smooth map $F : X \to Y$ the dual map $(D_pF)^\vee : T^*_{F(p)}Y \to T^*_p X$ is called the *pullback*, denoted by F^*.

Lemma 2.3.6. *For a smooth function g on Y we have $F^*\mathrm{d}g = \mathrm{d}(g \circ F)$.*

Proof. Given $[\gamma] \in T_pX$ we have

$$(F^*\mathrm{d}g)([\gamma]) = (\mathrm{d}g)(D_pF([\gamma])) = (\mathrm{d}g)([F \circ \gamma]) = (g \circ F \circ \gamma)^\bullet(0) = (\mathrm{d}(g \circ F))([\gamma]).$$ □

2.4. **Multilinear algebra.** *[Covered by handout of this subsection of the notes]*

Let U and V be finite-dimensional vector spaces over a field $\mathbb{K}$.

Definition 2.4.1. The *tensor product* $U \otimes V$ (or $U \otimes_{\mathbb{K}} V$) is the $\mathbb{K}$-vector space generated by symbols $u \otimes v$, with $u \in U$ and $v \in V$, modulo

$$(\lambda_1 u_1 + \lambda_2 u_2) \otimes v = \lambda_1(u_1 \otimes v) + \lambda_2(u_2 \otimes v)$$
$$u \otimes (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1(u \otimes v_1) + \lambda_2(u \otimes v_2)$$

for all u, u_1 and u_2 in U, v, v_1 and v_2 in V, and λ_1 and λ_2 in $\mathbb{K}$.

Warning! A general element of $U \otimes V$ is not of the form $u \otimes v$. It a sum of such things.

Lemma 2.4.2. *(i) If $\mathsf{e}_1, \ldots, \mathsf{e}_m$ is a basis for U and $\mathsf{f}_1, \ldots, \mathsf{f}_n$ is a basis for V then the $\mathsf{e}_i \otimes \mathsf{f}_j$ form a basis for $U \otimes V$, so $\dim(U \otimes V) = \dim U \dim V$.*

(ii) Tensor product is functorial*: if $\alpha : U_1 \to U_2$ and $\beta : V_1 \to V_2$ are linear maps then there is an induced linear map $\alpha \otimes \beta : U_1 \otimes V_1 \to U_2 \otimes V_2$ given by $u \otimes v \mapsto \alpha(u) \otimes \beta(v)$, extended linearly.*

(iii) (Universal property of tensor products) Giving a linear map from $U \otimes V$ is equivalent to giving a bilinear map from $U \times V$. □

Definition 2.4.3. The evaluation map $V^\vee \times V \to \mathbb{K}$, given by $(\theta, \mathsf{v}) \mapsto \theta(v)$, is bilinear so induces a linear map $V^\vee \otimes V \to \mathbb{K}$ called *contraction*. By functoriality, other tensor factors can 'come along for the ride', and we still call this contraction. For example

$$T \otimes V^\vee \otimes U \otimes V \otimes W \to T \otimes U \otimes W.$$

Definition 2.4.4. The rth *tensor power* of V is

$$V^{\otimes r} = \underbrace{V \otimes \cdots \otimes V}_{r}.$$

The rth *exterior power* of V is $\Lambda^r V = V^{\otimes r}/W$, where W is the subspace spanned by elements of the form $\cdots \otimes \mathsf{v} \otimes \cdots \otimes \mathsf{v} \otimes \cdots$. We write the image of $\mathsf{v}_1 \otimes \cdots \otimes \mathsf{v}_r$ in $\Lambda^r V$ as $\mathsf{v}_1 \wedge \cdots \wedge \mathsf{v}_r$.

Example 2.4.5. Let V be 2-dimensional, with basis $\mathsf{e}_1, \mathsf{e}_2$. Then

- $V^{\otimes 0} = \Lambda^0 V = \mathbb{K}$.
- $V^{\otimes 1} = \Lambda^1 V = V$.
- $V^{\otimes 2}$ is 4-dimensional with basis $\mathsf{e}_i \otimes \mathsf{e}_j$ for $i, j = 1, 2$. $\Lambda^2 V$ is 1-dimensional with basis $\mathsf{e}_1 \wedge \mathsf{e}_2 = -\mathsf{e}_2 \wedge \mathsf{e}_1$. This is because:
  - $\mathsf{e}_i \wedge \mathsf{e}_i = 0$, since $\mathsf{e}_i \otimes \mathsf{e}_i$ is in W.
  - $(\mathsf{e}_1 + \mathsf{e}_2) \wedge (\mathsf{e}_1 + \mathsf{e}_2) = 0$, for the same reason, so $\mathsf{e}_1 \wedge \mathsf{e}_2 + \mathsf{e}_2 \wedge \mathsf{e}_1 = 0$.
- For $r \geq 3$, $V^{\otimes r}$ is 2^r-dimensional whilst $\Lambda^r V = 0$.

Lemma 2.4.6. *The wedge product is graded-commutative: if $P \in \Lambda^r V$ and $Q \in \Lambda^s V$ then*

$$P \wedge Q = (-1)^{rs} Q \wedge P.$$

If V is n-dimensional, with basis $\mathsf{e}_1, \dots, \mathsf{e}_n$, then $\Lambda^r V$ has dimension $\binom{n}{r}$, with basis $\mathsf{e}_I := \mathsf{e}_{i_1} \wedge \cdots \wedge \mathsf{e}_{i_r}$, where I ranges over multi-indices *of length r, meaning tuples $(i_1, \dots, i_r)$ with $i_1 < \cdots < i_r$.* □

Geometrically $\mathsf{v}_1 \wedge \cdots \wedge \mathsf{v}_r$ represents the 'signed r-dimensional volume of the parallelepiped in V with edge vectors $\mathsf{v}_1, \dots, \mathsf{v}_r$'.

Exterior powers inherit functoriality from tensor powers: a linear map $\alpha : U \to V$ induces

$$\wedge^r \alpha : \Lambda^r U \to \Lambda^r V \quad \text{by} \quad \mathsf{u}_1 \wedge \cdots \wedge \mathsf{u}_r \mapsto \alpha(\mathsf{u}_1) \wedge \cdots \wedge \alpha(\mathsf{u}_r)$$

extended linearly.

Example 2.4.7. The space $\Lambda^n V$ is one-dimensional, so endomorphisms of it are just scalars. Given an endomorphism α of V, the scalar corresponding to $\wedge^n \alpha : \Lambda^n V \to \Lambda^n V$ is by definition $\det(\alpha)$. The volume meaning of $\Lambda^n V$ gives the interpretation of $\det(\alpha)$ as the volume scale factor.

2.5. **Tensors and forms.** Just as we can dualise vector bundles, we can apply any fibrewise algebraic operation that is functorial.

Example 2.5.1. Suppose E and F are vector bundles over B, trivialised over an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ with transition functions $g_{\beta\alpha}$ and $h_{\beta\alpha}$ respectively. Then $E \oplus F$ is defined to be the vector bundle over B with fibre $E_b \oplus F_b$ over each b, built using transition functions

$$g_{\beta\alpha} \oplus h_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(\operatorname{rk} E, \mathbb{R}) \times \mathrm{GL}(\operatorname{rk} F, \mathbb{R}) \subset \mathrm{GL}(\operatorname{rk} E + \operatorname{rk} F, \mathbb{R}).$$

Similarly $E \otimes F$ has fibre $E_b \otimes F_b$ and transition functions $g_{\beta\alpha} \otimes h_{\beta\alpha}$. Contraction defines a bundle morphism $E \otimes F^\vee \otimes F \to E$.

Example 2.5.2. If $F : X \to Y$ is a smooth map then DF is naturally a section of $T^*X \otimes F^*TY$: for each $p \in X$ we have $(T^*X \otimes F^*TY)_p = (T_pX)^\vee \otimes T_{F(p)}Y = \mathrm{Hom}_{\mathbb{R}}(T_pX, T_{F(p)}Y)$, using Sheet 2.

We can similarly construct tensor and exterior powers of a single bundle.

Definition 2.5.3. A *tensor (field) of type* (p, q) on X is a section of $(TX)^{\otimes p} \otimes (T^*X)^{\otimes q}$. An *$r$-form* is a section of $\Lambda^r T^*X$ (this coincides with our earlier definition when $r = 1$). The space of r-forms over open $U \subset X$ is written $\Omega^r(U)$.

A tensor of type:

- $(0, 0)$ is a section of $\underline{\mathbb{R}}$, i.e. a function, sometimes called a scalar field.

- $(1,0)$ is a vector field.
- $(0,1)$ is the same thing as a 1-form.
- $(0,q)$ is something which, at each point, 'eats q vectors multilinearly and spits out a scalar'.

An r-form eats r vector fields multilinearly and antisymmetrically (see below for formulae).

2.6. **Index notation.** From now on, local coordinates will be indexed by superscripts rather than subscripts.

In local coordinates $x^1, \dots, x^n$ a section of, say, $TX \otimes T^*X \otimes TX$ (which is a specific kind of tensor of type $(2,1)$) can be written uniquely as

$$\sum_{i,j,k} T^{i\ k}_{\ j}\, \partial_i \otimes \mathrm{d}x^j \otimes \partial_k$$

for locally defined smooth functions $T^{i\ k}_{\ j}$. Horizontal positions of indices on T reflect which tensor factor they refer to. Vertical positions denote TX vs T^*X. We will often use *summation convention* where each repeated index, once up and once down, is summed over, so

$$T = T^{i\ k}_{\ j}\, \partial_i \otimes \mathrm{d}x^j \otimes \partial_k.$$

We'll often write T as just $T^{i\ k}_{\ j}$, without the $\partial_i \otimes \mathrm{d}x^j \otimes \partial_k$.

Tensor product becomes juxtaposition:

$$(T^{i\ k}_{\ j}\, \partial_i \otimes \mathrm{d}x^j \otimes \partial_k) \otimes (S_{lm}\, \mathrm{d}x^l \otimes \mathrm{d}x^m) = (T^{i\ k}_{\ j} S_{lm})\, \partial_i \otimes \mathrm{d}x^j \otimes \partial_k \otimes \mathrm{d}x^l \otimes \mathrm{d}x^m$$

so $(T \otimes S)^{i\ k}_{\ j\ lm} = T^{i\ k}_{\ j} S_{lm}$. Contraction becomes summation, e.g. contraction of the third factor of T with the second factor of S in $T \otimes S$ is

$$(T^{i\ k}_{\ j} S_{lm}) \langle \partial_k, \mathrm{d}x^m \rangle\, \partial_i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^l = (T^{i\ k}_{\ j} S_{lm}) \delta_k^{\ m}\, \partial_i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^l = (T^{i\ k}_{\ j} S_{lk})\, \partial_i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^l,$$

or just $T^{i\ k}_{\ j} S_{lk}$.

Similarly, in local coordinates an r-form α can be written uniquely as

$$\sum_I \alpha_I\, \mathrm{d}x^I = \sum_I \alpha_I\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_r},$$

where the sum is over multi-indices. This can eat r vectors $v_{(1)}, \dots, v_{(r)}$ and spit out the number

$$\sum_{I, \sigma \in S_r} \varepsilon(\sigma) \alpha_I v_{(1)}^{i_{\sigma(1)}} \dots v_{(r)}^{i_{\sigma(r)}}.$$

This corresponds to viewing α as the tensor

$$\sum_{I, \sigma \in S_r} \varepsilon(\sigma) \alpha_I\, \mathrm{d}x^{i_{\sigma(1)}} \otimes \cdots \otimes \mathrm{d}x^{i_{\sigma(r)}}$$

of type $(0,r)$ and contracting with $v_{(1)}, \dots, v_{(r)}$.

Warning! Some people divide by $r!$ when summing over permutations.

We sometimes refer to the components of this tensor as $\alpha_{i_1 \dots i_r}$, and when the i_j form a multi-index, i.e. $i_1 < \cdots < i_r$, this agrees with our existing meaning of $\alpha_{i_1 \dots i_r} = \alpha_I$.

Warning! It is valid to write

$$\alpha = \alpha_{i_1 \dots i_r}\, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r}$$

in summation convention, but

$$\alpha_{i_1 \dots i_r}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_r} = r!\alpha.$$

Example 2.6.1. On $\mathbb{R}^2$ we have

$$\mathrm{d}x^1 \wedge \mathrm{d}x^2 = \mathrm{d}x^1 \otimes \mathrm{d}x^2 - \mathrm{d}x^2 \otimes \mathrm{d}x^1.$$

A general 2-form looks like $\alpha_{12}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 = \alpha_{12}\,\mathrm{d}x^1 \otimes \mathrm{d}x^2 + \alpha_{21}\,\mathrm{d}x^2 \otimes \mathrm{d}x^1$, where $\alpha_{21} = -\alpha_{12}$ and $\alpha_{11} = \alpha_{22} = 0$.

2.7. **Pushforward and pullback.** Fix manifolds X and Y. For any smooth map $F : X \to Y$ we can do the following:

- Given $p \in X$ we have a pushforward map
$$F_* : (T_pX)^{\otimes r} \to (T_{F(p)}Y)^{\otimes r}$$
defined by $(D_pF)^{\otimes r}$, i.e. applying D_pF on every tensor factor.
- Given $p \in X$ we have pullback maps
$$F^* : (T^*_{F(p)}Y)^{\otimes r} \to (T^*_pX)^{\otimes r} \quad \text{and} \quad F^* : \Lambda^r T^*_{F(p)}Y \to \Lambda^r T^*_pX$$
defined by $((D_pF)^\vee)^{\otimes r}$ and $\wedge^r((D_pF)^\vee)$.
- Given an r-form α on Y, we can pull back α to give an r-form $F^*\alpha$ on X, defined by $(F^*\alpha)_p = F^*(\alpha_{F(p)})$. We can do exactly the same if α is a tensor of type $(0, r)$ defined on Y.

