Rough paths notes

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Space of rough paths

Let V be a Banach space and $T > 0$. A V-valued α-Hölder rough path is a pair of functions

$$X : [0, T] \to V \text{ and } \mathbb{X} : [0, T]^2 \to V^{\otimes 2}$$

such that

$$\|X\|_\alpha = \sup_{s \neq t \in [0,T]} \frac{\|X_{s,t}\|}{|s-t|^\alpha}, \|\mathbb{X}\|_{2\alpha} = \sup_{s \neq t \in [0,T]} \frac{\|\mathbb{X}_{s,t}\|}{|s-t|^{2\alpha}} < \infty$$

and moreover, satisfy *Chen's relation*:

$$\delta \mathbb{X}_{s,u,t} = \mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t} \tag{1}$$

where we denote $X_{s,t} = X_t - X_s$.
Recall that the tensor norm is defined as

$$\|x\| = \inf \left\{ \sum \|v\| \|w\| : x = \sum v \otimes w \right\}.$$

Intuitively, the lift $\mathbb{X}_{s,t}$ is representing $\int_s^t X_{s,r} \otimes \mathrm{d}X_r$ and Chen's relation follows by asserting that

- the map $f :\mapsto \int f_r \mathrm{d}X_r$ is linear;
- $\int_s^t \mathrm{d}X_r = X_t - X_s$ and
- $\int_s^t f_r \mathrm{d}X_r = \int_s^u f_r \mathrm{d}X_r + \int_u^t f_r \mathrm{d}X_r$.

We remark that the lift $\mathbb{X}$ is not unique from Chen's relation. Indeed, if $\mathbb{X}$ is a lift, then so is $\mathbb{X}_{s,t} + F_t - F_s$. In fact, they are all of this form.
We say a rough path $(X, \mathbb{X})$ is weakly geometric if

$$\mathrm{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2} X_{s,t} \otimes X_{s,t}.$$

This relation is determined by integration by parts.
We introduce the following vocabulary:

- the map $X \mapsto (X, \mathbb{X})$ is called a rough path lift.
- $\mathscr{C}^{\infty}$ denotes the space of smooth rough paths.
- $\mathscr{L}(\mathscr{C}^{\infty}) \subseteq \mathscr{C}^{\infty}$ denotes the canonical lift of a smooth paths.
- $\mathscr{C}_g^{\alpha}$ the space of weakly geometric rough paths.
- $\mathscr{C}_g^{0,\alpha} = \overline{\mathscr{L}(\mathscr{C}^{\infty})}^{\mathscr{C}^{\alpha}}$.

For $\alpha < \beta$, we have the inclusion $\mathscr{C}_g^{\beta} \subseteq \mathscr{C}_g^{0,\alpha} \subseteq \mathscr{C}_g^{\alpha}$.

We would like to introduce a topology on the space of rough paths via a norm which makes the space of rough paths a linear subspace of $C^{\alpha} \oplus C_2^{2\alpha}$. By observing that the norm $\|X\|_{\alpha} + \|\mathbb{X}\|_{2\alpha}$ does not respect Chen's relation, we remark that this norm does not make the space of rough paths a linear subspace. Instead, define

$$|||(X, \mathbb{X})|||_{\alpha} = \|X\|_{\alpha} + \sqrt{\|\mathbb{X}\|_{2\alpha}}.$$

We also record the following (in-homogeneous) α-Hölder metric on $\mathscr{C}^{\alpha}([0,T],V)$:

$$\rho_{\alpha}(\mathsf{X}, \mathsf{Y}) = \sup_{s \neq t \in [0,T]} \frac{\|X_{s,t} - Y_{s,t}\|}{|s-t|^{\alpha}} + \sup_{s \neq t \in [0,T]} \frac{\|\mathbb{X}_{s,t} - \mathbb{Y}_{s,t}\|}{|s-t|^{2\alpha}}.$$

The space of rough paths can be alternatively described by a path taking values in a certain additive Lie group. Define

$$T^{(2)}(V) = \mathbb{R} \oplus V \oplus V^{\otimes 2}$$

with the multiplication

$$(a, b, c) \otimes (a', b', c') = (aa', ab' + a'b, ac + a'c' + b \otimes b').$$

$T^{(2)}(V)$ is known as the step-2 truncated tensor algebra over V. Then, considering

$$T_1^{(2)}(V) = \{1\} \oplus V \oplus V^{\otimes 2}$$

with the inherited multiplication from $T^{(2)}(V)$, we have that $T_1^{(2)}(V)$ is a Lie group with the identity $(1, 0, 0)$ and inverse given by

$$(1, b, c)^{-1} = (1, -b, b^{\otimes 2} - c).$$

Then, we consider paths taking values in $T_1^{(2)}(V)$ and in particular, interpret

$$t \mapsto \mathsf{X}_t = (1, X_t, \mathbb{X}_{0,t}) \in T_1^{(2)} V.$$

Chen's relation in this case becomes $\mathsf{X}_{s,u} \otimes \mathsf{X}_{u,t} = \mathsf{X}_{s,t}$ where $\mathsf{X}_{s,t} = \mathsf{X}_s^{-1} \otimes \mathsf{X}_t$.

