Chapter 6: Introduction to Financial Time Series (Part A: review)

Basic concepts:

1. Stationary time series, weakly stationary and strictly stationary

2. Auto-covariance function (ACVF) and autocorrelation function (ACF)

3. Hypothesis test of ACF

4. White noise (WN)

1 Theoretical definitions of time series

A time series $\{Z_t\}$ is a sequence of random variables indexed by time t:

$$\{...Z_1, Z_2, ...Z_{t-1}, Z_t, Z_{t+1}, ...\}.$$

A realization of a stochastic process (or time series) is the sequence of observed data

$$\{...Z_1 = z_1, Z_2 = z_2, ...Z_{t-1} = z_{t-1}, Z_t = z_t, Z_{t+1} = z_{t+1}, ...\}$$

Hereafter, we use $\{z_t\}$ for both theoretical time series and its observations/realizations.

Definition: Let $\{z_t\}$ be a time series (a sequence of random variables) with $Ez_t^2 < \infty$. The mean function of $\{z_t\}$ is

$$\mu_t = E(z_t) \qquad \text{(or denonted by } \mu_z(t))$$

The variance function

$$\sigma_t^2 = \mathbf{cov}(z_t) = E(z_t - \mu_t)^2$$

The covariance function of $\{z_t\}$

$$\gamma(r, s) \;=\; \mathbf{cov}(z_r, z_s) = E[(z_r - \mu_r)(z_s - \mu_s)] \qquad \text{(or denonted by } \gamma_z(r, s))$$

An important assumption behind the time series analysis

Pattern must repeatedly appear, i.e. "History repeats itself", so that

(a) we can observe the pattern;

(b) make inference about the pattern;

(c) and to make prediction of the time series based on the pattern

(d) In statistics, this is the assumption of "stationary time series"

Definition: $\{z_t\}$ is $\mathbf{strictly\ stationary}$ if for any given finite integer k, and for any set of subscripts $t_1, t_2, ..., t_k$ the joint distribution of $z_{t_1}, z_{t_2}, ..., z_{t_k}$ depends only on $t_1 - t_2, t_2 - t_3, ..., t_{k-1} - t_k$ but not directly on t_1, ..., t_k.

Example 1 . A few examples.

(a) If $\{Z_t\}$ is stationary, then (Z_6, Z_9, Z_{20}) and (Z_{21}, Z_{24}, Z_{35}) have the same distribution

(b) If $\{Z_t\}$ is an IID sequence, then it is strictly stationary.

(c) Let $\{Z_t\}$ be an iid sequence and let X independent of $\{Z_t\}$. Let $Y_t = Z_t + X$, Then the sequence $\{Y_t\}$ is strictly stationary.

A fact: for any function/transformation $g(.)$, $\{g(z_t)\}$ is also strictly stationary.

Definition: $\{z_t\}$ is $\mathbf{weakly\ stationary}$ (or covariance stationary, or stationary) if

(a) μ_t does not depend on t, denoted by μ or (μ_z to indicate expectation of z)

(b) $\gamma(r, s) = \gamma(|r - s|)$ (independent of r and s but dependent on their "lag" on the time axis). Alternatively, if $s = t$ and $r = t + h$

$$\gamma(t + h, t) = \gamma(h) = \gamma(-h) \quad \text{(or } \ \gamma_z(h) \text{)}$$

By the definition we have $\mathbf{cov}(z_t) = \gamma(0)$ (denoted by σ_z^2)

Example 2 If $\{z_t\}$ is stationary, then

$$\gamma(2) = \mathbf{cov}(z_1, z_3) = \mathbf{cov}(z_2, z_4) = ...$$

and

$$\gamma(3) = \mathbf{cov}(z_1, z_4) = \mathbf{cov}(z_2, z_5) = ...$$

A simple fact. If $\{Z_t\}$ is strictly stationary and $\mathbf{cov}(Z_t) < \infty$, then $\{Z_t\}$ is also weakly stationary. However, a (weakly) stationary time series is usually not strictly stationary

We usually refer "stationary" as weakly stationary later

Example 3 Let

$$y_t = \beta_0 + \beta_1 t + z_t$$

where z_t is stationary with mean 0 and $\beta_1 \neq 0$. Is y_t stationary?