Summary: we can push forward tensors with up-indices at a point, we can pull back forms or tensors with down-indices at a point, and we can pull back forms or tensors with down-indices over the whole manifold.

If F is actually a diffeomorphism then we can do the following:

- Given $p \in X$ we can push forward any tensor or form at p to $F(p)$ by applying D_pF on TX factors and $((D_pF)^\vee)^{-1}$ on T^*X factors.
- Given any tensor or form on X we can push it forward to a tensor or form on Y by doing this at each point.
- Given $q \in Y$ we can pull back any tensor or form from q to $F^{-1}(q)$, using $D_q(F^{-1})$ and $(D_{F^{-1}(q)}F)^\vee$.

Summary: we can push forward or pull back any tensor or form at a point, or over the whole manifold. Note that in this setting we have $(F^{-1})_* = F^*$ and $(F^{-1})^* = F_*$.

3. Differential forms

3.1. **Exterior derivative.** So far we can differentiate smooth functions and smooth maps, but we have no way to differentiate vector fields or general tensors. It turns out to be more natural to differentiate 1-forms than vector fields.

Let's naïvely try to differentiate a 1-form $\alpha = \alpha_i\,\mathrm{d}x^i$. We get $\frac{\partial \alpha_i}{\partial x^j}\,\mathrm{d}x^j \otimes \mathrm{d}x^i$. In different coordinates y^i we have $\alpha = \alpha'_i\,\mathrm{d}y^i$, where $\alpha'_i = \alpha_k \frac{\partial x^k}{\partial y^i}$. Using these coordinates the naïve derivative is
$$\begin{aligned}
\frac{\partial \alpha'_i}{\partial y^j}\,\mathrm{d}y^j \otimes \mathrm{d}y^i &= \frac{\partial}{\partial y^j}\left(\alpha_k \frac{\partial x^k}{\partial y^i}\right)\mathrm{d}y^j \otimes \mathrm{d}y^i \\
&= \left(\frac{\partial \alpha_k}{\partial y^j}\frac{\partial x^k}{\partial y^i} + \alpha_k \frac{\partial^2 x^k}{\partial y^j \partial y^i}\right)\mathrm{d}y^j \otimes \mathrm{d}y^i \\
&= \frac{\partial \alpha_i}{\partial x^j}\,\mathrm{d}x^j \otimes \mathrm{d}x^i + \alpha_k \frac{\partial^2 x^k}{\partial y^j \partial y^i}\,\mathrm{d}y^j \otimes \mathrm{d}y^i
\end{aligned}$$
So the naïve definition is coordinate-dependent through the term
$$\alpha_k \frac{\partial^2 x^k}{\partial y^j \partial y^i}\,\mathrm{d}y^j \otimes \mathrm{d}y^i.$$
But this is *symmetric*, so we can kill it by antisymmetrising. The cleanest way to do this is to replace $\otimes$ with $\wedge$.

Definition 3.1.1. The *exterior derivative* of a 1-form α is the 2-form $\mathrm{d}\alpha$ defined in local coordinates by

$$\mathrm{d}\alpha = \frac{\partial \alpha_i}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^i = \mathrm{d}\alpha_i \wedge \mathrm{d}x^i.$$

The above calculation shows that this is well-defined, independent of the choice of local coordinates. More generally, given a p-form $\alpha = \alpha_I\,\mathrm{d}x^I$ (implicitly summing over I) its *exterior derivative* is the $(p+1)$-form

$$\mathrm{d}\alpha := \frac{\partial \alpha_I}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^I = \mathrm{d}\alpha_I \wedge \mathrm{d}x^I.$$

You can check that this is similarly well-defined.

If we'd tried to do this for vector fields or for tensors with any up indices then it wouldn't have worked.

Proposition 3.1.2. *The exterior derivative has the following properties:*

(i) It is $\mathbb{R}$-linear.
(ii) On 0-forms it agrees with the differential.
(iii) It squares to zero, i.e. $\mathrm{d} \circ \mathrm{d} = 0$.
(iv) It satisfies the graded Leibniz rule: for any p-form α and q-form β

$$\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta.$$

(v) It commutes with pullback: $\mathrm{d}(F^*\alpha) = F^*(\mathrm{d}\alpha)$ for any smooth map $F : X \to Y$ and p-form α on Y.

Aisde: the first part allows us to show that not all 1-forms are differentials. For example, if $x\mathrm{d}y$ on $\mathbb{R}^2$ were of the form $\mathrm{d}f$ then we'd have $\mathrm{d}(x\mathrm{d}y) = \mathrm{d}^2 f = 0$, but we actually have $\mathrm{d}(x\mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}y \neq 0$.

Proof. (i) and (ii) are obvious.

(iii) For any p-form α, we have in local coordinates that

$$\mathrm{d}^2\alpha = \frac{\partial^2 \alpha_I}{\partial x^k \partial x^j}\,\mathrm{d}x^k \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I,$$

which vanishes since the partial derivative is symmetric in j, k.

(iv) In local coordinates we have

$$\begin{aligned}
\mathrm{d}(\alpha \wedge \beta) &= \mathrm{d}\big(\alpha_I \beta_J\,\mathrm{d}x^I \wedge \mathrm{d}x^J\big)\\
&= \big((\mathrm{d}\alpha_I)\beta_J + \alpha_I(\mathrm{d}\beta_J)\big) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J\\
&= (\mathrm{d}\alpha_I \wedge \mathrm{d}x^I) \wedge (\beta_J\,\mathrm{d}x^J)\\
&\qquad\qquad + (-1)^p\big(\alpha_I\,\mathrm{d}x^I\big) \wedge (\mathrm{d}\beta_J \wedge \mathrm{d}x^J)\\
&= (\mathrm{d}\alpha) \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta.
\end{aligned}$$

(v) Let $x^1, \dots, x^n$ be local coordinates on X about a point x, and let $y^1, \dots, y^m$ be local coordinates on Y about $F(x)$, so $\alpha = \alpha_I\,\mathrm{d}y^I$. Then F expresses the y^i as functions of the x^i, and near x we have

$$\begin{aligned}
F^*(\mathrm{d}\alpha) &= F^*(\mathrm{d}\alpha_I \wedge \mathrm{d}y^{i_1} \wedge \dots \wedge \mathrm{d}y^{i_r})\\
&= (F^*\mathrm{d}\alpha_I) \wedge (F^*\mathrm{d}y^{i_1}) \wedge \dots \wedge (F^*\mathrm{d}y^{i_r})\\
&= \mathrm{d}(\alpha_I \circ F) \wedge \mathrm{d}(y^{i_1} \circ F) \wedge \dots \wedge \mathrm{d}(y^{i_r} \circ F) \quad \text{by Lemma 2.3.6}\\
&= \mathrm{d}\big((\alpha_I \circ F)\,\mathrm{d}(y^{i_1} \circ F) \wedge \dots \wedge \mathrm{d}(y^{i_r} \circ F)\big) \quad \text{by (iii) and (iv)}\\
&= \mathrm{d}\big(F^*(\alpha_I\,\mathrm{d}y^I)\big)\\
&= \mathrm{d}(F^*\alpha).
\end{aligned}$$

□

In fact, these properties (with or without (v), since it follows from the others) characterise d uniquely among maps $\Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$.

Definition 3.1.3. A form α is *closed* if $\mathrm{d}\alpha = 0$ and *exact* if $\alpha = \mathrm{d}\beta$ for some form β. Exact forms are closed since $\mathrm{d}^2 = 0$.

3.2. **de Rham cohomology.** Forms detect topology.

Fix an n-manifold X. Write $Z^r(X)$ and $B^r(X)$ for the spaces of closed, respectively exact, r-forms on X. We just saw that $B^r(X) \subset Z^r(X)$.

Definition 3.2.1. The *rth de Rham cohomology group of X* is

$$H^r_{\mathrm{dR}}(X) \coloneqq Z^r(X)/B^r(X).$$

Note $H^r_{\mathrm{dR}}(X) = 0$ for $r > \dim X$. We set $H^r_{\mathrm{dR}}(X) = 0$ for $r < 0$. Given a closed form α, we write $[\alpha]$ for the cohomology class it represents. Say α and β are *cohomologous* if $[\alpha] = [\beta]$, i.e. if $\alpha = \beta + \mathrm{d}\gamma$ for some γ.

Example 3.2.2 (Trivial cases).
(i) The zeroth cohomology group of X is

$$\begin{aligned} H^0_{\mathrm{dR}}(X) &= \{f \in \Omega^0(X) : \mathrm{d}f = 0\}/0 \\ &= \{\text{locally constant functions on } X\} \\ &= \mathbb{R}^{\{\text{components of } X\}}. \end{aligned}$$

(ii) If X is a point then $\Omega^r(X) = 0$ for all $r \neq 0$. So

$$H^r_{\mathrm{dR}}(X) \cong \begin{cases} \mathbb{R} & \text{if } r = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.2.3 (First non-trivial case). We have

$$H^r_{\mathrm{dR}}(S^1) \cong \begin{cases} 0 & \text{if } r \neq 0, 1 \\ \mathbb{R} & \text{if } r = 0 \\ ? & \text{if } r = 1. \end{cases}$$

Since S^1 is 1-dimensional every 1-form is closed, so

$$H^1_{\mathrm{dR}}(S^1) = \{\text{1-forms}\}/\{\text{differentials}\} = \{f\,\mathrm{d}\theta\}\Big/\Big\{\frac{\partial g}{\partial \theta}\,\mathrm{d}\theta\Big\},$$

where f and g denote 2π-periodic functions. The map

$$\Omega^1(S^1) \to \mathbb{R} \quad \text{given by} \quad f\,\mathrm{d}\theta \mapsto \int_0^{2\pi} f(\theta)\,\mathrm{d}\theta$$

kills all differentials by the Fundamental Theorem of Calculus, so induces a map $I : H^1_{\mathrm{dR}}(S^1) \to \mathbb{R}$. This is linear and surjective (take $f = 1$). We claim I is injective and hence is an isomorphism. Suppose then that $I(f\,\mathrm{d}\theta) = 0$. We want to find g such that $f = \frac{\partial g}{\partial \theta}$. Define g by

$$g(\theta) = \int_0^\theta f(t)\,\mathrm{d}t.$$

This is 2π-periodic since $I(f\mathrm{d}\theta) = 0$.

Proposition 3.2.4 (Contravariant functoriality). *Given a smooth map*

$$F : X \to Y,$$

pullback F^* of forms descends to cohomology and induces a linear map

$$F^* : H^r_{\mathrm{dR}}(Y) \to H^r_{\mathrm{dR}}(X) \quad \textit{for each } r.$$

Proof. We saw already that F^* commutes with d. This means that it sends $Z^r(Y)$ to $Z^r(X)$ because

$$\text{if} \quad \mathrm{d}\alpha = 0 \quad \text{then} \quad \mathrm{d}F^*\alpha = F^*\mathrm{d}\alpha = 0.$$

And it sends $B^r(Y)$ to $B^r(X)$ because

$$\text{if} \quad \alpha = \mathrm{d}\beta \quad \text{then} \quad F^*\alpha = F^*\mathrm{d}\beta = \mathrm{d}F^*\beta.$$

Therefore it descends to the quotient Z^r/B^r. □

Proposition 3.2.5. *Wedge product of forms descends to cohomology and makes $H^\bullet_{\mathrm{dR}}(X)$ into an associative, graded-commutative, unital $\mathbb{R}$-algebra.*

Proof. Suppose $[\alpha] \in H^r_{\mathrm{dR}}(X)$ and $[\beta] \in H^s_{\mathrm{dR}}(X)$. We want to define $[\alpha] \wedge [\beta]$ to be $[\alpha \wedge \beta]$, so need to check that $\alpha \wedge \beta$ is closed and that its cohomology class depends only on the classes of α and β.

Well, we have $\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^r \alpha \wedge \mathrm{d}\beta = 0$.

And if $\alpha' = \alpha + \mathrm{d}\gamma$ and $\beta' = \beta + \mathrm{d}\delta$ then

$$\begin{aligned} \alpha' \wedge \beta' &= \alpha \wedge \beta + (\mathrm{d}\gamma) \wedge \beta' + \alpha \wedge \mathrm{d}\delta \\ &= \alpha \wedge \beta + \mathrm{d}(\gamma \wedge \beta') + (-1)^r \mathrm{d}(\alpha \wedge \delta). \end{aligned}$$ □

Since pullbacks commute with $\wedge$ and send 1 to 1, they automatically induce unital algebra homomorphisms on $H^\bullet_{\mathrm{dR}}(X)$.

The crucial feature of de Rham cohomology that makes it topological is the following.

Proposition 3.2.6. *If $F_0, F_1 : X \to Y$ are smooth maps which are* homotopic *then F_0^* and F_1^* induce the same map $H^\bullet_{\mathrm{dR}}(Y) \to H^\bullet_{\mathrm{dR}}(X)$.*

Proof. See Section 5. □

F_0 and F_1 being homotopic means there exists a *homotopy* between them, meaning a smooth map $F : X \times [0,1] \to Y$ such that $F(\cdot, 0) = F_0$ and $F(\cdot, 1) = F_1$. Strictly we haven't yet defined smooth maps from manifolds with boundary, but we'll do so shortly. Any reasonable definition will do here.

