

# Stochastic Localization and its Applications

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# 1 Introduction

The notion of stochastic localization was first introduced in the 2013 paper by Eldan [Eld13] in order to make progress regarding an isoperimetric problem known as the *Kannan-Lovász-Simonovitz* (KLS) conjecture. As it turns out, stochastic localization has been also useful in many other adjacent areas, in particular, in sampling and Markov mixing. This essay will provide an introduction to stochastic localization and describe its applications in Markov mixing and the KLS conjecture. Furthermore, using stochastic localization, this essay will also provide an alternative proof of a known bound for the log-Sobolev constant of log-concave measures.

Stochastic localization in its most general form describes a sequence of random measures (random variables taking values in the space of measures) which begins at a given measure and converges to dirac measures almost everywhere, namely it “localizes”, and moreover, satisfy a certain martingale condition. These sequences of random measures are useful in studying specific measures. Namely, by evolving the stochastic localization in time, particular properties of the structure collapses allowing us say something about them. On the other hand, the martingale property allows us to preserve these properties (possibly up to some time). Hence, by balancing the two, i.e. allowing the sequence to evolve so it is close to a dirac measure while not evolve too long such that we lose the properties of the original measure, we are able to obtain results about the original measure.

In our case, we will mostly focus on one specific type of stochastic localization known as the linear-tilt localization. The linear-tilt localization is a special case of stochastic localizations in which at each time step, the measure is “tilted” in a random direction. This random direction can be chosen in a variety of ways however one choice of interest is when the random direction is chosen according to a Wiener process. A stochastic localization constructed this way accumulates a Gaussian component which becomes more and more significant as the process evolves. This is particularly helpful as Gaussian measures are well understood and we can use properties of the Gaussian to obtain results about our original measure.

## 1.1 Structure of this essay

This essay consists of four sections. We now give a brief overview of them.

- The first section of this essay consists of setting up the theory of stochastic localization. We also introduce some specific examples of stochastic localizations which will be used in subsequent sections most importantly the discrete and continuous linear-tilt localization.
- We then move on to discuss the application of the stochastic localization to bounding Markov mixing time. Motivated by the Markov chain Monte Carlo, the Markov mixing time provides a quantitative measure on how fast a Markov chain converges to its stationary distribution. It turns out that the mixing time of Markov chains which kernels can be described using stochastic localizations is related to how the variance of the stochastic localization evolves. This method is then applied to the Glauber dynamics on the Boolean hypercube to obtain a mixing time bound to the Ising model.
- We also discuss a proof of Eldan’s original 2013 result reducing the KLS conjecture to another seemingly weaker conjecture known as the thin-shell conjecture. Roughly speaking, the KLS conjecture asserts that the class of measures known as the *isotropic log-concave* measures are concentrated in the similar way to the Gaussian. While we shall mostly

consider the KLS conjecture using the language of concentrations, it can be equivalently phrased with a more geometric point of view, that the maximum proportion of volume by surface area of a log-concave measure is bounded by a universal constant. As a result, the KLS conjecture has many important consequences in convex geometry. In particular, by noticing that any uniform measure on a convex body of unit volume is log-concave, the KLS conjecture also implies the Bourgain slicing conjecture which asserts that any convex body of unit volume in  $\mathbb{R}^n$  has a intersection with some hyperplane with volume bounded below by a universal constant. Moreover, the KLS conjecture has applications in bounding mixing times for ball walks (c.f. [LV16]), the central limit theorem for convex bodies (c.f. [GBVV14]) and directly implies the aforementioned thin-shell conjecture.

- Finally, inspired by the proof from the third section, we will use stochastic localization methods to provide a bound for the log-Sobolev constant of log-concave measures.

## 2 Stochastic Localization Scheme

In this section we introduce the notion of stochastic localization and provide some useful examples of them which are studied further in the subsequent sections.

We work in general Borel spaces  $(\mathcal{X}, \Sigma)$  for this section while restricting our focus to either the Euclidean space  $\mathbb{R}^n$  or the Boolean hypercube  $\{-1, 1\}^n$  in subsequent sections. We take  $(\Omega, \mathcal{F}, \mathbb{P})$  to be our underlying probability space and we introduce the notation  $\mathcal{M}(\mathcal{X})$  for the space of probability measures on  $\mathcal{X}$ .

**Definition 2.1** (Prelocalization process). Given  $\mu \in \mathcal{M}(\mathcal{X})$ , a measure-valued stochastic process  $(\mu_t)_{t \geq 0}$  is said to be a prelocalization of  $\mu$  if

L0  $\mu_0 = \mu$ .

L1 For all  $t \geq 0$ ,  $\mu_t$  is a probability measure almost everywhere, i.e.  $\mathbb{P}(\mu_t(\mathcal{X}) = 1) = 1$ .

L2 For all  $A \in \Sigma$ ,  $(\mu_t(A))_{t \geq 0}$  is a martingale with respect to the natural filtration of  $(\mu_t)$ .

**Definition 2.2** (Stochastic localization process, [CE22]). Given  $\mu \in \mathcal{M}(\mathcal{X})$ , a measure-valued stochastic process  $(\mu_t)_{t \geq 0}$  is said to be a stochastic localization of  $\mu$  if in addition to being a prelocalization of  $\mu$ ,  $(\mu_t)_{t \geq 0}$  also satisfies

L3 For all  $A \in \Sigma$ ,  $\mu_t(A)$  converges almost everywhere to 0 or 1 as  $t \rightarrow \infty$ .

We say a stochastic localization is discrete if  $t$  takes value in  $\mathbb{N}$  and continuous if  $t$  takes value in  $\mathbb{R}_{\geq 0}$ . For shorthand, we denote  $(\mu_k)_k$  for a discrete stochastic localization of  $\mu$ .

**Proposition 2.1.** Straightaway, by the martingale property, if  $(\mu_t)_{t \geq 0}$  is a stochastic localization of  $\mu$ , then

- $\mathbb{E}[\mu_t] = \mu$  for all  $t \geq 0$ .
- $\mu_t \rightarrow \delta_X$  almost everywhere as  $t \rightarrow \infty$  for some  $X \sim \mu$  (here weak and total variational convergence are equivalent and so  $\rightarrow$  can mean either).

*Proof.* The first statement is immediate as for all  $A \in \Sigma$ ,

$$\mathbb{E}[\mu_t](A) \triangleq \mathbb{E}[\mu_t(A)] = \mathbb{E}[\mu_0(A)] = \mu(A).$$

To prove the second statement, let us first parse what the claim is. Fixing a realization  $\omega$  of  $\mu_t$ , we have by (L3) that  $\mu_t$  converges to some Dirac measure based at some  $x_\omega \in \mathcal{X}$ . Thus, defining the random variable  $X : \omega \mapsto x_\omega$ , we need to show  $X \sim \mu$ . Indeed, by taking  $\phi : \mathcal{X} \rightarrow \mathbb{R}$  to be any bounded and continuous function, by the definition of  $X$

$$\int \phi(x) \mu_t(dx) \xrightarrow{\text{a.e.}} \int \phi(x) \delta_X(dx) = \phi(X) \text{ as } t \rightarrow \infty.$$

Thus, taking expectation on both sides, we have

$$\mathbb{E}[\phi(X)] = \mathbb{E}\left[\int \phi d\mu_t\right] = \int \phi d\mathbb{E}[\mu_t] = \int \phi d\mu$$

implying  $X \sim \mu$  as required. □

An example of a stochastic localization scheme is the coordinate by coordinate localization scheme on  $\mathcal{X} = \{-1, 1\}^n$ . This scheme relates to the Glauber dynamics for which the stochastic localization scheme provides a mixing bound. We will discuss this property further in section 3.3 while only considering its construction for now.

Given a probability measure  $\mu$  on  $\{-1, 1\}^n$ , we introduce the random variable  $X \sim \mu$ , and  $Y$  a uniform random variable over all permutations of  $[n] = \{1, \dots, n\}$  independent of  $X$ . Then, the coordinate by coordinate stochastic localization of  $\mu$  is the process  $(\mu_k)_k$  such that for all  $x \in \{-1, 1\}^n$ ,

$$\mu_k(x) = \mathbb{P}(X = x \mid X_{Y_1}, \dots, X_{Y_{n \wedge k}}).$$

Namely,  $\mu_k$  is the law of  $X$  conditioned on  $X_{Y_1}, \dots, X_{Y_k}$ .

$(\mu_k)_k$  is indeed a stochastic localization of  $\mu$ . It is clear that (L0) and (L1) are satisfied. By construction of  $(\mu_k)_k$ , denoting  $\mathcal{F}_k := \sigma(X_{Y_1}, \dots, X_{Y_{n \wedge k}})$ , we have by the tower property

$$\mathbb{E}[\mu_{k+1}(x) \mid \mathcal{F}_k] = \mathbb{E}[\mathbb{E}[\mathbb{P}(X = x \mid X) \mid \mathcal{F}_{k+1}] \mid \mathcal{F}_k] = \mathbb{E}[\mathbb{P}(X = x \mid X) \mid \mathcal{F}_k] = \mu_k(x)$$

implying  $(\mu_k(x))$  a martingale as required for (L2). Finally, it is clear that

$$\lim_{k \rightarrow \infty} \mu_k(x) = \mu_n(x) = \mathbb{P}(X = x \mid X) = \mathbf{1}_{\{X=x\}} \in \{0, 1\}$$

implying (L3).

An analogous construction of the coordinate by coordinate stochastic localization scheme in  $\mathbb{R}^n$  is the random subspace localization. Similar to before, for a probability measure  $\mu$  on  $\mathbb{R}^n$ , we introduce the random variable  $X \sim \mu$  and  $Y$  a uniform random variable on  $O(n)$  (so the column vectors  $\{Y_1, \dots, Y_n\}$  form an orthonormal basis of  $\mathbb{R}^n$ ) independent of  $X$ . Then, we define the random subspace stochastic localization of  $\mu$  as  $(\mu_k)_k$  where  $\mu_k$  is the law of  $X$  conditioned on  $\langle X, Y_1 \rangle, \dots, \langle X, Y_{n \wedge k} \rangle$ .

## 2.1 Linear-tilt localization schemes

An important class of stochastic localization schemes are the linear-tilt schemes. Introduced by Eldan in [Eld13], linear-tilt schemes has been vital in the recent progress regarding the KLS conjecture. More recently, a discrete version of the linear-tilt scheme was introduced in [CE22] and is used to provide a mixing bound for Glauber dynamics. We will in this section introduce this family of localizations and consider two specific examples of such linear-tilt schemes which are useful for our analysis later.

Informally, given a probability measure  $\mu$  on  $\mathcal{X} \subseteq \mathbb{R}^n$ , the linear-tilt scheme of  $\mu$  is constructed recursively in which at each step, we pick a random direction and multiply the density at this time with a linear function along this direction (i.e. a tilt along a random direction).

Let  $\mu$  be a probability measure on  $\mathcal{X} \subseteq \mathbb{R}^n$ , we introduce the following definition.

**Definition 2.3** (Barycenter). The barycenter of  $\mu$  with respect to the function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is

$$\bar{\mu}(F) := \int_{\mathcal{X}} x F(x) \mu(dx).$$

In the case that  $F = \text{id}$ , we simply write  $\bar{\mu} = \bar{\mu}(F) = \mathbb{E}_{X \sim \mu}[X]$ .

**Definition 2.4** (Linear-tilt localization). A measure-valued stochastic process  $(\mu_t)_{t \geq 0}$  is said to be a linear-tilt localization of the probability measure  $\mu$  if

1.  $\mu_t \ll \mu$  for each  $t \geq 0$ , and
2. denoting  $F_t := d\mu_t/d\mu$ , we have  $F_0 = 1$  and

$$dF_t(x) = \langle x - \bar{\mu}(F_t), dZ_t \rangle F_t(x) \quad (1)$$

for some stochastic process  $(Z_t)_{t \geq 0}$  such that  $\mathbb{E}[dZ_t | \mu_t] = 0$  for all  $t \geq 0$ .

It is clear that  $(\mu_t(A))_t$  is a martingale for all  $A \in \Sigma$  by observing that equation (1) has no drift term. Furthermore, since  $\mu_t(\mathcal{X}) = \int F_t d\mu$  is differentiable by construction,  $(\mu_t(\mathcal{X}))$  has zero quadratic variation and thus is constant in  $t$ . With this in mind, as  $\mu_0 = \mu$  is a probability measure, it follows:

**Proposition 2.2.** If  $(\mu_t)_t$  is a linear-tilt localization of  $\mu$ , then  $\mu_t$  is a probability measure for each  $t$ .

**Corollary 2.3.** A linear-tilt localization  $(\mu_t)_t$  of  $\mu$  is a prelocalization of  $\mu$ .

We remark that in general, a linear-tilt localization is not necessarily a stochastic localization as (L3) might not be satisfied. It is possible to impose sufficient conditions on  $(Z_t)$  for which (L3) holds, e.g. by requiring  $\|\text{Cov}(Z_t)\|_{\text{op}}$  to decrease sufficiently fast. However, for generality, we will not restrict ourselves to one of these conditions. Instead, we will consider (L3) case by case in the following examples of linear-tilt schemes.

### 2.1.1 Linear-tilt localization driven by a Wiener process

A natural choice of  $(Z_t)_{t \geq 0}$  is the standard Wiener process on  $\mathbb{R}^n$ . Denoting  $(W_t)_{t \geq 0}$  a standard Wiener process on  $\mathbb{R}^n$ , we define the random functions  $(F_t)_{t \geq 0}$  to be the solution of the following infinite system of SDEs (existence and uniqueness is established by theorem 5.2 in [Øks03]):

$$F_0 = 1, dF_t(x) = \langle x - \bar{\mu}(F_t), dW_t \rangle F_t(x), \quad (2)$$

for all  $x \in \mathbb{R}^n$ . We shall from this point forward denote the random variables  $a_t := \bar{\mu}(F_t)$ .

By applying Itô's formula, we make the following useful observation: for all  $x \in \mathbb{R}^n$ ,

$$d \log F_t(x) = \frac{dF_t(x)}{F_t(x)} - \frac{d[F(x)]_t}{2F_t(x)^2} = \langle x - a_t, dW_t \rangle - \frac{1}{2} \|x - a_t\|^2 dt \quad (3)$$

where the second equality follows by the construction of  $F$ . Hence, as  $\log F_0(x) = 0$ , we observe

$$\begin{aligned} \log F_t(x) &= \int_0^t \langle x - a_s, dW_s \rangle - \frac{1}{2} \int_0^t \|x - a_s\|^2 ds \\ &= \left( \langle x, W_t \rangle - \int_0^t \langle a_s, dW_s \rangle \right) - \left( \frac{t}{2} \|x\|^2 + \frac{1}{2} \int_0^t \|a_s\|^2 ds - \int_0^t \langle x, a_s \rangle ds \right) \\ &= - \left( \int_0^t \langle a_s, dW_s \rangle + \frac{1}{2} \|a_s\|^2 ds \right) + \langle x, a_t + W_t \rangle - \frac{t}{2} \|x\|^2. \end{aligned}$$

Thus, taking  $dz_t := \langle a_t, dW_t \rangle + \frac{1}{2} \|a_t\|^2 dt$  and  $v_t := a_t + W_t$ , we observe  $F_t(x)$  is a density function with a Gaussian component of the form

$$F_t(x) = e^{z_t + \langle x, v_t \rangle - \frac{t}{2} \|x\|^2}, \quad (4)$$

for given Itô processes  $(z_t), (v_t)$ .

With this formulation of  $F_t(x)$  in mind, it follows that  $F_t$  is non-negative, and so, we may define  $(\mu_t)_t$  to be the process such that  $d\mu_t = F_t d\mu$ . It is clear that  $(\mu_t)_t$  is a linear-tilt localization of  $\mu$  and so, is a prelocalization of  $\mu$ .

