May 6, 2025

p-Variation vs α-Hölder Continuity

For $\alpha \in (0,1]$ and $\gamma \in \mathscr{C}^{\alpha}([0,T],V)$, it is well-known that $\gamma \in \mathscr{C}^{p\text{-var}}([0,T],V)$ for $p = \alpha^{-1}$ and

$$\|\gamma\|_{p\text{-var}} \leq T^{\alpha}\|\gamma\|_{\alpha}.$$

Moreover, if γ is a continuous path of finite p-variation for some $p > 1$, we can reparametrize γ (so that its p-variation norm grows with constant speed) to obtain a p^{-1}-Hölder path. In this case, one can make sure that the reparametrized process has unit Hölder norm.

A question arises on whether or not, starting from a Hölder path, we can bound its corresponding p-variation norm from below by its Hölder norm. The answer is in general negative. Indeed, philosophically, the Hölder norm is a local property whilst the p-variation norm is global. Thus, the Hölder norm cannot capture the global behavior of the path and consequently the two norms cannot be equivalent in general. To be more precise, we know that the p-variation is invariant under reparametrization. Thus, if there exists non-decreasing non-negative function f_T for which

$$f_T(\|\gamma\|_{\alpha}) \leq \|\gamma\|_{p\text{-var}},$$

then, reparametrizing γ by increasing the speed on the interval for which it Hölder norm attains its maximum, and compensating by decreasing the speed on the rest of the interval, the left hand side can be made arbitrarily large whilst keeping the right hand side constant.

We remark that the above argument cannot be applied to the upper bound. More precisely, we cannot reparametrize a path γ to make its Hölder norm arbitrarily small whilst keeping its domain invariant.

Interpolation

By viewing L^{∞} as C^0 (here C^0 is not the space of continuous paths but the "Hölder space" with 0 Hölder exponents), we expect a interpolation inequality

$$L^0 \to L^p \to L^{\infty} = C^0 \to C^{\alpha} \to C^k \to C^{\infty}.$$

An example of this is from [HP10, Lemma A.3]:

$$\|f\|_{\infty} \leq 2\left(T^{-\frac{1}{2}}\|f\|_{L^2} \vee \|f\|_{L^2}^{\frac{2\alpha}{2\alpha+1}}\|f\|_{\alpha}^{\frac{1}{2\alpha+1}}\right)$$

for any $f \in \mathscr{C}^{\alpha}([0,T])$.

ϑ-Hölder Roughness and Control of the Gubinelli Derivative

We have a natural understanding of the Hölder roughness of a path. In particular, although we have the embeddings between Hölder spaces, we do not usually consider a constant path to be a $\frac{1}{2}$-Hölder path, nor a Brownian motion to be a $\frac{1}{3}$-Hölder path. Namely, we would like to work on the boundary of how rough the path is. In the context of rough path theory, working on the boundary guarantees uniqueness of the Gubinelli derivative.

For simplicity, let us consider the one-dimensional case: Let $(X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T])$ be such that, for some $s \in [0,T]$, there

$$\limsup_{t\to s} \frac{|X_{st}|}{|t-s|^\vartheta} = \infty$$

for some $\vartheta \in (\alpha, 2\alpha)$. Then, if $(Y, Y') \in \mathscr{D}_X^\alpha$, we have that

$$Y_s' = \frac{Y_{st}}{X_{st}} + \frac{R_{st}}{|t-s|^{2\alpha}} \frac{|t-s|^\vartheta}{X_{st}} |t-s|^{2\alpha-\vartheta}.$$

Thus, taking $t \to s$, we have that $Y_s' = \liminf_{t\to s} \frac{Y_{st}}{X_{st}} =: \partial_X Y_s$. Namely, Y_s' is uniquely determined by X and Y.

In general, for a rough path $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T])$ which is ϑ-Hölder rough, denoting $L_\vartheta(X)$ for its modulus of ϑ-Hölder roughness, we have the estimate [HP10, Proposition 1]

$$L_\vartheta(X)\|Y'\|_\infty \lesssim \|Y\|_\infty \varepsilon^{-\vartheta} + \|R^Y\|_{2\alpha}\varepsilon^{2\alpha-\vartheta}$$

for any $(Y, Y') \in \mathscr{D}_X^\alpha$ and $\varepsilon \in (0, \frac{T}{2}]$. Thus, in the case where $(Y, Y') \in \mathscr{D}_X^\alpha$ is $(\mathbf{X}, \varphi)$-coherent, $\varphi(Y), \partial_X\varphi(Y), \partial_X^2\varphi(Y)\ldots$ can all be controlled by Y and R^Y:

$$\begin{aligned}
\|\varphi(Y)\|_\infty &\lesssim (L_\vartheta(X))^{-1}(\|Y\|_\infty \varepsilon_0^{-\vartheta} + \|R^Y\|_{2\alpha}\varepsilon_0^{2\alpha-\vartheta}) \\
\|\partial_X\varphi(Y)\|_\infty &\lesssim (L_\vartheta(X))^{-1}(\|\varphi(Y)\|_\infty \varepsilon_1^{-\vartheta} + \|R^Y\|_{2\alpha}\varepsilon_1^{2\alpha-\vartheta}) \\
&\lesssim (L_\vartheta(X))^{-2}\|Y\|_\infty \varepsilon_0^{-\vartheta}\varepsilon_1^{-\vartheta} + (L_\vartheta(X))^{-1}\|R^Y\|_{2\alpha}(((L_\vartheta(X)))^{-1}\varepsilon_1^{-\vartheta}\varepsilon_0^{2\alpha-\vartheta} + \varepsilon_1^{2\alpha-\vartheta}) \\
&\ldots
\end{aligned}$$

for small $\varepsilon_0, \varepsilon_1, \ldots$. We can even bound the Hölder norm of $\partial_X\varphi(Y)$ since

$$\|\partial_X\varphi(Y)\|_\alpha \le \|R^{\varphi(Y)}\|_{2\alpha}T^\alpha + \|X\|_\alpha\|\partial_X^2\varphi(Y)\|_\infty.$$