Corollary 3.2.7. *If $F : X \to Y$ is a* homotopy equivalence, *i.e. there exists $G : Y \to X$ such that $G \circ F$ and $F \circ G$ are homotopic to id_X and id_Y respectively, then $F^* : H^\bullet_{\mathrm{dR}}(Y) \to H^\bullet_{\mathrm{dR}}(X)$ is an isomorphism.*

Proof. If such a G exists then Proposition 3.2.6 tells us that the maps $F^* : H^\bullet_{\mathrm{dR}}(Y) \to H^\bullet_{\mathrm{dR}}(X)$ and $G^* : H^\bullet_{\mathrm{dR}}(X) \to H^\bullet_{\mathrm{dR}}(Y)$ are inverse to each other. □

Example 3.2.8 (Poincaré Lemma)**.** For all n we have $H^\bullet_{\mathrm{dR}}(\mathbb{R}^n) \cong H^\bullet_{\mathrm{dR}}(\mathrm{pt})$.

3.3. **Orientations.** We want to define integrals of forms over manifolds, and orientations tell us how to attach signs to them, e.g. whether $\int_{\mathbb{R}}$ means $\int_{-\infty}^{\infty}$ or $\int_{\infty}^{-\infty}$.

Definition 3.3.1. An *orientation* of a vector space V is a non-zero element of $\Lambda^n V$ modulo rescaling by $\mathbb{R}_{>0}$. An *orientation* of a rank-k vector bundle $E \to B$ is a nowhere-zero section of $\Lambda^k E$ modulo rescaling by positive smooth functions. The bundle is *orientable* if admits an orientation (which is equivalent to $\Lambda^k E$ being trivial), and *non-orientable* otherwise. It is *oriented* if it is equipped with a choice of orientation.

E.g. any trivial bundle is orientable. The tautological bundle over $\mathbb{RP}^n$ is non-orientable (Sheet 2).

Definition 3.3.2. A manifold X is *orientable* if TX is orientable, and is *non-orientable* otherwise. An *orientation* of X is an orientation of TX. Say X is *oriented* if it's equipped with a choice of orientation.

E.g. S^n is orientable for all n. $\mathbb{RP}^n$ is orientable for some but not all n (Example Sheet 2).

Definition 3.3.3. A *volume form* on an n-manifold X is a nowhere-zero n-form.

A volume form ω determines an orientation of X by saying a basis $\mathsf{v}_1, \ldots, \mathsf{v}_n \in T_pX$ is positively oriented iff $\omega(\mathsf{v}_1, \ldots, \mathsf{v}_1) > 0$. Conversely, an orientation defines a choice of volume form up to rescaling by positive smooth functions.

3.4. **Partitions of unity.** These allow us to patch together local constructions.

Definition 3.4.1. Given a manifold X and open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, a *partition of unity subordinate to this cover* is a collection $\{\rho_\alpha : X \to [0,1]\}_{\alpha\in\mathcal{A}}$ of smooth functions such that:

- For each α the support of ρ_α is contained in U_α.
- Every point p in X has an open neighbourhood on which all but finitely many of the ρ_α vanish. (The collection is *locally finite*.)
- The sum $\sum_\alpha \rho_\alpha$ is the constant function 1. Note that by local finiteness the sum is finite near each point, so makes sense.

Lemma 3.4.2. *For any manifold and open cover, there exists a partition of unity subordinate to that cover.*

Proof. It's unenlightening so we refer to Lee (Theorem 2.23). □

3.5. **Integration.** Fix an oriented n-manifold X and a compactly supported n-form ω on X.

Definition 3.5.1. The *integral of ω over X*, denoted $\int_X \omega$, is defined as follows. Cover X by coordinate patches $\{U_\alpha\}$, and choose a partition of unity $\{\rho_\alpha\}$ subordinate to this cover. Each $\rho_\alpha\omega$ is a compactly supported n-form that can be written in terms of the local coordinates $x^1_\alpha, \dots, x^n_\alpha$ on U_α, say

$$\rho_\alpha\omega = f_\alpha \,\mathrm{d}x^1_\alpha \wedge \cdots \wedge \mathrm{d}x^n_\alpha.$$

Assuming that the x^i_α are ordered in a positively oriented way, we set

$$\int_X \omega = \sum_\alpha \int_{\mathbb{R}^n} f_\alpha \,\mathrm{d}x^1_\alpha \dots \mathrm{d}x^n_\alpha.$$

Here, the integral on the RHS is the usual integral of the compactly supported smooth function f_α with respect to Lebesgue measure.

Lemma 3.5.2. *This is independent of choices.*

Proof. If $\{V_\beta\}$ is another choice of cover, with coordinates y^i_β and partition of unity $\{\sigma_\beta\}$, such that $\sigma_\beta\omega = g_\beta \,\mathrm{d}y^1_\beta \wedge \cdots \wedge \mathrm{d}y^n_\beta$ then on $U_\alpha \cap V_\beta$ we have

$$\rho_\alpha g_\beta \,\mathrm{d}y^1_\beta \wedge \cdots \wedge \mathrm{d}y^n_\beta = \rho_\alpha\sigma_\beta\omega = \sigma_\beta f_\alpha \,\mathrm{d}x^1_\alpha \wedge \cdots \wedge \mathrm{d}x^n_\alpha.$$

Therefore

$$\sigma_\beta f_\alpha = \rho_\alpha g_\beta \det\left(\frac{\partial y^i_\beta}{\partial x^j_\alpha}\right)$$

and hence

$$\begin{aligned}
\sum_\alpha \int_{\mathbb{R}^n} f_\alpha \,\mathrm{d}x^1_\alpha \dots \mathrm{d}x^n_\alpha &= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} \sigma_\beta f_\alpha \,\mathrm{d}x^1_\alpha \dots \mathrm{d}x^n_\alpha \\
&= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} \rho_\alpha g_\beta \det\left(\frac{\partial y^i_\beta}{\partial x^j_\alpha}\right) \mathrm{d}x^1_\alpha \dots \mathrm{d}x^n_\alpha \\
&= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} \rho_\alpha g_\beta \,\mathrm{d}y^1_\beta \dots \mathrm{d}y^n_\beta \quad \text{by change of variables for Lebesgue integral} \\
&= \sum_\beta \int_{\mathbb{R}^n} g_\beta \,\mathrm{d}y^1_\beta \dots \mathrm{d}y^n_\beta.
\end{aligned}$$

□

Remark 3.5.3. (i) Since ω is compactly-supported, and our partitions of unity are locally finite, only finitely many f_α or g_β are non-zero. So all sums appearing in the definition and lemma are finite.

(ii) The orientedness of X enters the above proof by ensuring that the Jacobian determinant is positive. This is necessary because it's the *absolute value* of the determinant that appears in the change of variables formula for the Lebesgue integral.

3.6. **Stokes's theorem.** The fundamental theorem of calculus says that if X is a closed interval $[a, b]$, and f is a smooth function (0-form) on X, then

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = f(b) - f(a).$$

We can rewrite this in a coordinate-free way as

$$\int_X \mathrm{d}f = \int_{\partial X} f.$$

Stokes's theorem generalises this. First we need to make sense of manifolds-with-boundary.

Definition 3.6.1. A *(smooth) n-manifold-with-boundary* is defined in exactly the same way as an ordinary n-manifold, except that the codomain V of each chart $\varphi : U \xrightarrow{\sim} V$ is now an open set in $\mathbb{R}^n$ or $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. Given $p \in X$ and a chart φ containing it, say p is in the *boundary*, ∂X, if $V \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and $\varphi(p) \subset \{0\} \times \mathbb{R}^{n-1}$. Otherwise p is in the *interior*, $\mathring{X}$. This is independent of the choice of chart. Smooth functions on and maps between manifolds-with-boundary are defined in the obvious way.

Note: a map F from an open subset W of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ to $\mathbb{R}^m$ is smooth if there exists an open set W' in $\mathbb{R}^n$ containing W, and an extension $\overline{F}$ of F to W' which is smooth as a map $W' \to \mathbb{R}^m$.

Example 3.6.2.
(i) An ordinary n-manifold X is an n-manifold-with-boundary, with $\partial X = \emptyset$.
(ii) The closed unit ball $D^n : \{\mathsf{x} \in \mathbb{R}^n : \|\mathsf{x}\| \leq 1\}$ in $\mathbb{R}^n$ is an n-manifold with boundary. Its interior is the open ball $\{\|\mathsf{x}\| < 1\}$ and its boundary is S^{n-1}.
(iii) If X is a manifold-with-boundary and Y a manifold, then $X \times Y$ is naturally a manifold-with-boundary. Its interior is $\mathring{X} \times Y$ and its boundary is $(\partial X) \times Y$. (If X and Y both have non-empty boundary then $X \times Y$ is *not* a manifold with boundary: it has *corners*.)
(iv) If X is an n-manifold-with-boundary then $\mathring{X}$ and ∂X are respectively an n-manifold and an $(n-1)$-manifold (without boundary).

Aside. For $p \in \partial X$, if we tried to define T_pX in terms of curves $(-\varepsilon, \varepsilon) \to X$ based at p then we'd end up with a space of dimension $n-1$ rather than n (in fact, we'd get exactly $T_p\partial X$). This dimension thus gives a way to distinguish interior and boundary points independently of charts. To define T_pX correctly we should allow curves to be defined on $[0, \varepsilon)$ or $(-\varepsilon, 0]$ only. This is equivalent to considering curves in the codomains of charts which may 'cross the boundary' into $\mathbb{R}_{<0} \times \mathbb{R}^{n-1}$.

Definition 3.6.3. If X is oriented then we orient ∂X as follows. Given p is in ∂X, let $o_X \in \Lambda^n T_pX$ represent the orientation of X and let $\mathsf{n} \in T_pX$ denote any vector transverse to the boundary, pointing outwards. Then orient ∂X using the unique $o_{\partial X} \in \Lambda^{n-1} T_p\partial X$ satisfying

$$o_X = \mathsf{n} \wedge o_{\partial X}.$$

Theorem 3.6.4 (Stokes's Theorem). *If ω is a compactly supported $(n-1)$-form on X then*

$$\int_X \mathrm{d}\omega = \int_{\partial X} \omega.$$

(Formally the RHS means $\int_{\partial X} \iota^*\omega$, where $\iota : \partial X \hookrightarrow X$ is the inclusion.)

Proof. **Step 1:** Reduce to a coordinate patch.

Take a cover $\{U_\alpha\}$ of X by coordinate patches and let $\{\rho_\alpha\}$ be a partition of unity subordinate to this cover. We then have

$$\int_X \mathrm{d}\omega = \int_X \mathrm{d}\Big(\sum_\alpha \rho_\alpha \omega\Big) = \sum_\alpha \int_{U_\alpha} \mathrm{d}(\rho_\alpha \omega)$$

$$\int_{\partial X} \iota^*\omega = \int_{\partial X} \iota^*\Big(\sum_\alpha \rho_\alpha \omega\Big) = \sum_\alpha \int_{\partial U_\alpha} \iota^*(\rho_\alpha \omega).$$

So it's STP the result for each $\rho_\alpha \omega$, supported in a coordinate patch.

Step 2: Compute! Consider a compactly supported $(n-1)$-form

$$\omega = \sum_i \omega_i \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n$$

on $X = \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, WLOG oriented by $\partial_{x^1} \wedge \cdots \wedge \partial_{x^n}$. We have

$$\iota^* \omega = \omega_1 \, \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n.$$

The induced orientation on ∂X is $-\partial_{x^2} \wedge \cdots \wedge \partial_{x^n}$, since $-\partial_{x^1}$ is outward-pointing, so we get

$$\int_{\partial X} \iota^* \omega = -\int_{\{0\} \times \mathbb{R}^{n-1}} \omega_1 \, \mathrm{d}x^2 \dots \mathrm{d}x^n.$$

Meanwhile

$$\mathrm{d}\omega = \sum_{i,j} \frac{\partial \omega_i}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n = \sum_i (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

So

$$\begin{aligned} \int_X \mathrm{d}\omega = \sum_i (-1)^{i-1} \int_{\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}} \frac{\partial \omega_i}{\partial x^i} \mathrm{d}x^1 \dots \mathrm{d}x^n &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}_{\geq 0}} \frac{\partial \omega_1}{\partial x^1} \mathrm{d}x^1 \right) \mathrm{d}x^2 \dots \mathrm{d}x^n \\ &+ \sum_{i \geq 2} (-1)^{i-1} \int_{\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-2}} \left(\int_{\mathbb{R}} \frac{\partial \omega_i}{\partial x^i} \mathrm{d}x^i \right) \mathrm{d}x^1 \dots \widehat{\mathrm{d}x^i} \dots \mathrm{d}x^n \end{aligned}$$

Using the fundamental theorem of calculus, all terms vanish except the first, which is

$$-\int_{\{0\} \times \mathbb{R}^{n-1}} \omega_1 \, \mathrm{d}x^2 \dots \mathrm{d}x^n.$$

And this is what we want, so we're done. □

E.g. take $X = \{\mathsf{x} \in \mathbb{R}^2 : \|\mathsf{x}\| \leq a\}$ and $\omega = \mathrm{d}x \wedge \mathrm{d}y$. Then

$$\mathrm{Area}(X) = \int_X \omega = \frac{1}{2} \int_X \mathrm{d}(x \, \mathrm{d}y - y \, \mathrm{d}x) = \frac{1}{2} \int_{\partial X} x \, \mathrm{d}y - y \, \mathrm{d}x = \frac{1}{2} \int_{\partial X} r^2 \, \mathrm{d}\theta = \frac{a^2}{2} \int_{\partial X} \mathrm{d}\theta = \pi a^2.$$

Another way to think of the exterior derivative is 'the thing that makes Stokes true'.