The remainder of this section is devoted to showing  $(\mu_t)_t$  is furthermore a stochastic localization of  $\mu$  in the special case that it is log-concave (namely we will show (L3) for this special case), and prove some basic properties about this process useful for our analysis later.

**Definition 2.5** (Log-concave measure). A measure  $\mu$  on  $\mathbb{R}^n$  is said to log-concave if it is of the form  $d\mu = e^{-V} d\text{Leb}^n$  for some convex function  $V : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$

Straightaway, we observe that if  $\mu$  is log-concave,  $\mu_t$  has density of the form  $F_t e^{-V}$  with respect to the Lebesgue measure. Thus, as  $t$  increases, we observe that  $\mu_t$  becomes gradually more akin to a Gaussian measure. This fact is very beneficial for our analysis later.

To show  $(\mu_t)$  satisfies (L3) if  $\mu$  is log-concave, we study the limiting behavior of  $(\mu_t)$  as  $t \rightarrow \infty$  by considering their covariances:

$$A_t := \text{Cov}[\mu_t] = \int (x - a_t) \otimes (x - a_t) \mu_t(dx), \quad (5)$$

where  $\otimes$  denotes the Kronecker product. In particular, we will show  $(A_t)_{ij} \rightarrow 0$  for all  $i, j \in \{1, \dots, n\}$  as  $t \rightarrow \infty$  allowing us to conclude  $(\mu_t)$  converges weakly to some Dirac measure. Indeed, this is a direct consequence of the following lemma.

**Lemma 2.4** (Brascamp-Lieb, [BL76]). Given  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  convex and  $K > 0$ , if  $\nu$  is an isotropic (i.e.  $\mathbb{E}_{X \sim \nu}[X] = 0$  and  $\text{Cov}_{X \sim \nu}(X) = \text{id}_n$ ) probability measure on  $\mathbb{R}^n$  of the form

$$d\nu = Z e^{-V(x) - \frac{K}{2} \|x\|^2} d\text{Leb}^n(x)$$

with  $Z$  being the normalization constant, then  $\nu$  satisfy the Poincaré inequality. In particular, for all smooth  $\phi$ ,

$$\text{Var}_\nu[\phi] \leq K^{-1} \mathbb{E}_\nu[\|\nabla \phi\|^2].$$

With this lemma in mind, by taking  $\nu = \mu_t$  using equation (4) and defining  $\pi_i(x) := x_i$ , we have by the Cauchy-Schwarz inequality

$$(A_t)_{ij} \leq \sqrt{\text{Var}_{\mu_t}[\pi_i]} \sqrt{\text{Var}_{\mu_t}[\pi_j]} \leq \max_{k=1, \dots, n} \frac{1}{t} \int \|\nabla \pi_k\|^2 d\mu_t$$

Again, using equation (4), we note that any realizations of  $(F_t(x))$  is eventually decreasing in  $t$  for all  $x \neq 0$ , implying

$$\sup_{t>0} \max_{k=1, \dots, n} \int \|\nabla \pi_k\|^2 d\mu_t = \sup_{t>0} \max_{k=1, \dots, n} \int x_k^2 d\mu_t < \infty.$$

Thus, by taking  $t \rightarrow \infty$  we have  $(A_t)_{ij} \rightarrow 0$  for all  $i, j \in \{1, \dots, n\}$  as claimed and we have  $(\mu_t)$  satisfying (L3). Hence, the linear-tilt localization is indeed a stochastic localization of  $\mu$ .

**Corollary 2.5.**  $(\mu_t)$  converges set-wise to some Dirac measure almost everywhere. We denote this limiting (random) Dirac measure by  $\delta_{a_\infty}$  where  $a_\infty$  is some  $\mathbb{R}^n$ -valued random variable.



As a result of proposition 2.1, we also have the following corollaries.

**Corollary 2.6.** The massive point  $a_\infty$  of the limiting Dirac measure is the limit of  $a_t$  as  $t \rightarrow \infty$  and has law  $\mu$ .

**Corollary 2.7.** Taking  $\phi$  to be any continuous function, we define  $M_t = \int \phi d\mu_t$ . Then,  $(M_t)_{t \geq 0}$  is a martingale and

$$M_t \xrightarrow{\text{a.e.}} \phi(a_\infty) \sim \phi_* \mu \quad (6)$$

where  $\phi_* \mu$  denotes the push-forward measure of  $\mu$  along  $\phi$ .

### 2.1.2 Discrete time linear-tilt localization

We may construct an analogous version of the linear-tilt localization for discrete time. By utilizing the little- $o$  notation, equation (1) can be rewritten as

$$\frac{d\mu_{t+h}}{d\mu}(x) = \frac{d\mu_t}{d\mu}(x) + \langle x - \bar{\mu}(F_t), h dZ_t \rangle \frac{d\mu_t}{d\mu}(x) + o(h).$$

Hence, an discrete analog of the linear tilt localization is defined as the following.

**Definition 2.6** (Discrete time linear-tilt localization). Given a measure  $\mu \in \mathcal{M}(\mathcal{X})$ , the discrete time linear-tilt localization is the sequence of random measures  $(\mu_k)_k$  defined by  $\mu_0 = \mu$  and

$$d\mu_{k+1} = (1 + \langle x - \bar{\mu}_k, Z_k \rangle) d\mu_k \quad (7)$$

for some sequence of random variables such that  $\mathbb{E}[Z_k | \mu_k] = 0$  for all  $k \in \mathbb{N}$ .

The coordinate by coordinate localization can be formulated as a discrete time linear-tilt localization. Given  $\mu$  a probability measure on  $\{-1, 1\}^n$ , we recall that the coordinate by coordinate localization is defined by “pinning” an additional random coordinate after each time step. To phrase this as a linear-tilt localization, we take the random variables  $Z_k$  to be

$$Z_k := e_{Y_k} \cdot \begin{cases} \frac{1}{1 + (\bar{\mu}_k)_{Y_k}} & \text{with probability } \frac{1 + (\bar{\mu}_k)_{Y_k}}{2} \\ \frac{-1}{1 - (\bar{\mu}_k)_{Y_k}} & \text{with probability } \frac{1 - (\bar{\mu}_k)_{Y_k}}{2} \end{cases} \quad (8)$$

where  $e_1, \dots, e_n$  are the standard basis of  $\mathbb{R}^n$  and again  $Y$  is a uniform random variable over all permutations of  $[n]$ . Thus, the linear-tilt localization  $(\mu_k)$  given by this choice of  $(Z_k)$  is defined by

$$\mu_{k+1}(\sigma) = (1 + \|Z_k\|(\sigma_{Y_k} - (\bar{\mu}_k)_{Y_k}))\mu_k(\sigma)$$

for all  $k < n$ . Similar to before, we terminate the process at time  $n$  and so we extend the process to all times by taking  $\mu_k = \mu_{n \wedge k}$ .

Let us parse this definition to see why this is equivalent to the coordinate by coordinate localization. Taking  $k < n$ , we have at the  $k + 1$ -th step,  $Y_{k+1}$  chooses a random axis which had not been chosen before. Then,  $Z_k$  is chosen such that for each configuration  $\sigma$ , the probability of  $\sigma_{Y_k}$  being  $\pm 1$  is proportional the mass of  $\mu_k$  on  $\pm e_{Y_k}$ . This is precisely the steps needed to construct the coordinate by coordinate localization as conditioning on an additional axis in this case is simply a projection on to said axis.

### 3 Markov Mixing

An application of stochastic localizations is used to prove mixing bounds for Markov processes. Focusing on the framework established by Chen and Eldan in [CE22], we will present their prescribed method to bound the mixing time for a special class of Markov processes: namely the Markov chains associated with stochastic localizations. As a specific example, we will in this section also describe Chen and Eldan's [CE22] application of this framework to the Glauber dynamics and as a result, recover a version of the main theorem on spectral independence as presented by Anari, Liu and Oveis Gharan in [ALOG20].

#### 3.1 Mixing bounds

The motivation for Markov mixing bounds fundamentally comes from sampling. Suppose we wish to sample from some probability distribution  $\mu$ . A common method to achieve this to through the use of the Markov chain Monte Carlo (MCMC):

**Theorem 1** (MCMC). Given  $(X_n)$  an irreducible positively recurrent homogenous Markov process on  $\mathcal{X}$  with stationary distribution  $\mu$ , for any  $\phi : \mathcal{X} \rightarrow \mathbb{R}$  integrable,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \phi(X_k) = \int \phi d\mu$$

almost everywhere.

With this theorem in mind, MCMC allows us to sample  $\mu$  by sampling from a Markov chain instead. It is in general not difficult to come up with such Markov processes, although the difficulty often arises when we want to know its rate of convergence. This motivates the notion of mixing bounds which quantifies the time for which the Markov process takes before its law is approximately stationary.

**Definition 3.1** (Total variation mixing time). Given a probability measure  $\nu \in \mathcal{M}(\mathcal{X})$ , a Markov kernel  $K$  with stationary distribution  $\mu$  and some  $\epsilon > 0$ , the  $\epsilon$ -total variation mixing time is defined as

$$t_{\text{mix}}(P, \epsilon, \nu) := \inf\{t \geq 0 \mid \|K^t \nu - \mu\|_{\text{TV}} < \epsilon\},$$

Furthermore, we denote

$$t_{\text{mix}}(P, \epsilon) = \sup_{x \in \mathcal{X}} t_{\text{mix}}(P, \epsilon, \delta_x)$$

the worst mixing time starting at a point.

A standard method of analyzing the mixing times of Markov chains is through the use of the spectral gap.

**Theorem 2** ([LPW17]). Given a reversible and irreducible Markov chain with kernel  $K$  on the state space  $\mathcal{X}$  with stationary distribution  $\mu$ , denoting  $\mu_{\min} = \inf_{x \in \mathcal{X}} \mu(x)$ , we have

$$t_{\text{mix}}(K, \epsilon) \leq \left\lceil \frac{1}{\text{gap}(K)} \left( \frac{1}{2} \log \left( \frac{1}{\mu_{\min}} \right) + \log \left( \frac{1}{2\epsilon} \right) \right) \right\rceil,$$

where  $\text{gap}(K)$  is the spectral gap of  $K$  and is defined as

$$\text{gap}(K) := 1 - \sup\{\lambda \mid \lambda \text{ is an eigenvalue of } K, \lambda \neq 1\}.$$

It is not difficult to see that

$$\text{gap}(K) = \inf_{\substack{\phi: \mathcal{X} \rightarrow \mathbb{R} \\ \int \phi d\mu = 0}} 1 - \frac{\int \phi K \phi d\mu}{\int \phi^2 d\mu} = \inf_{\phi: \mathcal{X} \rightarrow \mathbb{R}} \frac{1}{2\text{Var}_\mu[\phi]} \int (\phi(x) - \phi(y))^2 K(x, dy) \mu(dx), \quad (9)$$

where  $\mu$  is the stationary measure of  $K$  and  $K\phi = \int \phi(y)K(\cdot, dy)$ . We will take equation (9) to be the defining property of the spectral gap in the case  $K$  is defined on a general state space. Nonetheless, as the inequality in theorem 2 is only meaningful whenever  $\mu_{\min} > 0$ , this theorem is only applicable for Markov chains on finite state space (and we can replace  $\inf$  with  $\min$ ).

### 3.1.1 Ising model and Glauber dynamics

The Ising model is a well-known measure in mathematical physics and is often used to model magnetism and gas behavior. While fundamentally, the Ising model is constructed using Bernoulli random variables, as these random variables interacts with one another, sampling the Ising model directly has exponential computational complexity increasing with the number of particles. Hence, to sample from the Ising model, one often instead applies MCMC to the Glauber dynamics which is a Markov chain with stationary measure being the Ising model. We will in this small section introduce these definitions and recall a known result [ALOG20] bounding the spectral gap and consequently the mixing time of the Glauber dynamics.

Heuristically, the Ising model measures the probability that a graph is in a specific configuration of up and down spins in which neighboring vertices are more likely to have the same spin. This “likeliness” is controlled by a parameter  $\beta$  in which a larger  $\beta$  means that neighboring vertices are more likely to align.

**Definition 3.2** (Ising model, [CE22]). Given a graph  $G = (V, E)$ ,  $\beta > 0$  and  $h \in \mathbb{R}$ , the Ising model on  $G$  with inverse temperature  $\beta$  and external field  $h$  is the probability measure  $\mu_{\beta, h}$  on  $\{-1, 1\}^V$  defined such that for all  $\sigma \in \{-1, 1\}^V$ ,

$$\mu_{\beta, h}(\sigma) := \frac{1}{Z} \exp \left\{ \beta \sum_{xy \in E} \sigma_x \sigma_y + h \sum_{x \in V} \sigma_x \right\}$$

where  $Z > 0$  is the normalizing constant.

To sample from the Ising model, one naive method is by simply starting from an arbitrary configuration  $\sigma_1 \in \{-1, 1\}^n$ , and at each time step, update the configuration according to the description of the Ising model. Namely, the configuration at the next time step either remains the same or change at one vertex. Furthermore, the probability of this occurring is weighted according to  $\mu$ . This procedure results in a Markov chain which turns out to have the Ising model as its stationary distribution. This Markov chain is known as the Glauber dynamics. Formally, it is defined as the following.

**Definition 3.3** (Glauber dynamics, [CE22]). Given a measure  $\mu$  of  $\{-1, 1\}^n$ , the Glauber dynamics of  $\mu$  is the Markov chain with kernel

$$K(\sigma^1, \sigma^2) := \mathbf{1}_{\{\|\sigma^1 - \sigma^2\|_1 = 1\}} \frac{1}{n} \frac{\mu(\sigma^2)}{\mu(\sigma^1) + \mu(\sigma^2)} + \mathbf{1}_{\{\sigma^1 = \sigma^2\}} \frac{1}{n} \sum_{\|\tilde{\sigma} - \sigma^1\|_1 = 1} \frac{\mu(\sigma^1)}{\mu(\sigma^1) + \mu(\tilde{\sigma})}, \quad (10)$$

where we define  $\|\sigma^1 - \sigma^2\|_1 := \frac{1}{2} \sum_{i=1}^n |\sigma_i^1 - \sigma_i^2|$  for all  $\sigma^1, \sigma^2 \in \{-1, 1\}^n$ .

Recently, Anari, Liu and Oveis Gharan in [ALOG20] were able to provide a bound for the spectral gap for a special class of Glauber dynamics satisfying a property known as spectral independence. We quickly recall their results here.

**Definition 3.4** (Pairwise influence matrix, [ALOG20]). Given a measure  $\mu$  on  $\{-1, 1\}^n$ , we define the pairwise influence matrix  $\Psi(\mu)$  such that it has entries

$$\Psi(\mu)_{i,j} := \mathbb{P}_{X \sim \mu}[X_i = 1 \mid X_j = 1] - \mathbb{P}_{X \sim \mu}[X_i = 1 \mid X_j = -1],$$

for all  $i \neq j$  and we set  $\Psi(\mu)_{i,i} = 0$  for all  $i$ .

**Definition 3.5** (Spectral independence, [ALOG20]). A measure  $\mu$  on  $\{-1, 1\}^n$  is said to be  $\eta$ -spectrally independent if the maximum eigenvalue of  $\Psi(\mu)$  is bounded above by  $\eta$ .

In addition, we say  $\mu$  is  $(\eta_0, \dots, \eta_{n-2})$ -spectrally independent if defining  $\mu_k$  to be the law of  $X$  conditioned on  $X_{i_1}, \dots, X_{i_k}$  for some  $X \sim \mu$  and any  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ ,  $\mu_k$  is  $\eta_k$ -spectrally independent for each  $k = 0, \dots, n-2$ .

