WIP Title

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2 The KLS and Thin-Shell Conjecture

While the KLS conjecture is often phrased as a isoperimetric problem, we will mainly consider the conjecture here as a problem regarding measures while providing some discussions regarding its equivalent formulations.

Mention we only work with Borel measures

2.1 Concentration

Definition 2.1 (Concentration, [Eld18]). Let μ be a measure on $\mathbb{R}^n$, then μ is said to be C-(inversely)-concentrated if for all 1-Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}_\mu[\phi] = \mathrm{Var}_{X\sim\mu}[\phi(X)] \leq C^2. \tag{1}$$

We denote the least possible such C by C^μ_{con}.

Heuristically, the concentration measures the relation between μ and the Euclidean metric by providing a numerical control for the variance of its norm. This is perhaps best illustrated by the following proposition.

Proposition 2.1. Let X be a $\mathbb{R}^n$-valued random variable. Then for all K-Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}[\phi(X)] \leq K^2\mathrm{Var}[\|X\|^2].$$

Proof. WLOG. by subtracting its expectation from X, we may assume $\mathbb{E}[X] = 0$. Let X' be a i.i.d. copy of X on the same probability space. Then for all K-Lipschitz function ϕ, we have

$$\begin{aligned}
2\mathrm{Var}[\phi(X)] &= \mathrm{Var}[\phi(X) - \phi(X')] && \text{(i.i.d.)}\\
&= \mathbb{E}[(\phi(X) - \phi(X'))^2] - \mathbb{E}[\phi(X) - \phi(X')]^2\\
&= \mathbb{E}[(\phi(X) - \phi(X'))^2] && \text{(identically distributed)}\\
&\leq K^2\mathbb{E}[\|X - X'\|^2] && \text{(as } \phi \text{ is } K\text{-Lipschitz)}\\
&= K^2\mathbb{E}[X^TX + X'^TX' - X^TX' - X'^TX]\\
&= 2K^2\mathrm{Var}[\|X\|^2] - 2K^2\mathrm{Cov}(X, X') = 2K^2\mathrm{Var}[\|X\|^2]. && \text{(independence)}
\end{aligned}$$

implying $\mathrm{Var}[\phi(X)] \leq K^2\mathrm{Var}[\|X\|^2]$ as claimed. □

With this proposition in mind, it is clear that for $\mathbb{R}$-valued random variables X, its law μ has concentration $C^\mu_{\text{con}} = \mathrm{Var}[X]$. Furthermore, by considering the projection maps, it follows that the standard Gaussian measure on $\mathbb{R}^n$ is 1-concentrated.

We note that the definition we are presenting here is slightly non-standard. However, utilising a remarkable result due to Milman, we show that this definition is equivalent to the following definitions in a specific sense.

Definition 2.2 (Exponential concentration, [Mil18]). Given a measure μ on $\mathbb{R}^n$, we say μ has exponential concentration if there exists some $c, D > 0$ such that for all 1-Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}, t > 0$, we have

$$\mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t) \leq ce^{-Dt}. \tag{2}$$

Fixing $c = 1$, we denote the largest possible D as $D^\mu_{\exp}$.

Definition 2.3 (First-moment concentration, [Mil18]). Again, given μ a measure on $\mathbb{R}^n$, we say μ has first-moment concentration if there exists some $D > 0$ such that for all 1-Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$, we have

$$\mathbb{E}_\mu[|\phi - \mathbb{E}_\mu[\phi]|] \leq \frac{1}{D}. \tag{3}$$

We denote the largest possible D by D^μ_{FM}.

