ESCUELA POLITÉCNICA NACIONAL FACULTAD DE INGENIERÍA DE SISTEMAS MÉTODOS NUMÉRICOS [Tarea 07] Ejercicios Unidad 03-B splines cúbicos

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Conjunto de Ejercicios

Repositorio de Github

1. Dados los puntos (0,1), (1,5), (2,3), determine el *spline cúbico*.

- $S(x)$ es un spline cúbico
  - $S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$

$S_0(0) = 1$: $a_0 = 1$

$S_0(1) = 5$: $a_0 + b_0 + c_0 + d_0 = 5$

$S_1(1) = 5$: $a_1 = 5$

$S_1(2) = 3$: $a_1 + b_1 + c_1 + d_1 = 3$

Las derivadas de los splines son:

$S_0'(x) = b_0 + 2c_0x + 3d_0x^2$

$S_0''(x) = 2c_0 + 6d_0x$

$S_1'(x) = b_1 + 2c_1(x - 1) + 3d_1(x - 1)^2$

$S_1''(x) = 2c_1 + 6d_1(x - 1)$

En $x = 1$:

$S_0'(1) = S_1'(1) \Rightarrow b_0 + 2c_0 + 3d_0 = b_1$

$S_0''(1) = S_1''(1) \Rightarrow 2c_0 + 6d_0 = 2c_1$

a) Frontera natural:

1. $a_0 = 1$ y $a_1 = 5$.
2. $a_0 + b_0 + c_0 + d_0 = 5$ y $c_0 = 0$:

$$1 + b_0 + d_0 = 5 \implies b_0 + d_0 = 4$$

3. $a_1 + b_1 + c_1 + d_1 = 3$:

$$5 + b_1 + c_1 + d_1 = 3 \implies b_1 + c_1 + d_1 = -2$$

4. $b_0 + 2c_0 + 3d_0 = b_1$ y $c_0 = 0$:

$$b_0 + 3d_0 = b_1$$

5. $2c_0 + 6d_0 = 2c_1$ y $c_0 = 0$:

$$6d_0 = 2c_1 \implies c_1 = 3d_0$$

6. $2c_1 + 6d_1 = 0$:

$$2(3d_0) + 6d_1 = 0 \implies 6d_0 + 6d_1 = 0 \implies d_0 = -d_1$$

Sustituyendo $d_1 = -d_0$ en $b_1 + c_1 + d_1 = -2$:

$$b_1 + 3d_0 - d_0 = -2 \implies b_1 + 2d_0 = -2$$

$b_1 = b_0 + 3d_0$ y $b_0 = 4 - d_0$:

$$(4 - d_0) + 3d_0 + 2d_0 = -2 \implies 4 + 4d_0 = -2 \implies d_0 = -\frac{3}{2}$$

Entonces:

$b_0 = 4 - d_0 = 4 - \left(-\frac{3}{2}\right) = \frac{11}{2}$

$b_1 = b_0 + 3d_0 = \frac{11}{2} + 3\left(-\frac{3}{2}\right) = \frac{11}{2} - \frac{9}{2} = 1$

$c_1 = 3d_0 = 3\left(-\frac{3}{2}\right) = -\frac{9}{2}$

$d_1 = -d_0 = \frac{3}{2}$

Spline cúbico con frontera natural:

$S_0(x) = 1 + \frac{11}{2}x - \frac{3}{2}x^3$ para $x \in [0, 1]$

$S_1(x) = 5 + (x-1) - \frac{9}{2}(x-1)^2 + \frac{3}{2}(x-1)^3$ para $x \in [1, 2]$

b) Frontera condicionada:

$a_0 = 1$ y $a_1 = 5$.

$b_0 = 2$.