[Answer. No, because

$$E(y_t) = \beta_0 + \beta_1 t$$

depends on time t.]

Example 4 Let

$$y_t = z_t + z_{t-1} + z_{t-2} + \cdots + z_1$$

where $z_t \sim N(0, \sigma^2)$. Then y_t is not stationary. Note that the model can be written as

$$y_t = y_{t-1} + z_t$$

which is called random walk. In financial market, the (daily, weekly, monthly) price of an asset is usually a random walk.

[a random walk is not stationary, because

$$\begin{aligned} \mathbf{cov}(y_t) &= \mathbf{cov}(z_t + z_{t-1} + z_{t-2} + \cdots + z_1) \\ &= \mathbf{cov}(z_t) + \mathbf{cov}(z_{t-1}) + \cdots + \mathbf{cov}(z_1) \\ &= t\sigma^2 \end{aligned}$$

depends on t.]

Definition: If $\{z_t\}$ is stationary, then

(a) $\gamma(h) = \mathbf{cov}(z_{t+h}, z_t)$ is called the **autocovariance function** (ACVF) of $\{z_t\}$,

(b)

$$\rho_z(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\mathbf{cov}(z_{t+h}, z_t)}{\mathbf{cov}(z_t)} = Corr(z_{t+h}, z_t)$$

is called the **autocorrelation function (ACF) at lag** h of $\{z_t\}$.

Basic properties of $\gamma(.)$

(a) $\gamma(0) \geq 0$.

(b) $|\gamma(h)| \leq \gamma(0)$ for all h

(c) $\gamma(h) = \gamma(-h)$.

[Proof: (a) $\gamma(0) = \mathbf{cov}(z_t) \geq 0$.

(b) Because

$$|\rho(h)| = \left|\frac{\gamma(h)}{\gamma(0)}\right| \leq 1, \quad \text{thus} \quad |\gamma(h)| \leq \gamma(0).$$

(c) Because

$$\gamma(h) = \mathbf{cov}(z_{t+h}, z_t) = \mathbf{cov}(z_t, z_{t+h}) = \gamma(-h).$$

Definition: If $\{z_t\}$ is a stationary with $Ez_t = 0$, $\gamma(0) = \sigma^2$ and

$$\gamma(h) = 0, \quad \text{for any} \quad h \neq 0$$

then we call $\{z_t\}$ **white noise** sequence, denoted by $z_t \sim WN(0, \sigma_z^2)$

Example 5 suppose $\{y_t\}$ is a stationary time series, and $\{\epsilon_t\}$ is a $WN(0, \sigma^2)$. It they are independent, which means $E\{y_t\epsilon_{t+k}\} = E\{y_t\}E\{\epsilon_{t+k}\}$ for all t and k, then

$$z_t = \{a + by_t^2\}^{1/2}\epsilon_t, \quad a \geq 0, b \geq 0,$$

is also a WN.

Example 6 suppose $\{\epsilon_t\}$ is a sequence of IID $N(0, \sigma^2)$ (thus it is WN). Let

$$z_t = \{a + bz_{t-1}^2\}^{1/2}\epsilon_t, \quad a \geq 0, b \geq 0.$$

Then $\{z_t\}$ is also a WN. One simple fact in this model is that z_{t-1} depends only on $\epsilon_{t-1}, \epsilon_{t-2}, \ldots$. This is an ARCH model (to be discussed later)

Example 7 We can build time series with white noise sequence. Suppose $z_t \sim WN(0, \sigma^2)$. let

$$y_t = z_t + \theta z_{t-1}$$

Then

$$E(y_t) = E(z_t + \theta z_{t-1}) = Ez_t + \theta E z_{t-1} = 0.$$

and

$$\begin{aligned}
\mathbf{cov}(y_t) &= \mathbf{cov}(z_t + \theta z_{t-1}) \\
&= \mathbf{cov}(z_t) + \theta^2 Var(z_{t-1}) \\
&= \sigma^2 + \theta^2\sigma^2 = (1+\theta^2)\sigma^2
\end{aligned}$$

what about $\mathbf{cov}(y_{t+h}, y_t)$?