3.7. **Applications of Stokes's theorem.**

Proposition 3.7.1 (Integration by parts)**.** *If α is a $(p-1)$-form on X, and β an $(n-p)$-form, and at least one of them is compactly supported, then*

$$\int_X (\mathrm{d}\alpha) \wedge \beta = \int_{\partial X} \alpha \wedge \beta + (-1)^p \int_X \alpha \wedge \mathrm{d}\beta$$

Proof. By Stokes we have

$$\int_X \mathrm{d}(\alpha \wedge \beta) = \int_{\partial X} \alpha \wedge \beta,$$

but by Leibniz we have

$$\int_X \mathrm{d}(\alpha \wedge \beta) = \int_X \big((\mathrm{d}\alpha) \wedge \beta + (-1)^{p-1} \alpha \wedge \mathrm{d}\beta \big).$$ □

Proposition 3.7.2. *If X is a compact oriented n-manifold (without boundary) then $\int_X$ defines a linear map*

$$\int_X : H^n_{\mathrm{dR}}(X) \to \mathbb{R}.$$

Proof. We just need to check that it's well-defined, i.e. that if $\alpha' = \alpha + \mathrm{d}\gamma$ then

$$\int_X \alpha' = \int_X \alpha.$$

And the difference between these is

$$\int_X \mathrm{d}\gamma,$$

which vanishes by Stokes since X has no boundary. □

Corollary 3.7.3. *If X is compact and orientable then $H^n_{\mathrm{dR}}(X) \neq 0$.*

Proof. Fix an orientation on X and let ω be a volume form representing the orientation. Then $[\omega]$ is a class in $H^n_{\mathrm{dR}}(X)$ and satisfies $\int_X [\omega] > 0$. □

4. Connections on vector bundles

Notation and terminology:

- Given a vector bundle $E \to B$, an *E-valued r-form* is a section of $E \otimes \Lambda^r T^* B$. At each point $b \in B$ it eats r vectors multilinearly and antisymmetrically, and spits out an element of E_b.
- Given a vector space V, a *V-valued form* on B is a section of $\underline{V} \otimes \Lambda^r T^* B$.
- Write $\Omega^r(E)$ for $\{E\text{-valued } r\text{-forms on } B\}$, and $\Gamma(E)$ for $\{\text{global sections of } E\} = \Omega^0(E)$.
- Write $\mathfrak{gl}(k, \mathbb{R})$ for the vector space of $k \times k$ real matrices.

4.1. **Connections.** Let $\pi : E \to B$ be a vector bundle of rank k and let s be a section. Under each trivialisation

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^k,$$

s becomes an $\mathbb{R}^k$-valued function which we denote by v_α. The naïve derivative is the $\mathbb{R}^k$-valued 1-form $\mathrm{d}\mathsf{v}_\alpha$, which we view as a local E-valued 1-form via Φ_α^{-1}. On overlaps we have $\mathsf{v}_\beta = g_{\beta\alpha}\mathsf{v}_\alpha$, so if we take the naïve derivative under the Φ_β trivialisation and then translate the answer to the Φ_α trivialisation, we get

$$g_{\beta\alpha}^{-1}\,\mathrm{d}(g_{\beta\alpha}\mathsf{v}_\alpha) = \mathrm{d}\mathsf{v}_\alpha + (g_{\beta\alpha}^{-1}\,\mathrm{d}g_{\beta\alpha})\mathsf{v}_\alpha.$$

So the answer is trivialisation-dependent via the action of the $\mathfrak{gl}(k, \mathbb{R})$-valued 1-form $g_{\beta\alpha}^{-1}\,\mathrm{d}g_{\beta\alpha}$ on v_α.

Definition 4.1.1. A *connection* $\mathcal{A}$ on E comprises a $\mathfrak{gl}(k, \mathbb{R})$-valued 1-form A_α on each trivialisation patch $U_\alpha \subset B$, such that on overlaps we have

$$A_\alpha = g_{\beta\alpha}^{-1}\,\mathrm{d}g_{\beta\alpha} + g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} \tag{4.1.1}$$

The *covariant derivative* of a section s with respect to a connection $\mathcal{A}$ is the E-valued 1-form $\mathrm{d}^{\mathcal{A}} s$ defined locally under Φ_α by $\mathrm{d}\mathsf{v}_\alpha + A_\alpha \mathsf{v}_\alpha$. This is independent of the choice of trivialisation by the above calculation. Say s is *horizontal* or *covariantly constant* with respect to $\mathcal{A}$ if $\mathrm{d}^{\mathcal{A}} s = 0$.

The A_α are called the *local connection 1-forms* of $\mathcal{A}$. Note that the zero section is always horizontal, but non-zero horizontal sections need not exist (even locally).

Example 4.1.2 (Trivial connection)**.** Given a global trivialisation Φ_α of E we can define $\mathcal{A}$ by $A_\alpha = 0$. For other trivialisations Φ_β, A_β is uniquely determined by (4.1.1). A section s is horizontal iff $\mathrm{pr}_2 \circ \Phi_\alpha(s)$ is locally constant.

Lemma 4.1.3. *For any connection $\mathcal{A}$ on $E \to B$, the covariant derivative*

$$\mathrm{d}^{\mathcal{A}} : \Gamma(E) \to \Omega^1(E)$$

is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$\mathrm{d}^{\mathcal{A}}(fs) = f\,\mathrm{d}^{\mathcal{A}} s + s \otimes \mathrm{d}f.$$

Conversely, any such $\mathbb{R}$-linear map satisfying the Leibniz rule is induced by a unique connection.

Proof. $\mathbb{R}$-linearity is clear. The Leibniz rule can be checked locally in a trivialisation Φ_α, where

$$\text{LHS} = \mathrm{d}(f\mathsf{v}_\alpha) + A_\alpha f \mathsf{v}_\alpha = f(\mathrm{d}\mathsf{v}_\alpha + A_\alpha \mathsf{v}_\alpha) + \mathsf{v}_\alpha \otimes \mathrm{d}f = \text{RHS}.$$

For the converse see Sheet 3. □

Example 4.1.4. Given a submanifold $\iota : X \hookrightarrow \mathbb{R}^N$, there is a trivial connection $\mathcal{A}_0$ on $\iota^* T\mathbb{R}^N$. Then

$$\mathcal{D} : \Gamma(TX) \hookrightarrow \Gamma(\iota^* T\mathbb{R}^N) \xrightarrow{\mathrm{d}^{\mathcal{A}_0}} \Omega^1(\iota^* T\mathbb{R}^N) \xrightarrow{\text{orthog proj } \iota^* T\mathbb{R}^N \to TX} \Omega^1(TX)$$

inherits $\mathbb{R}$-linearity and the Leibniz rule from $d^{\mathcal{A}_0}$ so defines a connection on TX.

Lemma 4.1.5 (Existence). *Any vector bundle E admits a connection.*

Proof. For each α define a $\mathfrak{gl}(k, \mathbb{R})$-valued 1-form on U_α by

$$A_\alpha = \sum_\gamma \rho_\gamma g_{\gamma\alpha}^{-1} \,\mathrm{d}g_{\gamma\alpha}.$$

On overlaps we have

$$\begin{aligned} g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} &= \sum_\gamma g_{\beta\alpha}^{-1} (\rho_\gamma g_{\gamma\beta}^{-1} \,\mathrm{d}g_{\gamma\beta}) g_{\beta\alpha} \\ &= \sum_\gamma \rho_\gamma g_{\gamma\alpha}^{-1} (\mathrm{d}(g_{\gamma\beta} g_{\beta\alpha}) - g_{\gamma\beta} \,\mathrm{d}g_{\beta\alpha}) \\ &= \sum_\gamma \rho_\gamma g_{\gamma\alpha}^{-1} \,\mathrm{d}g_{\gamma\alpha} - \sum_\gamma \rho_\gamma g_{\beta\alpha}^{-1} \,\mathrm{d}g_{\beta\alpha} \\ &= A_\alpha - g_{\beta\alpha}^{-1} \,\mathrm{d}g_{\beta\alpha}. \end{aligned}$$

So the A_α define a connection. □

4.2. **Connections vs** $\mathrm{End}(E)$.

Let E, U_α, and $g_{\beta\alpha}$ be as above. Let

$$\rho : \mathrm{GL}(k, \mathbb{R}) \to \mathrm{GL}(\mathfrak{gl}(k, \mathbb{R})) \cong \mathrm{GL}(k^2, \mathbb{R})$$

be the representation $\rho(A)(M) = AMA^{-1}$ for $A \in \mathrm{GL}(k, \mathbb{R})$ and $M \in \mathfrak{gl}(k, \mathbb{R})$.

Definition 4.2.1. $\mathrm{End}(E)$ is the vector bundle over B with total space

$$\coprod_{b \in B} \mathrm{End}(E_b).$$

It's trivialised over the U_α, with model fibres $\mathfrak{gl}(k, \mathbb{R}) \cong \mathbb{R}^{k^2}$ instead of $\mathbb{R}^k$, and with transition functions $\rho(g_{\beta\alpha})$. So a section M of $\mathrm{End}(E)$ is equivalent to a smooth map $M_\alpha : U_\alpha \to \mathfrak{gl}(k, \mathbb{R})$ for each α, satisfying $M_\beta = g_{\beta\alpha} M_\alpha g_{\beta\alpha}^{-1}$ on overlaps. (We could equally have defined $\mathrm{End}(E)$ as $E \otimes E^\vee$.)

Lemma 4.2.2 (Non-uniqueness). *If $\mathcal{A}$ is a connection on E and Δ is an $\mathrm{End}(E)$-valued 1-form then we can define a connection $\mathcal{A} + \Delta$ by local connection 1-forms $A_\alpha + \Delta_\alpha$. If $\mathcal{A}'$ is another connection then it can be written as $\mathcal{A} + \Delta$ for a unique such Δ. So the space of connections on E is naturally an affine space for $\Omega^1(\mathrm{End}(E))$, i.e. it carries a free transitive action of this vector space.*

Proof. The $A_\alpha + \Delta_\alpha$ satisfy

$$\begin{aligned} A_\alpha + \Delta_\alpha &= g_{\beta\alpha}^{-1} \,\mathrm{d}g_{\beta\alpha} + g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} + g_{\alpha\beta} \Delta_\beta g_{\alpha\beta}^{-1} \\ &= g_{\beta\alpha}^{-1} \,\mathrm{d}g_{\beta\alpha} + g_{\beta\alpha}^{-1} (A_\beta + \Delta_\beta) g_{\beta\alpha}, \end{aligned}$$

which is the required transformation law for a connection. Similarly $A'_\alpha - A_\alpha$ satisfies the transformation law for an $\mathrm{End}(E)$-valued 1-form Δ. Then $\mathcal{A}' = \mathcal{A} + \Delta$, and this Δ is obviously unique. □

4.3. **Curvature algebraically.**

Fix $E \to B$ with a connection $\mathcal{A}$.

Definition 4.3.1. The *exterior covariant derivative* $\mathrm{d}^{\mathcal{A}} : \Omega^\bullet(E) \to \Omega^{\bullet+1}(E)$ is the unique $\mathbb{R}$-linear extension of $\mathrm{d}^{\mathcal{A}} : \Gamma(E) \to \Omega^1(E)$ satisfying the Leibniz rule

$$\mathrm{d}^{\mathcal{A}}(\sigma \wedge \omega) = (\mathrm{d}^{\mathcal{A}}\sigma) \wedge \omega + (-1)^r \sigma \wedge \mathrm{d}\omega,$$

for E-valued r-forms σ and ordinary forms ω. In trivialisations an E-valued form σ becomes an $\mathbb{R}^k$-valued form σ_α, and

$$\mathrm{d}^{\mathcal{A}}\sigma = \mathrm{d}\sigma_\alpha + A_\alpha \wedge \sigma_\alpha.$$

Proposition 4.3.2. *There is a unique* $\mathrm{End}(E)$*-valued 2-form F such that for any E-valued form σ we have*

$$(\mathrm{d}^{\mathcal{A}})^2\sigma = F \wedge \sigma.$$

Proof. Locally in trivialisations $(\mathrm{d}^{\mathcal{A}})^2\sigma$ is

$$\begin{aligned} \mathrm{d}(\mathrm{d}\sigma_\alpha + A_\alpha \wedge \sigma_\alpha) + A_\alpha \wedge (\mathrm{d}\sigma_\alpha + A_\alpha \wedge \sigma_\alpha) &= (\mathrm{d}A_\alpha) \wedge \sigma_\alpha - A_\alpha \wedge \mathrm{d}\sigma_\alpha + A_\alpha \wedge \mathrm{d}\sigma_\alpha + A_\alpha \wedge A_\alpha \wedge \sigma_\alpha \\ &= (\mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha) \wedge \sigma_\alpha. \end{aligned}$$

The local $\mathfrak{gl}(k,\mathbb{R})$-valued 2-forms $F_\alpha := \mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha$ transform correctly to define an $\mathrm{End}(E)$-valued 2-form (Sheet 3). □

Definition 4.3.3. F is the *curvature* of $\mathcal{A}$. $\mathcal{A}$ is *flat* if $F = 0$.