$a_0 + b_0 + c_0 + d_0 = 5$:

$1 + 2 + c_0 + d_0 = 5 \implies c_0 + d_0 = 2$

$a_1 + b_1 + c_1 + d_1 = 3$:

$5 + b_1 + c_1 + d_1 = 3 \implies b_1 + c_1 + d_1 = -2$

$b_0 + 2c_0 + 3d_0 = b_1$:

$2 + 2c_0 + 3d_0 = b_1$

$2c_0 + 6d_0 = 2c_1$:

$c_1 = c_0 + 3d_0$

$b_1 + 2c_1 + 3d_1 = -1$:

$(2 + 2c_0 + 3d_0) + 2(c_0 + 3d_0) + 3d_1 = -1$

Simplificando:

$2 + 4c_0 + 9d_0 + 3d_1 = -1 \implies 4c_0 + 9d_0 + 3d_1 = -3$

$c_0 + d_0 = 2$ y $c_1 = c_0 + 3d_0$, y $b_1 + c_1 + d_1 = -2$:

$(2 + 2c_0 + 3d_0) + (c_0 + 3d_0) + d_1 = -2 \implies 2 + 3c_0 + 6d_0 + d_1 = -2$

$3c_0 + 6d_0 + d_1 = -4$

Resolviendo el sistema:

$c_0 = 2 - d_0$

Sustituyendo en $3c_0 + 6d_0 + d_1 = -4$:

$3(2 - d_0) + 6d_0 + d_1 = -4 \implies 6 + 3d_0 + d_1 = -4 \implies 3d_0 + d_1 = -10$

Sustituyendo en $4c_0 + 9d_0 + 3d_1 = -3$:

$4(2 - d_0) + 9d_0 + 3d_1 = -3 \implies 8 + 5d_0 + 3d_1 = -3 \implies 5d_0 + 3d_1 = -11$

Multiplicando la primera ecuación por 3:

$9d_0 + 3d_1 = -30$

Restando la segunda ecuación:

$4d_0 = -19 \implies d_0 = -\frac{19}{4}$

$d_1 = -10 - 3d_0 = -10 - 3(-\frac{19}{4}) = -10 + \frac{57}{4} = \frac{17}{4}$

$c_0 = 2 - d_0 = 2 - (-\frac{19}{4}) = \frac{27}{4}$

$c_1 = c_0 + 3d_0 = \frac{27}{4} + 3(-\frac{19}{4}) = \frac{27}{4} - \frac{57}{4} = -\frac{30}{4} = -\frac{15}{2}$

$b_1 = 2 + 2c_0 + 3d_0 = 2 + 2(\frac{27}{4}) + 3(-\frac{19}{4}) = 2 + \frac{54}{4} - \frac{57}{4} = 2 - \frac{3}{4} = \frac{5}{4}$

Spline cúbico con frontera condicionada:

$S_0(x) = 1 + 2x + \frac{27}{4}x^2 - \frac{19}{4}x^3 \quad \text{para } x \in [0, 1]$

$S_1(x) = 5 + \frac{5}{4}(x-1) - \frac{15}{2}(x-1)^2 + \frac{17}{4}(x-1)^3 \quad \text{para } x \in [1, 2]$

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2. Dados los puntos $(-1, 1)$ y $(1, 3)$, determine el spline cúbico sabiendo que: $f'(x_0) = 1$ y $f'(x_n) = 2$.

- Interpolación en los puntos:

  $S(-1) = 1 \implies a = 1$

  $S(1) = 3 \implies a + 2b + 4c + 8d = 3$

- Derivadas en los extremos:

  $S'(x) = b + 2c(x - x_0) + 3d(x - x_0)^2$

  $S'(-1) = 1 \implies b = 1$

  $S'(1) = 2 \implies b + 4c + 12d = 2$

Sustituyendo $a = 1$ y $b = 1$ en las ecuaciones:

1. $S(1) = 3$:
$$1 + 2(1) + 4c + 8d = 3 \implies 4c + 8d = 0 \quad (1)$$

2. $S'(1) = 2$:
$$1 + 4c + 12d = 2 \implies 4c + 12d = 1 \quad (2)$$

Restando (1) de (2):

$$(4c + 12d) - (4c + 8d) = 1 - 0 \implies 4d = 1 \implies d = \frac{1}{4}$$

Sustituyendo d en (1):

$$4c + 8\left(\frac{1}{4}\right) = 0 \implies 4c + 2 = 0 \implies c = -\frac{1}{2}$$

Spline cúbico final

$$S(x) = 1 + (x+1) - \frac{1}{2}(x+1)^2 + \frac{1}{4}(x+1)^3$$

Forma simplificada:

$$S(x) = \frac{1}{4}x^3 - \frac{1}{2}x^2 + \frac{5}{4}x + \frac{5}{4}$$

Verificación

1. En $x = -1$:
   - $S(-1) = 1$ (cumple).
   - $S'(-1) = 1$ (cumple).
2. En $x = 1$:
   - $S(1) = 3$ (cumple).
   - $S'(1) = 2$ (cumple).