**Theorem 3** ([ALOG20]). If  $\mu$  is a measure on  $\{-1, 1\}^n$  which is  $(\eta_0, \dots, \eta_{n-2})$ -spectrally independent, then the Glauber dynamics of  $\mu$  has spectral gap bound

$$\text{gap}(K) \geq \frac{1}{n} \prod_{k=0}^{n-2} \left(1 - \frac{\eta_k}{n-k-1}\right).$$

As an application of stochastic localizations, we will in section 3.3 recover an equivalent bound by using the theories established above. In particular, assuming the same assumptions as theorem 3, we will show that

$$\text{gap}(K) \geq \prod_{k=0}^{n-2} \left(1 - \frac{\eta_k + 1}{n-k}\right).$$

## 3.2 Dynamics of stochastic localizations

As alluded to in the introduction, one may associate a Markov process at each time step of a stochastic localization process for which the original process is stationary. We will in this section define these Markov processes and show that the Glauber dynamics can be constructed using this method.

**Definition 3.6** (Markov process associated with a stochastic localization, [CE22]). Let  $(\mu_t)_{t \geq 0}$  be a prelocalization of  $\mu$  such that  $\mu_t$  is absolutely continuous (almost everywhere) with respect to  $\mu$  for all  $t$ . For all  $\tau > 0$ , we define the dynamics associated with  $(\mu_t)_t$  at  $\tau$  to be the Markov process with kernel

$$K(x, A) := \mathbb{E}_{\mathbb{P}} \left[ \frac{d\mu_{\tau}}{d\mu}(x) \mu_{\tau}(A) \right]$$

for all  $x \in \mathcal{X}, A \in \Sigma$ .

As hinted by the notation, rather than a deterministic time  $\tau$ ,  $\tau$  can also be taken to be an appropriate stopping time. In this case, the theorems below will remain to hold by invoking the optional stopping theorem whenever necessary.

This construction indeed results in a kernel since for each  $x \in \mathcal{X}$ , as  $\mathbb{E}_{\mathbb{P}}[\mu_{\tau}] = \mu$  by (L2),

$$K(x, \Omega) = \mathbb{E}_{\mathbb{P}} \left[ \frac{d\mu_{\tau}}{d\mu}(x) \right] = \frac{d}{d\mu} \mathbb{E}_{\mathbb{P}}[\mu_{\tau}](x) = \frac{d\mu}{d\mu}(x) = 1$$

where the third equality follows by the uniqueness of the Radon-Nikodym derivative. In particular, for all  $A \in \Sigma$ , we have

$$\int_A \mathbb{E}_{\mathbb{P}} \left[ \frac{d\mu_{\tau}}{d\mu} \right] d\mu = \mathbb{E}_{\mathbb{P}} \left[ \int_A \frac{d\mu_{\tau}}{d\mu} d\mu \right] = \mathbb{E}_{\mathbb{P}}[\mu_{\tau}](A).$$

**Proposition 3.1.** The Markov process associated with a prelocalization  $(\mu_t)_{t \geq 0}$  of  $\mu$  is reversible and has stationary distribution  $\mu$ .

*Proof.* Taking  $\phi : \mathcal{X}^2 \rightarrow \mathbb{R}$  integrable, we have by Fubini's theorem

$$\begin{aligned} \int_{\mathcal{X}^2} \phi(x, y) K(x, dy) \mu(dx) &= \int \phi(x, y) \mathbb{E}_{\mathbb{P}} \left[ \frac{d\mu_{\tau}}{d\mu}(x) \mu_{\tau}(dy) \right] \mu(dx) \\ &= \mathbb{E}_{\mathbb{P}} \left[ \int \phi(x, y) \mu_{\tau}(dy) \frac{d\mu_{\tau}}{d\mu}(x) \mu(dx) \right] \\ &= \mathbb{E}_{\mathbb{P}} \left[ \int \phi(x, y) \mu_{\tau}(dy) \mu_{\tau}(dx) \right]. \end{aligned} \tag{11}$$

Similarly, by the same calculation,

$$\int_{\mathcal{X}^2} \phi(x, y) K(y, dx) \mu(dy) = \mathbb{E}_{\mathbb{P}} \left[ \int \phi(x, y) \mu_{\tau}(dy) \mu_{\tau}(dx) \right].$$

Thus,

$$\int_{\mathcal{X}^2} \phi(x, y) K(x, dy) \mu(dx) = \int_{\mathcal{X}^2} \phi(x, y) K(y, dx) \mu(dy)$$

for any integrable  $\phi : \mathcal{X}^2 \rightarrow \mathbb{R}$  implying  $K$  is reversible.

On the other hand, for all  $A \in \Sigma$ , we compute using the martingale property

$$\begin{aligned} K\mu(A) &= \int K(x, A) \mu(dx) = \int \mathbb{E}_{\mathbb{P}} \left[ \frac{d\mu_{\tau}}{d\mu}(x) \mu_{\tau}(A) \right] = \mathbb{E}_{\mathbb{P}} \left[ \mu_{\tau}(A) \int \frac{d\mu_{\tau}}{d\mu} d\mu \right] \\ &= \mathbb{E}_{\mathbb{P}}[\mu_{\tau}(A) \mu_{\tau}(\Omega)] = \mathbb{E}_{\mathbb{P}}[\mu_{\tau}(A)] = \mu(A) \end{aligned}$$

implying  $\mu$  is the stationary measure of  $K$ . □

**Proposition 3.2.** Taking  $K$  to be the kernel of the Markov process associated with a prelocalization  $(\mu_t)_{t \geq 0}$  of  $\mu$  at time  $\tau$ , we have

$$\text{gap}(K) = \inf_{\phi : \mathcal{X} \rightarrow \mathbb{R}} \frac{\mathbb{E}[\text{Var}_{\mu_{\tau}}[\phi]]}{\text{Var}_{\mu}[\phi]}.$$

*Proof.* By equation (11) (where we take  $\phi(x, y) = \phi(x)\phi(y)$ ), we have

$$\int_{\mathcal{X}^2} \phi(x)\phi(y)K(x, dy)\mu(dx) = \mathbb{E}_{\mathbb{P}} \left[ \left( \int_{\mathcal{X}} \phi d\mu_{\tau} \right)^2 \right], \quad (12)$$

for any integrable  $\phi : \mathcal{X} \rightarrow \mathbb{R}$ . On the other hand, we observe

$$\int_{\mathcal{X}^2} \phi(y)^2 K(x, dy)\mu(dx) = \int (K\phi^2)(x)\mu(dx) = \int \phi(x)^2 (K\mu)(dx) = \int \phi(x)^2 \mu(dx)$$

as  $\mu$  is the stationary measure of  $K$ . Thus, for any integrable  $\phi : \mathcal{X} \rightarrow \mathbb{R}$ , by substituting the above two equations, we have

$$\begin{aligned} & \frac{1}{2\text{Var}_{\mu}[\phi]} \int_{\mathcal{X}^2} (\phi(x) - \phi(y))^2 K(x, dy)\mu(dx) \\ &= \frac{1}{2\text{Var}_{\mu}[\phi]} \left( \int \phi(x)^2 \mu(dx) - 2 \int \phi(x)\phi(y)K(x, dy)\mu(dx) + \int \phi(y)^2 K(x, dy)\mu(dx) \right) \\ &= \frac{1}{\text{Var}_{\mu}[\phi]} \left( \int \phi^2 d\mu - \mathbb{E}_{\mathbb{P}} \left[ \left( \int_{\mathcal{X}} \phi d\mu_{\tau} \right)^2 \right] \right) \\ &= \frac{1}{\text{Var}_{\mu}[\phi]} \mathbb{E}_{\mathbb{P}} \left[ \int \phi^2 d\mu_{\tau} - \left( \int_{\mathcal{X}} \phi d\mu_{\tau} \right)^2 \right] = \frac{\mathbb{E}[\text{Var}_{\mu_{\tau}}[\phi]]}{\text{Var}_{\mu}[\phi]}. \end{aligned}$$

Hence, recalling the equivalent form of the spectral gap as described by (9), the result follows by taking infimum on both sides.  $\square$

With regards to theorem 2, in the case  $\mu$  has full support on the finite state space  $\mathcal{X}$  (e.g. in the setting of Glauber dynamics) the above proposition provides a method for computing an upper bound for the mixing time. In particular, should the prelocalization  $(\mu_k)_k$  satisfy

$$\mathbb{E}[\text{Var}_{\mu_{k+1}}[\phi] \mid \mu_k] \geq (1 - \epsilon)\text{Var}_{\mu_k}[\phi]$$

for given  $\epsilon > 0, k = 0, \dots, m-1$  and any integrable function  $\phi : \mathcal{X} \rightarrow \mathbb{R}$ , we have the telescoping product

$$\frac{\mathbb{E}[\text{Var}_{\mu_m}[\phi]]}{\text{Var}_{\mu}[\phi]} = \mathbb{E} \left[ \prod_{k=0}^{m-1} \frac{\mathbb{E}[\text{Var}_{\mu_{k+1}}[\phi] \mid \mu_k]}{\text{Var}_{\mu_k}[\phi]} \right] \geq (1 - \epsilon)^m.$$

Hence, we have the bound

$$\text{gap}(K)^{-1} \leq (1 - \epsilon)^{-m}$$

which immediately provides an upper bound for the mixing time of  $K$  in light of theorem 2. This motivates the following definition.

**Definition 3.7** (Discrete time approximate conservation of variance (ACV), [CE22]). A prelocalization process  $(\mu_k)_k$  is said to conserve  $(\eta_k)$ -variance up to time  $m$  if for any integrable function  $\phi : \mathcal{X} \rightarrow \mathbb{R}, 0 \leq k < m$ ,

$$\mathbb{E}[\text{Var}_{\mu_{k+1}}[\phi] \mid \mu_k] \geq (1 - \eta_k)\text{Var}_{\mu_k}[\phi].$$

By the same computation above, if  $(\mu_k)_k$  conserves  $(\eta_k)$ -variance up to time  $m$ , then it associated dynamics at time  $m$  has a spectral gap of at least  $\prod_{k=0}^{m-1} (1 - \eta_k)$ .

### 3.2.1 ACV of discrete time linear-tilt localization

In the case of the discrete time linear-tilt localization  $(\mu_k)_k$  of some  $\mu$ , we may compute an explicit value  $(\eta_k)$  for which  $(\mu_k)_k$  conserves  $(\eta_k)$ -variance. Indeed, given integrable  $\phi : \mathcal{X} \rightarrow \mathbb{R}$ , we have

$$\begin{aligned}
\mathbb{E}[\text{Var}_{\mu_{k+1}}[\phi] \mid \mu_k] &= \mathbb{E} \left[ \int \phi^2 d\mu_{k+1} - \left( \int \phi d\mu_{k+1} \right)^2 \mid \mu_k \right] \\
&= \int \phi^2 d\mu_k - \mathbb{E} \left[ \left( \int \phi(x)(1 + \langle x - \bar{\mu}_k, Z_k \rangle) \mu_k(dx) \right)^2 \mid \mu_k \right] \\
&= \int \phi^2 d\mu_k - \left( \int \phi d\mu_k \right)^2 - \mathbb{E} \left[ \left( \int \phi(x) \langle x - \bar{\mu}_k, Z_k \rangle \mu_k(dx) \right)^2 \mid \mu_k \right] \\
&= \text{Var}_{\mu_k}[\phi] - \text{Var} \left[ \int \phi(x) \langle x - \bar{\mu}_k, Z_k \rangle \mu_k(dx) \mid \mu_k \right]
\end{aligned} \tag{13}$$

where the third and last equality follows as  $\mathbb{E}[Z_k \mid \mu_k] = 0$ . Then, denoting

$$C_k := \text{Cov}[Z_k \mid \mu_k] \text{ and } U_k := \int \phi(x)(x - \bar{\mu}_k) \mu_k(dx),$$

equation (13) becomes

$$\mathbb{E}[\text{Var}_{\mu_{k+1}}[\phi] \mid \mu_k] = \text{Var}_{\mu_k}[\phi] - \|C_k^{1/2} U_k\|^2.$$

With this reduction, our goal is reduced to bounding  $\|C_k^{1/2} U_k\|^2$  by an expression of the form  $\eta_k \text{Var}_{\mu_k}[\phi]$  where  $\eta_k$  is independent of  $\phi$ . Indeed, taking  $\theta$  to be the normalized vector  $C_k^{1/2} U_k / \|C_k^{1/2} U_k\|$ , we have

$$\begin{aligned}
\|C_k^{1/2} U_k\|^2 &= \langle C_k^{1/2} U_k, \theta \rangle^2 = \left( \int \phi(x) \langle C_k^{1/2}(x - \bar{\mu}_k), \theta \rangle \mu_k(dx) \right)^2 \\
&= \left( \int \left( \phi(x) - \int \phi d\mu_k \right) \langle C_k^{1/2}(x - \bar{\mu}_k), \theta \rangle \mu_k(dx) \right)^2 \\
&\leq \left( \int \left( \phi(x) - \int \phi d\mu_k \right)^2 \mu_k(dx) \right) \int \langle C_k^{1/2}(x - \bar{\mu}_k), \theta \rangle^2 \mu_k(dx) \\
&= \text{Var}_{\mu_k}[\phi] \int \langle \theta, C_k^{1/2}(x - \bar{\mu}_k)^{\otimes 2} C_k^{1/2} \theta \rangle \mu_k(dx) = \text{Var}_{\mu_k}[\phi] \langle \theta, C_k^{1/2} \text{Cov}(\mu_k) C_k^{1/2} \theta \rangle \\
&\leq \text{Var}_{\mu_k}[\phi] \|C_k^{1/2} \text{Cov}(\mu_k) C_k^{1/2}\|_{\text{op}}
\end{aligned}$$

where the inequality is due to Cauchy-Schwarz. Hence, for any discrete linear-tilt localization  $(\mu_k)_k$ , we have

$$\mathbb{E}[\text{Var}_{\mu_{k+1}}[\phi] \mid \mu_k] \geq (1 - \|C_k^{1/2} \text{Cov}(\mu_k) C_k^{1/2}\|_{\text{op}}) \text{Var}_{\mu_k}[\phi] \tag{14}$$

for any integrable  $\phi : \mathcal{X} \rightarrow \mathbb{R}$  and thus, conserve  $(1 - \eta_k)$ -variance where we define  $\eta_k := \|C_k^{1/2} \text{Cov}(\mu_k) C_k^{1/2}\|_{\text{op}}$ .

### 3.2.2 ACV of linear-tilt localization driven by a Wiener process

In the setting of the linear-tilt localization driven by a Wiener process, a similar computation leveraging on the Cauchy-Schwarz inequality yields a useful inequality which turns out to be very useful with regards to the KLS conjecture. We will present the computation now while leaving its application to the KLS conjecture to section 4.

In the continuous time setting, it no longer made sense to analyze the difference of the variance between subsequent time steps. Instead, we will analyze its time derivative. In particular, recalling the definition of the martingale  $(M_t)$  from corollary 2.7, we may use the same method as the discrete time analysis to compute a bound for  $d[M_t]$ .

**Proposition 3.3.** Given the linear-tilt localization  $(\mu_t)_{t \geq 0}$  of  $\mu$  driven by a Wiener process as defined in section 2.1.1, for all isotropic  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ , we have

$$d[M]_t \leq \text{Var}_{\mu_t}[\phi] \|A_t\|_{\text{op}} dt,$$

where  $A_t := \int (x - a_t)^{\otimes 2} d\mu_t$  is the covariance matrix of  $\mu_t$  and  $M_t := \int \phi d\mu_t$ .

Before proving this proposition, we first recall that for a martingale  $(X_t)_t$ , its quadratic variation  $[X]_t$  is the unique adapted, continuous and non-decreasing process such that  $(X_t^2 - [X]_t)_t$  is a martingale. In particular, if  $X_0 = 0$ , we observe

$$\mathbb{E}[X_t^2] - \mathbb{E}[X]_t = \mathbb{E}[X_0^2] = 0$$

for all  $t$ . Thus,

$$\text{Var}[X_t] = \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 = \mathbb{E}[X]_t - \mathbb{E}[X_0]^2 = \mathbb{E}[X]_t.$$

Now, since in our case  $\phi$  is isotropic, we have  $M_0 = \int \phi d\mu = 0$  and so,  $\text{Var}[M_t] = \mathbb{E}[M]_t$ . Thus, by taking expectation on both sides, one observes the resemblance of the above equation with that of the discrete time ACV inequality.

