[Tarea 10] Ejercicios Unidad 04-C | Descomposición LU

CONJUNTO DE EJERCICIOS

1. Realice las siguientes multiplicaciones matriz-matriz:

a. $\begin{bmatrix} 2 & -3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & -3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 5 & -4 \\ -3 & 2 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 0 \\ 5 & 2 & -4 \end{bmatrix}\begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & -2 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 & 1 & 2 \\ -2 & 3 & 0 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -4 & 1 \\ 0 & 2 \end{bmatrix}$

Ejercicio 1.

a). $\begin{bmatrix} 2 & -3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2-6 & 10 \\ 3-2 & 15 \end{bmatrix} = \begin{bmatrix} -4 & 10 \\ 1 & 15 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 5 & -4 \\ -3 & 2 & 0 \end{bmatrix} = \begin{pmatrix} 2+9 & 10-6 & -8 \\ 3+3 & 15-2 & -12 \end{pmatrix} = \begin{pmatrix} 11 & 4 & -8 \\ 6 & 13 & -12 \end{pmatrix}$

c) $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 0 \\ 5 & 2 & -4 \end{bmatrix}\begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & -2 \end{bmatrix} = \begin{pmatrix} -3+2 & 2+3 & -4+3-2 \\ 3 & 4 & -8-3 \\ 2-8 & 5-12 & -10-2+8 \end{pmatrix} = \begin{pmatrix} -1 & 5 & -3 \\ 3 & 4 & -11 \\ -6 & -7 & -4 \end{pmatrix}$

d) $\begin{bmatrix} 2 & 1 & 2 \\ -2 & 3 & 0 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -4 & 1 \\ 0 & 2 \end{bmatrix} = \begin{pmatrix} 2-4 & -4+1+4 \\ -2-12 & +4+3 \\ 2+4 & -4-1+6 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -14 & 7 \\ 6 & 1 \end{pmatrix}$

2. Determine cuáles de las siguientes matrices son no singulares y calcule la inversa de esas matrices:

a. $\begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -4 & -2 \\ 2 & 1 & 1 & 5 \\ -1 & 0 & -2 & -4 \end{bmatrix}$ **d.** $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 9 & 11 & 1 & 0 \\ 5 & 4 & 1 & 1 \end{bmatrix}$

Ejercicio 2

a) $\begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix} = 4\begin{pmatrix} 0 & 7 \\ -1 & -3 \end{pmatrix} - 2\begin{pmatrix} 3 & 7 \\ -2 & -3 \end{pmatrix} + 6\begin{pmatrix} 3 & 0 \\ -2 & -1 \end{pmatrix}$

$= 4(7) + 2(-9+14) + 6(-3)$

$= 28 - 10 - 18$

$= 0 \rightarrow$ es singular.

b) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix} = 1(2) - 2(2+3)$

$= 2 - 10$

$= -8 \neq \rightarrow$ no singular.

$C = \begin{pmatrix} 1+1 & -(2+3) & 2-3 \\ -(2) & 1 & -(1-6) \\ -2 & +1 & 1-4 \end{pmatrix} = \begin{pmatrix} 2 & -5 & -1 \\ -2 & 1 & 5 \\ -2 & 1 & -3 \end{pmatrix}$

$\text{adj}(B) = \begin{pmatrix} 2 & -2 & -2 \\ -5 & 1 & 1 \\ -1 & 5 & -3 \end{pmatrix}$

$B^{-1} = \frac{1}{-8}\begin{pmatrix} 2 & -2 & -2 \\ -5 & 1 & 1 \\ -1 & 5 & -3 \end{pmatrix} = \begin{pmatrix} -1/4 & 1/4 & 1/4 \\ 5/8 & -1/8 & -1/8 \\ 1/8 & 5/8 & 3/8 \end{pmatrix} \approx \begin{pmatrix} -0.25 & 0.25 & 0.25 \\ 0.625 & -0.125 & -0.125 \\ 0.125 & 0.625 & 0.375 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -4 & -2 \\ 2 & 1 & 1 & 5 \\ -1 & 0 & -2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & -1 & 3 & 3 \\ 0 & 1 & -3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \det(C) = 0$

singular.

d) $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 9 & 11 & 1 & 0 \\ 5 & 4 & 1 & 1 \end{pmatrix}$ $\det = 4(7)(1)(1) = 28 \neq 0$

$\left(\begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 & 1 & 0 & 0 \\ 9 & 11 & 1 & 0 & 0 & 0 & 1 & 0 \\ 5 & 4 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & -1.5 & 1 & 0 & 0 \\ 0 & 11 & 1 & 0 & -2.25 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 & -0.25 & 0 & 0 & 1 \end{array}\right)$

$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -0.214 & 0.143 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.104 & -1.573 & 1 & 0 \\ 0 & 0 & 1 & 1 & -0.398 & -0.572 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -0.214 & 0.143 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.104 & -1.573 & 1 & 0 \\ 0 & 0 & 0 & 1 & -0.5 & 1.0 & -1 & 1 \end{array}\right)$

