Stat-581: Probability and Statistical Inference for Data Science
Quiz - 3
Sequential Probability Ratio Test - Bernoulli

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1. SPRT function for Bernoulli

```
strp_bernoulli<-function(alpha0 = 0.01, alpha1 = 0.01, p1 = 0.45, p2 = 0.55, bern_p =0.3){
  # alpha0 is Type1 error   alpha1 is Type2 error
  # p<=p1: NULL Hypothesis  p>=p1: Alternate Hypothesis
  S = 0
  log_likelihood = 0
  ## to keep track of number of steps required for convergence ##
  n_converge = 0
  # calculating threshold for stopping
  A = log(alpha1/(1 - alpha0))
  B = log((1 - alpha1)/alpha0)
  hypo_accepted = -1

  while(TRUE){
    n_converge = n_converge + 1
    # generating bernoulli RV with p = bern_p
    data_point = rbinom(1, 1, bern_p)
    # log-likelihood ratio
    log_likelihood = (data_point*p2 + (1-data_point)*(1-p2)) -
                     (data_point*p1 + (1-data_point)*(1-p1))
    # cumulative sum of the log-likelihood ratio
    S = S + log_likelihood
    # Stopping Rule #
    if(S>=B){
      #Accept H1
      hypo_accepted = 1
      break
    }
    if(S<=A){
      #Accept H0
      hypo_accepted = 0
      break
    }
}
  return(list(n_converge = n_converge, hypo_accepted = hypo_accepted))
}
```

Define a pair of hypotheses:

1. Null hypothesis H0: The p value of underlying Bernoulli random variable is p1=0.45
2. Alternate hypothesis H1: The p value of underlying Bernoulli random variable is p2=0.55

For a given sample from Bernoulli distribution, compute the log likelihood $\log\lambda_i$:

- Likelihood function for a Bernoulli sample
  $f_p(x) = p^x(1-p)^{1-x}$

- Log-Likelihood function for a Bernoulli sample
  $\log(f_p(x)) = xp + (1-x)(1-p)$

$$\begin{aligned} \mathit{log\lambda(x)} &= \log(f_{p2}(x)/f_{p1}(x)) \\ &= \log(f_{p2}(x)) - \log(f_{p1}(x)) \\ &= [xp_2 + (1-x)(1-p_2)] - [xp_1 + (1-x)(1-p_1)] \end{aligned}$$

Define thresholds:

alpha0 = desired type I error
alpha1 = desired type II error

$A = (alpha1/(1-alpha0))$
$B = ((1-alpha1)/alpha0)$

Given *alpha0* = 0.1 and *alpha1* = 0.1, *A* = -4.59512 and *B* = 4.59512

Cumulative sum of log likelihoods = S_i
Start with $S_0 = 0$

Generate a new bernoulli random variable x_i
$S_i = S_{i-1} + \log\lambda(x_i)$

Stopping conditions:

1. $A < S_i < B$: continue monitoring (critical inequality)
2. $S_i \leq B$: Accept H1
3. $S_i \geq A$: Accept H0

2. Simulation function for Bernoulli

```
simulate_strp_bernoulli <- function(bern_p = 0.3, nsim = 100){

  sum_n = 0
  H0_count = 0
  H1_count = 0

  ## Averaging over the STPR function
  for(i in c(1:nsim)){
    # calling the SRTP function which generates bernoulli variables with p = bern_p #
    strp_result = strp_bernoulli(bern_p=bern_p)

    ## if H0 is accepted
    if(strp_result$hypo_accepted == 0){
      H0_count = H0_count + 1
    }
    ## if H1 is true
    if(strp_result$hypo_accepted == 1){
      H1_count = H1_count + 1
    }

    sum_n = sum_n + strp_result$n_converge
  }
  avg_steps = sum_n/nsim

   return(list(avg_steps=avg_steps, H0_count=H0_count, H1_count=H1_count))
}
```

3. Test results including all output

a. Run it on a sequence of x's distributed Bernoulli .3, and .56

p_values	Average Steps for convergence	H0_count	H1_count
0.3	112.92	100	0
0.46	602.4	100	0
0.5	2021.58	55	45
0.54	597	0	100
0.56	484.64	0	100

b. What do you think it would do for .54 ? Try it. Why does it give the result you got?

Intuitively setting **p = 0.54** should generate random bernoulli variables having higher likelihood to Hypothesis 1 (p value of underlying Bernoulli random variable is 0.55) than Hypothesis 0 (The p value of underlying Bernoulli random variable is 0.45). This can be determined and verified by the SPRT function for Bernoulli defined above. We tested this by running the simulation 100 times and the results are shown in Table-1. Thus, for p = 0.54, hypothesis 1 is true as seen.

For **p = 0.5**, it has equal likelihood to both hypothesis 0 and hypothesis 1, since it is the midpoint of (0.45,0.55). Thus, the ratio of acceptance of hypothesis 0 to acceptance of hypothesis 1 is close to 1 i.e. both are accepted nearly equal number of times. For 10 trails, hypothesis 1 might be true 5 times and hypothesis 0 maybe true 5 times. The ratio of hypothesis 0 : hypothesis 1 might be 4:6 or the inverse. One of the sample results are shown in (Table 1) on simulating the SPRT bernoulli 100 times.