*Proof.* We compute

$$\begin{aligned} dM_t &= d \int \phi(x) F_t(x) \mu(dx) = \int \phi(x) \langle x - a_t, dW_t \rangle \mu_t(dx) \\ &= \left\langle \int \phi(x)(x - a_t) \mu_t(dx), dW_t \right\rangle = \langle U_t, dW_t \rangle \end{aligned}$$

where we denote  $U_t := \int \phi(x)(x - a_t) \mu_t(dx)$  as in the discrete time case. Hence, by considering the component-wise quadratic variation, we have

$$d[M]_t = \|U_t\|^2 dt. \tag{15}$$

The remainder of the calculation is quite similar to the discrete case. Indeed, denoting  $\theta$  the



vector  $U_t$  normalized to have norm 1, we obtain,

$$\begin{aligned}
d[M]_t &= \langle U_t, \theta \rangle^2 dt = \left\langle \int \left( \phi(x) - \int \phi d\mu_t \right) (x - a_t) \mu_t(dx), \theta \right\rangle^2 dt \\
&= \left( \int \left( \phi(x) - \int \phi d\mu_t \right) \langle x - a_t, \theta \rangle \mu_t(dx) \right)^2 dt \\
&\leq \text{Var}_{\mu_t}[\phi] \left( \int \langle \theta, (x - a_t)^{\otimes 2} \theta \rangle \mu_t(dx) \right) dt \\
&= \text{Var}_{\mu_t}[\phi] \langle \theta, A_t \theta \rangle dt \leq \text{Var}_{\mu_t}[\phi] \|A_t\|_{\text{op}} dt.
\end{aligned} \tag{16}$$

where the second equality by applying Fubini's theorem and noting that  $\int (x - a_t) \mu_t(dx) = 0$ . Moreover, the first inequality follows again by the Cauchy-Schwarz inequality.  $\square$

### 3.3 Glauber dynamics as an associated Markov process

Glauber dynamics can be constructed as an associated Markov chain of the coordinate by coordinate localization. Namely, taking  $\mu \in \mathcal{M}(\{-1, 1\}^n)$  and  $\tau = n - 1$ , we will show the associate Markov kernel

$$\mathbb{E} \left[ \frac{\mu_{n-1}(\sigma^1) \mu_{n-1}(\sigma^2)}{\mu(\sigma^1)} \right] = K(\sigma^1, \sigma^2)$$

for all  $\sigma^1, \sigma^2 \in \{-1, 1\}^n$  where  $K(\sigma^1, \sigma^2)$  as defined by equation (10).

First, taking  $\sigma^1, \sigma^2 \in \{-1, 1\}^n$  such that  $\|\sigma^1 - \sigma^2\| = 1$ , say  $\sigma^1$  and  $\sigma^2$  differs at the  $m$ -th coordinate, we have

$$\mathbb{E} \left[ \frac{\mu_{n-1}(\sigma^1) \mu_{n-1}(\sigma^2)}{\mu(\sigma^1)} \right] = \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ \frac{\mu_{n-1}(\sigma^1) \mu_{n-1}(\sigma^2)}{\mu(\sigma^1)} \mid Y_n = k \right].$$

Now, if  $Y_n = k \neq m$ , the configuration is fixed at the  $m$ -th coordinate. However, as  $\sigma^1$  and  $\sigma^2$  differs at the  $m$ -th coordinate, either  $\mu_{n-1}(\sigma^1) = 0$  or  $\mu_{n-1}(\sigma^2) = 0$ . Thus, all but the  $m$ -th term in the above sum vanishes and we have the kernel equals

$$\begin{aligned}
&= \frac{1}{n} \mathbb{E} \left[ \frac{\mu_{n-1}(\sigma^1) \mu_{n-1}(\sigma^2)}{\mu(\sigma^1)} \mid Y_n = m \right] \\
&= \frac{1}{n} \mathbb{E} \left[ \frac{\mu_{n-1}(\sigma^1) \mu_{n-1}(\sigma^2)}{\mu(\sigma^1)} \mathbf{1}_{\{\text{supp } \mu_{n-1} = \{\sigma^1, \sigma^2\}\}} \mid Y_n = m \right] \\
&= \frac{\mathbb{P}(\text{supp } \mu_{n-1} = \{\sigma^1, \sigma^2\})}{n} \mathbb{E} \left[ \frac{\mu_{n-1}(\sigma^1) \mu_{n-1}(\sigma^2)}{\mu(\sigma^1)} \mid Y_n = m, \text{supp } \mu_{n-1} = \{\sigma^1, \sigma^2\} \right]
\end{aligned}$$

Then, substituting in the equalities,

$$\mathbb{P}(\text{supp } \mu_{n-1} = \{\sigma^1, \sigma^2\}) = \mu(\sigma^1) + \mu(\sigma^2),$$

and

$$\begin{aligned}
& \mathbb{E}[\mu_{n-1}(\sigma^1)\mu_{n-1}(\sigma^2) \mid Y_n = m, \text{supp } \mu_{n-1} = \{\sigma^1, \sigma^2\}] \\
&= \mathbb{E}[\mu_{n-1}(\sigma^1)\mu_{n-1}(\sigma^2) \mid \text{supp } \mu_{n-1} = \{\sigma^1, \sigma^2\}] \\
&= \frac{\mu(\sigma^1)}{\mu(\sigma^1) + \mu(\sigma^2)} \frac{\mu(\sigma^2)}{\mu(\sigma^1) + \mu(\sigma^2)}
\end{aligned}$$

we have

$$\mathbb{E} \left[ \frac{\mu_{n-1}(\sigma^1)\mu_{n-1}(\sigma^2)}{\mu(\sigma^1)} \right] = \frac{\mu(\sigma^1) + \mu(\sigma^2)}{n\mu(\sigma^1)} \frac{\mu(\sigma^1)}{\mu(\sigma^1) + \mu(\sigma^2)} \frac{\mu(\sigma^2)}{\mu(\sigma^1) + \mu(\sigma^2)} = \frac{1}{n} \frac{\mu(\sigma^2)}{\mu(\sigma^1) + \mu(\sigma^2)}$$

which is precisely the kernel of the Glauber dynamics of two neighboring configurations. On the other hand, if  $\|\sigma^1 - \sigma^2\| > 1$ , then

$$\mathbb{E} \left[ \frac{\mu_{n-1}(\sigma^1)\mu_{n-1}(\sigma^2)}{\mu(\sigma^1)} \right] = 0$$

as  $\mu_{n-1}$  have fixed all but one coordinate. Thus, the dynamics associated with the coordinate is precisely the Glauber dynamics (we don't need to check the case  $\sigma^1 = \sigma^2$  as they are kernels and so  $K(\sigma, \sigma) = 1 - \sum_{\tilde{\sigma} \neq \sigma} K(\sigma, \tilde{\sigma})$ ).

Now, as the coordinate by coordinate localization is a discrete linear-tilt localization as presented by section 2.1.2, we have by equation (14) that the spectral gap of the Glauber dynamics is bounded below by

$$\text{gap}(K) \geq \prod_{k=0}^{n-2} (1 - \|C_k^{1/2} \text{Cov}(\mu_k) C_k^{1/2}\|_{\text{op}}), \quad (17)$$

where  $C_k := \text{Cov}[Z_k \mid \mu_k]$  with  $Z_k$  defined by equation (8). Furthermore, by observing from definition of  $Z_k$  that  $(Z_k)_i (Z_k)_j = 0$  for all  $i \neq j$ ,  $C_k$  has only non-zero entries on its diagonal for which

$$\begin{aligned}
(C_k)_{i,i} &= (\text{Cov}[Z_k \mid \mu_k, Y_k = i])_{i,i} \mathbb{P}(Y_k = i) \\
&= \frac{1}{n-k} \left( \frac{1}{(1 + (\bar{\mu}_k)_i)^2} \frac{1 + (\bar{\mu}_k)_i}{2} + \frac{1}{(1 - (\bar{\mu}_k)_i)^2} \frac{1 - (\bar{\mu}_k)_i}{2} \right) \\
&= \frac{1}{n-k} \frac{1}{1 - (\bar{\mu}_k)_i^2} = \frac{1}{n-k} (\text{Cov}(\mu_k)_{i,i})^{-1}.
\end{aligned}$$

Thus, by introducing the correlation matrix  $\text{Cor}(\mu)$  with entries

$$\text{Cor}(\mu)_{i,j} = \frac{\text{Cov}(\mu)_{i,j}}{\sqrt{\text{Cov}(\mu)_{i,i}} \sqrt{\text{Cov}(\mu)_{j,j}}},$$

we have  $C_k^{1/2} \text{Cov}(\mu_k) C_k^{1/2} = \frac{1}{n-k} \text{Cor}(\mu_k)$  and equation (17) becomes

$$\text{gap}(K) \geq \prod_{k=0}^{n-2} \left( 1 - \frac{\|\text{Cor}(\mu_k)\|_{\text{op}}}{n-k} \right). \quad (18)$$

This inequality allows us to directly recover a bound for the spectral gap should  $\mu$  be spectrally independent. Indeed, denoting the maximum eigenvalue of a matrix  $A$  by  $\rho(A)$ , as  $\text{Cor}(\mu_k)$  is symmetric, we have

$$\|\text{Cor}(\mu_k)\|_{\text{op}} = \rho(\text{Cor}(\mu_k)).$$

Furthermore, by observing

$$\Psi(\mu_k) = \text{Cov}(\mu_k) \text{diag}(\text{Cov}(\mu_k))^{-1} - \text{Id}_n,$$

we have

$$\|\text{Cor}(\mu_k)\|_{\text{op}} = \rho(\text{Cov}(\mu_k) \text{diag}(\text{Cov}(\mu_k))^{-1}) = \rho(\Psi(\mu_k) + \text{Id}_n) = \rho(\Psi(\mu_k)) + 1.$$

Thus, if  $\mu$  is a measure on  $\{-1, 1\}^n$  which is  $(\eta_0, \dots, \eta_{n-2})$ -spectrally independent, we have  $\rho(\Psi(\mu_k)) \leq \eta_k$  for all  $k = 0, \dots, n-2$  and so, by equation (17), we have

$$\text{gap}(K) \geq \prod_{k=0}^{n-2} \left(1 - \frac{\eta_k + 1}{n - k}\right). \quad (19)$$

This bound and theorem 3 are said to be equivalent as both equations provide polynomial lower bounds for the spectral gap in terms of the spectral independence coefficients.

## 4 The KLS and Thin-Shell Conjecture

In this section, we present the original context and motivation for the construction of stochastic localizations: the KLS conjecture. The KLS conjecture is a conjecture which roughly states that all log-concave measures on  $\mathbb{R}^n$  are concentrated in a way similar to a Gaussian measure. To tackle this problem, Eldan in [Eld13] introduced stochastic localizations in order to reduce the KLS conjecture to a seemingly weaker conjecture (up to a logarithmic constant): the thin-Shell conjecture. More recently, stochastic localization was applied by Chen in [Che20] to provide an almost constant lower bound for the KLS conjecture.

We will now describe a proof of Eldan's original reduction of the KLS conjecture to the thin-Shell conjecture. The method presented in this section is due to Lee and Vempala [LV16] and reformulated in the language of concentration by Eldan in [Eld18].

### 4.1 Concentration

Let us quickly introduce some preliminary definitions required to state the aforementioned conjectures.

**Definition 4.1** (Concentration, [Eld18]). Let  $\mu$  be a measure on  $\mathbb{R}^n$ , then  $\mu$  is said to be  $C$ -concentrated if for all 1-Lipschitz function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$\text{Var}_\mu[\phi] = \text{Var}_{X \sim \mu}[\phi(X)] \leq \frac{1}{C^2}. \quad (20)$$

We denote the largest possible such  $C$  by  $C_{\text{con}}^\mu$ .

Heuristically, the concentration is a quantity which inversely measures how “spread out” the density of  $\mu$  is in each direction. In the 1-dimensional case, the concentration of  $\mu$  is therefore simply the inverse of its variance. This is illustrated by the following proposition.

**Proposition 4.1.** Let  $X$  be a  $\mathbb{R}^n$ -valued random variable. Then for all  $K$ -Lipschitz function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$\text{Var}[\phi(X)] \leq K^2 \text{Var}[X].$$

*Proof.* We first prove the proposition in the case that  $\mathbb{E}[X] = 0$ .

Let  $X'$  be a i.i.d. copy of  $X$  on the same probability space. Then for all  $K$ -Lipschitz function  $\phi$ , we have

$$\begin{aligned} 2\text{Var}[\phi(X)] &= \text{Var}[\phi(X) - \phi(X')] && \text{(i.i.d.)} \\ &= \mathbb{E}[(\phi(X) - \phi(X'))^2] - \mathbb{E}[\phi(X) - \phi(X')]^2 \\ &= \mathbb{E}[(\phi(X) - \phi(X'))^2] && \text{(identically distributed)} \\ &\leq K^2 \mathbb{E}[\|X - X'\|^2] && \text{(as } \phi \text{ is } K\text{-Lipschitz)} \\ &= K^2 \mathbb{E}[X^T X + X'^T X' - X^T X' - X'^T X] \\ &= 2K^2 \text{Var}[X] - 2K^2 \text{Cov}(X, X') = 2K^2 \text{Var}[X]. && \text{(independence)} \end{aligned}$$

implying  $\text{Var}[\phi(X)] \leq K^2 \text{Var}[X]$  as claimed.

For general  $X$ , by defining  $\phi'(x) := \phi(x + \mathbb{E}[X])$ , we can apply the 0 mean case to  $X - \mathbb{E}[X]$  and  $\phi'$  to obtain

$$\text{Var}[\phi(X)] = \text{Var}[\phi'(X - \mathbb{E}[X])] \leq K^2 \text{Var}[X - \mathbb{E}[X]] = K^2 \text{Var}[X]$$

as required.  $\square$

With this proposition in mind, it is clear that for  $\mathbb{R}$ -valued random variables  $X$ , its law  $\mu$  has concentration  $C_{\text{con}}^\mu = \text{Var}[X]^{-1}$ .

We note that the definition we are presenting here is slightly non-standard. However, utilizing the following remarkable result due to Milman, we show that this definition is equivalent to the following more standard definitions whenever we are working with log-concave measures.

**Definition 4.2** (Exponential concentration, [Mil18]). Given a measure  $\mu$  on  $\mathbb{R}^n$ , we say  $\mu$  has exponential concentration if there exists some  $c, D > 0$  such that for all 1-Lipschitz function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $t > 0$ , we have

$$\mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t) \leq ce^{-Dt}. \quad (21)$$

Fixing  $c = 1$ , we denote the largest possible  $D$  as  $D_{\text{exp}}^\mu$ .

**Definition 4.3** (First-moment concentration, [Mil18]). Again, taking  $\mu$  a measure on  $\mathbb{R}^n$ , we say  $\mu$  has first-moment concentration if there exists some  $D > 0$  such that for all 1-Lipschitz function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ , we have

$$\mathbb{E}_\mu[|\phi - \mathbb{E}_\mu[\phi]|] \leq \frac{1}{D}. \quad (22)$$

We denote the largest possible  $D$  by  $D_{\text{FM}}^\mu$ .

It is clear that exponential concentration implies first-moment concentration. Indeed, if  $\mu$  has exponential concentration with constant  $D$  (taking  $c = 1$ ), then by the tail probability formula,

$$\mathbb{E}_\mu[|\phi - \mathbb{E}_\mu[\phi]|] = \int_0^\infty \mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t) dt \leq \int_0^\infty e^{-Dt} dt = \frac{1}{D}.$$

On the other hand, Milman showed that for log-concave measures on  $\mathbb{R}^n$ , exponential concentration and first-moment concentration are equivalent in the following sense.