(i) If $h = 1$, we have

$$\begin{aligned} \mathbf{cov}(y_{t+1}, y_t) &= E[(z_{t+1} + \theta z_t)(z_t + \theta z_{t-1})] \\ &= E[z_{t+1}(z_t + \theta z_{t-1})] + \theta E[z_t(z_t + \theta z_{t-1})] \\ &= 0 + \theta E(z_t^2) = \theta \sigma^2. \\ & \qquad (\because E(z_s z_t) = 0, \quad t \neq s) \end{aligned}$$

(ii) If $h > 1$

$$\begin{aligned} \mathbf{cov}(y_{t+h}, y_t) &= E[(z_{t+h} + \theta z_{t+h-1})(z_t + \theta z_{t-1})] \\ &= E[z_{t+h}(z_t + \theta z_{t-1})] + \theta E[z_{t+h-1}(z_t + \theta z_{t-1})] \\ &= 0 + \theta E[z_{t+h-1}(z_t + \theta z_{t-1})] \\ &= 0 + 0 = 0. \qquad \because z + h - 1 > t. \end{aligned}$$

Wold's Decomposition Theorem

Any weakly stationary time series $\{Y_t\}$ can be represented in the form

$$\begin{aligned} Y_t &= \mu + \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_2 \varepsilon_{t-2} + \ldots \\ &= \mu + \sum_{k=0}^{\infty} \phi_k \varepsilon_{t-k} \end{aligned}$$

where

$$\phi_0 = 1, \quad \sum_{k=1}^{\infty} \phi_k^2 < \infty$$

and

$$\{\varepsilon_t\} \text{ is WN}(0, \sigma^2).$$

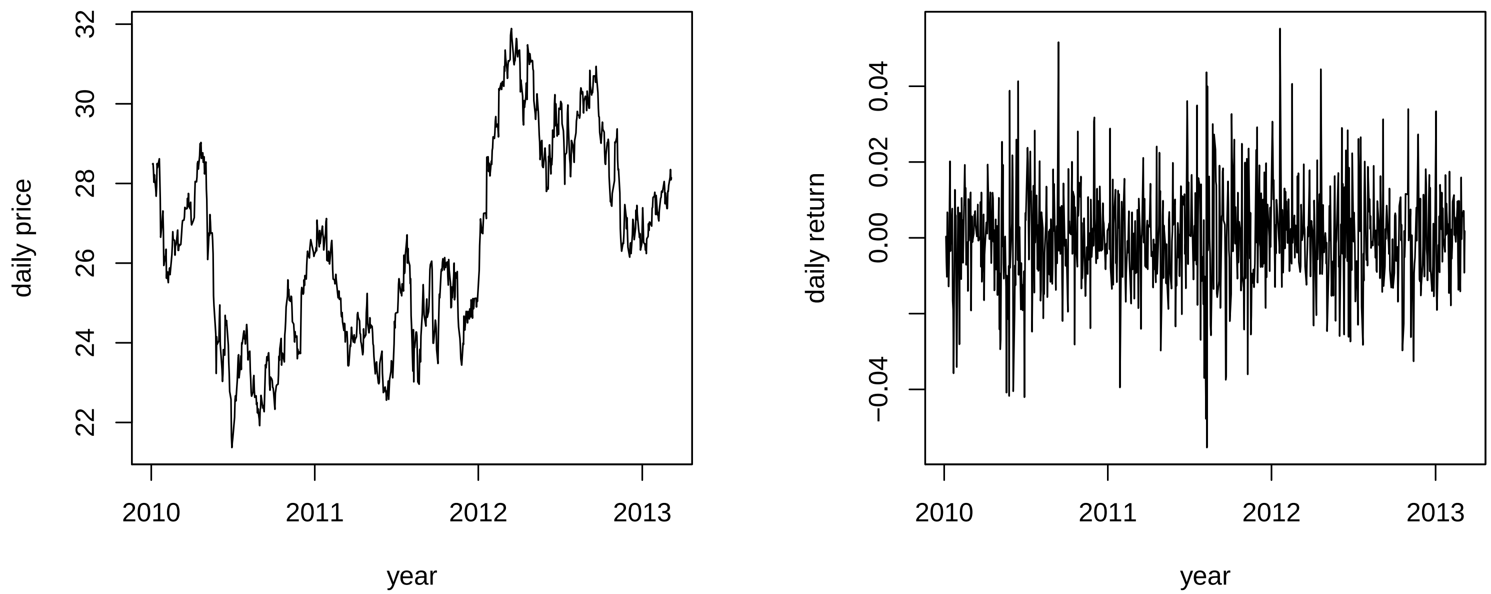
Properties

1. $EY_t = \mu$
2. $\gamma(0) = \mathbf{cov}(Y_t) = \sigma^2 \sum_{k=0}^{\infty} \phi_k^2$
3. when $h > 1$,