Nonetheless, this control is not sufficient for applications in [LY25, Section 4.4] where $\|Y\|_{\infty;I} < R(k+1)$ for some known constants R, k no matter what choice of ε we choose. In particular, for this application, we find that we will need to take ε large in order to control the term $\|Y\|_\infty\varepsilon^{-\vartheta}$. This motivates a long range version of ϑ-Hölder roughness: We say X has ϑ-long range variation if, for every $s \in \mathbb{R}_+$ and $r > 1$

$$\sup_{|t-s|\le r} |X_{st}| \ge U_\vartheta(X) r^\vartheta.$$

By the same arguments as in [HP13], the same bound holds assuming long range variation but now we allow for large $\varepsilon > 1$.

It is not difficult to see that a fractional Brownian motion B^H with Hurst parameter H satisfies the above definition for any $\vartheta < H$. This is in contrast to ϑ-Hölder roughness in which B^H is ϑ-Hölder rough for any $\vartheta > H$. This is unfortunate in the sense that, for applications in [LY25], we need ϑ-long range variation for $\vartheta > 1-\alpha$ with α being the Hölder exponent of the driving rough path which is not possible. Indeed, philosophically, we can already see that these controls cannot improve the result as the assumptions say nothing about φ.

Computation of the Symbol of the Leray Projector

Denote $\mathbb{T}^3 = [0, 2\pi]^3$ the 3-dimensional torus. For each $k \in \mathbb{Z}^3 \setminus \{0\} =: \mathbb{Z}^3_*$, there exists normal vectors $\varphi_{k,\pm} \in \mathbb{C}^3$ orthogonal to k such that

$$\nabla \times \varphi_{k,\pm} e^{ik\cdot x} = \pm \|k\| \varphi_{k,\pm} e^{ik\cdot x},$$

i.e. $\varphi_{k,\pm} e^{ik\cdot x}$ is an eigenfunction of the curl operator with eigenvalue $\pm\|k\|$. It is known that $\{\varphi_{k,\pm} e^{ik\cdot x}\}_{k\in\mathbb{Z}^3_*}$ forms an orthonormal basis of $L^2_\sigma(\mathbb{T}^3)$ (the divergence free subspace of L^2).

The Leray projector P is the orthogonal projection from L^2 to L^2_σ.

We want to compute the 3×3 matrix $P(k)$ such that for all $u \in L^2$,

$$Pu = P \sum_{k\in\mathbb{Z}^3_*} u_k e^{ik\cdot x} = \sum_{k\in\mathbb{Z}^3_*} e^{ik\cdot x} P(k) u_k.$$

By observing that $\nabla \cdot \left(u e^{ik\cdot x}\right) = e^{ik\cdot x}(ik \cdot u)$ for any $u \in \mathbb{C}^3$, we have that $ue^{ik\cdot x}$ is divergence free if and only if u is orthogonal to k and it is not difficult to see that $P(ue^{ik\cdot x}) = e^{ik\cdot x} u_k$ where u_k is the orthogonal projection of u onto the plane normal to k. Thus, we have that

$$P(k)v = v - \frac{k \cdot v}{\|k\|^2} k = \frac{1}{\|k\|^2} \begin{pmatrix} \|k\|^2 - k_1^2 & -k_1k_2 & -k_1k_3 \\ -k_2k_1 & \|k\|^2 - k_2^2 & -k_2k_3 \\ -k_3k_1 & -k_3k_2 & \|k\|^2 - k_3^2 \end{pmatrix} v.$$

We remark that, denoting J for the rotation matrix around the z-axis, and observing that the symbol for the curl operator:

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ (\nabla\times)(k) = i \begin{pmatrix} 0 & k_3 & -k_2 \\ -k_3 & 0 & k_1 \\ k_2 & -k_1 & 0 \end{pmatrix}$$

we compute

$$S(k) := P(k)JP(k) = i\frac{k_3}{\|k\|^2}(\nabla\times)(k).$$

Thus,

$$\begin{aligned} \frac{t}{\varepsilon} S \varphi_{k,\pm} e^{ik\cdot x} &= \frac{t}{\varepsilon} e^{ik\cdot x} S(k) \varphi_{k,\pm} = i \frac{t}{\varepsilon} \frac{k_3}{\|k\|^2} e^{ik\cdot x} (\nabla\times)(k) \varphi_{k,\pm} \\ &= i \frac{t}{\varepsilon} \frac{k_3}{\|k\|^2} \nabla \times \varphi_{k,\pm} e^{ik\cdot x} = \pm i \frac{t}{\varepsilon} \frac{k_3}{\|k\|} \varphi_{k,\pm} e^{ik\cdot x} \end{aligned}$$

and moreover, $e^{\frac{t}{\varepsilon}S} \varphi_{k,\pm} e^{ik\cdot x} = e^{\pm i \frac{t}{\varepsilon} \frac{k_3}{\|k\|}} \varphi_{k,\pm} e^{ik\cdot x}$.

References

[HP10] Martin Hairer and Natesh S. Pillai. Ergodicity of hypoelliptic sdes driven by fractional brownian motion, 2010.

[HP13] Martin Hairer and Natesh S. Pillai. Regularity of laws and ergodicity of hypoelliptic sdes driven by rough paths. *The Annals of Probability*, 41(4), July 2013.

[LY25] Xue-Mei Li and Kexing Ying. Strong completeness of sdes and non-explosion for rdes with coefficients having unbounded derivatives, 2025.