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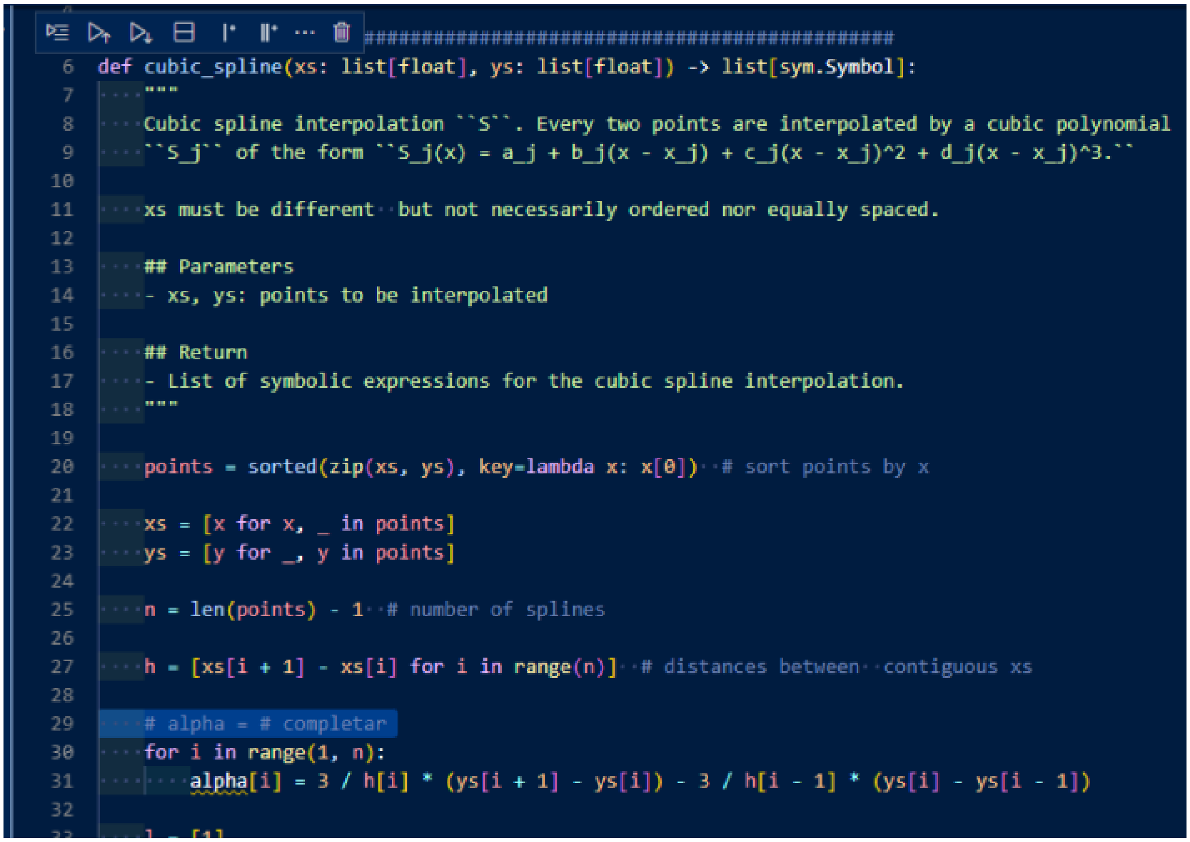
3. Diríjase al pseudocódigo del spline cúbico con frontera natural provisto en clase, en base a ese pseudocódigo complete la siguiente función:



```
##################################################
def cubic_spline(xs: list[float], ys: list[float]) -> list[sym.Symbol]:
    """
    Cubic spline interpolation ``S``. Every two points are interpolated by a cubic polynomial
    ``S_j`` of the form ``S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3.``

    xs must be different  but not necessarily ordered nor equally spaced.

    ## Parameters
    - xs, ys: points to be interpolated

    ## Return
    - List of symbolic expressions for the cubic spline interpolation.
    """

    points = sorted(zip(xs, ys), key=lambda x: x[0])  # sort points by x

    xs = [x for x, _ in points]
    ys = [y for _, y in points]

    n = len(points) - 1  # number of splines

    h = [xs[i + 1] - xs[i] for i in range(n)]  # distances between  contiguous xs

    # alpha = # completar
    for i in range(1, n):
        alpha[i] = 3 / h[i] * (ys[i + 1] - ys[i]) - 3 / h[i - 1] * (ys[i] - ys[i - 1])