Denoting $T_0^{(2)}(V) = V \oplus V^{\otimes 2}$, we define its Lie bracket by

$$[(b, c), (b', c')] = b \otimes b' - b' \otimes b.$$

This makes $T_0^{(2)}(V)$ a Lie algebra. Then, denoting $[V, V] = \overline{\langle [v, w] : v, w \in V \rangle}$, we define $g^{(2)}(V)$ to be the subalgebra of $T_0^{(2)}$ defined by

$$g^{(2)}(V) = V \oplus [V, V].$$

We remark that in $\mathbb{R}^d$, $[\mathbb{R}^d, \mathbb{R}^d]$ is the space of symmetric matrices and so is automatically closed. Define the map

$$\exp : T_0^{(2)}(V) \to T_1^{(2)}(V) : (b, c) \mapsto \left(1, b, c + \frac{1}{2} b \otimes b\right).$$

Then, denoting $G^{(2)}(V) = \exp\big(g^{(2)}(V)\big) \leq T_1^{(2)}(V)$, it is easy the check that paths taking values in $G^{(2)}(V)$ corresponds to the weakly geometric rough paths.

Integration of one-forms

Integration of rough paths is motivated by Taylor expansion. Suppose $F : V \to \mathscr{L}(V, W) \in C^{1+}$, and $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$, then Taylor expansion gives

$$F(X_r) \approx F(X_s) + DF(X_r) X_{s,r}$$

with $DF(X_t) \in \mathscr{L}(V, \mathscr{L}(V, W)) = \mathscr{L}(V \otimes V, W)$. Then, the Riemann-Stieltjes sum can be interpreted as

$$\int_s^t F(X_r) \mathrm{d}\mathbf{X}_r = \lim_{|\mathscr{P}| \to 0} \sum_{[u,v] \in \mathscr{P}} \big(F(X_u) X_{u,v} + DF(X_r) \mathbb{X}_{s,r}\big) \tag{2}$$

where we hope the higher order terms vanished. This turns out to be the case for $\alpha \in (1/3, 1/2]$.

Lemma 1. Let $F : V \to \mathscr{L}(V, W) \in C_b^2$ and $\mathbf{X} = (X, \mathbb{X})$ be an α-Hölder rough path with $\alpha > 1/3$. Then, denoting $Y = F(X)$, $Y' = DF(X)$ and $R_{s,t}^Y = Y_{s,t} - Y_s' X_{s,t}$, then

$$Y, Y' \in C^\alpha \text{ and } R^Y \in C_2^{2\alpha}.$$

Moreover, $\|Y\|_\alpha \leq \|DF\|_\infty \|X\|_\alpha$, $\|Y'\|_\alpha \leq \|D^2 F\|_\infty \|X\|_\alpha$ and $\|R^Y\|_{2\alpha} \leq \frac{1}{2} \|D^2 F\|_\infty \|X\|_\alpha^2$.

The above lemma allows us to apply the sewing lemma with increments

$$\Xi_{s,t} = Y_s X_{s,t} + Y_s' \mathbb{X}_{s,t}$$

resulting in the following theorem.

Theorem 1 (Lyons)**.** Let $F : V \to \mathscr{L}(V, W) \in C_b^2$ and $(X, \mathbb{X})$ be an α-Hölder rough path with $\alpha > 1/3$. Then, the limit (2) exists and we have the bound

$$\left\| \int_s^t F(X_r) \mathrm{d}\mathbf{X}_r - F(X_s) X_{s,t} - DF(X_s) \mathbb{X}_{s,t} \right\| \lesssim_\alpha \|F\|_{C_b^2} (\|X\|_\alpha^3 + \|X\|_\alpha \|\mathbb{X}\|_{2\alpha}) |t - s|^{3\alpha}.$$

Since fundamentally, the above theorem relies solely on Lemma 1, it is natural to make the following definition for the space of integrands.