It is clear that exponential concentration implies first-moment concentration. Indeed, if μ has exponential concentration with constant D (taking $c = 1$), then by the tail probability formula,

$$\mathbb{E}_\mu[|\phi - \mathbb{E}_\mu[\phi]|] = \int_0^\infty \mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t)\mathrm{d}t \leq \int_0^\infty e^{-Dt}\mathrm{d}t = \frac{1}{D}.$$

On the other hand, Milman showed that for log-concave measures (namely, measures of the form $\mathrm{d}\mu = \exp(-H)\mathrm{dLeb}^n$ for some convex function $H : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$) on $\mathbb{R}^n$, exponential concentration and first-moment concentration are equivalent in the following sense.

Theorem 1 (Milman, [Mil08]). For all log-concave measure μ on $\mathbb{R}^n$, μ has exponential concentration if and only if μ has first-moment concentration. Furthermore, $D^\mu_{\exp} \simeq D^\mu_{\mathrm{FM}}$ where we write $A \simeq B$ if there exists universal constants $C_1, C_2 > 0$ such that $C_1 A \leq B \leq C_2 A$.

With this theorem in mind, we establish the following correspondence.

Proposition 2.2. For all measures μ on $\mathbb{R}^n$, we have

$$\text{Exponentially concentrated} \implies \text{Concentrated} \implies \text{First-moment concentrated}$$

and $D^\mu_{\exp} \leq \sqrt{2}(C^\mu_{\mathrm{con}})^{-1}$ and $(2C^\mu_{\mathrm{con}})^{-1} \leq D^\mu_{\mathrm{FM}}$. Hence, if μ is log-concave, $D^\mu_{\exp} \simeq D^\mu_{\mathrm{FM}} \simeq (C^\mu_{\mathrm{con}})^{-1}$.

Proof. Assume first that μ is C-concentrated. Then by the Chebyshev inequality, we have

$$\mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t) \leq \frac{1}{t^2}\mathrm{Var}_\mu[\phi] \leq \frac{C^2}{t^2},$$

for all 1-Lipschitz ϕ. Thus, by tail probability,

$$\begin{aligned}
\mathbb{E}_\mu[|\phi - \mathbb{E}_\mu[\phi]|] &= \int_0^\infty \mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t)\mathrm{d}t \\
&\leq \inf_{a>0}\left\{\int_0^a \mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t)\mathrm{d}t + C^2\int_a^\infty \frac{1}{t^2}\mathrm{d}t\right\} \\
&\leq \inf_{a>0}\left\{a + \frac{C^2}{a}\right\} = 2C,
\end{aligned}$$

implying μ is first-moment concentrated with respect to the constant $(2C)^{-1}$.

On the other hand, if μ is exponential concentration with some constant D, then again by the tail probability,

$$\mathrm{Var}_\mu[\phi] = \int_0^\infty \mu((\phi - \mathbb{E}_\mu[\phi])^2 \geq t)\mathrm{d}t \leq \int_0^\infty e^{-D\sqrt{t}}\mathrm{d}t = \frac{2}{D^2}$$

implying μ is $\sqrt{2}D^{-1}$-concentrated. □

2.2 The KLS and thin-shell conjecture

Informally, the KLS conjecture suggests that any log-concave measure on $\mathbb{R}^n$ admits the same concentration as that of the Gaussian measure. However, unlike the Gaussian, as the concentration of measures is not invariant under linear transformations, it is clear that the KLS conjecture would not hold without a suitable normalization. This leads us to the following formulation of the KLS conjecture.

Conjecture 1 (Kannan-Lovász-Simonovitz, [Eld18]). Denoting $\mathscr{M}_{\text{con}}$ the set of all log-concave probability measures μ on $\mathbb{R}^n$ satisfying $\text{Var}_\mu[T] \leq 1$ for all 1-Lipschitz linear maps $T : \mathbb{R}^n \to \mathbb{R}$, there exists a *universal* constant C such that for all $\mu \in \mathscr{M}_{\text{con}}$, μ is C-concentrated.

We remark that C is universal in the sense that it does not depend on any parameter and in particular is independent of the dimension n.

