Notes on Signature Transform

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We introduce the following notations:

- For $p \geq 1$, $a < b \in \mathbb{R}_+$ and V a normed space, denote $C_0^{p\text{-var}}([a,b],V) = C_0^{p\text{-var}}$ for the space of functions x with finite p-variation and is such that $x_0 = 0$. We omit the 0 subscript from the notation in the case the latter condition is removed. When $p = 1$, we simply write C^1.
- For V a vector space, denote $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ and $T((V)) = \prod_{k=0}^{\infty} V^{\otimes k}$. $T(V)$ is referred to as the tensor algebra while $T((V))$ is referred to as the extended tensor algebra. Note that $T(V)$ can be viewed as a subspace of $T((V))$ for which all but finitely may terms are zero.
- We use the concise notation $v_1 v_2 = v_1 \cdot v_2$ for the tensor product $v_1 \otimes v_2$ whenever there is no ambiguity.

In the case $p \in [1,2)$, using the theory of Young integration, we can define iterated integrals of the form

$$S(x)^{(n)}_{[a,b]} := \int_{0<t_1<\cdots<t_n<T} \mathrm{d}x_{t_1} \cdots \mathrm{d}x_{t_n}.$$

Definition 1. The signature transform (ST) is the mapping $S = S_{[a,b]} : C^{p\text{-var}}([a,b],V) \to T((V))$ defined by

$$S(x) = S(x)_{[a,b]} = \left(1,\ S(x)^{(1)}_{[a,b]},\ S(x)^{(2)}_{[a,b]},\ \ldots\right).$$

Moreover, we define the truncated signature transform (TST) $S_n(x) = S_n(x)_{[a,b]} \in T(V)$ by simply setting all but the first $n+1$ terms to zero, i.e. the last non-zero term is $S(x)^{(n)}_{[a,b]}$.

The signature occurs naturally in the context of controlled differential equations. Suppose $x \in C^{p\text{-var}}$ for some $p \in [1,2)$ and $T(y) : V \to V$ is a linear map. Then, performing Picard iteration on the differential equation $\mathrm{d}y_t = T(y_t)\mathrm{d}x_t$ with initial condition y_0:

$$y_t^{(0)} = y_0,\ y_t^{(k+1)} = y_0 + \int_0^t T(y^{(k)})\mathrm{d}x_t,$$

it is easy to see by induction that

$$y_t^{(k)} = \sum_{i=0}^{k} T^{\otimes i}(y_0) S(x)^{(i)}_{[0,t]}$$

where $T^{\otimes i}(y)$ denotes the i-fold multilinear map defined by

$$T^{\otimes 0}(y) = y,\ T^{\otimes (k+1)}(y)(v_1,\ldots,v_{k+1}) = T(T^{\otimes k}(y)(v_1,\ldots,v_k))(v_{k+1}).$$

Theorem 1. The signature transform is invariant under reparametrization, i.e. for $x \in C^{p\text{-var}}([a,b],V)$, $\lambda : [c,d] \to [a,b]$ a continuous, monotone surjection, $S(x)_{[a,b]} = S(x \circ \lambda)_{[c,d]}$.

Definition 2. For $x \in C^1$, we define the reparametrization of x with constant speed x^* to be the path $x^* = x \circ \lambda : [0, \|x\|_1] \to V$ where

$$\lambda(t) := \inf\{s \geq a : \|x\|_{1;[a,s]} > t\}.$$

As a consequence of the above proposition, $S(x) = S(x^*)$.

Definition 3. For $x \in C^{p\text{-var}}([a,b])$, $y \in C^{p\text{-var}}([b,c])$, we define the concatenation of x and y to be the path $x * y \in C^{p\text{-var}}([a,c])$ defined by

$$(x * y)_t = \begin{cases} x_t & t \in [a,b), \\ y_t - y_b + x_b & t \in [b,c]. \end{cases}$$

Theorem 2 (Chen's relation)**.** For $x \in C^{p\text{-var}}([a,b])$, $y \in C^{p\text{-var}}([b,c])$, $S(x * y) = S(x)S(y)$.

Chen's relation is incredibly useful in actual computations. Suppose we observe a time series $(t_i, x_i)_{i=1}^n$ and by linear interpolation, we obtain a path $\gamma \in C^1([0,1],V)$. We would like to compute the signature of γ.