```

Figure 1: image.png

```
S_0(x) = 1.5*x**3 - 0.5*x - 5
Simplificado: 1.5*x**3 - 0.5*x - 5

S_1(x) = 4.0*x - 1.5*(x - 1)**3 + 4.5*(x - 1)**2 - 8.0
Simplificado: -1.5*x**3 + 9.0*x**2 - 9.5*x - 2.0
```

4. Usando la función anterior, encuentre el spline cúbico para:

- xs = [1, 2, 3]
- ys = [2, 3, 5]

```
S_0(x) = 0.75*x + 0.25*(x - 1)**3 + 1.25
Simplificado: 0.25*x**3 - 0.75*x**2 + 1.5*x + 1.0

S_1(x) = 1.5*x - 0.25*(x - 2)**3 + 0.75*(x - 2)**2
Simplificado: -0.25*x**3 + 2.25*x**2 - 4.5*x + 5.0
```

5. Usando la función anterior, encuentre el spline cúbico para:

- xs = [0, 1, 2, 3]
- ys = [-1 ,1, 5, 2]

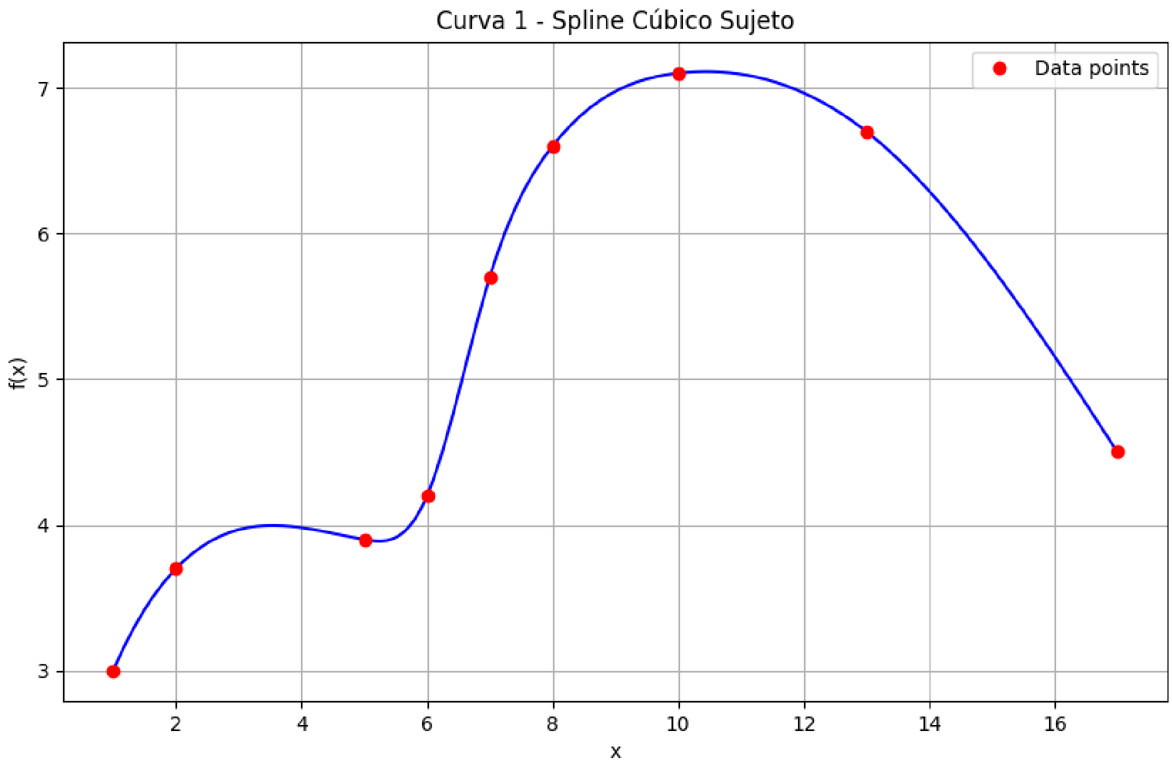
```
S_0(x) = 1.0*x**3 + 1.0*x - 1
Simplificado: 1.0*x**3 + 1.0*x - 1

S_1(x) = 4.0*x - 3.0*(x - 1)**3 + 3.0*(x - 1)**2 - 3.0
Simplificado: -3.0*x**3 + 12.0*x**2 - 11.0*x + 3.0

S_2(x) = 1.0*x + 2.0*(x - 2)**3 - 6.0*(x - 2)**2 + 3.0
Simplificado: 2.0*x**3 - 18.0*x**2 + 49.0*x - 37.0
```

