Chapter 7 Conditional Heteroscedastic Models (Part C)

1 Review of Estimation of Covariance matrix[1]

(a) Suppose considering p assets and their returns $R_t = (r_{t,1}, ..., r_{t,p})^\top$, with observations

$$\begin{array}{cccc} r_{1,1} & r_{1,2} & \dots & r_{1,p} \\ r_{2,1} & r_{2,2} & \dots & r_{2,p} \\ r_{3,1} & r_{3,2} & \dots & r_{2,p} \\ \vdots & \vdots & \vdots & \vdots \\ r_{T,1} & r_{T,2} & \dots & r_{T,p} \end{array}$$

The unconditional covariance matrix is defined as

$$\Sigma = Cov(R_t)$$

[1] This is a hot research topis in Statistical research! We here only discuss one estimation method.

(b) We actually calculate it by

$$\hat{S} = T^{-1} \sum_{t=1}^{T} (R_t - \bar{\mu})(R_t - \bar{\mu})^\top$$

where

$$\bar{\mu} = T^{-1} \sum_{t=1}^{T} R_t$$

(Please prove it) or Simply (because $\bar{\mu}$ is very small)

$$\hat{S} = T^{-1} \sum_{t=1}^{T} R_t R_t^\top$$

Shrinkage estimation: Shrink S towards a biased estimator with lower variance,

$$S^* = (1 - \lambda)S + \lambda S_{diag},$$

where S_{diag} is the diagonals of S, to improves performance in high-dimensional settings. The choice of $\lambda = 0.1$ can be used or cross-validation

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R: A number of Estimators for the covariance matrix

library(fPortfolio)

covEstimator(x)      # the sample mean and covariance matrix

mcdEstimator(x)      # the MCD estimators of mean and covariance
                      matrix (not discussed)

shrinkEstimator(x)      # the shrink estimators of mean and
                      covariance matrix
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*Please note that, x must be a time series data, otherwise, use timeSeries(x) to
transform the formate of the data.
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Example 1.1 This is a simulation, where the covariance matrix is actually known, data are generated from $N(0, \Sigma)$. The shrink estimator usually have smaller estimation error than the sample covariance matrix; see code03shrinkCov.R.

(c) Shrink estimator 2: shrinking to the first factor is helpful in financial data. By the CAPM, we have

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{i,t}$$

or

$$R_t = \alpha + \beta R_{Mt} + \mathcal{E}_t$$

where $\alpha = (\alpha_1, ..., \alpha_p)^\top$ and $\beta = (\beta_1, ..., \beta_p)^\top$ and $\mathcal{E}_t = (\varepsilon_{1,t}, ..., \varepsilon_{p,t})^\top$.

(i) Because we assume R_{Mt} is independent of $\mathcal{E}_t$, it follows

$$\Sigma_R = \mathbf{cov}(\mathbf{R_t}) = \beta\beta^\top \sigma^{\mathbf{2}}_{\mathbf{M,t}} + \mathbf{cov}(\mathcal{E}_\mathbf{t})$$

(ii) If the model is correct, then $\varepsilon_{1,t}, ..., \varepsilon_{p,t}$ are independent. We have

$$\Sigma_{\mathcal{E}} = \mathbf{cov}(\mathcal{E}_\mathbf{t}) = \mathbf{diag}(\sigma^{\mathbf{2}}_{\mathbf{1}}, ..., \sigma^{\mathbf{2}}_{\mathbf{p}})$$

which is a diagonal matrix.

(iii) The final estimator is

$$\alpha\beta\beta^\top\sigma_M^2 + (1-\alpha)\Sigma_R$$

or

$$\beta\beta^\top\sigma_M^2 + \alpha\Sigma_{\mathcal{E}} + (1-\alpha)D_{\mathcal{E}}$$

where D is the diagonal of $\Sigma_{\mathcal{E}}$. I personally recommend $w = 1/\sqrt{T}$.

Example 1.2 *Consider the DJ30 components, we estimate the conditional covariance matrix by exponential smoothing*

see factorCovEstimate.R.

Multivariate GARCH

(a) Weighted covariance estimation

$$\hat{S} = \frac{\sum_{t=1}^{T} w_t R_t R_t^\top}{\sum_{t=1}^{T} w_t}$$

A special case is

$$\hat{S}_t = (1-\lambda) \sum_{i=1}^{\infty} \lambda^{i-1} R_{t-i} R_{t-i}^\top$$

Usually, $\lambda = 0.94$. [This leads to the Multivariate GARCH model discussed later]

(b) It is well known that when p is large, the estimator $\hat{S}$ can be improved using the shrinking methods.

We can follow suit.