Conjecture 2 (Thin-shell, [Eld13]). Taking $\mathscr{M}_{\text{con}}$ as above, there exists a universal constant C such that for all $\mu \in \mathscr{M}_{\text{con}}$, we have

$$\sqrt{\text{Var}_\mu[\|\cdot\|]} \leq C.$$

As the norm function is 1-Lipschitz, it is *a priori* that the thin-shell conjecture is weaker than that of the KLS conjecture. On the other hand, as we shall describe in the next section, as a consequence of the stochastic localisation scheme, Eldan [Eld13] provides a reduction of the KLS conjecture to the thin-shell conjecture up to logarithmic factors.

Theorem 2 (Eldan, [Eld13]). Denoting $\mathscr{M}_{\text{con}}$ as above, we define

$$C_{\text{con}} := \inf\{C \mid \forall \mu \in \mathscr{M}_{\text{con}}, \mu \text{ is } C\text{-concentrated}\},$$

and

$$C_{\text{TS}} := \inf\left\{C \mid \forall \mu \in \mathscr{M}_{\text{con}}, \sqrt{\text{Var}_\mu[\|\cdot\|]} \leq C\right\} = \sup_{\mu \in \mathscr{M}_{\text{con}}} \sqrt{\text{Var}_\mu[\|\cdot\|]},$$

we have,

$$C_{\text{TS}} \leq C_{\text{con}} \leq C_{\text{TS}} \log n.$$

2.2.1 Equivalent formulation of the KLS conjecture

While the formulation of the KLS conjecture above is very useful, as mentioned previously, the KLS conjecture has several equivalent formulations; we will quickly present them here. However, before stating these formulations, let us quickly complete theorem 1 with two additional equivalences.

Definition 2.4 (Cheeger's inequality, [Mil08]). Given a measure μ on $\mathbb{R}^n$, we recall the Minkowski's boundary measure defined by

$$\mu^+(A) := \liminf_{\epsilon \downarrow 0} \frac{\mu(A_\epsilon) - \mu(A)}{\epsilon}$$

where $A_\epsilon := \{x \in \mathbb{R}^n \mid \text{dist}(x, A) \leq \epsilon\}$ is the ϵ-thickening of some Borel set A. Then, we say μ satisfy Cheeger's inequality if there exists some D such that for all A satisfying $\mu(A) \leq \frac{1}{2}$,

This definition is from Eldan not Milman, can we show equivalence?

$$\mu(A) \leq D\mu^+(A).$$

We denote the largest such D by D_C^μ.

Theorem 3 (Milman, [Mil08]). For all log-concave measure μ on $\mathbb{R}^n$, the following are equivalent

- μ has exponential concentration with constant D^{μ}_{exp}.
- μ has first-moment concentration with constant D^{μ}_{FM}.
- μ satisfy the Cheeger's inequality with constant D^{μ}_{C}.
- μ satisfy the Poincaré inequality: there exists some $D > 0$ such that for all differentiable $\phi : \mathbb{R}^n \to \mathbb{R}$ satisfying $\int \phi \mathrm{d}\mu = 0$, we have

  $$D \cdot \mathrm{Var}_{\mu}[\phi] \leq \int \|\nabla \phi\|^2 \mathrm{d}\mu.$$

  We denote the largest such D by D^{μ}_{P}.

Furthermore, $D^{\mu}_{\text{exp}} \simeq D^{\mu}_{\text{FM}} \simeq D^{\mu}_{\text{P}} \simeq D^{\mu}_{\text{C}}$.

With this theorem and proposition 2.2 in mind, it is clear that the KLS conjecture can be instead formulated with any of these inequalities instead.

Conjecture 3 (KLS, [Eld13]). Denoting $\mathscr{M}_{\text{iso}}$ the set of all log-concave probability measures μ on $\mathbb{R}^n$ satisfying $\int x\mu(\mathrm{d}x) = 0$ (namely μ is isotropic), there exists a *universal* constant D such that for all $\mu \in \mathscr{M}_{\text{iso}}$, μ satisfy the Cheeger's inequality with constant D.