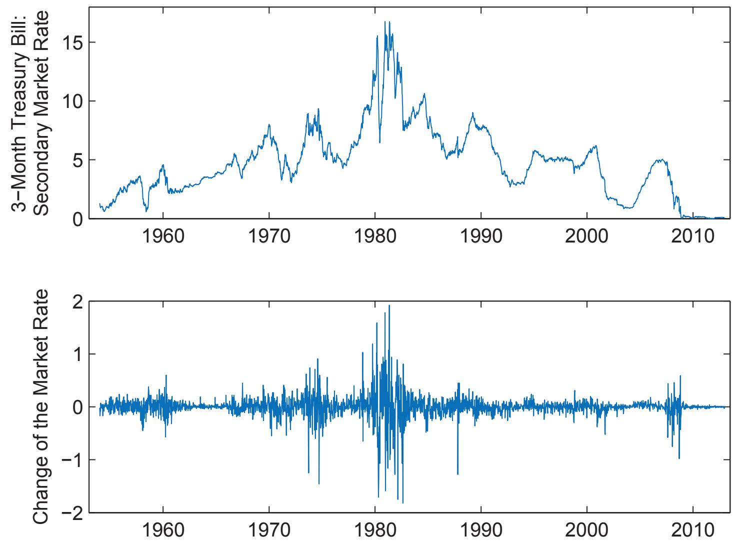
$$\begin{aligned}
\gamma(h) &= E\{(Y_t - \mu)(Y_{t-h} - \mu)\} \\
&= E\{[\varepsilon_t + \phi_1 \varepsilon_{t-1} + ... + \phi_h \varepsilon_{t-h} + ...] \\
&\qquad \times [\varepsilon_{t-h} + \phi_1 \varepsilon_{t-h-1} + ...]\} \\
&= \sigma^2 \{\phi_h + \phi_1 \phi_{h+1} + ...\} \\
&= \sigma^2 \sum_{j=0}^{\infty} \phi_j \phi_{h+j}
\end{aligned}$$

How does a stationary time series look like

In every ("large enough") time window, the sample mean and variance must be similar.



The left time series is not, but the right is.



Weekly Secondary Market Rate 3-Month Treasury Bill (1954 to 2012)

The first time series is not stationary, but interestingly the second is weakly stationary!

How to get the stationary time series from non-stationary time series

In finance data, the stationary can usually be obtained by taking the first-difference

$$z_t = x_t - x_{t-1}$$

If z_t is stationary, then x_t is integrated of order 1.

Example 8 The random walk is integrated of order 1.

We can also consider second-difference

$$z_t = (x_t - x_{t-1}) - (x_{t-1} - x_{t-2}) = x_t - 2x_{t-1} + x_{t-2}$$

If z_t is stationary (only after being taken second-difference), then x_t is integrated of order 2.

2 The sample ACVF, ACF

Suppose we have observations $z_1, z_2, \cdots, z_n$. Suppose that $\{z_t\}$ is stationary, then we have to esitamte μ_z, $\gamma(h)$ and $\rho_z(h)$, with $h = 0, 1, 2, \cdots$.

Sample mean $\bar{z} = \frac{\sum_{t=1}^{n} z_t}{n}$

Sample variance $\hat{\sigma}_z^2 = \frac{1}{n}\sum_{t=1}^{n}(z_t - \bar{z})^2 = \hat{\gamma}(0)$

Sample autocovariance function at lag h, SACVF

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(z_t - \bar{z})(z_{t+h} - \bar{z})$$

Sample autocorrelation coefficient function at lag h, SACF

$$\hat{\rho}(h)(\text{ or } r_h) = \frac{\sum_{t=1}^{n-h}(z_t - \bar{z})(z_{t+h} - \bar{z})}{\sum_{t=1}^{n}(z_t - \bar{z})^2} = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$$