Let us first compute the signature of a linear path. Suppose $x_a, v \in V$ and $x_t = x_a + \frac{t-a}{b-a} v$. Then, we can compute

$$S(x)^{(k)}_{[a,b]} = \frac{v^{\otimes k}}{(b-a)^k} \int_{a<t_1<\cdots<t_k<b} \mathrm{d}t_1 \cdots \mathrm{d}t_k = \frac{v^{\otimes k}}{k!}.$$

Thus, $S(x) = \exp_\otimes(v) = \sum_{i=0}^{\infty} \frac{1}{k!} v^{\otimes i}$.

Now, going back to γ. As γ is the concatenation of linear paths of the form $x^i_t : [t_i, t_{i+1}] \to V$, it follows by Chen's relation that

$$S(\gamma)_{[t_1,t_n]} = \exp_\otimes(x_2 - x_1) \otimes \cdots \otimes \exp_\otimes(x_n - x_{n-1}).$$

Consider now the ordinary differential equation $\mathrm{d}y = y\mathrm{d}x$ which has solution $y = y_0 e^x$. In the context of controlled differential equations, as indicated by the linear case, the signature plays the role of the exponential function. In particular, we have the following theorem.

Theorem 3 (CDE formulation of ST)**.** Let $x \in C^{p\text{-var}}$ for some $p \in [1,2)$ and $E \subseteq T((V))$ be a Banach subalgebra containing $\mathsf{S}_p := S(C^{p\text{-var}})_{[a,b]} \subseteq T((V))$. Then, for any $y \in E$, $y_t := yS(x)_{[a,t]}$ is the unique solution to the controlled differential equation

$$\mathrm{d}y_t = y_t \mathrm{d}x_t \text{ with } y_0 = y.$$

Definition 4. For $x \in C^{p\text{-var}}$, define the time reversal of x to be $\overleftarrow{x}$ defined by

$$\overleftarrow{x}_t = x_{a+b-t} \text{ for all } t \in [a,b].$$

Via the CDE formulation of the signature transform, it is easy to see that the time reversal of a path inverts the signature of said path.

Proposition 0.1. For $x \in C^{p\text{-var}}$, $S(\overleftarrow{x})S(x) = S(x)S(\overleftarrow{x}) = 1$.

Definition 5. Define the shuffle operation $\sqcup\!\sqcup : T(V) \times T(V) \to T(V)$ by

- for $f \in V$ and $r \in \mathbb{R}$, $f \sqcup\!\sqcup r = r \sqcup\!\sqcup f = rf$,
- for $f = f_- \otimes a \in V^{\otimes k}, g = g_- \otimes b \in V^{\otimes l}$,

$$f \sqcup\!\sqcup g = (f_- \sqcup\!\sqcup g) \otimes a + (f \sqcup\!\sqcup g_-) \otimes b.$$

For x differentiable and $f, g \in V^*$, we observe by the integration by parts formula that

$$f(x_{ab})g(x_{ab}) = \int_a^b f(x_{as})\mathrm{d}g(x_s) + \int_a^b g(x_{as})\mathrm{d}f(x_s)$$

where we have denoted $x_{at} := x_t - x_a$. Thus, by observing

$$\left\langle f, S(x)^{(1)}_{[a,b]} \right\rangle = f\left(\int_a^b \mathrm{d}x_s\right) = f(x_{ab})$$

and moreover, as $f \sqcup\!\sqcup g = g \otimes f + f \otimes g$, we have

$$\begin{aligned} \left\langle f \sqcup\!\sqcup g, S(x)^{(2)}_{[a,b]} \right\rangle &= \left\langle g \otimes f, \int_a^b \int_a^s \mathrm{d}x_r \otimes \mathrm{d}x_s \right\rangle + \left\langle f \otimes g, \int_a^b \int_a^s \mathrm{d}x_r \otimes \mathrm{d}x_s \right\rangle \\ &= \int_a^b g(x_{as})\mathrm{d}f(x_s) + \int_a^b f(x_{as})\mathrm{d}g(x_s). \end{aligned}$$

Thus, the above equality can be alternatively written as

$$\left\langle f \sqcup\!\sqcup g, S(x)^{(2)}_{[a,b]} \right\rangle = \left\langle f, S(x)^{(1)}_{[a,b]} \right\rangle \left\langle g, S(x)^{(1)}_{[a,b]} \right\rangle.$$

Theorem 4 (Shuffle identity)**.** For $x \in C^{p\text{-var}}$ where $p \in [1,2)$ and $f, g \in T((V))^*$, we have

$$\left\langle f \sqcup\!\sqcup g, S(x)_{[a,b]} \right\rangle = \left\langle f, S(x)_{[a,b]} \right\rangle \left\langle g, S(x)_{[a,b]} \right\rangle.$$

An important consequence of the shuffle identity is the following proposition.

Proposition 0.2. S_p is linearly independent in $T((V))$.