Application in Portfolio

Example 1.3 *Consider the DJ30 components, we estimate the conditional covariance matrix by exponential smoothing*

$$H_t = \{\alpha^1 u_{t-1} u_{t-1}^\top + \alpha^2 u_{t-2} u_{t-2}^\top + ...\}/\{\alpha^1 + \alpha^2 + ...\}$$

Based on H_t, we can construct portfolios. Here, we only consider the minimum risk portfolio $w^\top R_t$, where

$$\min_w w^\top H_t w,$$

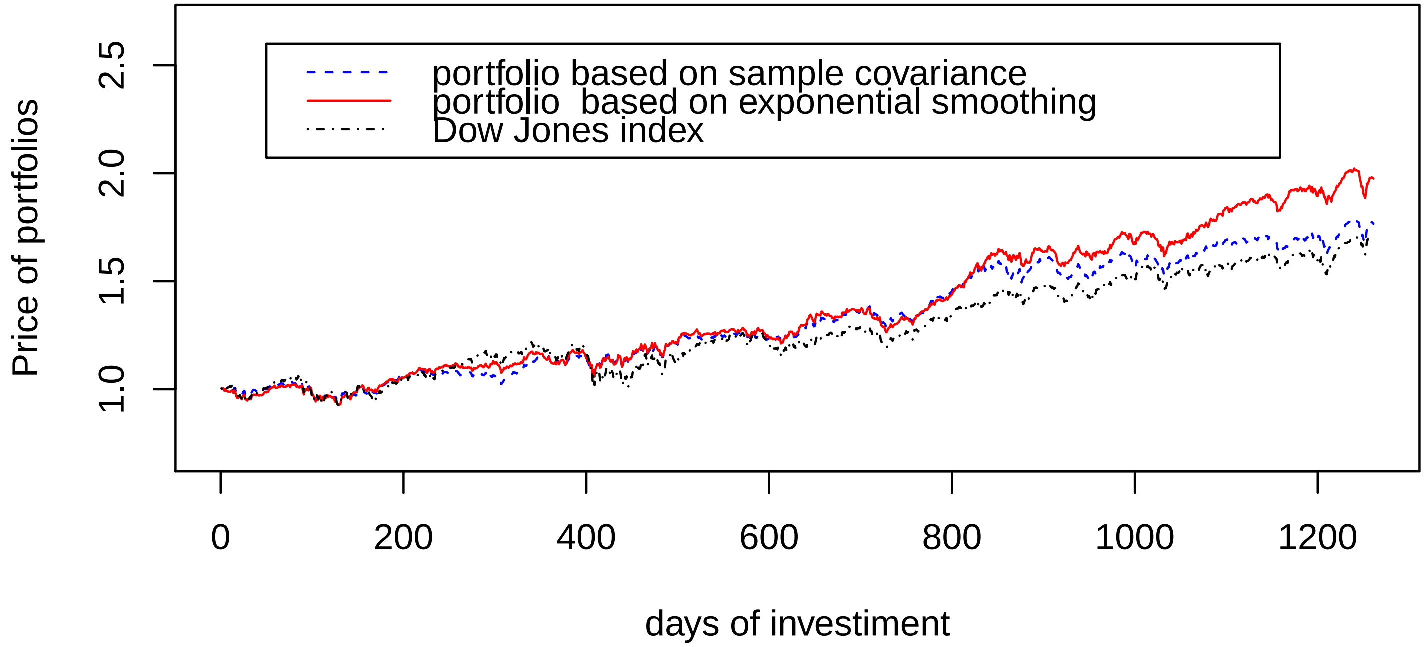
with $w = (w_1, ..., w_{30})^\top$,

$$w_1, ..., w_{30} \geq 0, \qquad w_1 + ... + w_{30} = 1.$$

Based on all the data starting from 2000 to 2014, the portfolio based on sample covariance matrix (of the past 500 days) has bigger volatility than those based on the exponential smoothing, respectively

0.00686939 and 0.00675209

The figure below shows the prices of the portfolios. see myportfolio.R.



(d) The Conditional covariance matrix is defined as

$$H_t = Cov(R_t | R_s, s < t) = (\sigma_{ij,t})_{1 \le i,j \le p}$$

(e) We assume

$$R_t = \mu_t + u_t$$

where

$$u_t = R_t - \mu_t = H_t^{1/2} \xi_t.$$

where ξ_t is multivariate with mean 0 and variance-covariance matrix I (identity matrix).

(f) The dynamic conditional correlation (DCC) GARCH

$$H_t = (1 - \alpha - \beta) H_0 + \alpha u_{t-1} u_{t-1}^\top + \beta H_{t-1}$$

where H_0 is the unconditional covariance matrix of R_t, $\alpha > 0, \beta > 0$ and $\alpha + \beta \le 1$.

(g) Note that DCC is also a generalized form of exponential smoothing. There are many models for the conditional covariance matrix

(h) The constant conditional correlation GARCH model by Bollerslev (1990) is often used. For example, the CCC(1,1) model is given by

$$H_t = diag(\sigma_{1,t}, ..., \sigma_{p,t}) C diag(\sigma_{1,t}, ..., \sigma_{p,t})$$

Or

$$\sigma_{ij,t} = \sigma_{i,t}\sigma_{j,t}\rho_{ij}$$

where $C = (\rho_{ij})_{1 \le i,j \le p}$ is a constant coefficient correlation matrix, and

$$\sigma^2_{k,t} = \omega_k + \alpha_k u^2_{k,t-1} + \beta_k \sigma^2_{k,t-1}$$

which is the GARCH(1,1) model.

Example 1.4 *Consider the DJ30 components, we use CCC model to predict the conditional variance matrix, and apply portfolio theory to find the minimum risk portfolio. The following plot is a comparison of the performance (assuming that there is not transaction fee ...); see ccc.R.*