Example 9 $y_t = \{$ 0.8445, -0.3338, 0.4706, 0.1755, -0.711, -0.4352, -0.0405, 1.4258, -1.8739, 2.2478, -3.3019, 2.486, -1.1905, 1.0113, -0.2929, 0.7905, 0.3213, -0.3139, -0.6062, 0.3438$\}$. We have $n = 20$, and

$$\bar{x} = (0.8445 - 0.3338 + ... + 0.3438)/20 = 0.05$$

sample ACVF is

$$\hat{\gamma}(0) = \hat{\sigma}_x^2 = \frac{1}{20}\sum_{t=1}^{n}(x_t - \bar{x})^2 = 1.6646$$

$$\begin{aligned}
\hat{\gamma}(1) &= \frac{1}{20}\sum_{t=1}^{n-1}(x_t - \bar{x})(x_{t+1} - \bar{x}) = -1.3641 \\
\hat{\gamma}(2) &= \frac{1}{20}\sum_{t=1}^{n-2}(x_t - \bar{x})(x_{t+2} - \bar{x}) = 1.0670
\end{aligned}$$

t	x_t	x_{t-1}	x_{t-2}	$\cdots$	x_{t-10}	$\cdots$	x_{t-18}	x_{t-19}
1	0.8445	NA	NA	$\cdots$	NA	$\cdots$	NA	NA
2	-0.3338	0.8445	NA	$\cdots$	NA	$\cdots$	NA	NA
3	0.4706	-0.3338	0.8445	$\cdots$	NA	$\cdots$	NA	NA
4	0.1755	0.4706	-0.3338	$\cdots$	NA	$\cdots$	NA	NA
5	-0.711	0.1755	0.4706	$\cdots$	NA	$\cdots$	NA	NA
6	-0.4352	-0.711	0.1755	$\cdots$	NA	$\cdots$	NA	NA
7	-0.0405	-0.4352	-0.711	$\cdots$	NA	$\cdots$	NA	NA
8	1.4258	-0.0405	-0.4352	$\cdots$	NA	$\cdots$	NA	NA
9	-1.8739	1.4258	-0.0405	$\cdots$	NA	$\cdots$	NA	NA
10	2.2478	-1.8739	1.4258	$\cdots$	NA	$\cdots$	NA	NA
11	-3.3019	2.2478	-1.8739	$\cdots$	0.8445	$\cdots$	NA	NA
12	2.486	-3.3019	2.2478	$\cdots$	-0.3338	$\cdots$	NA	NA
13	-1.1905	2.486	-3.3019	$\cdots$	0.4706	$\cdots$	NA	NA
14	1.0113	-1.1905	2.486	$\cdots$	0.1755	$\cdots$	NA	NA
15	-0.2929	1.0113	-1.1905	$\cdots$	-0.711	$\cdots$	NA	NA
16	0.7905	-0.2929	1.0113	$\cdots$	-0.4352	$\cdots$	NA	NA
17	0.3213	0.7905	-0.2929	$\cdots$	-0.0405	$\cdots$	NA	NA
18	-0.3139	0.3213	0.7905	$\cdots$	1.4258	$\cdots$	NA	NA
19	-0.6062	-0.3139	0.3213	$\cdots$	-1.8739	$\cdots$	0.8445	NA
20	0.3438	-0.6062	-0.3139	$\cdots$	2.2478	$\cdots$	-0.3338	0.8445

$$\hat{\gamma}(3) = -0.8559 \quad \hat{\gamma}(4) = 0.5497 \quad \hat{\gamma}(5) = -0.3338 \quad \hat{\gamma}(6) = 0.1656$$

$$\hat{\gamma}(7) = -0.0520 \quad \hat{\gamma}(8) = -0.0178 \quad \hat{\gamma}(9) = 0.0943 \quad \hat{\gamma}(10) = -0.1344$$

and the SACF are $r(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$:

$$r(0) = 1, \quad r(1) = -0.8195, \quad r(2) = 0.6410,$$

$$r(3) = -0.5142, r(4) = 0.3302, r(5) = -0.2005,$$

$$r(6) = 0.0995, r(7) = -0.0312, \; r(8) = -0.0107,$$

$$r(9) = 0.0567, \; r(10) = -0.0808$$

3 Testing Hypothesis $H_0 : \rho(h) = 0, \quad h \geq 1$

Suppose that $z_1, \cdots, z_n$ is a realization (sample, observations) from a stationary time series. The SACF at lag h is