Definition 1. Let $X \in C^\alpha([0, T], V)$ and $Y \in C^\alpha([0, T], \bar{W})$. We say Y is controlled by X if there exists some $Y' \in C^\alpha([0, T], \mathscr{L}(V, \bar{W}))$ and $R^Y \in C_2^{2\alpha}$ with $R_{s,t}^Y = Y_{s,t} - Y_s' X_{s,t}$. We write the space of all such pairs of (Y, Y') as $\mathscr{D}_X^{2\alpha}([0, T], \bar{W})$.

We endow $\mathscr{D}_X^{2\alpha}([0, T], \bar{W})$ with the seminorm

$$\|Y, Y'\|_{X, 2\alpha} = \|Y'\|_\alpha + \|R^Y\|_{2\alpha}.$$

Theorem 2 (Gubinelli). Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha([0.T], V)$ and $(Y, Y') \in \mathscr{D}_X^{2\alpha}([0,T], \mathscr{L}(V,W))$ where $\alpha \in (1/3, 1/2]$. Then, the limit

$$\int_s^t Y_r \mathrm{d}\mathbf{X}_r = \lim_{|\mathscr{P}|\to 0} \sum_{[u,v]\in\mathscr{P}} \left(Y_u X_{u,v} + Y_u' \mathbb{X}_{u,v} \right)$$

exists. Moreover, we have the bound

$$\left\| \int_s^t Y_r \mathrm{d}\mathbf{X}_r - Y_s X_{s,t} - Y_s' \mathbb{X}_{s,t} \right\| \lesssim_\alpha (\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \|Y'\|_\alpha)|t-s|^{3\alpha}.$$

Finally, the map

$$\Phi : \mathscr{D}_X^{2\alpha}([0,T], \mathscr{L}(V,W)) \to \mathscr{D}_X^{2\alpha}([0,T], W) : (Y, Y') \mapsto \left(\int_0^\cdot Y_r \mathrm{d}\mathbf{X}_r, Y \right)$$

is a continuous linear map between Banach spaces with the bound

$$\|\Phi(Y,Y')\|_{X,2\alpha} \leq \|Y\|_\alpha + \|Y'\|_\infty \|\mathbb{X}\|_{2\alpha} + C_\alpha T^\alpha (\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \|Y'\|_\alpha).$$

Moreover, if $(X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T], V)$ and $(Y, Y') \in \mathscr{D}_X^{2\alpha}([0,T], \mathscr{L}(\bar{V}, \bar{W}))$, $(Z, Z') \in \mathscr{D}_Y^{2\alpha}([0,T], \bar{V})$, then defining

$$\Xi_{s,t} = Y_s Z_{s,t} + Y_s' Z_s' X_{s,t}$$

where we interpret $Y_s' \in \mathscr{L}(V, \mathscr{L}(\bar{V}, \bar{W})) \simeq \mathscr{L}(V \otimes \bar{V}, \bar{W})$, we may apply the sewing lemma in order to make sense of $\int_s^t Y_r \mathrm{d}Z_r$.

Reduced rough paths

Suppose now we look at rough integration against gradient vector fields, i.e. $\int_0^T DF(X_s)\mathrm{d}\mathbf{X}_s$ for some $F \in \mathscr{C}_b^1(V,W)$. In this case, the corresponding integral is given by

$$\int_0^T DF(X_s)\mathrm{d}\mathbf{X}_s = \sum_{(s,t)\in\mathscr{P}} \left(DF(X_s)X_{s,t} + D^2F(X_s)\mathbb{X}_{s,t} \right) + o(|\mathscr{P}|).$$

In this case, as $D^2F(X_s) \in \mathscr{L}(V \otimes V, W)$ is symmetric, it annihilates any antisymmetric parts of $\mathbb{X}$. Thus, denoting $\mathbb{S} + \mathbb{A} = \mathbb{X}$ for the symmetric and any antisymmetric part of $\mathbb{X}$, we have that $D^2F(X_s)\mathbb{X}_{s,t} = D^2F(X_s)\mathbb{S}_{s,t}$ and

$$\int_0^T DF(X_s)\mathrm{d}\mathbf{X}_s = \sum_{(s,t)\in\mathscr{P}} \left(DF(X_s)X_{s,t} + D^2F(X_s)\mathbb{S}_{s,t} \right) + o(|\mathscr{P}|). \tag{3}$$