Example 4.3.4. (i) Trivial connections are flat. Sheet 3: if $\mathcal{A}$ is flat then it's locally trivial.
(ii) Take $E = \underline{\mathbb{R}}^2 \to \mathbb{R} \times S^1$ with $\mathcal{A}$ given by

$$A_\alpha = f \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathrm{d}x + g \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{d}\theta$$

in the standard trivialisation. Then

$$\begin{aligned} F_\alpha &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathrm{d}f \wedge \mathrm{d}x + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{d}g \wedge \mathrm{d}\theta + fg \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \mathrm{d}x \wedge \mathrm{d}\theta + fg \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathrm{d}\theta \wedge \mathrm{d}x \\ &= \left[\frac{\partial f}{\partial \theta} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\partial g}{\partial x} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - 2fg \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \mathrm{d}x \wedge \mathrm{d}\theta. \end{aligned}$$

As defined, it's not clear what F means geometrically. We will see shortly that it is the local obstruction to the existence of horizontal sections. More precisely, given a point $p \in B$ and a vector $\mathsf{v}_0 \in E_p$, one can try to move v_0 horizontally (with respect to $\mathcal{A}$) around a small rectangle in B with one corner at p. The result will be an element $\mathsf{v}_1 \in E_p$, which will generally be different from v_0. This difference is captured by $-F(p)\mathsf{v}_0$, as made precise in Proposition 4.5.1. If there were a local horizontal section s defined near p, with $s(p) = \mathsf{v}_0$, then v_1 would have to agree with v_0, since 'moving v_0 horizontally' around a path in B just amounts to 'following the values of s over that path'. In this case we must therefore have $F(p)\mathsf{v}_0 = 0$. Our next task is to formalise these notions.

4.4. **Parallel transport.** Fix a vector bundle $E \to [0,1]$ with connection $\mathcal{A}$.

Lemma 4.4.1. *For each $s_0 \in E_0$ there exists a unique horizontal section s of E with $s(0) = s_0$. This s depends linearly on s_0.*

Definition 4.4.2. The *parallel transport* of s_0 from 0 to 1 is the element $s(1) \in E_1$. Since s depends linearly on s_0, parallel transport defines a linear map $E_0 \to E_1$.

Proof of Lemma 4.4.1. Locally the horizontal section equation is

$$\mathrm{d}\mathsf{v}_\alpha + A_\alpha \mathsf{v}_\alpha = 0.$$

Writing $A_\alpha = M_\alpha \otimes \mathrm{d}t$, where M_α is a $\mathfrak{gl}(k,\mathbb{R})$-valued function on $U_\alpha \subset [0,1]$, this becomes

$$\frac{\mathrm{d}\mathsf{v}_\alpha}{\mathrm{d}t} + M_\alpha \mathsf{v}_\alpha = 0.$$

By standard ODE theory, horizontal sections with given initial conditions exist locally and are unique (locally, hence globally). Since the equation is linear, solutions depend linearly on initial conditions.

It remains to prove global existence. For each $p \in [0,1]$ we can construct a local fibrewise basis of horizontal sections. By compactness of $[0,1]$ there exist $a_0 = 0 < a_1 < \cdots < a_N = 1$ such that on each

$[a_i, a_{i+1}]$ we have such a local fibrewise basis $s_i^1, \ldots, s_i^k$. The given $s_0 \in E_0$ can be written uniquely as

$$s_0 = \sum_{j=1}^{k} \lambda_{0j} s_0^j(0).$$

Then $\sum_j \lambda_{0j} s_0^j$ gives a solution s over $[a_0, a_1]$. Next write $s(a_1)$ as a linear combination $\sum_j \lambda_{1j} s_1^j(a_1)$. Then $\sum_j \lambda_{1j} s_1^j$ extends s over $[a_1, a_2]$. Keep going up to $[a_{N-1}, a_N]$. □

Now suppose $E \to B$ is over a general base, and $\gamma : [0,1] \to B$ is a smooth map. Recall that $\gamma^* E$ is trivialised over the $\gamma^{-1} U_\alpha$ with transition functions $\gamma^* g_{\beta\alpha}$. Given a connection $\mathcal{A}$ on E, we can thus define a connection $\gamma^* \mathcal{A}$ on $\gamma^* E$ via the $\mathfrak{gl}(k, \mathbb{R})$-valued 1-forms $\gamma^* A_\alpha$.

Definition 4.4.3. Given $s_0 \in E_{\gamma(0)}$, the *horizontal lift of* γ (with respect to $\mathcal{A}$) starting at s_0 is the unique horizontal section s of $\gamma^* E$ starting at s_0. *Parallel transport along* γ is the linear map $\mathcal{P}_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ given by $s_0 \mapsto s(1)$. If γ is a loop ($\gamma(0) = \gamma(1)$) then $\mathcal{P}_\gamma$ is the *holonomy* or *monodromy* of $\mathcal{A}$ around γ.

Example 4.4.4. (i) Consider S^2 with the 'orthogonally project from $\mathbb{R}^3$' connection. The horizontal lift of a path γ starting at $\mathsf{v}_0 \in T_{\gamma(0)} S^2$ is the unique map $\mathsf{v} : [0,1] \to TS^2$ such that:

- $\mathsf{v}(t) \in T_{\gamma(t)} S^2 \subset \mathbb{R}^3$ for all t
- $\dot{\mathsf{v}}(t)$, as a vector in $\mathbb{R}^3$, is orthogonal to $T_{\gamma(t)} S^2$ for all t (so its orthogonal projection, i.e the covariant derivative, is 0)
- $\mathsf{v}(0) = \mathsf{v}_0$.

(ii) For $\underline{\mathbb{R}}^2 \to \mathbb{R} \times S^1$ with

$$A_\alpha = f \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathrm{d}x + g \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{d}\theta,$$

parallel transport of v_0 along $\gamma(t) = (t, 0)$ satisfies $\dot{\mathsf{v}} + f \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \mathsf{v} = 0$ and $\mathsf{v}(0) = \mathsf{v}_0$, so

$$\mathsf{v}(t) = \begin{pmatrix} e^{-\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix} \mathsf{v}_0 \quad \text{where} \quad \lambda = \int_0^t f(x, 0)\, \mathrm{d}x.$$

Similarly, the monodromy around $\gamma(t) = (0, 2\pi t)$ is

$$\mathcal{P}_\gamma = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \quad \text{where} \quad \varphi = \int_0^{2\pi} g(0, \theta)\, \mathrm{d}\theta.$$

4.5. **Curvature geometrically.** Fix $E \to B$ with connection $\mathcal{A}$. Fix also a point $p \in B$, and a trivialisation Φ_α and choice of local coordinates x^i about p. We'll write A_α as $A_i \, \mathrm{d}x^i$ locally, where each A_i is a $\mathfrak{gl}(k, \mathbb{R})$-valued function. Similarly let F_α be $F_{ij} \, \mathrm{d}x^i \otimes \mathrm{d}x^j$.

For small a and b we can parallel transport from p distance a in the x^i-direction, then distance b in the x^j-direction, then back round the other two sides of the rectangle. Call the four segments γ_1, γ_2, γ_3, and γ_4 respectively. We obtain a map

$$\mathcal{P}_{a,b} = \mathcal{P}_{\gamma_4} \circ \mathcal{P}_{\gamma_3} \circ \mathcal{P}_{\gamma_2} \circ \mathcal{P}_{\gamma_1} : E_p \to E_p$$

Proposition 4.5.1. *Under the trivialisation* Φ_α *we have*

$$\left.\frac{\partial^2 \mathcal{P}_{a,b}}{\partial a \, \partial b}\right|_{a=b=0} = -F_{ij}(p).$$

Proof. The easiest way to prove this rigorously is using the Lie derivative (see Sheet 3). We'll give a direct sketch proof here, glossing over some analytic niceties. You can either think about how to fill these in (see Nicolaescu Proposition 3.3.14 if you want to check), or just view this as a heuristic argument that can be made precise later.

Under Φ_α the parallel transport equation in the x^i direction is $\dot{\mathsf{v}}_\alpha + A_i \mathsf{v}_\alpha = 0$. So

$$\mathcal{P}_{\gamma_1} = I - aA_i(p) + \ldots,$$

where ... denotes higher order terms that will wash out. Similarly

$$\mathcal{P}_{\gamma_2} = I - bA_j(q) + \cdots = I - b\left(A_j(p) + a\frac{\partial A_j}{\partial x^i}(p)\right) + \ldots,$$

where $q = \gamma_1(1) = \gamma_2(0)$. So

$$\mathcal{P}_{\gamma_2} \circ \mathcal{P}_{\gamma_1} = I - aA_i(p) - bA_j(p) - ab\left(\frac{\partial A_j}{\partial x^i}(p) - A_j(p)A_i(p)\right) + \ldots.$$

Doing similarly for γ_3 and γ_4, we get

$$\mathcal{P}_{a,b} = I - ab\left(\frac{\partial A_j}{\partial x^i}(p) - \frac{\partial A_i}{\partial x^j}(p) + A_i(p)A_j(p) - A_j(p)A_i(p)\right) + \cdots = I - abF_{ij}(p) + \ldots. \qquad \square$$

Corollary 4.5.2. *Fix $p \in B$ and $\mathsf{v}_0 \in E_p$. If $F(p)\mathsf{v}_0 \neq 0$, then there does not exist a local horizontal section s of E near p with $s(p) = \mathsf{v}_0$.*

Proof. If such a section s existed then for any path γ starting at p (and remaining sufficiently close to p) the section $s \circ \gamma$ would be *a* horizontal lift of γ starting at v_0, and therefore *the* horizontal lift. Hence the parallel transport of v_0 along γ is simply $s(\gamma(1))$. Applying this to γ_1 from Proposition 4.5.1, and then to γ_2, γ_3, and γ_4 in turn, we see that $\mathcal{P}_{a,b}\mathsf{v}_0 = \mathsf{v}_0$ for all small a and b, so $F(p)\mathsf{v}_0$ would be 0. $\square$

Example 4.5.3. Consider $\underline{\mathbb{R}} \to \mathbb{R}^2$ with $A_\alpha = C\, x^1\, \mathrm{d}x^2$, where C is constant, and let $p = 0$. Then

$$\mathcal{P}_{\gamma_1} = \mathcal{P}_{\gamma_3} = \mathcal{P}_{\gamma_4} = 1 \in \mathrm{GL}(1, \mathbb{R}) \quad \text{and} \quad \mathcal{P}_{\gamma_2} = e^{-Cab}.$$

So

$$\mathcal{P}_{a,b} = e^{-Cab} \quad \text{and hence} \quad \left.\frac{\partial^2 \mathcal{P}_{a,b}}{\partial a\, \partial b}\right|_{a=b=0} = -C.$$

Compare with

$$F_\alpha = C\, \mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

We can explicitly see the obstruction to existence of a horizontal section here, since if s (given by a smooth function v_α under our trivialisation) were horizontal then we'd have

$$\mathrm{d}\mathsf{v}_\alpha = -Cx^1\mathsf{v}_\alpha\, \mathrm{d}x^2, \quad \text{i.e.} \quad \frac{\partial \mathsf{v}_\alpha}{\partial x^1} = 0 \quad \text{and} \quad \frac{\partial \mathsf{v}_\alpha}{\partial x^2} = -Cx^1\mathsf{v}_\alpha.$$

This is consistent with symmetry of mixed partial derivatives only if $C = 0$ or $\mathsf{v}_\alpha = 0$.

Example 4.5.4. Consider $\underline{\mathbb{R}} \to S^1$ with $A_\alpha = C\, \mathrm{d}\theta$. Local horizontal sections exist and are of the form $Ke^{-C\theta}$. If $C \neq 0$ then there is no global horizontal section due to non-trivial monodromy around S^1.

Summary: curvature is the local obstruction to the existence of horizontal sections, monodromy is the global obstruction.

5. Flows and Lie derivatives

5.1. **Flows.** Fix an n-manifold X. Given a vector field v on X, and a point $p \in X$ one can try to flow along v from p, i.e. try to solve the ODE

$$\dot{\gamma}(t) = \mathsf{v}(\gamma(t)) \quad \text{with initial condition} \quad \gamma(0) = p.$$

Standard ODE theory says that for ε sufficiently small this has a unique solution $\gamma : (-\varepsilon, \varepsilon) \to X$, which depends smoothly on p, called an *integral curve* of v. Assembling these solutions for all p defines the flow of v.

Definition 5.1.1 (Non-standard)**.** A *flow domain* is an open neighbourhood U of $X \times \{0\}$ in $X \times \mathbb{R}$ such that for each $p \in X$ the set $U \cap (\{p\} \times \mathbb{R})$ is connected (i.e. is an open interval around 0).

Definition 5.1.2. A *local flow* of v is a smooth map $\Phi : U \to X$, where U is a flow domain, satisfying

- $\Phi(-, 0) = \mathrm{id}_X$.

- $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(p,t) = \mathsf{v}(\Phi(p,t))$ for all $(p,t) \in U$.

We'll write Φ^t for $\Phi(-,t)$.

The above ODE discussion shows that local flows always exist, and are unique in the sense that any two local flows coincide on the intersection of their domains.

Lemma 5.1.3. *Any local flow $\Phi : U \to X$ of v satisfies $\Phi^s \circ \Phi^t = \Phi^{s+t}$ wherever this makes sense. So in particular $\Phi^t = (\Phi^{t/N})^{\circ N}$ and $\Phi^{-t} = (\Phi^t)^{-1}$ where these make sense.*

Proof. Fix p and t and let $q = \Phi^t(p)$. Then the paths

$$\gamma_1(s) := \Phi^{s+t}(p) \quad \text{and} \quad \gamma_2(s) := \Phi^s(q)$$

are integral curves of v beginning at q. They must therefore coincide, by uniqueness. □

A vector field is *complete* if it admits a global flow, i.e. a flow defined on $X \times \mathbb{R}$. Not all vector fields are complete (e.g. $x^2\partial_x$ on $\mathbb{R}$) but those with compact support are: you can build a local flow Φ on $(-\varepsilon, \varepsilon) \times X$ for some $\varepsilon > 0$ and then define Φ^t to be $(\Phi^{t/N})^{\circ N}$ for $N \gg 0$.