**Theorem 4** (Milman, [Mil08]). For all log-concave measure  $\mu$  on  $\mathbb{R}^n$ ,  $\mu$  has exponential concentration if and only if  $\mu$  has first-moment concentration. Furthermore,  $D_{\text{exp}}^\mu \simeq D_{\text{FM}}^\mu$  where we write  $A \simeq B$  if there exists universal constants  $C_1, C_2 > 0$  such that  $C_1 A \leq B \leq C_2 A$ .

With this theorem in mind, we establish the following correspondence.

**Proposition 4.2.** For all measures  $\mu$  on  $\mathbb{R}^n$ , we have

$$\text{Exponentially concentrated} \implies \text{Concentrated} \implies \text{First-moment concentrated}$$

and  $D_{\text{exp}}^\mu \leq \sqrt{2} C_{\text{con}}^\mu$  and  $C_{\text{con}}^\mu / 2 \leq D_{\text{FM}}^\mu$ . Hence, if  $\mu$  is log-concave,  $D_{\text{exp}}^\mu \simeq D_{\text{FM}}^\mu \simeq C_{\text{con}}^\mu$ .

*Proof.* Assume first that  $\mu$  is  $C$ -concentrated. Then by the Chebyshev inequality, we have

$$\mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t) \leq \frac{1}{t^2} \text{Var}_\mu[\phi] \leq \frac{1}{C^2 t^2},$$

for all 1-Lipschitz  $\phi$ . Thus, by tail probability,

$$\begin{aligned}\mathbb{E}_\mu[|\phi - \mathbb{E}_\mu[\phi]|] &= \int_0^\infty \mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t) dt \\ &\leq \inf_{a>0} \left\{ \int_0^a \mu(|\phi - \mathbb{E}_\mu[\phi]| \geq t) dt + \frac{1}{C^2} \int_a^\infty \frac{1}{t^2} dt \right\} \\ &\leq \inf_{a>0} \left\{ a + \frac{1}{aC^2} \right\} = \frac{2}{C},\end{aligned}$$

implying  $\mu$  is first-moment concentrated with respect to the constant  $C/2$ .

On the other hand, if  $\mu$  is exponential concentrated with some constant  $D$ , then again by the tail probability,

$$\text{Var}_\mu[\phi] = \int_0^\infty \mu((\phi - \mathbb{E}_\mu[\phi])^2 \geq t) dt \leq \int_0^\infty e^{-D\sqrt{t}} dt = \frac{2}{D^2}$$

implying  $\mu$  is  $D/\sqrt{2}$ -concentrated.  $\square$

## 4.2 Example: concentration of the Gaussian

Given a class of measures, e.g. measures of a certain distribution, it is in general *not* true that the concentration coefficient of measures of said class is dimension invariant. However, this turns out to be the case for the class of standard Gaussian measures.

**Theorem 5** (Concentration of Gaussian measures). Denoting  $\gamma^n$  the standard Gaussian measure on  $\mathbb{R}^n$ ,  $\gamma^n$  is  $C$ -concentrated for some constant  $C$  which is independent of  $n$ . That is, for all 1-Lipschitz  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\text{Var}_{\gamma^n}[\phi] \leq C^{-2}$ .

This fact motivates the KLS conjecture which hypothesized that this invariance holds for a larger class of measures known as the log-concave measures. We will for completeness give a brief proof (based on theorem 1.7.1 in [Bog98]) of the above theorem taking  $C = 2/\pi$  (although one can further show Gaussian measures are 1-concentrated).

To prove this theorem we first observe the following elementary property of the Gaussian measure.

**Lemma 4.3.** For all  $x \in \mathbb{R}^n$ , we have  $\mathbb{E}_{\gamma^n}[|\langle x, \cdot \rangle|^2] = \|x\|^2$ .

*Proof.* Defining  $f_x := \langle x, \cdot \rangle$ , we have  $\mathbb{E}_{\gamma^n}[|\langle x, \cdot \rangle|^2] = \mathbb{E}_{(f_x)_* \gamma^n}[|\cdot|^2]$  where  $(f_x)_* \gamma^n$  is the push-forward of  $\gamma^n$  along  $f_x$ . Since  $f_x$  is linear, it follows that  $(f_x)_* \gamma^n \sim \mathcal{N}(0, \|x\|^2)$ . Hence, the result follows as  $\mathbb{E}_{(f_x)_* \gamma^n}[|\cdot|^2] = \text{Var}_{(f_x)_* \gamma^n}[\text{id}] = \|x\|^2$ .  $\square$

With this in mind, fixing a smooth 1-Lipschitz function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  (we can assume smoothness since any 1-Lipschitz function can be uniformly approximated by smooth 1-Lipschitz functions), we will now attempt to bound  $\int \int |\phi(x) - \phi(y)|^2 \gamma^n(dx) \gamma^n(dy)$  by first bounding it by the inte-

gral of an inner product. In particular, for all  $x, y \in \mathbb{R}^n$ , we observe

$$\begin{aligned} |\phi(x) - \phi(y)| &= \left| \int_0^{\pi/2} \partial_\theta \phi(x \sin \theta + y \cos \theta) d\theta \right| \leq \int_0^{\pi/2} |\partial_\theta \phi(x \sin \theta + y \cos \theta)| d\theta \\ &= \int_0^{\pi/2} |\langle \nabla \phi(x \sin \theta + y \cos \theta), x \cos \theta - y \sin \theta \rangle| d\theta. \end{aligned}$$

Then, by rescaling  $d\theta$ , we may apply Jensen's inequality resulting in

$$\begin{aligned} |\phi(x) - \phi(y)|^2 &\leq \left( \int_0^{\pi/2} |\langle \nabla \phi(x \sin \theta + y \cos \theta), x \cos \theta - y \sin \theta \rangle| d\theta \right)^2 \\ &\leq \frac{\pi}{2} \int_0^{\pi/2} |\langle \nabla \phi(x \sin \theta + y \cos \theta), x \cos \theta - y \sin \theta \rangle|^2 d\theta \end{aligned}$$

Thus, we have

$$\begin{aligned} &\int \int |\phi(x) - \phi(y)|^2 \gamma^n(dx) \gamma^n(dy) \\ &\leq \frac{\pi}{2} \int_0^{\pi/2} d\theta \int \int |\langle \nabla \phi(x \sin \theta + y \cos \theta), x \cos \theta - y \sin \theta \rangle|^2 \gamma^n(dx) \gamma^n(dy). \end{aligned}$$

Now, by substituting  $u = x \sin \theta + y \cos \theta$ ,  $v = x \cos \theta - y \sin \theta$  (which Jacobian has determinant 1), we have

$$\begin{aligned} \int \int |\phi(x) - \phi(y)|^2 \gamma^n(dx) \gamma^n(dy) &\leq \frac{\pi}{2} \int_0^{\pi/2} d\theta \int \int |\langle \nabla \phi(u), v \rangle|^2 \gamma^n(du) \gamma^n(dv) \\ &\leq \frac{\pi}{2} \int_0^{\pi/2} d\theta \int \|\nabla \phi(u)\|^2 \gamma^n(du) \end{aligned}$$

where the second inequality is due to lemma 4.3. Hence, as  $\phi$  is 1-Lipschitz,  $\|\nabla \phi(u)\| \leq 1$  for all  $u$  and thus,

$$\int \int |\phi(x) - \phi(y)|^2 \gamma^n(dx) \gamma^n(dy) \leq \frac{\pi}{2} \int_0^{\pi/2} d\theta \int d\gamma^n = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}.$$

The immediate consequence of this bound is that there exists some  $z \in \mathbb{R}^n$  such that  $\int |\phi(x) - z|^2 \gamma^n(dx) \leq \pi^2/4$ . Indeed, if no such  $z$  exists, then  $\int |\phi(x) - \phi(y)|^2 \gamma^n(dx) > \pi^2/4$  for all  $y$ , implying  $\int \int |\phi(x) - \phi(y)|^2 \gamma^n(dx) \gamma^n(dy) > \pi^2/4$  which contradicts the above bound. Hence, choosing such a  $z$ , we conclude the proof of theorem 5 since

$$\text{Var}_{\gamma^n}[\phi] = \min_{w \in \mathbb{R}^n} \int |\phi(x) - w|^2 \gamma^n(dx) \leq \int |\phi(x) - z|^2 \gamma^n(dx) \leq \left(\frac{\pi}{2}\right)^2$$

as required.

### 4.3 The KLS and thin-shell conjecture

Finally, with all the set-up and motivation in mind, we may now state the KLS conjecture. As alluded to in the previous section, heuristically, the KLS conjecture suggests that any log-concave measure on  $\mathbb{R}^n$  admits concentration roughly similar to that of the Gaussian measure. Namely, we would like to say something of the form “all log-concave measures are concentrated by a constant which is independent of the measure”. However, as currently stated, this statement is obviously false as spreading out the measure decreases the concentration. Furthermore, we need to be careful since, unlike the Gaussian, the concentration of general log-concave measures are not invariant under linear functions. Hence, it is clear that the KLS conjecture would not hold without a suitable normalization. This leads us to the following formulation of the KLS conjecture.

**Conjecture 1** (Kannan-Lovász-Simonovitz, [Eld18]). Denoting  $\mathcal{M}_{\text{iso}}^n$  the set of all isotropic log-concave probability measures  $\mu$  on  $\mathbb{R}^n$  (recall that  $\mu$  is isotropic if  $\mathbb{E}_{X \sim \mu}[X] = 0$  and  $\text{Cov}_{X \sim \mu}(X) = \text{id}_n$ ), there exists a *universal* constant  $C$  (i.e. does not depend on any parameter and in particular is independent of the dimension  $n$ ) such that for all  $\mu \in \mathcal{M}_{\text{con}}^n$ ,  $\mu$  is  $C$ -concentrated.

To address our concern that in general the concentration of log-concave measures are not invariant under linear maps, we observe that all isotropic measures  $\mu$  satisfies  $\text{Var}_{\mu}[T] \leq 1$  for all 1-Lipschitz linear maps  $T : \mathbb{R}^n \rightarrow \mathbb{R}$ . Indeed, if  $T : \mathbb{R}^n \rightarrow \mathbb{R}$  is a 1-Lipschitz linear function, i.e. it is of the form  $v \mapsto w^T v + d$  for some  $w \in S^{n-1}$  and  $d \in \mathbb{R}$ , then we have

$$\text{Var}_{\mu}[T] = \text{Var}_{X \sim \mu} \left[ \sum_{i=1}^n w_i X_i + d \right] = \sum_{i,j=1}^n w_i w_j \text{Cov}_{X \sim \mu}(X_i, X_j) = \sum_{i=1}^n w_i^2 = 1,$$

as  $\text{Cov}_{X \sim \mu}(X) = \text{id}$ .

With the above property, we may observe a trivial bound on the concentration of  $\mu$ : Taking  $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$  to be the projection into the  $i$ -th coordinate, as  $\pi_i$  is linear and 1-Lipschitz for all  $i$ , we have  $\text{Var}_{X \sim \mu}[X_i] \leq 1$ . Thus,

$$\begin{aligned} \text{Var}_{X \sim \mu}[X] &= \mathbb{E}_{X \sim \mu}[X^T X] - \mathbb{E}_{X \sim \mu}[X]^T \mathbb{E}_{X \sim \mu}[X] \\ &= \sum_{i=1}^n (\mathbb{E}_{X \sim \mu}[X_i^2] - (\mathbb{E}_{X \sim \mu}[X_i])^2) = \sum_{i=1}^n \text{Var}_{\mu}[\pi_i] \leq n. \end{aligned}$$

Thus, by proposition 4.1, we have that  $\text{Var}_{\mu}[\phi] \leq n$  for all 1-Lipschitz  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  implying  $\mu$  is  $n^{-1/2}$ -concentrated<sup>1</sup>.

**Conjecture 2** (Thin-shell, [Eld13]). Taking  $\mathcal{M}_{\text{iso}}^n$  as above, there exists a universal constant  $C$  such that for all  $\mu \in \mathcal{M}_{\text{con}}^n$ , we have

$$\sqrt{\text{Var}_{\mu}[\|\cdot\|]} \leq \frac{1}{C}.$$

Since the norm function is 1-Lipschitz, it is clear that the thin-shell conjecture is weaker than that of the KLS conjecture. On the other hand, as we shall describe in the following subsections, as a consequence of the theory of stochastic localization, it is possible to reduce the KLS conjecture to the thin-shell conjecture up to a logarithmic factor.

<sup>1</sup>This is also a consequence of theorem 2 in [LV16].



**Theorem 6** (Eldan, [Eld13]). Denoting  $\mathcal{M}_{\text{iso}}^n$  as above, we define

$$C_{\text{con}}^n := \sup \{C \mid \forall \mu \in \mathcal{M}_{\text{con}}^n, \mu \text{ is } C\text{-concentrated}\},$$

and

$$C_{\text{TS}}^n := \sup \{C \mid \forall \mu \in \mathcal{M}_{\text{con}}^n, \sqrt{\text{Var}_{\mu}[\|\cdot\|]} \leq C^{-1}\},$$

we have,

$$C_{\text{con}}^n \leq C_{\text{TS}}^n \leq C_{\text{con}}^n \log n.$$

The stochastic localization scheme has been wildly successful in making progress towards the KLS conjecture. Modifying the original arguments by Eldan, Lee and Vempala in [LV16] obtained the bound  $C_{\text{con}}^n \lesssim n^{-1/4}$ . Further modifying their arguments, a recent breakthrough by Chen in [Che20] improves the bound providing the following theorem.

**Theorem 7** (Chen, [Che20]).  $\log C_{\text{con}}^n \lesssim \sqrt{\log n \log \log n}$  and so  $C_{\text{con}}^n = n^{-o(1)}$ .

#### 4.3.1 Equivalent formulation of the KLS conjecture

While we have formulated the KLS conjecture using the language of concentration, the conjecture can be instead formulated in several equivalent ways. As demonstrated by proposition 4.2, it is clear that the KLS conjecture can be equivalently formulated using the exponential or first moment concentration. Moreover, two other common formulation of the KLS conjecture involves the Cheeger constant and the Poincaré inequality. For completeness of this exposition, we will now also briefly present the latter two equivalent formulations here.

The Cheeger constant is a quantity relating to the proportion of the volume of a set with its surface area. This quantity arises naturally when studying the isoperimetric problem. The isoperimetric problem is the problem in finding the set of unit volume with minimum surface area. In the case of  $\mathbb{R}^n$  equipped with the Lebesgue measure, we have known since the ancient Greeks that the solution is the unit ball. With this in mind, the KLS conjecture can be viewed geometrically as a natural extension of the isoperimetric problem replacing the Lebesgue measure with general log-concave measures on  $\mathbb{R}^n$ .

**Definition 4.4** (Minkowski's boundary measure). Given a measure  $\mu$  on  $\mathbb{R}^n$  and a Borel set  $A \subseteq \mathbb{R}^n$ , the Minkowski's boundary measure of  $A$  is defined as

$$\mu^+(\partial A) := \limsup_{\epsilon \downarrow 0} \frac{\mu(A_\epsilon) - \mu(A)}{\epsilon}.$$

where  $A_\epsilon := \{x \in \mathbb{R}^n \mid \text{dist}(x, A) \leq \epsilon\}$  is the  $\epsilon$ -thickening of  $A$ .

A similar definition for the boundary measure  $\mu^-$  can be given by taking the liminf instead. This does not matter in our case as any convex subsets of  $\mathbb{R}^n$  are regular and the two definitions coincide.

The isoperimetric problem for the measure  $\mu$  then becomes the problem of finding the set  $A$  of a fixed volume with minimum boundary measure.

**Definition 4.5** (Cheeger's inequality, [Mil08]). Given a probability measure  $\mu$  on  $\mathbb{R}^n$ , we say  $\mu$  satisfy Cheeger's inequality if there exists some  $D$  such that for all  $A$  with  $\mu(A) \leq 1/2$ ,

$$D\mu(A) \leq \mu^+(A).$$

We call the largest such  $D$  the Cheeger's constant and denote it by  $D_C^\mu$ .