Consequently, the rough integral in this case is completely determined by the symmetric part of the lift. With this in mind, we call a pair $(X, \mathbb{S})$ a *reduced rough path* if they satisfy the same Hölder condition as a rough path while $\mathbb{S}$ satisfies the reduced Chen's relation:

$$\delta\mathbb{S}_{s,u,t} = \mathbb{S}_{s,t} - \mathbb{S}_{s,u} - \mathbb{S}_{u,t} = \mathrm{Sym}(X_{s,u} \otimes X_{u,t})$$

It is easy to check that, for any $\gamma : [0,T] \to V \otimes V$ which is 2α-Hölder,

$$\mathbb{S}_{s,t} = \frac{1}{2}(X_{s,t} \otimes X_{s,t} + \gamma_{s,t})$$

is a reduced rough path. Moreover, it turns out all reduced rough paths are of this form.

Definition 2. For a reduced rough path $\mathsf{X} = (X, \mathbb{S})$, we define the bracket

$$[\mathsf{X}] : t \in [0,T] \mapsto [\mathsf{X}]_t = X_{0,t} \otimes X_{0,t} - 2\mathbb{S}_{0,t} \in \mathrm{Sym}(V \otimes V).$$

It is clear that $[\mathsf{X}]$ defines a 2α-Hölder path in $\mathrm{Sym}(V \otimes V)$ and $[\mathsf{X}]_{s,t} = X_{s,t} \otimes X_{s,t} - 2\mathbb{S}_{s,t}$.

Theorem 3 (Itô's formula for rough integrals)**.** For $F : V \to W$ in $\mathscr{C}_b^3$ and $\mathsf{X} = (X, \mathbb{S})$ a α-reduced rough path with $\alpha > \frac{1}{3}$, then

$$\mathrm{d}F(X_t) = DF(X_t)\mathrm{d}\mathsf{X}_t + \frac{1}{2}D^2F(X_t)\,\mathrm{d}\,[\mathsf{X}]_t$$

with the first integral defined by Equation (3) and the second integral interpreted as a Young integral.

For Brownian motion, the bracket coincides with the quadratic variation. Indeed, let $\mathsf{B} = (B, \mathbb{B}^{\text{Itô}})$ be the Itô lift of a Brownian motion in $\mathbb{R}$, by the classical Itô's formula, we have

$$\mathrm{Sym}(\mathbb{B}^{\text{Itô}}_{0,t}) = \mathbb{B}^{\text{Itô}}_{0,t} = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

Thus,

$$[\mathsf{B}]_t = B_t^2 - 2\mathbb{B}^{\text{Itô}}_{0,t} = t = \langle B \rangle_t.$$

Stability

Given $(X, \mathbb{X}), (\tilde{X}, \tilde{\mathbb{X}}) \in \mathscr{C}^\alpha$ and $(Y, Y') \in \mathscr{D}_X^{2\alpha}$, $(\tilde{Y}, \tilde{Y}') \in \mathscr{D}_{\tilde{X}}^{2\alpha}$, we define the following notion of "distance":

$$\|Y, Y'; \tilde{Y}, \tilde{Y}'\|_{X,\tilde{X},2\alpha} = \|Y' - Y\|_{2\alpha} + \|R^Y - R^{\tilde{Y}}\|_{2\alpha}.$$

While obviously, this is not a metric as (Y, Y') and $(\tilde{Y}, \tilde{Y}')$ are not necessarily in the same space, even in the case where $X = \tilde{X}$, this distance still does not differentiate between Y and $Y + c$. Nonetheless, by considering the canonical embedding

$$\iota_X : \mathscr{D}_X^{2\alpha} \to C^\alpha \oplus C^{2\alpha} : (Y, Y') \mapsto (Y', R^Y),$$

which is an injection if we fix $Y_0 = \xi$ (since then $Y_t = \xi + R^Y_{0,t} + Y'_0 X_{0,t}$), we have that

$$\|Y, Y'; \tilde{Y}, \tilde{Y}'\|_{X,\tilde{X},2\alpha} = \|\iota_X(Y, Y') - \iota_{\tilde{X}}(\tilde{Y}, \tilde{Y}')\|_{\alpha,2\alpha}.$$

To this end, we have a bound of the form

$$\|\Phi(Y, Y'); \Phi(\tilde{Y}, \tilde{Y}')\|_{X,\tilde{X},2\alpha} \le C(\rho_\alpha(\mathsf{X}, \tilde{\mathsf{X}}) + |Y'_0 - \tilde{Y}'_0| + T^\alpha \|Y, Y'; \tilde{Y}, \tilde{Y}'\|_{X,\tilde{X},2\alpha})$$