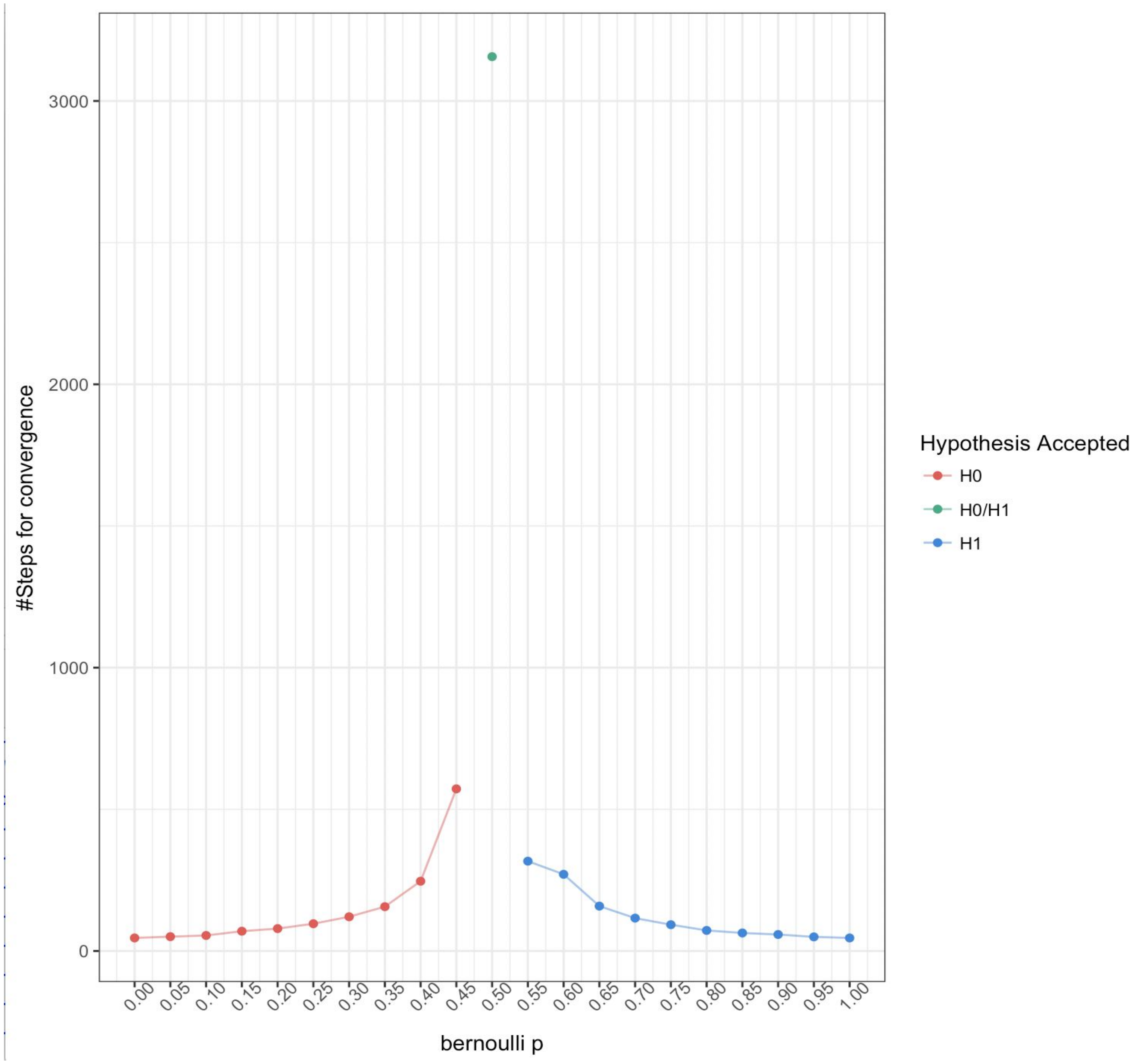
Extended Analysis:

Running the simulation for all the values of p ranging from 0 to 1 and running the simulation 100 times for each p value.

Data Frame:

	bern_p	avg_steps_to_converge	count_H0	count_H1	final_hypothesis_accepted
1	0.00	46.0	10	0	H0
2	0.05	50.6	10	0	H0
3	0.10	54.8	10	0	H0
4	0.15	69.8	10	0	H0
5	0.20	79.0	10	0	H0
6	0.25	96.2	10	0	H0
7	0.30	120.8	10	0	H0
8	0.35	156.4	10	0	H0
9	0.40	246.2	10	0	H0
10	0.45	572.2	10	0	H0
11	0.50	3156.4	5	5	H0/H1
12	0.55	317.0	0	10	H1
13	0.60	270.6	0	10	H1
14	0.65	158.4	0	10	H1
15	0.70	116.0	0	10	H1
16	0.75	92.8	0	10	H1
17	0.80	72.6	0	10	H1
18	0.85	63.6	0	10	H1
19	0.90	58.2	0	10	H1
20	0.95	49.8	0	10	H1
21	1.00	46.0	0	10	H1

Plot:



Key Observations:

1. The number of simulations is maximum for p = 0.5. This is because the Bernoulli random variable with p = 0.5 is equidistant from the set of both the Bernoulli random variable with p = 0.45 and p = 0.55. Thus, the value of *Si* will keep on fluctuating until it converges to either H0 or H1. This drastically increases the number of simulations needed for convergence.

2. In general, the p values ranging from (0.45, 0.55) require greater number of simulations. This can be evidently seen from the graph.
3. For p=0 and p=1, the bernoulli random variable becomes deterministic.
    a. p=0: x=0 => log likelihood = (1-0.55) - (1-0.45) = -0.1
    b. p=1: x=1 => log likelihood = 0.55 - 0.45 = 0.1

    Hence the S_i will uniformly go down below A= -4.5 or rises above B=4.5 by taking a constant 46 number of steps.

4. Hypothesis result:

    For p < 0.5 : hypothesis 0 is accepted
    For p > 0.5 : hypothesis 1 is accepted
    For p = 0.5 : hypothesis 0 and hypothesis 1 are accepted approximately in 1:1 ratio (accepted equal number of times)