5.2. **The Lie derivative.** Fix a vector field v with local flow Φ.

Definition 5.2.1. The *Lie derivative* of a tensor T along v, denoted $\mathcal{L}_{\mathsf{v}}T$, is

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi^t)^* T.$$

This is a tensor of the same type as T and depends on Φ only via v. It measures how $T_{\Phi^t(p)}$ differs from the transport of T_p along the flow.

Lemma 5.2.2. *For general t we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^t)^*T = (\Phi^t)^*\mathcal{L}_{\mathsf{v}}T.$$

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^t)^*T = \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0} (\Phi^{t+h})^*T = \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0} (\Phi^h \circ \Phi^t)^*T = (\Phi^t)^* \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0} (\Phi^h)^*T = (\Phi^t)^*\mathcal{L}_{\mathsf{v}}T. \qquad \square$$

Lemma 5.2.3. *For a function f on X we have $\mathcal{L}_{\mathsf{v}}f = \mathrm{d}f(\mathsf{v})$, whilst for a 1-form α we have*

$$\mathcal{L}_{\mathsf{v}}\alpha = \left(v^j \frac{\partial \alpha_i}{\partial x^j} + \alpha_j \frac{\partial v^j}{\partial x^i}\right) \mathrm{d}x^i$$

in local coordinates.

Proof. At an arbitrary point p in X we have

$$\mathcal{L}_{\mathsf{v}}f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi^t)^*f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\Phi^t(p)) = \mathrm{d}f(\mathsf{v}(p)).$$

For the second part, on a coordinate patch we have

$$\begin{aligned}
\mathcal{L}_{\mathsf{v}}\alpha &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\alpha_i \circ \Phi^t)\, \mathrm{d}(x^i \circ \Phi^t) \\
&= (\mathcal{L}_{\mathsf{v}}\alpha_i)\, \mathrm{d}x^i + \alpha_i\, \mathrm{d}(\mathcal{L}_{\mathsf{v}}x^i) \\
&= v^j \frac{\partial \alpha_i}{\partial x^j}\, \mathrm{d}x^i + \alpha_i\, \mathrm{d}v^i.
\end{aligned} \qquad \square$$

You can find formulae for the Lie derivatives of other tensors from the above using the following.

Lemma 5.2.4. *For a vector field w and 1-form α we have*

$$\mathcal{L}_{\mathsf{v}}(w^i\alpha_i) = (\mathcal{L}_{\mathsf{v}}\mathsf{w})^i\alpha_i + w^i(\mathcal{L}_{\mathsf{v}}\alpha)_i.$$

For tensors S and T of arbitrary types we have

$$\mathcal{L}_{\mathsf{v}}(S \otimes T) = (\mathcal{L}_{\mathsf{v}}S) \otimes T + S \otimes (\mathcal{L}_{\mathsf{v}}T).$$

Proof. Pullback commutes with contraction and $\otimes$, then use the usual product rule. ☐

Corollary 5.2.5. *For vector fields* v *and* w *we have*

$$\mathcal{L}_{\mathsf{v}}\mathsf{w} = \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}\right) \frac{\partial}{\partial x^i}.$$

Proof. For any 1-form α we know from Lemma 5.2.4 that

$$\mathcal{L}_{\mathsf{v}}(w^i \alpha_i) = (\mathcal{L}_{\mathsf{v}}\mathsf{w})^i \alpha_i + w^i (\mathcal{L}_{\mathsf{v}}\alpha)_i.$$

Expressing the first and last terms using Lemma 5.2.3 gives

$$v^j \frac{\partial (w^i \alpha_i)}{\partial x^j} = (\mathcal{L}_{\mathsf{v}}\mathsf{w})^i \alpha_i + w^i \Big(v^j \frac{\partial \alpha_i}{\partial x^j} + \alpha_j \frac{\partial v^j}{\partial x^i}\Big).$$

Now expand the LHS using Leibniz, and cancel the $v^j w^i \frac{\partial \alpha_i}{\partial x^j}$ from both sides. The result is the equation we want to prove, contracted with α. Since it holds for all α we can remove this contraction. ☐

Definition 5.2.6. The *Lie bracket* of vector fields v and w is

$$[\mathsf{v}, \mathsf{w}] := \mathcal{L}_{\mathsf{v}}\mathsf{w} = -\mathcal{L}_{\mathsf{w}}\mathsf{v}.$$

This makes the space of vector fields on X into a *Lie algebra*: a vector space equipped with a bilinear bracket operation that is *alternating* ($[\mathsf{v}, \mathsf{v}] = 0$ for all v), and satisfies the *Jacobi identity*

$$[\mathsf{u}, [\mathsf{v}, \mathsf{w}]] + [\mathsf{v}, [\mathsf{w}, \mathsf{u}]] + [\mathsf{w}, [\mathsf{u}, \mathsf{v}]] = 0$$

for all u, v, and w.

Another important property of the Lie derivative is diffeomorphism-invariance.

Lemma 5.2.7. *If $F : X \to Y$ is a diffeomorphism,* v *is a vector field on Y, and T is a tensor on Y, then*

$$F^*(\mathcal{L}_{\mathsf{v}}T) = \mathcal{L}_{F^*\mathsf{v}}(F^*T).$$

Proof. Let Φ be a local flow of v. Then $F^{-1} \circ \Phi \circ F$ is a local flow of $F^*\mathsf{v}$ so

$$F^* \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi^t)^* T = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi^t \circ F)^* T = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (F^{-1} \circ \Phi^t \circ F)^*(F^*T) = \mathcal{L}_{F^*\mathsf{v}} F^*T.$$ ☐

5.3. **Homotopy invariance of de Rham cohomology.** Exterior and Lie derivatives are related in the following way.

Definition 5.3.1. Given a vector field v and r-form α, $\iota_{\mathsf{v}}\alpha$ or $\mathsf{v} \lrcorner \alpha$ is the $(r-1)$-form given by

$$(\iota_{\mathsf{v}}\alpha)_{i_1 \dots i_{r-1}} = v^j \alpha_{j i_1 \dots i_{r_1}}.$$

Proposition 5.3.2 (Cartan's magic formula)**.**

$$\mathcal{L}_{\mathsf{v}}\alpha = \mathrm{d}(\iota_{\mathsf{v}}\alpha) + \iota_{\mathsf{v}}(\mathrm{d}\alpha).$$

Proof. Example Sheet 3. ☐

Recall Proposition 3.2.6: If $F_0, F_1 : X \to Y$ are smooth maps which are *homotopic* then F_0^* and F_1^* induce the same map $H^\bullet_{\mathrm{dR}}(Y) \to H^\bullet_{\mathrm{dR}}(X)$. We can now prove this.

Proof. Let $F = F_t : [0,1] \times X \to Y$ be a homotopy from F_0 to F_1, and let Φ be the flow of ∂_t on $[0,1] \times X$, which is just translation in the $[0,1]$-direction. Write $i_t : X \to [0,1] \times X$ for the inclusion of

$\{t\} \times X$. Then $F_t = F \circ i_t = F \circ \Phi^t \circ i_0$ so for any r-form α on Y we have

$$\begin{aligned} F_1^*\alpha - F_0^*\alpha &= \int_0^1 \Big(\frac{\mathrm{d}}{\mathrm{d}t} F_t^*\alpha \Big) \, \mathrm{d}t \\ &= \int_0^1 \Big(\frac{\mathrm{d}}{\mathrm{d}t} i_0^*(\Phi^t)^* F^*\alpha \Big) \, \mathrm{d}t \\ &= \int_0^1 i_0^*(\Phi^t)^* \big(\mathcal{L}_{\partial_t}(F^*\alpha) \big) \, \mathrm{d}t = \int_0^1 i_t^* \mathcal{L}_{\partial_t}(F^*\alpha) \, \mathrm{d}t. \end{aligned}$$

Now suppose α is closed. Then Cartan's magic formula gives

$$\mathcal{L}_{\partial_t}(F^*\alpha) = \mathrm{d}(\iota_{\partial_t} F^*\alpha) + \iota_{\partial_t}\mathrm{d}(F^*\alpha) = \mathrm{d}(\iota_{\partial_t} F^*\alpha).$$

Hence

$$\begin{aligned} F_1^*\alpha - F_0^*\alpha &= \int_0^1 i_t^* \mathrm{d}(\iota_{\partial_t} F^*\alpha) \, \mathrm{d}t \\ &= \mathrm{d} \int_0^1 i_t^*(\iota_{\partial_t} F^*\alpha) \, \mathrm{d}t \end{aligned}$$

So $F_1^*\alpha - F_0^*\alpha$ is exact, and hence $[F_0^*\alpha] = [F_1^*\alpha]$. □

6. Foliations and Frobenius integrability

6.1. **Foliations.** If $F : X \to Y$ is a submersion then X naturally decomposes into 'slices' $F^{-1}(q)$ which are all submanifolds of dimension $\dim X - \dim Y$. A k-foliation is a structure on X which locally decomposes it into slices like this, of dimension k, but where the slices may not globally fit together into submanifolds.

Example 6.1.1. Given any $\alpha \in \mathbb{R}$ we can locally slice up $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ into curves of slope α. The local slices fit together into submanifolds if and only if α is rational.

Definition 6.1.2. An atlas $\{\varphi_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha\}$ on X is *(k-)foliated* if transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ locally have the form

$$(\mathsf{x} \in \mathbb{R}^k, \mathsf{y} \in \mathbb{R}^{n-k}) \mapsto (\zeta(\mathsf{x}, \mathsf{y}), \eta(\mathsf{y})).$$

Two foliated atlases are *equivalent* if their union is also foliated. A *foliation* on X is an equivalence class of foliated atlas. We usually write coordinates arising from a foliated atlas as $x^1, \dots, x^k, y^1, \dots, y^{n-k}$.

The local slices are given by $\mathsf{y} =$ constant. In an overlapping chart, the y coordinates may be mixed up, but their level sets will locally be the same.

Example 6.1.3. If $F : X \to Y$ is a submersion then the foliation into slices $F^{-1}(q)$ is defined by charts in which F is projection onto the last $n-k$ coordinates.

6.2. **Distributions.** Fix an n-manifold X.

Definition 6.2.1. A *k-plane distribution* D on X is a rank-k subbundle of TX.

Example 6.2.2. $\langle \partial_x, \partial_y \rangle$ and $\langle \partial_x + y\,\partial_z, \partial_y \rangle$ are 2-plane distributions on $\mathbb{R}^3$, which we will keep as our running examples. They can also be described as $\ker \mathrm{d}z$ and $\ker(\mathrm{d}z - y\,\mathrm{d}x)$ respectively. Any k-plane distribution is locally the kernel of $n-k$ fibrewise linearly independent 1-forms.

Example 6.2.3. If X is k-foliated, with local coordinates $x^1, \dots, x^k, y^1, \dots, y^{n-k}$, then $\langle \partial_{x^1}, \dots, \partial_{x^k} \rangle$ is a k-plane distribution on X—the *tangent distribution* of the foliation.

Definition 6.2.4. A distribution is *integrable* if it arises from a foliation in this way.

Integrability is automatic if $k = 1$, since locally $D = \langle \mathsf{v} \rangle$ for a vector field v and then X is foliated by integral curves of v.

6.3. **Frobenius integrability.** Fix a k-plane distribution D on X.

Theorem 6.3.1 (Frobenius integrability)**.** *D is integrable iff it's closed under the Lie bracket, i.e. for all vector fields v and w lying in D the Lie bracket $[\mathsf{v}, \mathsf{w}]$ lies in D.*

Proof. Both conditions are local, so it suffices to prove they are equivalent locally about each point p.

Suppose D is integrable. Then about p there exist local coordinates

$$x^1, \dots, x^k, y^1, \dots, y^{n-k}$$

such that $D = \langle \partial_{x^1}, \dots, \partial_{x^k} \rangle$. This is manifestly closed under the Lie bracket.

Conversely, assume D is closed under $[\cdot, \cdot]$. About p we can take local coordinates

$$s^1, \dots, s^k, t^1, \dots, t^{n-k}$$

in which $p = 0$, and such that at p we have $D = \langle \partial_{s^1}, \dots, \partial_{s^k} \rangle$. By shrinking the coordinate patch we may assume that for $i = 1, \dots, k$ there exist (unique) smooth a_{ij} such that

$$\mathsf{v}_i := \partial_{s^i} + \sum_j a_{ij}\, \partial_{t^j} \quad \text{lies in } D.$$

We then have $D = \langle \mathsf{v}_1, \dots, \mathsf{v}_k \rangle$, so it suffices to construct coordinates $x^1, \dots, x^k, y^1, \dots, y^{n-k}$ so that $\partial_{x^i} = \mathsf{v}_i$, and this is what we'll do.

Let Φ_i be a local flow of v_i, and define F from an open neighbourhood of 0 in $\mathbb{R}^n$ to an open neighbourhood of p in X by

$$F : (x^1, \dots, x^k, y^1, \dots, y^{n-k}) \mapsto \Phi_1^{x^1} \circ \cdots \circ \Phi_k^{x^k}(\mathsf{s} = 0, \mathsf{t} = \mathsf{y}).$$

This satisfies $D_0F(\partial_{x^i}) = \mathsf{v}_i(p) = \partial_{s^i}$ and $D_0F(\partial_{y^i}) = \partial_{t^i}$, so D_0F is an isomorphism and hence F is a parametrisation about p.

