Notes on the Cameron-Martin Space

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Abstract theory

We look at Gaussian measures on (mostly infinitely dimensional) Banach spaces. While on $\mathbb{R}^d$, the notion of a Gaussian measure can be easily defined by specifying its density with respect to the Lebesgue measure, this is clearly not case for infinite dimensional spaces. Instead, since for X a separable Banach space, any Borel measure μ on X is uniquely determined by its projections on its linear functionals, we define a Gaussian measure on X by requiring that its projections are real Gaussians.

We will take X to be a separable Banach space and H a Hilbert space from this point forward. We denote $\mathscr{M}(X)$ and $\mathscr{M}(H)$ to be the space of Borel probability measures on X and H respectively.

Lemma 0.1. For $\mu, \nu \in \mathscr{M}(X)$, $\mu = \nu$ if and only if for all $l \in X^*$, we have that $l_*\mu = l_*\nu$.

Definition 0.1. $\mu \in \mathscr{M}(X)$ is a Gaussian measure if for all $l \in X^*$, denoting $l_*\mu$ the push-forward measure of μ along l, we have that $l_*\mu$ is a Gaussian measure on $\mathbb{R}$. We say μ is centered if for all $l \in X^*$, $l_*\mu$ is centered.

We focus on the case where μ is centered as we can always translate μ if needed.

Definition 0.2. For $\mu \in \mathscr{M}(X)$ a Gaussian measure, we define its covariance by

$$C_\mu : X^* \times X^* \to \mathbb{R} : (l, l') \mapsto \int_X l(x)l'(x)\mu(\mathrm{d}x).$$

By the property of the integral, it is clear that the covariance operator is a symmetry, positive semi-definite bilinear form.

Definition 0.3. For $\mu \in \mathscr{M}(X)$ a Gaussian measure, we define its Fourier transform by

$$\hat{\mu}(l) = \int_X e^{il(x)}\mu(\mathrm{d}x) = \exp\left(-\frac{1}{2}C_\mu(l, l)\right)$$

where the second equality is due to properties of the real Gaussian. Similar to the finite dimensional case, μ is uniquely determined by its Fourier transform (this is a general property not just for Gaussians).

The covariance of μ can be viewed in many ways most notably,

$$C_\mu : X^* \to X : l \mapsto \int_X x l(x) \mu(\mathrm{d}x)$$

for which the integral is well-defined (by Fernique, see below) as a Bochner integral. We then have $C_\mu(l, l') = l'(C_\mu l)$.

An important theorem for the Gaussian measure is Fernique's theorem which shows that the Gaussian has finite moments of all orders (via exponential Chebyshev).

Theorem 1 (Fernique's Theorem)**.** For $\mu \in \mathscr{M}(X)$ a Gaussian measure, there exists a constant $\alpha > 0$ such that

$$\int_X \exp\left(\alpha \|x\|^2\right) \mu(\mathrm{d}x) < \infty.$$

As a consequence of Fernique's, any Gaussian measure has finite variance and thus, the covariance operator is bounded.

Proposition 0.1. For $\mu \in \mathscr{M}(H)$ a Gaussian measure, viewing $C_\mu : H \to H$ applying Riesz representation whenever needed, we have

$$\int_H \|x\|^2 \mu(\mathrm{d}x) = \operatorname{tr} C_\mu = \sum_{k=1}^{\infty} \langle C_\mu(e_k), e_k \rangle$$

where $(e_k)_{k=1}^{\infty}$ is an orthonormal basis of H. Namely, C_μ is positive, self-adjoint and trace class. Conversely, if C is a positive, self-adjoint and trace class operator, then there exists a Gaussian measure μ on H such that $C = C_\mu$.

While so far, the definition and the behavior of the Gaussian on Banach space is seemingly intuitive, this cannot be further from the truth should we look a bit closer. While, in the finite dimensional case, the Gaussian measures are always equivalent to the Lebesgue measure, and consequently each other, in infinite dimensions, Gaussians a easily mutually singular. As we shall see, in fact with probability one, any translation of the Gaussian will be mutually singular to the original Gaussian. The Cameron-Martin space is then the linear subspace of X with zero measure for which translation dose not make the Gaussian measure singular. While it has measure zero, it turns out that the Cameron-Martin space characterized the Gaussian measure completely.

Definition 0.4. For $\mu \in \mathscr{M}(X)$ a Gaussian measure, we define

$$\mathring{H}_\mu = \{h \in X : \exists h^* \in X^*, C_\mu(h^*) = h\} = C_\mu(X^*),$$

where we equip it with the inner product $\langle h, k \rangle_\mu = C_\mu(h^*, k^*)$. Then, the Cameron-Martin space H_μ of μ is defined as the closure of $\mathring{H}_\mu$ under its norm $\|\cdot\|_\mu$.

Despite the norm under which the Cameron-Martin space is completed over is different, the Cameron-Martin space is always a subspace of the original Banach space.