With these definitions, the KLS conjecture can be equivalently reformulated as the following.

**Conjecture 3** (KLS, [Eld13]). Denoting as before  $\mathcal{M}_{\text{iso}}^n$  the set of all log-concave and isotropic probability measures  $\mu$  on  $\mathbb{R}^n$ , there exists a *universal* constant  $D$  such that for all  $\mu \in \mathcal{M}_{\text{iso}}^n$ ,  $\mu$  satisfy the Cheeger's inequality with constant  $D$ .

The equivalence of the reformulation follows by completing theorem 4.2 with two additional equivalences.

**Theorem 8** (Milman, [Mil08]). For all log-concave measure  $\mu$  on  $\mathbb{R}^n$ , the following are equivalent

- $\mu$  has exponential concentration with constant  $D_{\text{exp}}^\mu$ .
- $\mu$  has first-moment concentration with constant  $D_{\text{FM}}^\mu$ .
- $\mu$  satisfy the Cheeger's inequality with constant  $D_C^\mu$ .
- $\mu$  satisfy the Poincaré inequality: there exists some  $D > 0$  such that for all smooth  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying  $\int \phi d\mu = 0$ , we have

$$D\text{Var}_\mu[\phi] \leq \mathbb{E}_\mu[\|\nabla\phi\|^2] = \int \|\nabla\phi\|^2 d\mu.$$

We denote the largest such  $D$  by  $D_P^\mu$ .

Furthermore,  $D_{\text{exp}}^\mu \simeq D_{\text{FM}}^\mu \simeq D_P^\mu \simeq (D_C^\mu)^2$ .

With this theorem and proposition 4.2 in mind, it is clear that the KLS conjecture can be instead formulated with any of these inequalities instead. It is also worthwhile to pay attention to the formulation using the Poincaré inequality as it is relevant to the subsequent section on the log-Sobolev inequality.

**Conjecture 4** (KLS, [Eld13]). Denoting  $\mathcal{M}_{\text{iso}}^n$  as above, there exist a *universal* constant  $D$  such that for all  $\mu \in \mathcal{M}_{\text{iso}}^n$ ,  $\mu$  satisfy the Poincaré inequality with constant  $D$ .

#### 4.4 Reduction of KLS to thin-shell

We will now present a proof of theorem 6. As a high level overview, recall that the linear-tilt localization of a given measure is a measure-valued martingale for which the original measure is recovered in the limit. Then, as the concentration of the measure relates to the covariance of said measure, we will stop the martingale before the covariance grows too large. This allows us to analyze the martingale in a more tractable manner. However, as the sequence is a martingale, some properties are invariant in time and hence allowing us to conclude that these properties also hold for the original measure.

We recall the goal of theorem 6 is to control  $\text{Var}_\mu[\phi]$  by a logarithmic factor of  $\text{Var}_\mu[\|\cdot\|]$ . We will assume  $\mu$  is supported on  $B_n(0) \subseteq \mathbb{R}^n$  with  $B_n(0)$  the ball at the origin of radius  $n$ . The reason for this is due to a concentration bound for log-concave measures where one may show most of their densities lie within a compact support. As a result, the region outside said compact

set only contributes a bounded amount (in fact, it decreases in  $n$ ) to the variance, and does not affect our computation (c.f. [Kla06]). Thus, we also have

$$\text{supp } \mu_t = \text{supp } F_t \mu \subseteq \text{supp } \mu \subseteq B_n(0)$$

for all  $t > 0$ .

Let us fix  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  some 1-Lipschitz function and let  $(M_t)$  be the martingale as described in corollary 2.7, we have  $\text{Var}_\mu[\phi] = \text{Var}[\phi(a_\infty)]$  where  $a_\infty \sim \mu$ . Then, for all  $t > 0$ , by the law of total variance and the martingale property we have

$$\begin{aligned} \text{Var}_\mu[\phi] &= \text{Var}[M_\infty] = \text{Var}[\mathbb{E}[M_\infty | \mu_t]] + \mathbb{E}[\text{Var}[M_\infty | \mu_t]] \\ &= \text{Var}[M_t] + \mathbb{E}[\text{Var}[M_\infty | \mu_t]] \end{aligned} \quad (23)$$

where we introduce the notation  $\text{Var}[X | \mathcal{G}] := \mathbb{E}[(X - \mathbb{E}[X | \mathcal{G}])^2 | \mathcal{G}]$  for some random variable  $X$  and sub- $\sigma$ -algebra  $\mathcal{G}$ . Furthermore, by replacing  $\phi$  in corollary 2.7 with  $\phi^2$  and denoting the resulting martingale  $N_t := \int \phi^2 d\mu_t$ , we obtain  $N_t \rightarrow \phi(a_\infty)^2$  and hence,

$$\begin{aligned} \text{Var}[M_\infty | \mu_t] &= \mathbb{E}[\phi(a_\infty)^2 | \mu_t] - M_t^2 = N_t - M_t^2 \\ &= \int \phi^2 d\mu_t - \left( \int \phi d\mu_t \right)^2 = \text{Var}_{\mu_t}[\phi] \end{aligned}$$

Combining this with equation (23), we obtain

$$\text{Var}_\mu[\phi] = \text{Var}[M_t] + \mathbb{E}[\text{Var}_{\mu_t}[\phi]]. \quad (24)$$

for any  $t \geq 0$ . Furthermore, by applying the optional stopping theorem, the same equality holds when we take  $t$  to be a stopping time.

At this point, by recalling proposition 3.3, we recognize that the first term  $\text{Var}[M_t] = \mathbb{E}[M_t]$  is controlled by the operator norm of  $A_t$  with  $A_t$  being the covariance matrix of  $\mu_t$ . Thus, to bound the first term, the idea is to choose an appropriate stopping time  $\tau$  to stop the process before  $\|A_t\|_{\text{op}}$  grows too large. On the other hand, for the given  $\tau$ , by plugging in equation (4) into lemma 2.4, the second term  $\text{Var}_{\mu_\tau}[\phi]$  is then bounded by  $\tau^{-1}$  for which the expectation can be bounded explicitly.

We dedicate the remainder of this section to describe said procedure in detail.

#### 4.4.1 Analysis of the covariance matrix

As demonstrated in section 2.1.1, we know the limiting behavior of the covariance matrices, namely  $A_t \rightarrow 0$  point-wise as  $t \rightarrow \infty$ . This was important for us to establish the existence of the limit of  $(a_t)$  and  $(M_t)$ . However, as shown above, we now require some quantitative bounds for the operator norm of  $A_t$ . For this purpose, we first compute some useful properties of  $A_t$ .

Observing

$$\int dF_t(x) \mu(dx) = \int \langle x - a_t, dW_t \rangle \mu_t(dx) = \left\langle \int x \mu_t(dx) - a_t, dW_t \right\rangle = 0,$$

we have

$$\begin{aligned} da_t &= d \int x F_t(x) \mu(dx) = \int x dF_t(x) \mu(dx) = \int (x - a_t) dF_t(x) \mu(dx) \\ &= \int (x - a_t) \langle x - a_t, dW_t \rangle F_t(x) \mu(dx) = \int (x - a_t)^{\otimes 2} dW_t \mu_t(dx) = A_t dW_t \end{aligned} \quad (25)$$

where the second to last equality used the fact that  $v \langle v, w \rangle = v^{\otimes 2} w$  for any appropriate  $v, w$ . Similarly, computing using Itô's formula, we have

$$\begin{aligned} dA_t &= d \int (x - a_t)^{\otimes 2} F_t(x) \mu(dx) \\ &= \int (x - a_t)^{\otimes 2} dF_t(x) + F_t(x) d(x - a_t)^{\otimes 2} \\ &\quad - 2(x - a_t) \otimes d[a_t, F_t(x)]_t + F_t(x) d[a_t]_t \mu(dx). \end{aligned} \quad (26)$$

The second term vanishes as

$$\int F_t(x) d(x - a_t)^{\otimes 2} \mu(dx) = -2da_t \otimes \overbrace{\int (x - a_t) \mu_t(dx)}^{=0} = 0.$$

Also, by equation (25),  $da_t = A_t dW_t$  implying  $d[a_t]_t = A_t^2 dt$ . Finally, as both  $(a_t)$  and  $(F_t(x))$  are martingales,  $d[a_t, F_t(x)]_t = F_t(x) A_t x dt$  and the third term becomes

$$\begin{aligned} -2 \int (x - a_t) \otimes d[a_t, F_t(x)]_t \mu_t(dx) &= -2A_t \left( \int (x - a_t) \otimes x \mu_t(dx) \right) dt \\ &= -2A_t \left( \overbrace{\int (x - a_t)^{\otimes 2} \mu_t(dx)}^{A_t} + \overbrace{\int (x - a_t) \mu_t(dx) \otimes a_t}^{=0} \right) dt \\ &= -2A_t^2 dt. \end{aligned}$$

Hence, combining these and equation (2) together in (26), we have

$$dA_t = \int (x - a_t)^{\otimes 2} \langle x - a_t, dW_t \rangle \mu_t(dx) - A_t^2 dt$$

However, since we wish to bound  $A_t$  from above, as the drift term  $-A_t^2 dt$  only contributes negatively, an upper bound for the process of the form  $\int (x - a_t)^{\otimes 2} \langle x - a_t, dW_t \rangle \mu_t(dx)$  is also sufficient for  $A_t$ . Hence, we proceed by ignoring the drift term and redefine the process  $A_t$  such that

$$dA_t = \int (x - a_t)^{\otimes 2} \langle x - a_t, dW_t \rangle \mu_t(dx). \quad (27)$$

With this justification, we now proceed to bound the operator norm of this new  $A_t$ . In particular, as  $A_t$  is symmetric, we recall that  $\|A_t\|_{\text{op}} = \max_{i=1, \dots, n} \lambda_i(t) = \|(\lambda_i(t))_{i=1}^n\|_{\infty}$  where  $\lambda_i(t)$  denotes the distinct eigenvalues of  $A_t$ . Hence, it suffices to find a bound for the potential

$$\Phi^\alpha(t) = \sum_{i=1}^n |\lambda_i(t)|^\alpha = \|(\lambda_i(t))_{i=1}^n\|_\alpha^\alpha \quad (28)$$

for some  $\alpha > 0$ . Furthermore, as  $A_t$  is positive semi-definite,  $\lambda_i(t) \geq 0$  for all  $i = 1, \dots, n$  and thus we have  $\Phi^\alpha(t) = \sum_{i=1}^n \lambda_i(t)^\alpha$ . Again, to proceed, we will attempt to compute  $d\Phi^\alpha(t)$  at some  $t = t_0 > 0$  utilizing the following simple lemma.

**Lemma 4.4.** If  $A = [a_{ij}]$  is a diagonal matrix with distinct eigenvalues  $\lambda_i, \dots, \lambda_n$ , then for all  $i, j, k, l, m \in 1, \dots, n$ , we have

- $\frac{\partial \lambda_i}{\partial a_{jk}} = \delta_{ij} \delta_{ik}$ ;
- whenever  $i \neq j$ ,  $\frac{\partial^2 \lambda_i}{\partial a_{ij}^2} = 2(\lambda_i - \lambda_j)^{-1}$ ;
- and for  $j \neq l, k \neq m$  or  $i \neq j$  and  $i \neq k$ ,  $\frac{\partial^2 \lambda_i}{\partial a_{jk} \partial a_{lm}} = 0$ ,

where  $\delta_{ij}$  denotes the Kronecker delta function.

As this lemma requires the matrix to be diagonal, denoting  $e_1, \dots, e_n$  as the normalized eigenbasis of  $A_{t_0}$  (they are in fact orthonormal as  $A_{t_0}$  is positive semi-definite), we will consider  $A_t$  with respect to this basis by considering the entries

$$a_{ij}(t) := \langle e_i, A_t e_j \rangle.$$

Using equation (27), we compute

$$\begin{aligned} da_{ij}(t) &= \left\langle e_i, \left( \int (x - a_t)^{\otimes 2} \langle x - a_t, dW_t \rangle \mu_t(dx) \right) e_j \right\rangle \\ &= \left\langle \int \langle e_i, (x - a_t)^{\otimes 2} e_j \rangle (x - a_t) \mu_t(dx), dW_t \right\rangle = \langle \xi_{ij}, dW_t \rangle \end{aligned}$$

where we introduce the notation  $\xi_{ij} = \int \langle e_i, (x - a_t)^{\otimes 2} e_j \rangle (x - a_t) \mu_t(dx)$ . Thus, combining this with lemma 4.4, denoting  $\lambda_i = \lambda_i(t_0)$ , we have by Itô's formula

$$\begin{aligned} d\lambda_i(t) &= \sum_{j,k=1}^n \frac{\partial \lambda_i}{\partial a_{jk}} da_{jk}(t) + \frac{1}{2} \sum_{j,k=1}^n \sum_{l,m=1}^n \frac{\partial^2 \lambda_i}{\partial a_{jk} \partial a_{lm}} d[a_{jk}, a_{lm}]_t \\ &= \langle \xi_{ii}, dW_t \rangle + \sum_{j \neq i} \frac{d[a_{ij}]_t}{\lambda_i - \lambda_j} = \langle \xi_{ii}, dW_t \rangle + \sum_{j \neq i} \frac{\|\xi_{ij}\|^2}{\lambda_i - \lambda_j} dt. \end{aligned} \tag{29}$$

at  $t = t_0$ . As a result, it is also clear that  $d[\lambda_i(t)]_{t_0} = \|\xi_{ii}\|^2 dt$ .

Again applying Itô's formula, we may finally compute

$$\begin{aligned}
d\Phi^\alpha(t) &= \sum_{i=1}^n \frac{\partial \Phi^\alpha}{\partial \lambda_i} \Big|_{t=t_0} d\lambda_i(t) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \Phi^\alpha}{\partial \lambda_i \partial \lambda_j} \Big|_{t=t_0} d[\lambda_i, \lambda_j]_t \\
&= \alpha \sum_{i=1}^n \lambda_i^{\alpha-1} d\lambda_i(t) + \frac{1}{2} \alpha(\alpha-1) \sum_{i=1}^n \lambda_i^{\alpha-2} d[\lambda_i(t)]_t \\
&= \alpha \sum_{i=1}^n \lambda_i^{\alpha-1} \left( \langle \xi_{ii}, dW_t \rangle + \sum_{j \neq i} \frac{\|\xi_{ij}\|^2}{\lambda_i - \lambda_j} dt \right) + \frac{1}{2} \alpha(\alpha-1) \sum_{i=1}^n \lambda_i^{\alpha-2} d[\lambda_i(t)]_t \\
&= \alpha \sum_{i \neq j} \lambda_i^{\alpha-1} \frac{\|\xi_{ij}\|^2}{\lambda_i - \lambda_j} dt + \frac{1}{2} \alpha(\alpha-1) \sum_{i=1}^n \lambda_i^{\alpha-2} \|\xi_{ii}\|^2 dt + \left\langle \underbrace{\alpha \sum_{i=1}^n \lambda_i^{\alpha-1} \xi_{ii}}_{=: v_t}, dW_t \right\rangle \\
&= \frac{1}{2} \alpha \sum_{i \neq j} \|\xi_{ij}\|^2 \frac{\lambda_i^{\alpha-1} - \lambda_j^{\alpha-1}}{\lambda_i - \lambda_j} dt + \frac{1}{2} \alpha(\alpha-1) \sum_{i=1}^n \lambda_i(t)^{\alpha-2} \|\xi_{ii}\|^2 dt + \langle v_t, dW_t \rangle \\
&\leq \frac{1}{2} \alpha(\alpha-1) \sum_{i \neq j} \|\xi_{ij}\|^2 (\lambda_i \vee \lambda_j)^{\alpha-2} dt + \frac{1}{2} \alpha(\alpha-1) \sum_{i=1}^n \lambda_i(t)^{\alpha-2} \|\xi_{ii}\|^2 dt + \langle v_t, dW_t \rangle \\
&= \frac{1}{2} \alpha(\alpha-1) \sum_{i,j=1}^n \|\xi_{ij}\|^2 (\lambda_i \vee \lambda_j)^{\alpha-2} dt + \langle v_t, dW_t \rangle \leq \alpha^2 \sum_{i,j=1}^n \|\xi_{ij}\|^2 \lambda_i^{\alpha-2} dt + \langle v_t, dW_t \rangle,
\end{aligned}$$

where the first inequality holds as

$$\frac{\lambda_i^{\alpha-1} - \lambda_j^{\alpha-1}}{\lambda_i - \lambda_j} = \lambda_i^{\alpha-2} + \lambda_i^{\alpha-3} \lambda_j + \cdots + \lambda_i^{\alpha-2} \leq (\alpha-1)(\lambda_i \vee \lambda_j)^{\alpha-2}.$$

Thus, we have shown

$$d\Phi^\alpha(t) \leq \alpha^2 \sum_{i=1}^n \lambda_i(t)^{\alpha-2} \sum_{j=1}^n \|\xi_{ij}\|^2 dt + \langle v_t, dW_t \rangle \quad (30)$$

where  $v_t := \alpha \sum_{i=1}^n \lambda_i^{\alpha-1} \xi_{ii}$ .