$$r_h = \frac{\sum_{t=1}^{n-h}(z_t - \bar{z})(z_{t+h} - \bar{z})}{\sum_{t=1}^{n}(z_t - \bar{z})^2}$$

where $\bar{z} = \sum_{i=1}^{n} z_t/n$ The <u>*standard error of* r_h</u> is

$$s_{r_h} = \left(\frac{1 + 2\sum_{j=1}^{h-1}\rho(j)^2}{n}\right)^{1/2} \approx \left(\frac{1 + 2\sum_{j=1}^{h-1} r_j^2}{n}\right)^{1/2}$$

Under $H_0' : \rho(h) = 0$ for all $h > 0$,

$$s_{r_h} = (1/n)^{1/2}$$

The <u>2-σ criteria</u>: If $|r_h| < 2s_{r_h}$, then accept H_0; otherwise reject H_0[1]

[1] Under H_0, $P(\rho - 1.96 * s_{r_h} < r_h < \rho + 1.96 * s_{r_h}) = P(-1.96 * s_{r_h} < r_h < 1.96 * s_{r_h}) = 0.95$

For the above example, the ACF are

h	ACF(h)	s_{r_h}	$2s_{r_h}$
0	1	–	–
1	-0.8195	0.2236	0.4472
2	0.6410	0.2891	0.5782
3	-0.5142	0.3227	0.6454
4	0.3302	0.3426	0.6851
5	-0.2005	0.3504	0.7009
6	0.0995	0.3533	0.7066
7	-0.0312	0.3540	0.7080
8	-0.0107	0.3541	0.7081
9	0.0567	0.3541	0.7081
10	-0.0808	0.3543	0.7086

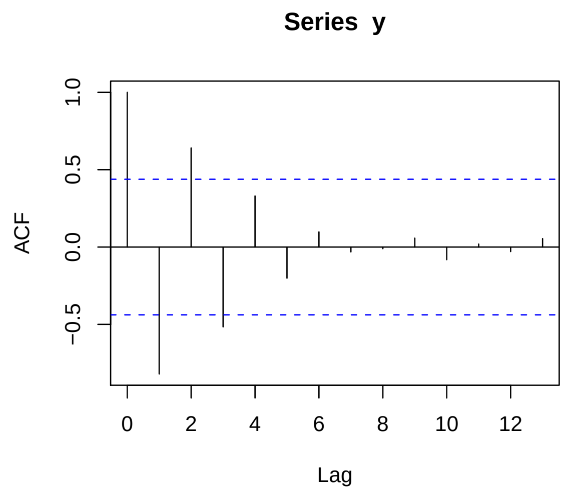
The autocorrelation coefficients at lag 1, 2 are significantly not zeros, and all the others are not significantly different from zero using the 2-σ criterion.

R code:

```
x = c(0.8445, -0.3338, 0.4706, 0.1755, -0.7110, -0.4352, -0.0405, 1.4258, -1.8739, 2.2478, -3.3019, 2.4860, -1.1905,
1.0113, -0.2929, 0.7905, 0.3213, -0.3139, -0.6062, 0.3438)
```

```
r = acf(x)
```
[2]

```
r$acf
```