Theorem 2 (Cameron-Martin)**.** Let $\mu \in \mathscr{M}(X)$ be a Gaussian. For $h \in X$, denoting $\tau_h : x \in X \mapsto x + h$, we have that $\tau_h^* \mu \ll \mu$ if and only if $h \in H_\mu$. In this case,

$$\frac{\mathrm{d}\tau_h^* \mu}{\mathrm{d}\mu} = \exp\left(h^*(x) - \frac{1}{2}\|h\|_\mu^2\right).$$

Proposition 0.2.
$$H_\mu = \bigcap_{V \le X; \mu(V)=1} V,$$
and $\mu(H_\mu) = 0$ if and only if $\dim(X) = \infty$.

As a consequence, if $A \in \mathscr{L}(X)$ a linear isomorphism, denoting $\nu = A_*\mu$, we have that
$$H_\nu = \bigcap_{V \le X; \nu(V)=1} V = \bigcap_{V \le X; \mu(A^{-1}V)=1} V = \bigcap_{V \le X; \mu(V)=1} AV = A(H_\mu)$$

Examples

Let μ be the Gaussian measure on $\mathscr{C}([0,1],\mathbb{R})$ with covariance $C_\mu(\delta_t, \delta_s) = s \wedge t$. μ corresponds to the law of the Brownian motion on $[0,1]$ and we call it the Wiener measure. We have that $H_\mu = H_0^{1,2}([0,1])$ - the space of all absolutely continuous functions h with $h(0) = 0$ and $\dot{h}(s) \in L^2([0,1])$. Moreover, for $h \in H_\mu$, $h^*(k) = \int_0^1 \dot{h}(s)\dot{k}(s)\mathrm{d}s$. As this defines a bounded linear functional from H_μ, we can almost surely uniquely extend h^* to a measurable element of $C_\mu(\delta_t, \delta_s)^*$. To see this, it suffices to check that
$$h_t = \delta_t(h) = C_\mu(\delta_t, h^*) = h^*(C_\mu(\delta_t)).$$
We see that
$$C_\mu(\delta_t)_s = \left(\int \omega \delta_t(\omega)\mu(\mathrm{d}\omega)\right)_s = \int \omega_s \omega_t \mu(\mathrm{d}\omega) = C_\mu(\delta_t, \delta_s) = s \wedge t.$$
Thus, $\dot{C}_\mu(\delta_t)_s = \mathbb{1}_{[0,t]}(s)$ and hence,
$$h^*(C_\mu(\delta_t)) = \int_0^t \dot{f}_s \mathrm{d}s = f_t$$
as required.

As an application, the Cameron-Martin theorem allows us to bound the probability that a Brownian motion stays near a given function. In particular, for $f \in H_0^{1,2}([0,1])$, denoting μ_W for the Wiener measure, we have that
$$\mathbb{P}(|B_t - f|_\infty \le \epsilon) = \tau_{-f}^* \mu_W(\overline{B}_\epsilon(0)) = e^{-\frac{1}{2}\|f\|^2_{H_0^{1,2}([0,1])}} \int_{\overline{B}_\epsilon(0)} e^{-f^*(\omega)} \mu_W(\mathrm{d}\omega).$$
This is then bounded above by
$$e^{-\frac{1}{2}\|f\|^2_{H_0^{1,2}([0,1])} + \epsilon \|f^*\|_{X^*}} \mathbb{P}(|B_t|_\infty \le \epsilon).$$
On the other hand, using the inequality $e^x \ge 1 + x$, we have that
$$\int_{\overline{B}_\epsilon(0)} e^{-f^*(\omega)} \mu_W(\mathrm{d}\omega) \ge \int_{\overline{B}_\epsilon(0)} (1 + -f^*(\omega))\mu_W(\mathrm{d}\omega) \ge \mathbb{P}(|B_t|_\infty \le \epsilon) - \int_{\overline{B}_\epsilon(0)} f^*(\omega)\mu_W(\mathrm{d}\omega)$$

Now, since $f^*_* \mu_W \sim \mathcal{N}(0, \|f\|^2_{H_W})$, the last integral is zero by spherical symmetry. Consequently, have the bound

$$\mathbb{P}(|B_t - f|_\infty \leq \epsilon) \in e^{-\frac{1}{2}\|f\|^2_{H_0^{1,2}([0,1])}} \left[\mathbb{P}(|B_t|_\infty \leq \epsilon), e^{\epsilon \|f^*\|_{X^*}} \mathbb{P}(|B_t|_\infty \leq \epsilon)\right]$$

and thus,

$$\lim_{\epsilon \downarrow 0} \frac{\mathbb{P}(|B_t - f|_\infty \leq \epsilon)}{\mathbb{P}(|B_t|_\infty \leq \epsilon)} = e^{-\frac{1}{2}\|f\|^2_{H_0^{1,2}([0,1])}}.$$