By recalling that our goal is to bound  $\|A_t\|_{\text{op}}$  from above (c.f. equation (24) and (16)), we may assume without loss of generality that  $\|A_t\|_{\text{op}} \geq 1$ . Thus, applying the reverse Cauchy-Schwarz inequality to equation (30), we have

$$\begin{aligned}
d\Phi^\alpha(t) &\leq 2\alpha^2 \sum_{i=1}^n \lambda_i(t)^{\alpha-2} \sum_{j=1}^n \|\xi_{ij}\|^2 dt + \langle v_t, dW_t \rangle \\
&\leq 2\alpha^2 \|A_t\|_{\text{op}}^2 \sum_{i=1}^n \lambda_i(t)^{\alpha-2} \sum_{j=1}^n \|\xi_{ij}\|^2 dt + \langle v_t, dW_t \rangle \\
&\lesssim 2\alpha^2 \sum_{i=1}^n \lambda_i(t)^\alpha \sum_{j=1}^n \|\xi_{ij}\|^2 dt + \langle v_t, dW_t \rangle.
\end{aligned}$$

Thus, defining  $K_t := \sup_i \sum_{j=1}^n \|\xi_{ij}\|^2$ , we have the asymptotic bound

$$d\Phi^\alpha(t) \lesssim 2\alpha^2 K_t \Phi^\alpha(t) dt + \langle v_t, dW_t \rangle. \quad (31)$$

#### 4.4.2 Stopping the process early

As outlined in the beginning of this section, we will stop the process early in order to provide a bound for the right hand side of equation (24). By observing equation (16), we hypothesize that we should stop the process once  $\|A_t\|_{\text{op}}$  grows too large. As a result we define the stopping time

$$\tau := \inf\{t > 0 \mid \|A_t\|_{\text{op}} > 2\} \wedge 1.$$

By the optional stopping theorem, proposition 3.3 and the fact that we had assumed  $\mu$  is supported in a ball of radius  $n$ , we have

$$\begin{aligned} [M]_\tau &= \int_0^\tau d[M]_t \leq \int_0^\tau \overbrace{\text{Var}_{\mu_t}[\phi]}^{\leq t^{-1} \wedge n^2} \overbrace{\|A_t\|_{\text{op}}}^{\leq 2} dt \\ &\leq 2 \int_0^\tau t^{-1} \wedge n^2 dt \leq 2 \int_0^1 t^{-1} \wedge n^2 dt = 2 + 4 \log n. \end{aligned}$$

Combining this with equation (24) (where we recall that  $\text{Var}[M_t] = \mathbb{E}[M]_t$ ), we obtain

$$\text{Var}_\mu[\phi] \leq 2 + 4 \log n + \mathbb{E}[\tau^{-1}], \quad (32)$$

and it remains to find an upper bound for  $\mathbb{E}[\tau^{-1}]$ . Observing that  $t < \tau$  whenever  $\Phi^\alpha(t) < 2^\alpha$ , we define  $\sigma$  the first time for which the potential  $\Phi^\alpha(t)$  reaches  $2^\alpha$ , namely

$$\sigma := \inf\{t > 0 \mid \Phi^\alpha(t) = 2^\alpha\},$$

we have  $\sigma^{-1} \geq \tau^{-1}$  and so it suffices to bound  $\sigma$  from below.

For simplicity, we will ignore the martingale term in equation (31) and regard it as an ODE. Then, by Gronwall's inequality, if we can find some constant  $K$  such that  $K_t \leq K$  for all  $t \leq \tau$ , we have the bound

$$S_t \leq n e^{2\alpha^2 K t}.$$

Thus, substituting  $\sigma$  into the above, we have

$$2^\alpha = S_\sigma \leq n e^{2\alpha^2 K \sigma}$$

implying

$$\frac{\alpha \log 2 - \log n}{2\alpha^2 K} \leq \sigma \leq \tau.$$

Then, taking  $\alpha = 10K \log n$ , it is easy to check that

$$\frac{1}{10K \log n} \leq \frac{\alpha \log 2 - \log n}{2\alpha^2 K}$$

implying  $\mathbb{E}[\tau^{-1}] \leq 10K \log n$ . Of course, this deduction only holds while ignoring the stochastic term  $\langle v_t, dW_t \rangle$ . Nonetheless, this can be justified as one can show that  $\|v_t\|_2$  is bounded  $\alpha \Phi^\alpha(t)$  and so the same analysis holds by applying the stochastic Gronwall's inequality (c.f. second part of lemma 34 in [LV18]).

Finally, to find a bound for  $(K_t)$ , we employ the following lemma.

**Lemma 4.5** (Lemma 1.6 in [Eld13]). Denoting  $C_{\text{TS}}^n$  as in theorem 6, there exists a constant  $C$  such that for any log-concave, isotropic probability measure  $\mu$ , we have

$$\sup_{\theta \in S^{n-1}} \sum_{i,j=1}^n \mathbb{E}_{X \sim \mu} [\langle X, e_i \rangle \langle X, e_j \rangle \langle X, \theta \rangle]^2 \leq C \sum_{k=1}^n \frac{1}{k(C_{\text{TS}}^n)^2},$$

where  $\{e_1, \dots, e_n\}$  is any orthonormal basis on  $\mathbb{R}^n$ .

Recalling that

$$\xi_{ij} = \mathbb{E}_{X+a_t \sim \mu_t} [\langle e_i, X^{\otimes 2} e_j \rangle X] = \mathbb{E}_{X+a_t \sim \mu_t} [\langle X, e_i \rangle \langle X, e_j \rangle X],$$

we have by Parseval's identity

$$\begin{aligned} K_t &= \sup_i \sum_{j=1}^n \|\xi_{ij}\|^2 = \sup_i \sum_{j=1}^n \|\mathbb{E}_{X+a_t \sim \mu_t} [\langle X, e_i \rangle \langle X, e_j \rangle X]\|^2 \\ &= \sup_i \sum_{j=1}^n \sum_{k=1}^n \langle \mathbb{E}_{X+a_t \sim \mu_t} [\langle X, e_i \rangle \langle X, e_j \rangle X], e_k \rangle^2 \\ &= \sup_i \sum_{j,k=1}^n \mathbb{E}_{X+a_t \sim \mu_t} [\langle X, e_i \rangle \langle X, e_j \rangle \langle X, e_k \rangle]^2 \\ &\leq \sup_{\theta \in S^{n-1}} \sum_{i,j=1}^n \mathbb{E}_{X+a_t \sim \mu} [\langle X, e_i \rangle \langle X, e_j \rangle \langle X, \theta \rangle]^2. \end{aligned}$$

We note that we cannot direct apply lemma 4.5 at this point since the measure  $\mu_t$  might not be isotropic. Hence, to be able to use the lemma, we need to normalize the covariance of  $\mu_t$ . Namely, taking  $X + a_t \sim \mu_t$ , we define  $Y = A^{-1/2}X$  which by construction is isotropic. Thus, by observing that

$$\mathbb{E}_{X+a_t \sim \mu} [\langle X, e_i \rangle \langle X, e_j \rangle \langle X, \theta \rangle]^2 \leq \|A_t\|_{\text{op}}^3 \mathbb{E}_{X+a_t \sim \mu} [\langle Y, e_i \rangle \langle Y, e_j \rangle \langle Y, \theta \rangle]^2,$$

we have

$$\begin{aligned} K_t &\leq \sup_{\theta \in S^{n-1}} \sum_{i,j=1}^n \mathbb{E}_{X+a_t \sim \mu} [\langle X, e_i \rangle \langle X, e_j \rangle \langle X, \theta \rangle]^2 \\ &\leq \|A_t\|_{\text{op}}^3 \sup_{\theta \in S^{n-1}} \sum_{i,j=1}^n \mathbb{E}_{X+a_t \sim \mu} [\langle Y, e_i \rangle \langle Y, e_j \rangle \langle Y, \theta \rangle]^2 \leq 8C \sum_{k=1}^n \frac{1}{k(C_{\text{TS}}^n)^2} \end{aligned} \tag{33}$$

where the last inequality follows as  $\|A_t\|_{\text{op}} \leq 2$  for all  $t < \tau$ .

At last, combining equation (32) and (33), we have

$$\text{Var}_{\mu}[\phi] \leq 2 + \log n \left( 4 + 80C \sum_{k=1}^n \frac{1}{k} (C_{\text{TS}}^n)^{-2} \right) = \Theta_n((C_{\text{TS}}^n)^{-1} \log n)^2$$

implying there exists a constant  $R > 0$  such that for all 1-Lipschitz  $\phi$ ,  $\sqrt{\text{Var}_{\mu}[\phi]} \leq R(C_{\text{TS}}^n)^{-1} \log n$ , i.e.  $\mu$  is  $(R \log n)^{-1} C_{\text{TS}}^n$ -concentrated and so,  $C_{\text{TS}}^n \lesssim C_{\text{con}}^n \log n$  as required.



## 5 Log-Sobolev Inequality via Stochastic Localization

In this section, we will take a look at an application of the stochastic localization technic to prove a version of the log-Sobolev inequality for the log-concave measures. The log-Sobolev inequality is a class of inequalities central to the concentration of measures. Using stochastic localization, we will in this section bound the log-Sobolev constant for log-concave measures supported in a ball of fixed diameter. We take heavy inspiration from the proof as presented in section 4 where we reduced the KLS conjecture to the thin-shell conjecture.

### 5.1 Entropy and log-Sobolev inequalities

Heuristically, similar to that of the variance, the entropy of a random variable is a measure of its uncertainty or randomness. Formally, the entropy is defined as the following.

**Definition 5.1** (Entropy). Given  $\phi : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$  and a measure  $\mu$ , we define the entropy of  $\phi$  with respect to  $\mu$  to be

$$\text{Ent}_\mu[\phi] := \mathbb{E}_\mu \left[ \phi \log \left( \frac{1}{\mathbb{E}_\mu[\phi]} \phi \right) \right] = \int \phi \log \phi d\mu - \int \phi d\mu \log \left( \int \phi d\mu \right)$$

with the convention that  $0 \log 0 = 0$ .

The log-Sobolev inequality can be then formulated as the following.

**Definition 5.2** (Log-Sobolev inequality, [LV16]). For a given measure  $\mu$  on  $\mathbb{R}^n$ ,  $\mu$  is said to satisfy the log-Sobolev inequality with log-Sobolev constant  $\rho_\mu$  if  $\rho_\mu$  is the largest  $\rho$  such that for all smooth  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\int \phi^2 d\mu = 1$ , we have

$$\frac{\rho}{2} \text{Ent}_\mu[\phi^2] \leq \mathbb{E}_\mu[\|\nabla \phi\|^2] = \int \|\nabla \phi\|^2 d\mu.$$

The log-Sobolev inequality is an incredibly useful inequality while studying the concentration of measures. In particular, should a measure satisfy a log-Sobolev equality, by using a well-known method known as Herbst's argument, one can then obtain the exponential concentration of said measure via the Chernoff bound.

We observe the similarity of the log-Sobolev inequality with that of the Poincaré inequality. Remarkably, a similar inequality to the Cheeger's inequality can be defined and a correspondence similar to the correspondence between the Poincaré constant and the Cheeger's constant can be made for the log-Sobolev constant and the log-Cheeger's constant.

**Definition 5.3** (Log-Cheeger's inequality). Given a probability measure  $\mu$  on  $\mathbb{R}^n$ , we say  $\mu$  satisfies the log-Cheeger's inequality if there exists some  $C$  such that for all  $A$  with  $\mu(A) \leq 1/2$ ,

$$C\mu(A)\sqrt{-\log(\mu(A))} \leq \mu^+(A).$$

We call the least such  $C$  the log-Cheeger's constant and denote it by  $C_\mu$ . It is known that  $\rho_\mu \simeq C_\mu^2$ .

It does not come as a surprise that as with the Poincaré inequality, the standard Gaussian measure  $\gamma^n$  in  $\mathbb{R}^n$  satisfies the log-Sobolev inequality with log-Sobolev constant  $\rho_{\gamma^n} = 1$  (c.f. [Gro75]).

However, in contrast to the Poincaré inequality where the KLS conjecture suggests that the Poincaré constant is bounded below by a universal constant, it is known that the log-Sobolev constant cannot be bounded below by a universal constant. Nonetheless, we are interested in the log-Sobolev constant of log-concave measure. More specifically, we will study the log-Sobolev constant for log-concave measures supported in a ball of fixed diameter to obtain the following result.

**Theorem 9** ([LV16]). For any isotropic log-concave measure  $\mu$  on  $\mathbb{R}^n$  with support on a ball of diameter  $D$ ,  $\mu$  has log-Sobolev constant  $\rho_\mu \gtrsim D^{-1}$ .

## 5.2 Entropy of bounded support

Before moving directly to proving theorem 9, let us first attempt to gain some intuition in how the entropy of a general measure with bounded support behaves. In contrast to the variance, where the variance of a measure with bounded support is trivially bounded by the radius squared of said support, the bound for the entropy is less clear. Nonetheless, a similar result can be established for general measures with bounded support.

**Lemma 5.1.** For  $\mu$  a measure on  $\mathbb{R}^n$  supported in a ball of diameter  $D \geq \epsilon$  for some fixed  $\epsilon > 0$ , there exists a constant  $C_\epsilon$  (depending only on  $\epsilon$ ) such that

$$\text{Ent}_\mu[\phi^2] \leq C_\epsilon D^2$$

for all smooth 1-Lipschitz functions  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ .

**Definition 5.4** (Sub-Gaussian measures in  $\mathbb{R}$ ). A measure  $\mu$  on  $\mathbb{R}$  is said to be sub-Gaussian with parameter  $\nu > 0$  if

$$\psi_\mu(\lambda) = \log m_\mu(\lambda) = \log \mathbb{E}_{X \sim \mu}[e^{\lambda X}] \leq \frac{\lambda^2 \nu}{2}$$

for all  $\lambda \in \mathbb{R}$ . In this case, we denote  $\mu \in \mathcal{G}(\nu)$ .

**Lemma 5.2** (Hoeffding's lemma, [Jog23]). For a measure  $\mu$  on  $\mathbb{R}$  with support of diameter  $D$ , we have  $\mu \in \mathcal{G}(D^2/4)$ .