Why are the normalisations equivalent?

Similarly, we may also reformulate the KLS conjecture using the constant provided by the Poincaré inequality.

Conjecture 4 (KLS, [Eld13]). Denoting $\mathscr{M}_{\text{iso}}$ as above, there exist a *universal* constant D such that for all $\mu \in \mathscr{M}_{\text{iso}}$, μ satisfy the Poincaré inequality with constant D.

By observing that $\mathrm{Var}_{\mu}[\phi] = \int \phi^2 \mathrm{d}\mu$ for all differentiable ϕ satisfying $\int \phi \mathrm{d}\mu = 0$, we remark that the Poincaré inequality can be alternatively written as

$$\frac{\int \phi^2 \mathrm{d}\mu}{\int \|\nabla \phi\|^2 \mathrm{d}\mu} \leq D^{-1}.$$

3 The Stochastic Localisation Scheme

We will in this section provide a description of the stochastic localisation scheme introduced by Eldan [Eld13] and describe its application in reducing (up to a logarithmic factor) the KLS conjecture into the thin-shell conjecture, i.e. we will describe a proof of 2.

3.1 Construction and basic properties

Definition 3.1 (Barycenter)**.** Given a (probability) measure μ on $\mathbb{R}^n$, we define its barycenter with respect to the function $F : \mathbb{R}^n \to \mathbb{R}$ to be

$$\bar{\mu}(F) := \int_{\mathbb{R}^n} xF(x)\mu(\mathrm{d}x).$$

Given the above definition, we now define the following construction central to the stochastic localisation scheme. Let $(W_t)_{t\geq 0}$ be a Wiener process, we define the random functions $(F_t)_{t\geq 0}$ to be the solution of the following infinite system of SDEs:

$$F_0 = 1, \mathrm{d}F_t(x) = \langle x - \bar{\mu}(F_t), \mathrm{d}W_t\rangle F_t(x), \tag{4}$$

for all $x \in \mathbb{R}^n$. We shall from this point forward denote the random variables $a_t := \bar{\mu}(F_t)$. By applying Itô's formula, we make the following useful observation: for all $x \in \mathbb{R}^n$,

Existence and uniqueness of F.

$$\mathrm{d}\log F_t(x) = \frac{\mathrm{d}F_t(x)}{F_t(x)} - \frac{\mathrm{d}[F(x)]_t}{2F_t(x)^2} = \langle x - a_t, \mathrm{d}W_t\rangle - \frac{1}{2}\|x - a_t\|^2\mathrm{d}t \tag{5}$$

where the second equality follows by the construction of F. Hence, as $\log F_0(x) = 0$, we observe

$$\begin{aligned}
\log F_t(x) &= \int_0^t \langle x - a_s, \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^t \|x - a_s\|^2\mathrm{d}s \\
&= \left(\langle x, W_t\rangle - \int_0^t \langle a_s, \mathrm{d}W_s\rangle\right) - \left(\frac{t}{2}\|x\|^2 + \frac{1}{2}\int_0^t \|a_s\|^2\mathrm{d}s - \int_0^t \langle x, a_s\rangle\mathrm{d}s\right) \\
&= -\left(\int_0^t \langle a_s, \mathrm{d}W_s\rangle + \frac{1}{2}\|a_s\|^2\mathrm{d}s\right) + \langle x, a_t + W_t\rangle - \frac{t}{2}\|x\|^2.
\end{aligned}$$

Thus, taking $\mathrm{d}z_t := \langle a_t, \mathrm{d}W_t\rangle + \frac{1}{2}\|a_t\|^2\mathrm{d}t$ and $v_t := a_t + W_t$, we observe $F_t(x)$ is of the form

$$F_t(x) = e^{z_t + \langle x, v_t\rangle - \frac{t}{2}\|x\|^2}, \tag{6}$$

for given Itô processes $(z_t), (v_t)$. With this formulation of $F_t(x)$ in mind, it follows F_t is non-negative, and so, we may define the measure-valued random variable μ_t such that $\mu_t = F_t\mu$, i.e. they have Radon-Nikodym derivative $\mathrm{d}\mu_t/\mathrm{d}\mu = F_t$.

