False Proof of the KLS Conjecture?

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The definitions and conjectures are copied from section 4.1 of wisdom.weizmann.ac.il/ ronene/GFANotes.pdf.

Definition 1 (Concentration of measures)**.** If μ is a probability measure on $\mathbb{R}^n$, we say μ is C-concentrated if for all 1-Lipschitz $\phi : \mathbb{R}^n \to \mathbb{R}$, we have

$$\mathrm{Var}_\mu[\phi] = \mathrm{Var}_{X\sim\mu}[\phi(X)] \le C^2.$$

Conjecture 1 (KLS)**.** Given μ a log-concave measure on $\mathbb{R}^n$ (or equivalently, a uniform measure over a convex body $K \subseteq \mathbb{R}^n$), if for all linear 1-Lipschitz function $T : \mathbb{R}^n \to \mathbb{R}$ one have $\mathrm{Var}_\mu[T] \le 1$, then μ is C-concentrated for some *universal constant* C.

Lemma 1. Let X be a $\mathbb{R}^n$-valued random variable. Then for all K-Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}[\phi(X)] \le K^2\mathrm{Var}[X].$$

Proof. WLOG. by subtracting its expectation from X, we may assume $\mathbb{E}[X] = 0$. Let X' be a i.i.d. copy of X on the same probability space. Then for all K-Lipschitz function ϕ, we have

$$\begin{aligned}
2\mathrm{Var}[\phi(X)] &= \mathrm{Var}[\phi(X) - \phi(X')] && \text{(i.i.d.)}\\
&= \mathbb{E}[(\phi(X) - \phi(X'))^2] - \mathbb{E}[\phi(X) - \phi(X')]^2\\
&= \mathbb{E}[(\phi(X) - \phi(X'))^2] && \text{(identically distributed)}\\
&\le K^2\mathbb{E}[\|X - X'\|^2] && \text{(as } \phi \text{ is } K\text{-Lipschitz)}\\
&= K^2\mathbb{E}[X^TX + X'^TX' - X^TX' - X'^TX]\\
&= 2K^2\mathrm{Var}[X] - 2K^2\mathrm{Cov}(X, X') = 2K^2\mathrm{Var}[X]. && \text{(independence)}
\end{aligned}$$

implying $\mathrm{Var}[\phi(X)] \le K^2\mathrm{Var}[X]$ as claimed. □