For any smooth  $\phi$  with support on  $D$ , we know that  $\phi^2$  has Lipschitz coefficient at most  $2D\text{Lip}(\phi)$ . Thus, for our measure  $\mu$  on  $\mathbb{R}^n$  which has support on a ball of diameter  $D$  and any smooth 1-Lipschitz  $\phi$ , we have that the push-forward measure of  $\mu$  along  $\phi^2$ ,  $(\phi^2)_*\mu$  is a measure on  $\mathbb{R}$  with support with diameter at most  $2D^2$ . Consequently, Hoeffding's lemma tells us that  $(\phi^2)_*\mu \in \mathcal{G}(D^4)$ . With this in mind, we can now prove lemma 5.1.

*Proof of lemma 5.1.* For ease of notation, let us denote  $f = \phi^2$ . Consider

$$\exp(\text{Ent}_\mu[f]) = \exp(\mathbb{E}_\mu[f \log f] - \mathbb{E}_\mu[f] \log \mathbb{E}[f]).$$

By considering that the function  $x \mapsto -x \log x$  has its maximum at  $e^{-1}$ , we have

$$-\mathbb{E}_\mu[f] \log \mathbb{E}[f] \leq e^{-1}$$

and thusly  $\exp(\text{Ent}_\mu[f]) \leq e^{e^{-1}} \exp(\mathbb{E}_\mu[f \log f])$  and we attempt to bound  $\exp(\mathbb{E}_\mu[f \log f])$ . By Jensen's inequality, we have

$$\exp(\mathbb{E}_\mu[f \log f]) \leq \mathbb{E}_\mu[\exp(f \log f)] = \mathbb{E}_\mu[f e^f].$$

Now, by Taylor expansion,

$$\mathbb{E}_\mu[f e^f] = \mathbb{E}_\mu \left[ f \sum_{n=0}^{\infty} \frac{f^n}{n!} \right] = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \mathbb{E}[f^n] \quad (34)$$

since the convergence is absolute.

On the other hand, by the above remark,  $f_*\mu$  is a sub-Gaussian measure with parameter  $D^4$ . Thus, we can obtain a bound on the moments of  $\mu$ . Indeed, since

$$\psi_{f_*\mu}(\lambda) = \log m_{f_*\mu}(\lambda) \leq \frac{\lambda^2 D^4}{2},$$

we have  $m_{f_*\mu}(\lambda) \leq e^{\lambda^2 D^4/2}$  for all  $\lambda \in \mathbb{R}$  where  $m_{f_*\mu}$  is the moment generating function of  $f_*\mu$ . Hence, Taylor expanding both sides, we obtain

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \partial_\lambda^n m_{f_*\mu}(0) \leq \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{n!} D^{4n}$$

for all  $\lambda \in \mathbb{R}$ . Thus, we must have

$$\mathbb{E}_\mu[f^{2n}] = \partial_\lambda^{2n} m_{f_*\mu}(0) \leq D^{4n}$$

and

$$\mathbb{E}_\mu[f^{2n+1}] = \partial_\lambda^{2n+1} m_{f_*\mu}(0) \leq 0.$$

Substituting the above into equation (34), we obtain

$$\mathbb{E}_\mu[f e^f] \leq \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} D^{4n} = D^2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} (D^2)^{2n-1} = D^2 \sinh D^2.$$

Thus,

$$\text{Ent}_\mu[f] \leq \log(e^{e^{-1}} \mathbb{E}_\mu[f e^f]) \leq e^{-1} + \log(D^2) + \log(\sinh D^2).$$

Now, since  $\log x + \log(\sinh x) \leq 2x$  and as we had assumed  $D \geq \epsilon$ , we have  $e^{-1} \leq e^{-1} \epsilon^{-2} D^2$ . Hence,

$$\text{Ent}_\mu[f] \leq (2 + e^{-1} \epsilon^{-2}) D^2 = C_\epsilon D^2$$

as desired.  $\square$

Unfortunately, this approach cannot result in a log-Sobolev inequality as the Lipschitz condition on  $\phi$  is too strong. Nonetheless, by restricting to the class of log-concave measures, Kannan, Lovász and Montenegro [LV07] were able to provide a bound for the log-Sobolev inequality of the same order.

**Lemma 5.3** ([LV07]). For  $\nu$  an isotropic log-concave measure on  $\mathbb{R}^n$  supported in a ball of diameter  $D$ , we have

$$\text{Ent}_\nu[\phi^2] \lesssim D^2 \mathbb{E}_\nu[\|\nabla \phi\|^2]$$

for all smooth functions  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ .

### 5.3 Log-Sobolev inequality via stochastic localization

As promised in the introduction, we will in this section present a modified proof of theorem 9. We attempt to analyze the log-Sobolev inequality using the same method as prescribed by section 4.4. As it turns out, while the initial steps can be followed verbatim, this method runs into issues when trying to bound the entropy of the martingale term resulted from the localization. In particular, applying the Cauchy-Schwarz inequality no longer provides the desired integral as it had done for the variance case. Instead, we will consider a different martingale  $N_t := \int \phi^2 \log \phi^2 d\mu_t$  and utilize the Burkholder-Davis-Gundy inequality.

Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  be smooth. Similar to before, we express the entropy of  $\phi^2$  with respect to  $\mu$  as the entropy of the martingale terms, in particular, denoting the martingale  $M_t := \int \phi^2 d\mu_t$ , we will show that

$$\text{Ent}_\mu[\phi^2] = \text{Ent}[M_t] + \mathbb{E}[\text{Ent}_{\mu_t}[\phi^2]], \quad (35)$$

In fact, a more general fact is true. Namely, for any real-valued function  $\Psi$ , defining the  $\Psi$ -entropy of  $\phi$  with respect to  $\mu$  as

$$\text{Ent}_\mu^\Psi[\phi] = \mathbb{E}_\mu[\Psi(\phi)] - \Psi(\mathbb{E}_\mu[\phi]),$$

we have that for  $\phi$  satisfying  $\Psi(\mathbb{E}_\mu[\phi]) = 0$ ,

$$\text{Ent}_\mu^\Psi[\phi] = \text{Ent}^\Psi[M_t] + \mathbb{E}[\text{Ent}_{\mu_t}^\Psi[\phi]], \quad (36)$$

where  $M_t := \int \phi d\mu_t$  is a martingale. This generalizes both (35) and (24) by taking  $\Psi(x) = x \log x$  and  $\Psi(x) = x^2$  respectively.

The derivation is rather simple. By definition

$$\text{Ent}_{\mu_t}^\Psi[\phi] = \mathbb{E}_{\mu_t}[\Psi(\phi)] - \Psi(\mathbb{E}_{\mu_t}[\phi]) = \mathbb{E}_{\mu_t}[\Psi(\phi)] - \Psi(M_t).$$

Thus, taking expectation on both sides,

$$\mathbb{E}_\mu[\Psi(\phi)] = \mathbb{E}[\mathbb{E}_{\mu_t}[\Psi(\phi)]] = \mathbb{E}[\Psi(M_t)] + \mathbb{E}[\text{Ent}_{\mu_t}^\Psi(\phi)]. \quad (37)$$

However, since  $\Psi(\mathbb{E}[\phi]) = 0$  by assumption, we have

$$\text{Ent}_\mu^\Psi[\phi] = \mathbb{E}_\mu[\Psi(\phi)] - \Psi(\mathbb{E}_\mu[\phi]) = \mathbb{E}_\mu[\Psi(\phi)],$$

and since  $M_t$  is a martingale, we also have  $\mathbb{E}[M_t] = \mathbb{E}[M_0] = \mathbb{E}_\mu[\phi]$ ,

$$\text{Ent}^\Psi[\Psi(M_t)] = \mathbb{E}[\Psi(M_t)] - \Psi(\mathbb{E}[M_t]) = \mathbb{E}[\Psi(M_t)] - \Psi(\mathbb{E}_\mu[\phi]) = \mathbb{E}[\Psi(M_t)].$$

Finally, substituting the above two equations into (37), we recover equation (36) as required.

With this in mind, we recall that in the reduction of the KLS conjecture, we proceeded to bound the two terms on the right hand side by using proposition 3.3 and the Brascamp-Lieb inequality respectively. We cannot do this for general  $\Psi$ . However, in the case that  $\Psi(x) = x \log x$ , the term  $\mathbb{E}[\text{Ent}_{\mu_t}[\phi^2]]$  can be bounded using the following lemma.

**Lemma 5.4** (Equation 2.17, [Led99]). Let  $U : \mathbb{R}^n \rightarrow \mathbb{R}$  be a twice differentiable function such that there exists some  $c > 0$  so that  $U''(x) \geq c \text{Id}_n$  for any  $x \in \mathbb{R}^n$ . Then, for all smooth functions  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ , we have

$$\text{Ent}_\nu[\phi^2] \leq \frac{2}{c} \mathbb{E}_\nu[\|\nabla \phi\|^2]$$

where  $d\nu = e^{-U} d\text{Leb}^n$  and  $U''$  denotes the Hessian of  $U$ .

Recalling equation (4), by substituting  $U(x) = -z_t - \langle v_t, x \rangle + \frac{t}{2}\|x\|^2 + V(x)$  we observe that  $U''(x) \geq \frac{t}{2}\text{Id}_n + V''(x) \geq \frac{t}{2}\text{Id}_n$  as  $V$  is convex. Thus, by the above lemma, as  $d\mu_t = e^{-U} d\text{Leb}^n$ ,  $\mu_t$  satisfies the log-Sobolev inequality of the form

$$\text{Ent}_{\mu_t}[\phi^2] \leq 4t^{-1} \mathbb{E}_{\mu_t}[\|\nabla \phi\|^2]. \quad (38)$$

On the other hand, we cannot use the same approach to bound  $\text{Ent}[M_t]$ . Indeed, while for proposition 3.3, we were able to invoke a clever application of the Cauchy-Schwarz inequality in order to bound  $[M]_t$  by  $\text{Var}_{\mu_t}[M_t]$  and the operator norm of the covariance matrix, the extra logarithmic term causes problem when we attempt to bound the entropy. Instead, we shall bound  $\text{Ent}[M_t]$  by an application of the Burkholder-Davis-Gundy inequality.

**Lemma 5.5** (Burkholder-Davis-Gundy, [Low10]). For all  $1 \leq p < \infty$ , there exists some constant  $C_p$  such that for any martingale  $(X_t)_{t \geq 0}$ , we have

$$\mathbb{E} \left[ \left( \sup_{s \leq t} X_s \right)^p \right] \leq C_p \mathbb{E} \left[ [X]_t^{p/2} \right]$$

for all  $t \geq 0$ .

With the above lemma in mind, denoting  $\psi = \phi^2 \log \phi^2$  and the martingale  $N_t = \int \psi d\mu_t$ , we have

$$\begin{aligned} \text{Ent}[M_t] &= \mathbb{E}[M_t \log M_t] - \mathbb{E}[M_t] \overbrace{\log \mathbb{E}[M_t]}^{=0} \\ &= \mathbb{E}[M_t \log M_t] = \mathbb{E}[\mathbb{E}_{\mu_t}[\phi^2] \log \mathbb{E}_{\mu_t}[\phi^2]] \\ &\leq \mathbb{E}[\mathbb{E}_{\mu_t}[\phi^2 \log \phi^2]] = \mathbb{E}[N_t] \\ &\leq C_1 \mathbb{E} \left[ [N]_t^{1/2} \right]. \end{aligned} \quad (39)$$

where the first inequality follows by Jensen's inequality. Now, similar to the variance case, we focus on bounding  $[N]_t$ . Similar to the proof of proposition 3.3, introducing the notation  $U_t := \int \psi(x)(x - a_t)\mu_t(dx)$ , we obtain

$$\begin{aligned} dN_t &= d \int \psi(x) F_t(x) \mu_t(dx) = \int \psi(x) \langle x - a_t, dW_t \rangle \mu_t(dx) \\ &= \left\langle \int \psi(x)(x - a_t)\mu_t(dx), dW_t \right\rangle = \langle U_t, dW_t \rangle. \end{aligned}$$

Hence, as  $\int (x - a_t)\mu_t(dx) = 0$ , we have

$$\begin{aligned} d[N]_t &= \|U_t\|^2 dt = \left\| \int \psi(x)(x - a_t)\mu_t(dx) \right\|^2 dt \\ &= \left\| \int (\psi(x) - M_t \log M_t)(x - a_t)\mu_t(dx) \right\|^2 dt. \end{aligned}$$

Now, by recalling that any linear tilt localization satisfies  $\mu_t \ll \mu$  and moreover  $\mu$  support in a ball of diameter  $D$ ,  $\mu_t$  must therefore also has support in a ball of diameter  $D$ . Thus,  $\|x - a_t\|_{L^\infty(\mu_t)} \leq$

$D$  for any  $t \geq 0$ . Then, we have

$$\begin{aligned}
d[N]_t &= \left\| \int (\psi(x) - M_t \log M_t)(x - a_t) \mu_t(dx) \right\|^2 dt \\
&\leq D^2 \left\| \int (\psi(x) - M_t \log M_t) \mu_t(dx) \right\|^2 dt \\
&= D^2 \left| \mathbb{E}_{\mu_t}[\phi^2 \log \phi^2] - \mathbb{E}_{\mu_t}[\phi^2] \log \mathbb{E}_{\mu_t}[\phi^2] \right|^2 dt \\
&= D^2 \text{Ent}_{\mu_t}[\phi^2]^2 dt.
\end{aligned}$$

Combining this with equation (39), we obtain

$$\text{Ent}[M_t] \lesssim \mathbb{E} \left[ \int_0^t d[N]_s^{1/2} \right] \leq \mathbb{E} \left[ \int_0^t D \text{Ent}_{\mu_s}[\phi^2] ds \right]. \quad (40)$$

At this point, as  $\mu_t$  is only supported in a ball of diameter  $D$ , we can utilize lemma 5.3 for bounding the log-Sobolev constant. Hence, combining this with lemma 5.4, we have that  $\text{Ent}_{\mu_t}[\phi^2] \lesssim (t^{-1} \wedge D) \mathbb{E}_{\mu_t}[\|\nabla \phi\|^2]$ . Thus, by Fubini's theorem, equation (40) becomes

$$\begin{aligned}
\text{Ent}[M_t] &\lesssim \mathbb{E} \left[ D \int_0^t (s^{-1} \wedge D^2) \mathbb{E}_{\mu_s}[\|\nabla \phi\|^2] ds \right] \\
&= D \int_0^t (s^{-1} \wedge D^2) \mathbb{E}_{\mu}[\|\nabla \phi\|^2] ds \\
&= D \mathbb{E}_{\mu}[\|\nabla \phi\|^2] \left( 1 + \int_{D^{-2}}^t s^{-1} ds \right) \\
&= D \mathbb{E}_{\mu}[\|\nabla \phi\|^2] (1 + \log t + 2 \log D)
\end{aligned}$$

for all  $t > D^{-2}$ .

Combining this with equation (35), we obtain

$$\begin{aligned}
\text{Ent}_{\mu}[\phi^2] &\leq \text{Ent}[M_t] + \mathbb{E}[\text{Ent}_{\mu_t}[\phi^2]] \\
&\lesssim \mathbb{E}_{\mu}[\|\nabla \phi\|^2] (D(1 + \log t + \log D) + 4t^{-1})
\end{aligned}$$

for all  $t > D^{-2}$ . Minimizing this equation in  $t$ , we find that it is minimized when  $t = 4D^{-1}$  resulting in

$$\text{Ent}_{\mu}[\phi^2] \lesssim (2 + \log 4) D \mathbb{E}_{\mu}[\|\nabla \phi\|^2]$$

whenever  $4D^{-1} \geq D^{-2}$  which holds if and only if  $D \geq 1/4$ . Thus,

$$\text{Ent}_{\mu}[\phi^2] \lesssim D \mathbb{E}_{\mu}[\|\nabla \phi\|^2]$$

where for  $D \leq 1$ ,  $D^{-2} \geq D^{-1}$  and so the conclusion remains to hold by lemma 5.3. Hence, we bounded the log-Sobolev constant with  $\rho_{\mu} \gtrsim D^{-1}$  recovering the result of theorem 9.

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