Oscillations on a spherical lens

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This writeup covers the theoretical background and procedure for the experiment of releasing a ball on top of a spherical lens.

1 Introduction

This is a preliminary experiment to get the student used on how to manipulate the experimental apparatus and analysis tools necessary to conduct the following experiments. The experiment consists of releasing a spherical mass on top of a spherical lens. The moviment of the ball will be similar to that of a pendulum and so can be used to study the local gravitational acceleration, if the radius of curvature of the lens is known. Otherwise, the latter can be estimated by using $g = 9.81$ m/s^2. The student will need to learn how to operate the high speed camera to record multiple launches and later analyse the footage using the program ImageJ, which can track the particle's movement.

In the following section we discuss the theoretical side that will aid in the analysis of the experiment. In §3 we give more details on the experimental procedure and analysis.

2 Theory

In what follows we will assume that the ball does not slip during its movement. As you perform the experiment, you will notice that a non negligible damping is present. Also, it may be necessary to include corrections to the small angle approximation for a better result, depending on the lens diameter and its radius. We will first derive the full nonlinear equation without damping, then solve it in the small angle approximation with damping and later discuss corrections to that approximation.

2.1 Equation of motion

Include picture with forces and everything. Let m be the mass of the ball and r its radius. From the figure, the tangential net force yields

$$m\dot{v} = -mg\sin\theta + f\,, \tag{2.1}$$

where f is the friction between the ball and the lens. The net torque allows us to find the friction:

$$I\dot{\omega} = -fr\,. \tag{2.2}$$

It should be noted that the moment of inertia of a uniform solid sphere is given by $I = \frac{2}{5}mr^2$. Besides these two equations, we have some other constraints. First, from the nonslip condition, we find

$$v = \omega r\,. \tag{2.3}$$

The other condition is for parametrizing the movement in terms of $\theta(t)$:

$$v = (R - r)\dot{\theta}\,. \tag{2.4}$$

Putting all these equations together and solving for $\ddot{\theta}$ yields

$$\ddot{\theta} = -\frac{5g}{7(R-r)}\sin\theta\,, \tag{2.5}$$

where we used the expression for the moment of inertia $I = \frac{2}{5}mr^2$.

This is the same equation that one would obtain for a pendulum of length $l = \frac{7}{5}(R - r)$ under gravitational acceleration g. Thus, as done in the introductory lab course, this may be used to measure the value of g. The idea is to restrict attention to small angles, such that $\sin\theta \approx \theta$. The resulting equation is that of an harmonic oscillator, $\ddot{\theta} = -\omega_0^2\theta$, with frequency

$$\omega_0 = \sqrt{\frac{5g}{7(R-r)}}\,. \tag{2.6}$$

By measuring periods, one can determine g, given R and r.[1]

2.2 Damping

In practice, we need to include damping to get a more accurate result. Including a velocity damping, we may write the equation of motion in the small angle approximation as

$$\ddot{\theta} + 2\beta\ddot{\theta} + \omega_0^2\theta = 0,, \tag{2.7}$$

where β is the damping parameter. The general solution for this equation [**include reference**] is given by

$$\theta(t) = Ae^{-\beta t}\cos(\omega t + \phi)\,, \tag{2.8}$$

where A and ϕ are parameters to be determined, for example, from the initial conditions, and the frequency

$$\omega = \sqrt{\omega_0^2 - \beta^2}\,. \tag{2.9}$$

[1]The radius R should be available from the lens specifications while r can be measured directly.

The validity of these equations assume $\omega_0 > \beta$, which is observed in the experiment. For completeness, the initial conditions are related to the parameters as

$$A^2 = \theta_0^2 + \left(\frac{\dot{\theta}_0 + \beta\theta_0}{\omega}\right)^2 , \quad \tan\phi = -\frac{\dot{\theta}_0 + \beta\theta_0}{\omega\theta_0} . \tag{2.10}$$

In practice, however, all parameters (A, β, ω, ϕ) are to be treated as free and will be determined by appropriate fitting procedure to best match solution (2.8) with the experimental data.

2.3 Nonlinear corrections

The small angle approximation is very good if the motion occurs near the center of the lens. If its radius of curvature is much bigger than its diameter, the approximation is still valid for motions near the border of the lens. However, in the case the radius and diameter are of comparable order, higher order corrections to the small angle approximations may be valuable.

In practice, a direct solution of the exact equation would be ideal, but that is too complicated in this case. A simpler approach is to consider the next term in the expansion of the sine function:

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \ldots . \tag{2.11}$$

The resulting nonlinear equation of motion,

$$\ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2\theta = \frac{\omega_0^3}{6}\theta^3 , \tag{2.12}$$

can then be solved perturbatively. It seems unclear what the perturbation parameter is in this case, since ω_0 is not assumed to be small. Such parameter is in fact the amplitude of the motion, A in (2.8). This will be more clear in the following.

Consider a general solution of the form $\theta = \theta^{(0)} + \theta^{(1)} + \theta^{(2)} + \ldots$, where the superscript indicates the order in perturbation theory. That is, $\theta^{(0)}$ is the solution of the unperturbed equation (2.7):

$$\theta^{(0)}(t) = Ae^{-\beta t}\cos(\omega t + \phi) , \tag{2.13}$$

Let's look at the first order correction only. Plugging the general solution up to first order in (2.12) leads to the following equation for $\theta^{(1)}$:

$$\ddot{\theta}^{(1)} + 2\beta\dot{\theta}^{(1)} + \omega_0^2\theta^{(1)} = \frac{\omega_0^3}{6}\left((\theta^{(0)})^3 + (\theta^{(1)})^3\right) , \tag{2.14}$$

Notice that $\theta^{(0)}$ is proportional to A and so the first term in the RHS of (2.14) determines that the leading contribution to $\theta^{(1)}$ is of order A^3. We may then neglect the second term in the RHS of that equation and treat it as a damped oscillator with an external force. We only need to solve for the particular solution, since the transient is already represented by the $\theta^{(0)}$ solution.

The equation to be solved is

$$\begin{aligned}\ddot{\theta}^{(1)} + 2\beta\dot{\theta}^{(1)} + \omega_0^2\theta^{(1)} &= \frac{\omega_0^3}{6}(\theta^{(0)})^3 = \frac{A\omega_0^3}{6}e^{-3\beta t}\cos^3(\omega t + \phi) \\ &= \frac{A\omega_0^3}{24}e^{-3\beta t}\left[3\cos(\omega t + \phi) + \cos(3\omega t + 3\phi)\right] .\end{aligned} \tag{2.15}$$

This equation can be solved by the standard procedure of turning the cosine into complex exponentials, solving the subsequent equation with an exponential solution and taking its real part as the solution of the original equation. This can be done separetely for each factor of frequency ω and 3ω separately. The solution is

$$\theta^{(1)}(t) = \mathrm{Re}\left[\frac{\frac{3\omega_0^2A^3}{24}e^{-3\beta t + i(\omega t+\phi)}}{(i\omega - 3\beta)^2 + 2\beta(i\omega - 3\beta) + \omega_0^2} + \frac{\frac{\omega_0^2A^3}{24}e^{-3\beta t + 3i(\omega t+\phi)}}{(3i\omega - 3\beta)^2 + 2\beta(3i\omega - 3\beta) + \omega_0^2}\right] . \tag{2.16}$$

Notice that no extra parameter is introduced. If deemed necessary, the student may consider higher order corrections and follow the same approach, observing and collecting the like powers of the amplitude as the perturbation parameter. The subsequent equation will always be that of a damped and forced harmonic oscillator.

3 Experiment and analysis