3. Resuelva los sistemas lineales 4 x 4 que tienen la misma matriz de coeficientes:

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 6, \\ x_1 \quad - x_3 + x_4 &= 4, \\ 2x_1 + x_2 + 3x_3 - 4x_4 &= -2, \\ -x_2 + x_3 - x_4 &= 5; \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 1, \\ x_1 \quad - x_3 + x_4 &= 1, \\ 2x_1 + x_2 + 3x_3 - 4x_4 &= 2, \\ -x_2 + x_3 - x_4 &= -1. \end{aligned}$$

Ejercicio 3

a)
$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 6 \\ x_1 \quad - x_3 + x_4 &= 4 \\ 2x_1 + x_2 + 3x_3 - 4x_4 &= -2 \\ -x_2 + x_3 - x_4 &= 5 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 6 \\ 1 & 0 & -1 & 1 & 4 \\ 2 & 1 & 3 & -4 & -2 \\ 0 & -1 & 1 & -1 & 5 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 6 \\ 0 & 1 & -3 & 2 & -2 \\ 0 & 3 & -1 & -2 & -14 \\ 0 & -1 & 1 & -1 & 5 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 6 \\ 0 & 1 & -3 & 2 & -2 \\ 0 & 0 & 8 & -8 & -8 \\ 0 & 0 & -2 & 1 & 3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 6 \\ 0 & 1 & -3 & 2 & -2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 3 & 5 \end{array}\right) \quad \det = 3$$

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & -4 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \quad \text{inversa} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$x = A^{-1} b_1$

$$xb_1 = \begin{pmatrix} 8 \\ -9 \\ 11 \\ 7 \end{pmatrix}$$

$x = A^{-1} b_2$

$$xb_2 = \begin{pmatrix} 4 \\ -2 \\ 2 \\ 5 \end{pmatrix}$$

4. Encuentre los valores de A que hacen que la siguiente matriz sea singular.

$$A = \begin{bmatrix} 1 & -1 & \alpha \\ 2 & 2 & 1 \\ 0 & \alpha & -\frac{3}{2} \end{bmatrix}.$$

Ejercicio 4.

$$A = \begin{pmatrix} 1 & -1 & \alpha \\ 2 & 2 & 1 \\ 0 & \alpha & -3/2 \end{pmatrix} = 1\begin{pmatrix} 2 & 1 \\ \alpha & -3/2 \end{pmatrix} + 1\begin{pmatrix} 2 & 1 \\ 0 & -3/2 \end{pmatrix} + \alpha\begin{pmatrix} 2 & 2 \\ 0 & \alpha \end{pmatrix}$$

$$= 1(-3-\alpha) + 1(-3) + \alpha(2\alpha - 2)$$

$$= -3 - \alpha - 3 + 2\alpha^2 - 2\alpha$$

$$= 2\alpha^2 - \alpha - 6 = 0 \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{1 \pm \sqrt{1 - 4(2)(-6)}}{2(2)} = \frac{1 \pm \sqrt{49}}{4} = \frac{1 \pm 7}{4}$$

$\alpha = \frac{8}{4}$ $\quad \alpha = 2 \quad \alpha_1 = -\frac{3}{2} \quad \det(0)$

5. Resuelva los siguientes sistemas lineales:

$$\textbf{a.}\quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\textbf{b.}\quad \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

Ejercicio 5.

a. $$\underset{A}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}} \underset{B}{\begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad A \cdot B\, x = b$$

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 6-2 & -2+1 \\ -2 & -3 & 1+3 \end{pmatrix} \to \begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 1 \\ -2 & -3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -2 & 3 & -5 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

$3x_3 = 3$

$x_3 = 1$

$-2x_2 + 3(1) = -5$

$x_2 = 4$

$2x_1 + 3(4) - (1) = 2$

$x_1 = -4{,}5$

b. $$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 2 \\ -1 & -1+1 & -1+2 \\ 3 & 3+2 & 3+4-1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ -1 & 0 & 1 \\ 3 & 5 & 6 \end{pmatrix} \to \left(\begin{array}{c|c} 1\ \ 0\ \ -1\ \ -1 & -3 \end{array}\right) \quad -2 \quad 0$$

$x = -3 \quad x = 3 \quad x_3 = 1$

6. Factorice las siguientes matrices en la descomposición LU mediante el algoritmo de factorización LU con $l_{ii} = 1$ para todas las i.

a. $\begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} 1.012 & -2.132 & 3.104 \\ -2.132 & 4.096 & -7.013 \\ 3.104 & -7.013 & 0.014 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1.5 & 0 & 0 \\ 0 & -3 & 0.5 & 0 \\ 2 & -2 & 1 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 2.1756 & 4.0231 & -2.1732 & 5.1967 \\ -4.0231 & 6.0000 & 0 & 1.1973 \\ -1.0000 & -5.2107 & 1.1111 & 0 \\ 6.0235 & 7.0000 & 0 & -4.1561 \end{bmatrix}$

Ejercicio 6.

a.