Is $e^{z_t + \langle x, v_t\rangle}$ itself a Itô process?

Proposition 3.1. For all $t \geq 0$, μ_t is a probability measure almost everywhere (a.e.), i.e. $\mathbb{P}(\mu_t(\mathbb{R}^n) = 1) = 1$.

Proof. As μ is a probability measure, it suffices to show $\partial_t \mu_t(\mathbb{R}^n) = 0$. To prove this, we first consider the discrete stochastic integral on the lattice $\Lambda = \mathbb{Z}^n \mathrm{d}t$ for some $\mathrm{d}t > 0$. Then, constructing $\mu_t^{\mathrm{d}t}$ on Λ via the same process as μ_t, for all $t = k\mathrm{d}t \in \Lambda$,

$$\begin{aligned}\partial_t \mu_t^{\mathrm{d}t}(\mathbb{R}^n) &= \int_{\mathbb{R}^n} \mathrm{d}F_t(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}^n} \langle x - a_t, \mathrm{d}W_t \rangle F_t(x)\mu(\mathrm{d}x) \\ &= \left\langle \int_{\mathbb{R}^n} (x - a_t)F_t(x)\mu(\mathrm{d}x), \mathrm{d}W_t \right\rangle = \langle a_t - a_t\mu_t(\mathbb{R}^n), \mathrm{d}W_t \rangle = 0\end{aligned}$$

by induction on k. However, by the very construction of the stochastic integral, the densities of $\mu_t^{\mathrm{d}t}$: $F_t^{\mathrm{d}t}$ converges a.e. to F_t as $\mathrm{d}t \to 0$ implying $\mu_t^{\mathrm{d}t} \to \mu_t$ weakly and so, $\mu_t^{\mathrm{d}t}(\mathbb{R}^n) \to \mu_t(\mathbb{R}^n)$ resulting in $\mu_t(\mathbb{R}^n) = 1$ as required. □

We would also like to study the limiting behavior of (μ_t) as $t \to \infty$. To achieve this, we will consider the covariance matrices

$$A_t := \mathrm{Cov}[\mu_t] = \int (x - a_t) \otimes (x - a_t)\mu_t(\mathrm{d}x), \tag{7}$$

where $\otimes$ denotes the Kronecker product. In particular, we will show $(A_t)_{ij} \to 0$ for all $i, j \in \{1, \cdots, n\}$ as $t \to \infty$ allowing us to conclude (μ_t) converges weakly to some Dirac measure. Indeed, this is a direct consequence of the following lemma.

Lemma 3.2 (Brascamp-Lieb inequality, [BL76])**.** Given $V : \mathbb{R}^n \to \mathbb{R}$ convex and $K > 0$, if ν is an isotropic probability measure on $\mathbb{R}^n$ of the form

$$\mathrm{d}\nu = Ze^{-V(x) - \frac{1}{2K}\|x\|^2}\mathrm{dLeb}^n$$

with Z being the normalization constant, then ν satisfy the Poincaré inequality, i.e. for all differentiable ϕ,

$$K\mathrm{Var}_\nu[\phi] \le \int \|\nabla\phi\|^2 \mathrm{d}\nu.$$

Proof of lemma 3.2. TODO. Maybe go in appendix? □

With this lemma in mind, by taking $\nu = \mu_t$ using equation 6 and defining $\phi_{ij}(x) := x_i x_j$, we obtain

$$(A_t)_{ij} = \mathrm{Var}_{\mu_t}[\phi_{ij}] \le t^{-1} \int \|\nabla\phi_{ij}\|^2 \mathrm{d}\mu_t.$$