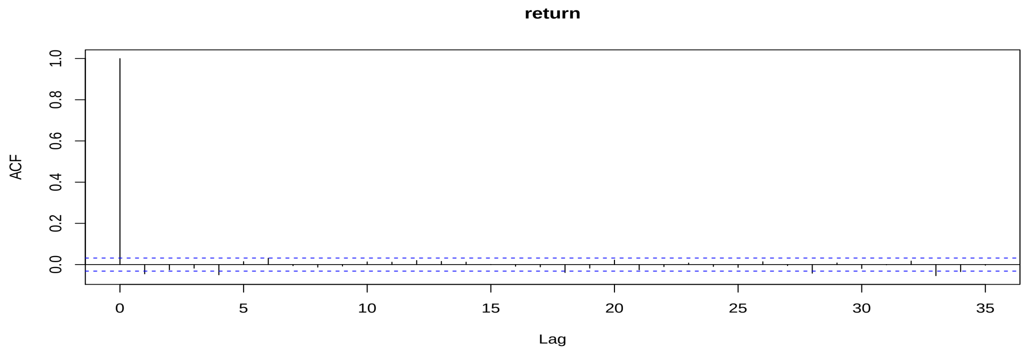


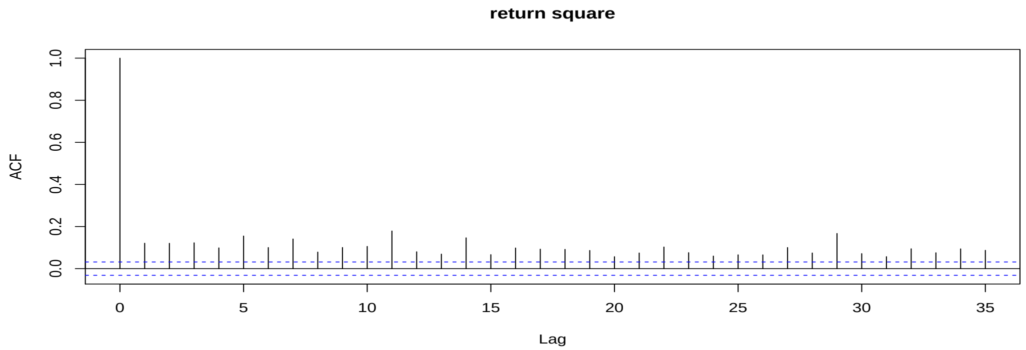
The autocorrelation coefficients at lag 1, 2 are significantly not zeros, and all the others are not significantly different from zero using the 2-σ criterion

Example 10 The daily returns of MSFT from 3/1/2010 to present should be reject to be white noise; see code code05C.R We can find the return itself shows not significant non-zero ACF; but the square of

[2] For different statistical packages, please note that when n is small, the calculation results may be different due to different approximation

the returns do; see the two SACF plot below





4 Test of White Noise

- White Noise means no information is contained that can make (linear) prediction of the time series.
- Thus we need to check particularly whether the residuals of a time series model is White Noise
- It is easy to see that if y_t is WN, then the autocorrelation function $\rho(k)$ at all lags are 0, i.e.

$$\rho(k) = 0, \quad \text{for all } k$$

- We sometimes further check the ACF of $\{y_t^2\}$, to see whether it can be predicted.

The Ljung-Box test is a type of statistical test of whether a number of autocorrelations of a time series are different from zero.

$$H_0\text{: } \rho(k) = 0, k = 1, ..., h.$$

Suppose the time series is $\{y_1, y_2, ..., y_n\}$ with sample autocorrelation coefficients $r(k)$. The test statistic is:

$$Q(h) = n(n+2)\sum_{k=1}^{h}\frac{\{r(k)\}^2}{n-k}$$

where n is the sample size, $r(k)$ is the sample autocorrelation at lag k, and h is the number of lags being tested. Under H_0,

$$Q(h) \sim \chi^2(h).$$

For significance level α, the critical region for rejection of the hypothesis of randomness is rejected if the calculated value (Ljung-Box statistic)

$$Q^*(h) > \chi^2_{1-\alpha}(h)$$

where $\chi^2_{1-\alpha}(h)$ is the $(1-\alpha)$-quantile of the chi-square distribution with h degrees of freedom.

The bigger the probability of $\chi^2(h) > Q^*(h)$ is, the smaller $Q^*(h)$ is, the more likely H_0 is correct.

$$p - value = P(\chi^2(h) > Q^*(h))$$

if $p - value > \alpha$, then we accept H_0.