It remains to show that $\partial_{x^i} = \mathsf{v}_i$ (so far we only saw this holds at p itself). If the Φ_j commuted with each other for small times then we'd be done, since then we'd have

$$\begin{aligned}
\partial_{x^i} &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_1^{x^1} \circ \cdots \circ \Phi_i^{x^i+t} \circ \cdots \circ \Phi_k^{x^k}(\mathsf{s} = 0, \mathsf{t} = \mathsf{y}) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_i^{x^i+t} \circ \Phi_1^{x^1} \circ \cdots \circ \Phi_k^{x^k}(\mathsf{s} = 0, \mathsf{t} = \mathsf{y}) \\
&= \mathsf{v}_i.
\end{aligned}$$

To prove that the Φ_j do indeed commute for small times, it suffices (Example Sheet 3) to show that $[\mathsf{v}_i, \mathsf{v}_j] = 0$ for all i and j. And to prove the latter, note that by closure of D under the bracket there exist (unique) smooth b_{ijl} such that

$$[\mathsf{v}_i, \mathsf{v}_j] = \sum_l b_{ijl} \mathsf{v}_l.$$

By equating coefficients of ∂_{s^l} on both sides we get that all b_{ijl} vanish, so we're done. □

Example 6.3.2. On $\mathbb{R}^3$, $\langle \partial_x, \partial_y \rangle$ is integrable since it's tangent to the foliation by $z = \text{const}$. However, $D := \langle \partial_x + y\,\partial_z, \partial_y \rangle$ is not integrable, since $[\partial_x + y\,\partial_z, \partial_y] = -\partial_z$, which is not in D. We can also argue directly: if D were locally tangent to $f = \text{const}$ then we'd have

$$\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial z} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0,$$

so

$$0 = \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial z} + y\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial f}{\partial z},$$

and hence also

$$\frac{\partial f}{\partial x} = -y\frac{\partial f}{\partial z} = 0,$$

so f must be constant.

7. CONNECTIONS ON VECTOR BUNDLES WITH EXTRA STRUCTURE

7.1. **The tangent bundle.** Suppose $\mathcal{A}$ is a connection on $E = TX \to X$. Given a choice of local coordinates, $x^1, \ldots, x^n$ on X, we get a trivialisation Φ_α of E by $\partial_{x^1}, \ldots, \partial_{x^n}$. We call this a *coordinate trivialisation* and typically write the components of the local connection 1-form as $\Gamma^i{}_{jk}$, where k is the 1-form index and the i and j are the $\mathfrak{gl}(n, \mathbb{R})$ indices. For a vector field v we then have

$$(\mathrm{d}^{\mathcal{A}}\mathsf{v})^i = \mathrm{d}v^i + \Gamma^i{}_{jk}v^j\,\mathrm{d}x^k.$$

Warning! The $\Gamma^i{}_{jk}$ do not transform like the components of a tensor of type $(1, 2)$. But the space of connections is an affine space for $\mathrm{End}(E)$-valued 1-forms, i.e. sections of

$$E \otimes E^\vee \otimes T^*X = TX \otimes T^*X \otimes T^*X,$$

i.e. tensors of type $(1, 2)$.

Definition 7.1.1. The *solder form* is the E-valued 1-form θ given by the fibrewise identity map under the identifications

$$E \otimes T^*X = TX \otimes T^*X = \mathrm{End}(TX).$$

In coordinate trivialisations it is $\mathsf{e}_i \otimes \mathrm{d}x^i$, where e_i is the ith standard basis vector in $\mathbb{R}^k$. The *torsion* T of $\mathcal{A}$ is the E-valued 2-form $\mathrm{d}^{\mathcal{A}}\theta$, given in coordinate trivialisations by

$$\mathrm{d}(\mathsf{e}_i \otimes \mathrm{d}x^i) + A_\alpha \wedge (\mathsf{e}_l \otimes \mathrm{d}x^l) = \Gamma^i{}_{lk}\,\mathsf{e}_i \otimes \mathrm{d}x^k \wedge \mathrm{d}x^l.$$

$\mathcal{A}$ is *torsion-free* if $T = 0$, i.e. iff $\Gamma^i{}_{jk}$ is symmetric in j and k. Sometimes such connections are called *symmetric*. See Sheet 4 for interpretations of torsion.

Proposition 7.1.2 ((First) Bianchi identity)**.** *We have* $\mathrm{d}^{\mathcal{A}}T = F \wedge \theta$.

Proof. We have $\mathrm{d}^{\mathcal{A}}T = (\mathrm{d}^{\mathcal{A}})^2\theta = F \wedge \theta$. □

Definition 7.1.3. A curve γ in X is a *geodesic* (with respect to $\mathcal{A}$) if $\dot\gamma$ is horizontal as a section of γ^*TX, i.e. iff the *geodesic equation* $\ddot\gamma^i + \Gamma^i{}_{jk}\dot\gamma^j\dot\gamma^k = 0$ holds.

Note also that:

- By the associated bundle construction (Sheet 3 Q4.), $\mathcal{A}$ induces connections T^*X and on all bundles of tensors and forms. The covariant derivative of θ as a tensor of type $(1, 1)$ is automatically 0.
- The curvature F is an $\mathrm{End}(TX)$-valued 2-form, which we can view as a tensor $F^i{}_{jkl}$ of type $(1, 3)$ which is antisymmetric in k and l.
- People often write the covariant derivative $\mathrm{d}^{\mathcal{A}}$ (or even $\mathcal{A}$ itself) as ∇, and its contraction with a vector (or vector field) v as ∇_{v}.

7.2. **Orthogonal vector bundles.** Let $E \to B$ be a vector bundle of rank k.

Definition 7.2.1. An *inner product* on E is a section g of $(E^\vee)^{\otimes 2}$ that is fibrewise symmetric and positive definite as a fibrewise bilinear form on E.

Lemma 7.2.2. *E admits an inner product.*

Proof. Cover E by trivialisations Φ_α. Define an inner product g_α on $E|_{U_\alpha}$ by taking the standard inner product on $\mathbb{R}^k$ under Φ_α. Now take a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$ and let $g = \sum_\alpha \rho_\alpha g_\alpha$. □

Definition 7.2.3. An *orthogonal vector bundle* is a vector bundle E equipped with an inner product g. An *othogonal trivialisation* is a trivialisation in which g becomes the standard inner product on $\mathbb{R}^k$.

Fix an inner product g on E.

Lemma 7.2.4. *E can be covered by orthogonal trivialisations. (I.e. g is locally trivial.)*

Proof. Locally we can trivialise E by a fibrewise basis of sections $s_1, \ldots, s_k$. By applying the Gram–Schmidt process fibrewise, we may assume the s_i are orthonormal. Then the corresponding trivialisation is orthogonal. □

Definition 7.2.5. A connection $\mathcal{A}$ on E is *orthogonal* if g is covariantly constant using the induced connection on $(E^\vee)^{\otimes 2}$.

Lemma 7.2.6. *Orthogonal connections exist and form an affine space for $\Omega^1(\mathfrak{o}(E))$, where $\mathfrak{o}(E)$ denotes the bundle of skew-adjoint endomorphisms of the fibres of E.*

Proof. Example Sheet 4. □

Lemma 7.2.7. *The curvature of an orthogonal connection on E is an $\mathfrak{o}(E)$-valued 2-form.*

Proof. Example Sheet 4. □

8. Riemannian geometry

8.1. Riemannian metrics.

Fix an n-manifold X.

Definition 8.1.1. A *(Riemannian) metric* on X is an inner product on $TX \to X$. A *Riemannian manifold* (X, g) is a manifold X equipped with a Riemannian metric g.

Since every vector bundle admits an inner product, every manifold admits a Riemannian metric.

Given a metric g_{ij} on X, write g^{ij} for the dual metric on T^*X. This satisfies (and is defined by) $g^{ij} = g^{ji}$ and $g^{ij} g_{jk} = \delta^i{}_k$. We denote contraction with g_{ij} or g^{ij} by raising or lowering indices:

$$g_{ij} T^{kj}{}_l = T^k{}_{il} \quad \text{and} \quad g^{ij} S_{jk} = S^i{}_k.$$

This corresponds to applying the isomorphism $TX \to T^*X$ induced by g_{ij} or the inverse isomorphism $T^*X \to TX$ induced by g^{ij}. Note that a section $T^i{}_j$ of $\mathrm{End}(TX)$ lies in $\mathfrak{o}(TX)$ iff

$$T^i{}_j g_{ik} + g_{ji} T^i{}_k = 0 \quad \text{i.e.} \quad T_{ij} = -T_{ji}.$$

When writing component expressions, we write $\mathrm{d}x^i \mathrm{d}x^j$ for $\frac{1}{2}(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i)$. E.g. the standard metric g_{Eucl} on $\mathbb{R}^n$ is $\sum (\mathrm{d}x^i)^2$.

8.2. The Levi-Civita connection.

Let (X, g) be a Riemannian manifold.

Theorem 8.2.1 (Fundamental Theorem of Riemannian Geometry)**.** *(X, g) admits a unique torsion-free orthogonal connection.*

Proof. We'll prove the more general statement that the map

$$\{\text{orthogonal connections on } TX\} \to \Omega^2(TX),$$

given by sending a connection to its torsion, is a bijection.

Fix an orthogonal connection $\mathcal{A}_0$ on TX. Every other orthogonal connection $\mathcal{A}$ can be written uniquely as $\mathcal{A}_0 + \Delta$ for an $\mathfrak{o}(TX)$-valued 1-form Δ. We'll actually consider the map $\Delta \mapsto T_{\mathcal{A}_0+\Delta} - T_{\mathcal{A}_0}$ instead, since this is linear. It sends Δ to $\Delta^i{}_{kj} - \Delta^i{}_{jk} = \Delta \wedge \theta$, so it's induced by a bundle morphism $F : \mathfrak{o}(TX) \otimes T^*X \to TX \otimes \Lambda^2 T^*X$ given by fibrewise wedging with θ. It's enough to show that F is an isomorphism, which we can check fibrewise.

Both $\mathfrak{o}(TX) \otimes T^*X$ and $TX \otimes \Lambda^2 T^*X$ have rank $n\binom{n}{2}$, as they are

$$\{\Delta^i{}_{jk} : \Delta_{ijk} = -\Delta_{jik}\} \quad \text{and} \quad \{T^i{}_{jk} : T^i{}_{jk} = -T^i{}_{kj}\}.$$

So it suffices to check that F is fibrewise injective, i.e. that if Δ satisfies $\Delta_{ijk} = -\Delta_{jik}$ and ${\Delta^i}_{kj} - {\Delta^i}_{jk} = 0$ then $\Delta = 0$. But if these two conditions hold then

$$\Delta_{ijk} = -\Delta_{jik} = -\Delta_{jki} = \Delta_{kji} = \Delta_{kij} = -\Delta_{ikj} = -\Delta_{ijk}. \qquad \square$$

Definition 8.2.2. This is the *Levi-Civita* connection. Its components ${\Gamma^i}_{jk}$ are called *Christoffel symbols*, and are given explicitly by

$$\Gamma_{ijk} = \tfrac{1}{2}(\partial_j g_{ik} + \partial_k g_{ji} - \partial_i g_{jk}).$$

Proposition 8.2.3. *If $\iota : X \to \mathbb{R}^N$ is an embedded submanifold then X inherits a metric $\iota^* g_{\mathrm{Eucl}}$ and hence a Levi-Civita connection. This coincides with the 'orthogonally project from $\iota^* T\mathbb{R}^N$' connection, so the latter depends on ι only via $\iota^* g_{\mathrm{Eucl}}$.*

Proof. Example Sheet 4. $\square$

8.3. **The Riemann tensor.** Fix a Riemannian metric (X, g) with its Levi-Civita connection ∇.

Definition 8.3.1. The curvature of ∇ is the *Riemann tensor* ${R^i}_{jkl}$. This is an $\mathfrak{o}(TX)$-valued 2-form, viewed as a tensor of type $(1, 3)$.

It has the following properties:

- ${R^i}_{jkl} = -{R^i}_{jlk}$ since it's a 2-form.
- $R_{ijkl} = -R_{jikl}$ since it's $\mathfrak{o}(TX)$-valued.
- First Bianchi identity: $R \wedge \theta = 0$, which amounts to

$$ {R^i}_{jkl} + {R^i}_{klj} + {R^i}_{ljk} = 0.$$

- Second Bianchi identity: $\mathrm{d}^\nabla R = 0$.

The first and fourth hold for any connection on TX. The second uses orthogonality of ∇, and the third uses torsion-freeness.

8.4. **Hodge theory.** Let (X, g) be an oriented Riemannian n-manifold. The metric g induces inner products on $\Lambda^p T^* X$ for all p: if $\alpha^1, \dots, \alpha^n$ are a local fibrewise orthonormal basis of 1-forms then the α^I are a local fibrewise orthonormal basis of p-forms. There is thus a distinguished positively oriented unit volume form ω. Given a p-form β there is a unique $(n-p)$-form $\star\beta$ such that for all p-forms α we have

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega.$$

Explicitly, $\star\alpha^I = \pm\alpha^J$, where $J = \{1, \dots, n\} \setminus I$. Assuming the α^i are positively oriented, the sign is $+$ iff I, J is an even permutation of $\{1, \dots, n\}$.