Then, again using equation 6, we note that any realizations of $(F_t(x))$ is eventually decreasing in t if $x \neq 0$. Hence, $\sup_t \int \|\nabla\phi_{ij}\|^2 \mathrm{d}\mu_t = \sup_t \int (x_i^2 + x_j^2)\mu_t(\mathrm{d}x) < \infty$. Thus, by taking $t \to \infty$ we have $(A_t)_{ij} \to 0$ for all $i, j \in \{1, \cdots, n\}$ as claimed.

Corollary 3.3. (μ_t) converges weakly to some Dirac measure almost everywhere. We denote this limiting (random) Dirac measure by δ_{a_∞} where a_∞ is some $\mathbb{R}^n$-valued random variable.

Since convergence implies relatively compact, applying the Dunford-Pettis theorem it follows that any realizations of (F_t) is uniformly integrable. Thus, we can make the following deductions about a_∞.

Corollary 3.4. The massive point a_∞ of the limiting Dirac measure is the limit of a_t as $t \to \infty$ and has law μ.

Proof. Since (F_t) is uniformly integrable we have convergence of means almost everywhere, namely

$$a_t = \int x\mu_t(\mathrm{d}x) \xrightarrow{a.e.} \int x\delta_{a_\infty}(\mathrm{d}x) =: a_\infty \text{ as } t \to \infty$$

implying that a_t converges a.e. to a_∞ as $t \to \infty$ as required.
Furthermore, taking $\phi : \mathbb{R}^n \to \mathbb{R}$ to be any bounded continuous function, we have

$$\int \phi(x)\mu(\mathrm{d}x) = \lim_{t\to\infty} \int \phi(x)\mu_t(\mathrm{d}x).$$

Then, taking expectation on both sides, we obtain

$$\begin{aligned}
\int \phi(x)\mu(\mathrm{d}x) &= \mathbb{E}\left[\lim_{t\to\infty} \int \phi(x)\mu_t(\mathrm{d}x)\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\int \phi(x)\mu_t(\mathrm{d}x)\right] && \text{(Dominated convergence)} \\
&= \lim_{t\to\infty} \mathbb{E}[\phi(a_t)] && \text{(LOTUS. theorem)} \\
&= \mathbb{E}[\phi(a_\infty)]. && \text{(Dominated convergence \& continuity of } \phi)
\end{aligned}$$

Thus, $\mathbb{E}_\mu[\phi] = \mathbb{E}[\phi(a_\infty)]$ for all bounded continuous ϕ implying $a_\infty \sim \mu$. $\square$

Proposition 3.5. For all $x \in \mathbb{R}^n$, $(F_t(x))_{t\geq 0}$ is a martingale. Furthermore, for any continuous bounded $\phi : \mathbb{R}^n \to \mathbb{R}$, defining the process $M_t := \int \phi \mathrm{d}\mu_t$, $(M_t)_{t\geq 0}$ is also a martingale.

Proof. By its very construction, $(F_t(x))$ is a martingale by observing equation 4 has no drift term. Now, for all $s \leq t$ we have

$$\begin{aligned}
\mathbb{E}[M_t \mid \mathscr{F}_s] &= \mathbb{E}\left[\int \phi(x)F_t(x)\mu(\mathrm{d}x) \middle| \mathscr{F}_s\right] \\
&= \int \phi(x)\mathbb{E}[F_t(x) \mid \mathscr{F}_s]\mu(\mathrm{d}x) = \int \phi(x)F_s(x)\mu(\mathrm{d}x) = M_s
\end{aligned}$$

implying (M_t) is also a martingale. $\square$

Using the same proof as corollary 3.4, we observe M_t converges a.e. to some M_∞ with law $\phi_*\mu$.

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