$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix}$

$u_{11} = a_{11} = 2$

$l_{21} = \dfrac{a_{21}}{u_{11}} = \dfrac{3}{2}$

$l_{31} = \dfrac{a_{31}}{u_{11}} = \dfrac{3}{2}$

$u_{12} = a_{12} = -1$

$u_{22} = a_{22} - l_{21} u_{12} = 3 - \left(\dfrac{3}{2}\right)(-1) = \dfrac{9}{2}$

$l_{32} = \dfrac{a_{32} - l_{31} u_{12}}{u_{22}} = \dfrac{3 - \left(\frac{3}{2}\right)(-1)}{\frac{9}{2}} = 1$

$u_{13} = a_{13} = 1$

$u_{23} = a_{23} - l_{21} u_{13} = 9 - 3/2(1) = 15/2$

$u_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23} = 5 - (3/2)(1) - (1)(15/2) = -4$

$L = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{pmatrix}$

b.

$\begin{pmatrix} 1.012_{11} & -2.132_{12} & 3.104_{13} \\ -2.132_{21} & 4.096_{22} & -7.013_{23} \\ 3.104_{31} & -7.013_{32} & 0.014_{33} \end{pmatrix}$

$u_{11} = a_{11} = 1.012$

$l_{21} = \dfrac{a_{21}}{u_{11}} = \dfrac{-2.132}{1.012} = -2.1067$

$l_{31} = \dfrac{a_{31}}{u_{11}} = \dfrac{3.104}{1.012} = 3.0672$

$u_{12} = -2.132$

$l_{22} = a_{22} - l_{21} u_{12}$

$4.096 + 2.1067 - 2.132$

$= -0.3969$

$u_{13} = a_{13} = 3.104$

$u_{23} = a_{23} - l_{21} u_{13} = -7.013 - (-2.1067)(304) = -0.4757$

$u_{33} = a_{33} l_{31} u_{13} \quad -(-2.1067)(3.104) - (-0.0003)(-0.4756$

$L = \begin{pmatrix} 1 & 0 & 0 \\ -2.1067 & 1 & 0 \\ 3.0672 & -0.0003 & 1 \end{pmatrix} \begin{pmatrix} 1.012 & -2.132 & 3.104 \\ 0 & -0.3969 & -0.4756 \\ 0 & 0 & -9.5125 \end{pmatrix}$

7. Modifique el algoritmo de eliminación gaussiana de tal forma que se pueda utilizar para resolver un sistema lineal usando la descomposición LU y, a continuación, resuelva los siguientes sistemas lineales.

a.
$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1, \\ 3x_1 + 3x_2 + 9x_3 &= 0, \\ 3x_1 + 3x_2 + 5x_3 &= 4. \end{aligned}$$

b.
$$\begin{aligned} 1.012x_1 - 2.132x_2 + 3.104x_3 &= 1.984, \\ -2.132x_1 + 4.096x_2 - 7.013x_3 &= -5.049, \\ 3.104x_1 - 7.013x_2 + 0.014x_3 &= -3.895. \end{aligned}$$

c.
$$\begin{aligned} 2x_1 \qquad\qquad\qquad &= 3, \\ x_1 + 1.5x_2 \qquad\quad &= 4.5, \\ -3x_2 + 0.5x_3 \quad &= -6.6, \\ 2x_1 - 2x_2 + x_3 + x_4 &= 0.8. \end{aligned}$$

d.
$$\begin{aligned} 2.1756x_1 + 4.0231x_2 - 2.1732x_3 + 5.1967x_4 &= 17.102, \\ -4.0231x_1 + 6.0000x_2 \qquad\qquad + 1.1973x_4 &= -6.1593, \\ -1.0000x_1 - 5.2107x_2 + 1.1111x_3 \qquad\qquad &= 3.0004, \\ 6.0235x_1 + 7.0000x_2 \qquad\qquad - 4.1561x_4 &= 0.0000. \end{aligned}$$

Ejercicio 7.

a. $2x_1 - x_2 + x_3 = -1$
$3x_1 + 3x_2 + 9x_3 = 0$
$3x_1 + 3x_2 + 5x_3 = 4.$

$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix}$ $L = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix}$ $U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{pmatrix}$

$Ly = b$ $\begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$

$y_1 = -1$
$\frac{3}{2} y_1 + y_2 = 0 \Rightarrow y_2 = \frac{3}{2}$
$\frac{3}{2} y_1 + y_2 + y_3 = 4 \Rightarrow y_3 = 4$

Resolver $Ux = y$

$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3/2 \\ 4 \end{pmatrix}$

$-4x_3 = 4 \Rightarrow x_3 = -1$
$\frac{9}{2} x_2 + \frac{15}{2}(1) = \frac{3}{2} \Rightarrow x_2 = 2$
$2x_1 - (2) + -1 \Rightarrow x_1 = 1$

$(1, 2, -1)$