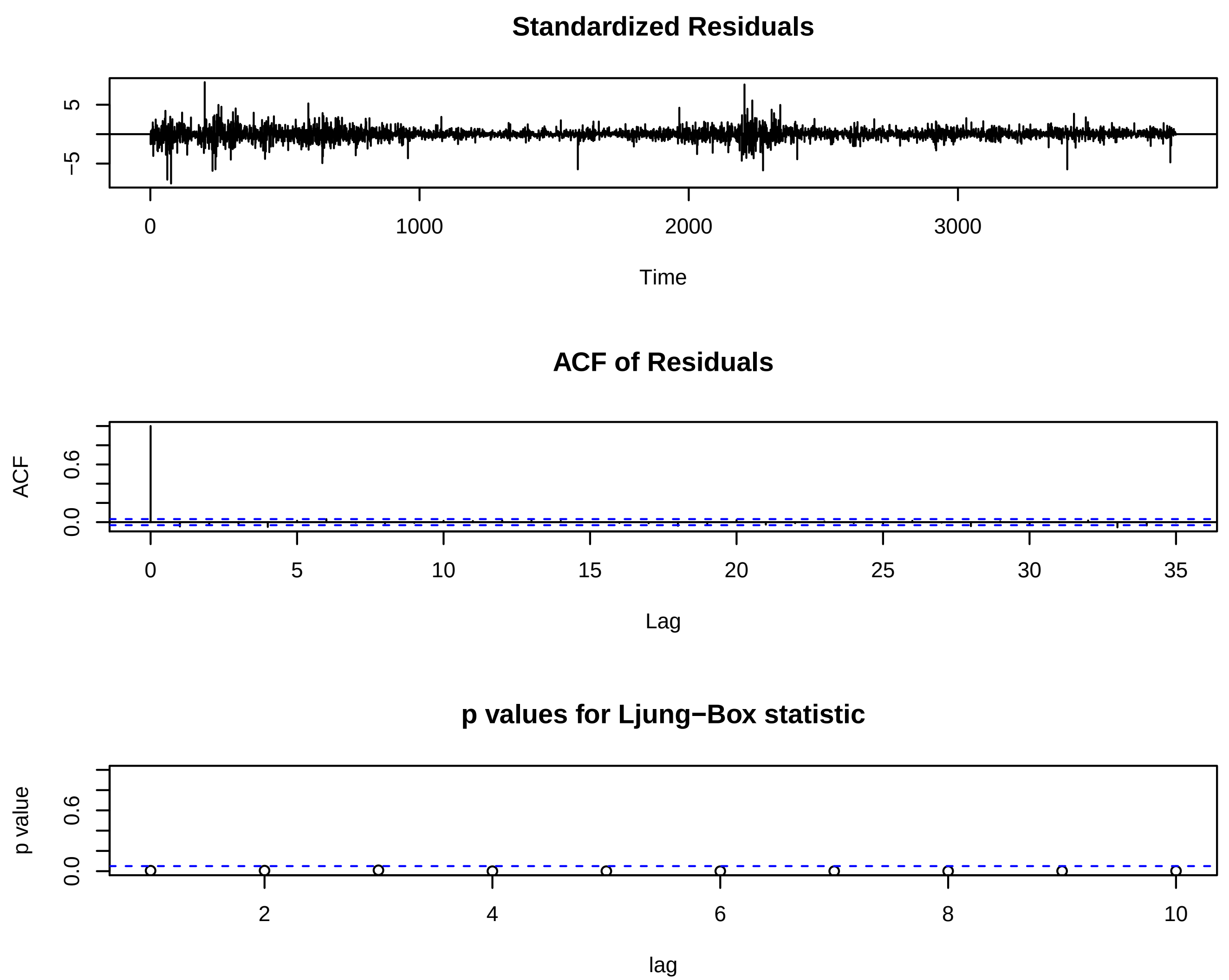
```
R:

Box.test([time series], lag = ??, type = "Ljung")

tsdiag([fitted model])      # to test whether the residuals

                                # of the fitted model is white noise
```

Example 11 . With significant level $\alpha = 0.05$, the daily returns of MSFT from 1/1/2000 to present should be reject to be white noise; the sqaured return (after removing the mean) is also rejected as White noise; see code code05Ab.R;

The output of Box.test;

```
> Box.test(r, lag=10, type="Ljung")

        Box-Ljung test

data:  r
X-squared = 27.1166, df = 10, p-value = 0.002496

>
> Box.test(r^2, lag=10, type="Ljung")

        Box-Ljung test

data:  r^2
X-squared = 516.7883, df = 10, p-value < 2.2e-16
```

The output for return r based tsdiag();


Standardized Residuals
5
−5
0
1000
2000
3000
Time
ACF of Residuals
ACF
0.0
0.6
0
5
10
15
20
25
30
35
Lag
p values for Ljung−Box statistic
p value
0.0
0.6
2
4
6
8
10
lag

The output for return square r^2 based tsdiag();