Interpolation space

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Throughout this note, we take L the generator of a analytic semigroup S on the Banach space $\mathscr{B}$ which satisfies $\|S(t)\| \leq Me^{-wt}$ for some $w > 0$ so that the resolvent of L, $\rho(L)$ contains the right half of the complex plane (recall $R_\lambda = \int_0^\infty e^{-\lambda t} S(t)\,\mathrm{d}t$ which is well-defined for all $\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > 0$).

For $\alpha > 0$, by viewing the formal expression $S(t) = e^{tL}$, by making an appropriate substitution, we have the following computation

$$\int_0^\infty t^{\alpha-1} e^{tL} \mathrm{d}t = (-L)^{-\alpha} \int_0^\infty t^{\alpha-1} e^{-t} \mathrm{d}t = (-L)^{-\alpha} \Gamma(\alpha).$$

This motivates the following definition:

$$(-L)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t) \mathrm{d}t.$$

In the case where $\alpha = 1$, we see that the above definition coincides with $R_0 = (-L)^{-1}$ as expected. Moreover, by observing

$$(-L)^{-1} = (-L)^{-1+\alpha}(-L)^{-\alpha}$$

as $(-L)^{-1}$ is injective, it follows that $(-L)^{-\alpha}$ is injective for $\alpha \in (0, 1]$. This argument can be extended in the case where $\alpha > 1$ and we have that $(-L)^{-\alpha}$ is injective for all $\alpha > 0$. Consequently, we can define $(-L)^\alpha$ to be the inverse of $(-L)^{-\alpha}$ for $\alpha > 0$ where $\mathscr{D}((-L)^\alpha) = \mathscr{R}((-L)^{-\alpha})$. With this, we define the interpolation spaces.

Definition. For $\alpha > 0$, we define the interpolation space

$$\mathscr{B}_\alpha = \mathscr{D}((-L)^\alpha) = \mathscr{R}((-L)^{-\alpha})$$

equipped with the norm $\|x\|_\alpha = \|(-L)^\alpha x\|$. On the other hand, we define $\mathscr{B}_{-\alpha}$ to be the completion of $\mathscr{B}$ under the norm $\|x\|_{-\alpha} = \|(-L)^{-\alpha} x\|$.

We have the following useful properties.

Proposition.

- For all $\alpha \geq \beta$ (regardless of sign), we have $\mathscr{B}_\alpha \subseteq \mathscr{B}_\beta$.
- For all $\alpha > 0, t > 0$, $S(t)\mathscr{B} \subseteq \mathscr{B}_\alpha$ and

$$\|(-L)^\alpha S(t)\|_{\mathscr{B} \to \mathscr{B}} = \|S(t)\|_{\mathscr{B} \to \mathscr{B}_\alpha} \leq \frac{C_\alpha}{t^\alpha}.$$

For this, first consider integer α and use the “identity”

$$\frac{1}{2\pi i}\int_{\gamma_{\phi,b}} e^{tz} R_z \mathrm{d}z = \int_{\gamma_{\phi,b}} \frac{e^{tz}}{z-L} \mathrm{d}z = e^{tL} = S(t).$$

- For all $t \in (0,1]$, $\alpha \in (0,1)$ and $x \in \mathscr{B}_\alpha$,

$$\|S(t)x - x\| \leq C_\alpha t^\alpha \|x\|_\alpha.$$