Definition 8.4.1. The map $\star : \Omega^p(X) \to \Omega^{n-p}(X)$ is the *Hodge star operator*. Considering its action on the α^I shows that it's a fibrewise linear isometry and squares to $(-1)^{p(n-p)} \mathrm{id}_{\Omega^p(X)}$.

Example 8.4.2. Take $\mathbb{R}^3$ with the standard metric and orientation, so $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. Then

$$\star\,\mathrm{d}x^1 = \mathrm{d}x^2 \wedge \mathrm{d}x^3 \quad \text{and} \quad \star(\mathrm{d}x^1 \wedge \mathrm{d}x^2) = \mathrm{d}x^3$$

and cyclically.

Assume from now on the X is compact. This lets us define an inner product on each $\Omega^p(X)$ by

$$\langle \alpha, \beta \rangle_X = \int_X \langle \alpha, \beta \rangle \omega = \int_X \alpha \wedge \star\beta.$$

For any $(p-1)$-form α and p-form β we then have

$$\begin{aligned}
\langle \mathrm{d}\alpha, \beta\rangle_X &= \int_X (\mathrm{d}\alpha) \wedge \star\beta \\
&= \int_X \Big(\mathrm{d}(\alpha \wedge \star\beta) - (-1)^{p-1}\alpha \wedge (\mathrm{d}\star\beta)\Big) \quad \text{by Leibniz} \\
&= (-1)^p \int_X \alpha \wedge (\mathrm{d}\star\beta) \quad \text{by Stokes} \\
&= (-1)^p \langle \alpha, \star^{-1}\mathrm{d}\star\beta\rangle_X.
\end{aligned}$$

So $\delta \coloneqq (-1)^p \star^{-1}\mathrm{d}\star = (-1)^{np+n+1}\star\mathrm{d}\star$ is adjoint to d.

Definition 8.4.3. $\delta : \Omega^\bullet(X) \to \Omega^{\bullet-1}(X)$ is the *codifferential* (d is the *differential*). α is *coclosed* if $\delta\alpha = 0$, *coexact* if $\alpha = \delta\beta$ for some β.

Notice $\delta^2 = -\star^{-1}\mathrm{d}\star\star^{-1}\mathrm{d}\star = -\star\mathrm{d}^2\star = 0$.

Definition 8.4.4. The *Laplace–Beltrami operator* $\Omega^\bullet(X) \to \Omega^\bullet(X)$ is

$$\Delta \coloneqq \mathrm{d}\delta + \delta\mathrm{d} = (\mathrm{d}+\delta)^2.$$

A form satisfying $\Delta\alpha = 0$ is *harmonic*. This is equivalent to α being closed and coclosed (Sheet 4). We denote the space of harmonic p-forms by $\mathcal{H}^p(X)$.

Theorem 8.4.5. *The map $\mathcal{H}^p(X) \to H^p_{\mathrm{dR}}(X)$, $\alpha \mapsto [\alpha]$ is an isomorphism, i.e. every de Rham cohomology class has a unique harmonic representative.*

Intuition: $\mathcal{H}^p(X) = \ker\Delta = (\ker\mathrm{d}) \cap (\ker\delta) = (\ker\mathrm{d}) \cap (\operatorname{im}\mathrm{d})^\perp \xrightarrow{\sim} \ker\mathrm{d}/\operatorname{im}\mathrm{d} = H^p_{\mathrm{dR}}(X)$.

To give a rigorous proof we'll use the following analytic result.

Theorem 8.4.6 (Hodge Decomposition)**.** *For all p the space $\mathcal{H}^p(X)$ of harmonic forms is finite-dimensional and we have orthogonal decompositions*

$$\begin{aligned}
\Omega^p(X) &= \mathcal{H}^p(X) \oplus \Delta\Omega^p(X) \\
&= \mathcal{H}^p(X) \oplus \mathrm{d}\delta\Omega^p(X) \oplus \delta\mathrm{d}\Omega^p(X) \\
&= \mathcal{H}^p(X) \oplus \mathrm{d}\Omega^{p-1}(X) \oplus \delta\Omega^{p+1}(X).
\end{aligned}$$

Proof. See Section 10.4.3 in Nicolaescu. □

Proof of Theorem 8.4.5. We have an orthogonal Hodge decomposition

$$\Omega^p(X) = \mathcal{H}^p(X) \oplus \mathrm{d}\Omega^{p-1}(X) \oplus \delta\Omega^{p+1}(X).$$

It suffices to show $\ker\mathrm{d} = \mathcal{H}^p(X) \oplus \mathrm{d}\Omega^{p-1}(X)$.

LHS $\supset$ RHS: Harmonic forms and exact forms are both closed.

LHS $\subset$ RHS: Write the RHS as $\delta\Omega^{p+1}(X)^\perp$. Then it suffices to show that $\delta\Omega^{p+1}(X)$ is orthogonal to $\ker\mathrm{d}$. For all $\alpha \in \ker\mathrm{d}$ and $\beta \in \Omega^{p+1}(X)$ we have

$$\langle \alpha, \delta\beta\rangle = \langle \mathrm{d}\alpha, \beta\rangle = 0.$$ □

9. Lie groups and principal bundles

9.1. **Lie groups and Lie algebras.**

Definition 9.1.1. A *Lie group* is a manifold G which also has a group structure, such that the multiplication $m : G \times G \to G$, $(g,h) \mapsto gh$ and inversion $i : G \to G$, $g \mapsto g^{-1}$ maps are smooth. An *embedded Lie subgroup* of a Lie group G is a subgroup H which is also a submanifold. The restrictions of the group operations from G to H are smooth so H is a Lie group.

Example 9.1.2. $\mathrm{GL}(n, \mathbb{R})$ is a Lie group and $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(n)$, and $\mathrm{SO}(n)$ are embedded Lie subgroups. Similarly, $\mathrm{GL}(n, \mathbb{C})$ is a Lie group and $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are embedded Lie subgroups. (See Sheet 1 Q9.)

Definition 9.1.3. Each $g \in G$ gives rise to diffeomorphisms L_g, R_g and C_g of G, defined by

$$L_g(h) = gh, \quad R_g(h) = hg, \quad \text{and} \quad C_g(h) = ghg^{-1} \quad \text{for all } h \in G.$$

These are *left-translation*, *right-translation*, and *conjugation* by g, and are diffeomorphisms as they are inverted by $L_{g^{-1}}$, $R_{g^{-1}}$, and $C_{g^{-1}}$. A tensor T on G is *left-invariant* if $L_g^* T = T$ for all g in G. Equivalently $(L_g)_* T = T$ for all g. Similarly for *right-invariant* or *conjugation-invariant*. It is *bi-invariant* if it's left- and right-invariant.

Lemma 9.1.4. *For any point h in G the 'evaluate at h' map*

$$\{\textit{left-invariant tensors on } G \textit{ of type } (p, q)\} \to \{\textit{tensors at } h \textit{ of type } (p, q)\}$$

sending T to T_h is a linear isomorphism. Similarly for right-invariant.

Proof. If T is left-invariant then for all $g \in G$ we have $T_g = (L_{gh^{-1}})_* T_h$. So T is completely determined by T_h and hence the map is injective. Given T_h, the same formula defines a left-invariant extension of T_h to all of G, so the map is surjective. Similarly for right-invariant. □

Definition 9.1.5. The *Lie algebra* of G, denoted $\mathfrak{g}$, is the tangent space $T_e G$ of G at the identity.

Example 9.1.6. The Lie algebras of $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, and $\mathrm{O}(n)$ are:

- $\mathfrak{gl}(n, \mathbb{R})$.
- $\mathfrak{sl}(n, \mathbb{R}) = \{A \in \mathfrak{gl}(n, \mathbb{R}) : D_I \det(A) = 0, \text{i.e. } \operatorname{tr} A = 0\}$.
- $\mathfrak{o}(n) = \{A \in \mathfrak{gl}(n, \mathbb{R}) : A^T = -A\}$.

For $\xi \in \mathfrak{g}$, let l_ξ denote the corresponding left-invariant vector field (from Lemma 9.1.4).

Lemma 9.1.7. *The Lie bracket of left-invariant vector fields is left-invariant.*

Proof. Given left-invariant vector fields v and w, for all $g \in G$ we have

$$(L_g)^*[\mathsf{v}, \mathsf{w}] = [(L_g)_* \mathsf{v}, (L_g)_* \mathsf{w}] = [\mathsf{v}, \mathsf{w}],$$

where the first equality uses diffeomorphism-invariance of the Lie derivative. □

Definition 9.1.8. The *Lie bracket* on $\mathfrak{g}$ is defined by $[\xi, \eta] = \zeta$, where ζ is the unique element of $\mathfrak{g}$ such that $[\mathsf{l}_\xi, \mathsf{l}_\eta] = \mathsf{l}_\zeta$ (this makes sense by Lemmas 9.1.4 and 9.1.7). This operation is bilinear, alternating, and satisfies the Jacobi identity since the Lie bracket of vector fields does. So it makes $\mathfrak{g}$ into a Lie algebra.

9.2. **Lie group actions.**

Definition 9.2.1. A left action of G on a manifold X is *smooth* if the action map $\sigma : G \times X \to X$, $(g, x) \mapsto gx$ is smooth. Similarly for right actions. E.g. G acting on itself by translation or conjugation, $\mathrm{GL}(n, \mathbb{R})$ acting on $\mathbb{R}^n$, $\mathrm{O}(n)$ acting on S^{n-1}.

Example 9.2.2. The *adjoint representation* is the action of G on $\mathfrak{g}$ given by $(g, \xi) \mapsto \mathrm{Ad}_g(\xi) := (C_g)_* \xi$.

Definition 9.2.3. Given a left action $\sigma : G \times X \to X$, the *infinitesimal action* of ξ on $x \in X$ is

$$\xi \cdot x := D_{(e,x)} \sigma(\xi, 0) = [\gamma(t) x],$$

where γ is any curve representing ξ. Similarly $x \cdot \xi = [x \gamma(t)]$ for a right action.

9.3. **Principal bundles.** Fix a Lie group G.

Definition 9.3.1. A *(principal) G-bundle* $\pi : P \to B$ is defined the same way as a vector bundle but trivialisations are

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$$

and on overlaps $\Phi_\beta \circ \Phi_\alpha^{-1}$ has the form

$$\begin{aligned}(U_\alpha \cap U_\beta) \times G &\to (U_\alpha \cap U_\beta) \times G \\ (b, g) &\mapsto (b, g_{\beta\alpha}(b)g),\end{aligned}$$

for some (necessarily smooth) transition map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$.

Example 9.3.2. If $\pi : E \to B$ is a vector bundle of rank k then the *frame bundle* $\pi_F : F(E) \to B$ is a principal $\mathrm{GL}(k, \mathbb{R})$-bundle, where

$$F(E)_b \coloneqq \{\text{ordered bases in } E_b\}.$$

Similarly, if E is an orthogonal vector bundle then the orthonormal frame bundle $F_{\mathrm{O}}(E) \to B$ is a principal $\mathrm{O}(k)$-bundle.

Note that:

- Most definitions carry over from vector bundles, e.g. sections, pullbacks, construction by gluing. But there is no analogue of the zero section.
- A G-bundle P carries a right G-action, defined by right translation in trivialisations, i.e

  $$\Phi_\alpha^{-1}(b, x)g \coloneqq \Phi_\alpha^{-1}(b, xg).$$

  The action is free and transitive on each fibre.
- Sections s of P over open $U \subset B$ correspond to trivialisations Φ of P over U: Φ defines s by $s(p) = \Phi^{-1}(p, e)$, whilst s defines Φ by $\Phi(s(p)g) = (p, g)$.

9.4. **Connections and curvature.** Fix a G-bundle P, and let $R_g : P \to P$ denote the right G-action.

Definition 9.4.1. A *connection* $\mathcal{A}$ on P is a $\mathfrak{g}$-valued 1-form on P satisfying:

- $\mathcal{A}_p(p \cdot \xi) = \xi$ for all $p \in P$ and all $\xi \in \mathfrak{g}$.
- $R_g^* \mathcal{A} = \mathrm{Ad}_{g^{-1}} \mathcal{A}$ for all $g \in G$.

Given a local section s_α over U_α, or equivalently a trivialisation Φ_α over U_α, the associated *local connection 1-form* is the $\mathfrak{g}$-valued 1-form A_α on U_α given by $s_\alpha^* \mathcal{A}$.

Lemma 9.4.2. *On overlaps we have $A_\alpha = \mathrm{Ad}_{g_{\beta\alpha}^{-1}} A_\beta + (L_{g_{\beta\alpha}^{-1}})_* \mathrm{d}g_{\beta\alpha}$. Conversely any collection of $\mathfrak{g}$-valued 1-forms on B related in this way define a connection on P.*

Proof. Sheet 4. □

So a connection on a vector bundle $E \to B$ is equivalent to a connection on $F(E) \to B$.

Definition 9.4.3. The *curvature* of a connection $\mathcal{A}$ is the $\mathfrak{g}$-valued 2-form $\mathcal{F}$ on P given by $\mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}]$, where for $\mathfrak{g}$-valued p- and q-forms $\sigma = \sum_i \xi_i \otimes \sigma_i$ and $\tau = \sum_j \eta_j \otimes \tau_j$ we define

$$[\sigma \wedge \tau] = \sum_{i,j} [\xi_i, \eta_j] \otimes (\sigma_i \wedge \tau_j).$$

$\mathcal{A}$ is *flat* iff $\mathcal{F} = 0$.