We are interested in the question "to what extent does the signature determine a path on $C_0^{p\text{-var}}$". It is clear that the signature does not determine the path completely as we saw that, for $x \in C_0^{p\text{-var}}$,

$$S(x * \overleftarrow{x}_{\cdot+(b-a)}) = S(x)S(\overleftarrow{x}_{\cdot+(b-a)}) = S(x)S(\overleftarrow{x}) = 1$$

while $x * \overleftarrow{x}_{\cdot+(b-a)}$ is not the constant path for any non-constant x.
In the finite dimensional case, one can show that any two paths which agrees on a one dimensional projection and have the same signature must be the same path. T
By defining the equivalence relation on $C_0^{p\text{-var}}$ by $S(x) = S(y)$, one can view S_p as a quotient of $C_0^{p\text{-var}}$. We introduce some terminologies for dealing with this quotient.

- We call S_p the space of *unparametrized paths*.

- For any $x \in C_0^{p\text{-var}}$, we denote $[x] \in \mathsf{S}_p$ for the equivalence class containing x under this equivalence relation.
- For $x \in C_0^{p\text{-var}}$ with $x = [1]$, we refer to x as a *tree like path*.
- For $x \in C_0^{p\text{-var}}$, we call $\|x\|_{p\text{-var}}$ its *length* and we say x is tree reduced if $\|x\|_{p\text{-var}} = \inf_{x' \in [x]} \|x'\|_{p\text{-var}}$.

For $p = 1$, there exists a unique (up to reparametrization) tree reduced path in each equivalence class.

Since by Chen's relation, the signature of a concatenated path is only dependent on the signature of its constituents, concatenation can be lifted to the quotient space. It is easy to see that S_p forms a group with this operation.

Definition 6. For any $f \in T((V))^*$, define $\Phi_f : \mathsf{S}_p \to \mathbb{R}$ by

$$\Phi_f([x]) = \langle f, S(x) \rangle .$$

Proposition 0.3. The set $\Phi_{T((V))^*} = \{\Phi_f : f \in T((V))^*\}$ is a unital subalgebra of $\mathbb{R}^{\mathsf{S}_p}$ which separates points.

Consequently, by Stone-Weierstrass, if S_p is equipped with a topology for which $K \subseteq \mathsf{S}_p$ is a compact subset, and $\Phi_{T((V))^*}|_K = \{\Phi_{f|_K} : f \in T((V))^*\} \subseteq C(K, \mathbb{R})$, then it is dense (with respect to $\|\cdot\|_\infty$). This type of result is referred to as *uniform approximation with signatures*.
For $x \in C_0^{p\text{-var}}$ and some vector field $f(y) : V \to V$, we define the Itô-Lyons map

$$\Phi : x \mapsto y, \text{ such that } \mathrm{d}y_t = f(y_t)\mathrm{d}x_t \text{ with initial condition } y_a.$$

We are interested in whether or not we can lift Φ through the quotient so that it is instead of a map from S_p. We will assume $f \in C^\infty$ and $p = 1$. Moreover, suppose that there exists some $C > 0$ such that for all $k \in \mathbb{N}$, $\|f^{(k)}\|_\infty \le C^k$.

Proposition 0.4 (Factorial decay of the signature)**.** For $x \in C^1$ and $k \in \mathbb{N}$,

$$\|S(x)^{(k)}\|_{V^{\otimes k}} \le \frac{1}{k!}\|x\|_1^k.$$

Let y be the solution of the CDE $\mathrm{d}y_t = f(y_t)\mathrm{d}x_t$ with initial condition y_a, we define

$$y_t^N = y_a + \sum_{k=1}^{N} f^{(k)}(y_a)S(x)^{(k)}.$$

Then, by induction and Taylor theorem, it is easy to see that

$$y_t = y_t^N + \int_{a<t_1<\cdots<t_{N+1}<t} f^{(N+1)}(y_{t_1})\mathrm{d}x_{t_1}\cdots\mathrm{d}x_{t_{N+1}}.$$

Thus, by the factorial decay of the signature, it follows that

$$\|y_t - y_t^N\| = \left\| \int_{a<t_1<\cdots<t_{N+1}<t} f^{(N+1)}(y_{t_1})\mathrm{d}x_{t_1}\cdots\mathrm{d}x_{t_{N+1}} \right\| \le \frac{C^{N+1}}{(N+1)!}\|x\|_1^{N+1} \to 0$$

as $N \to \infty$. Hence, as y^N is uniquely determined by the signature of x and the initial condition, it follows that Φ is invariant over any equivalence class and we may lift it to a map on S_p.