4. Use la función `cubic_spline_clamped`, provista en el enlace de Github, para graficar los datos de la siguiente tabla.

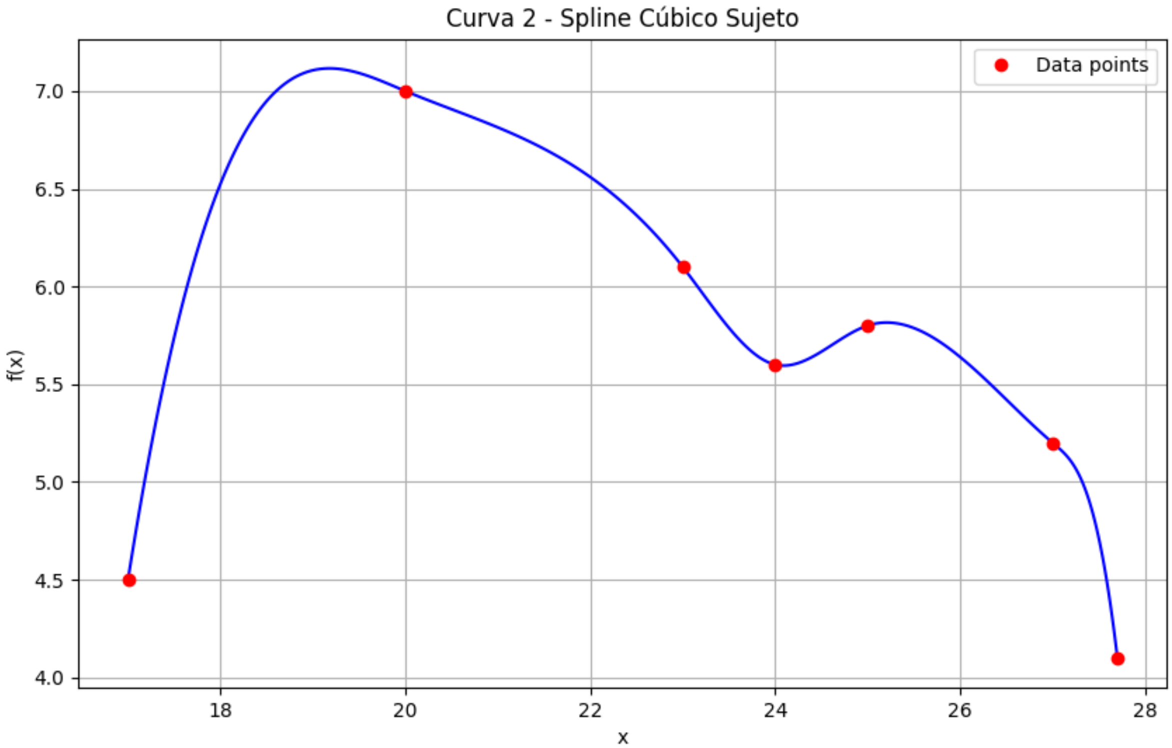
Curva 1				Curva 2				Curva 3			
i	x_i	$f(x_i)$	$f'(x_i)$	i	x_i	$f(x_i)$	$f'(x_i)$	i	x_i	$f(x_i)$	$f'(x_i)$
0	1	3.0	1.0	0	17	4.5	3.0	0	27.7	4.1	0.33
1	2	3.7		1	20	7.0		1	28	4.3	
2	5	3.9		2	23	6.1		2	29	4.1	
3	6	4.2		3	24	5.6		3	30	3.0	−1.5
4	7	5.7		4	25	5.8					
5	8	6.6		5	27	5.2					
6	10	7.1		6	27.7	4.1	−4.0				
7	13	6.7									
8	17	4.5	−0.67								

Figure 2: image.png

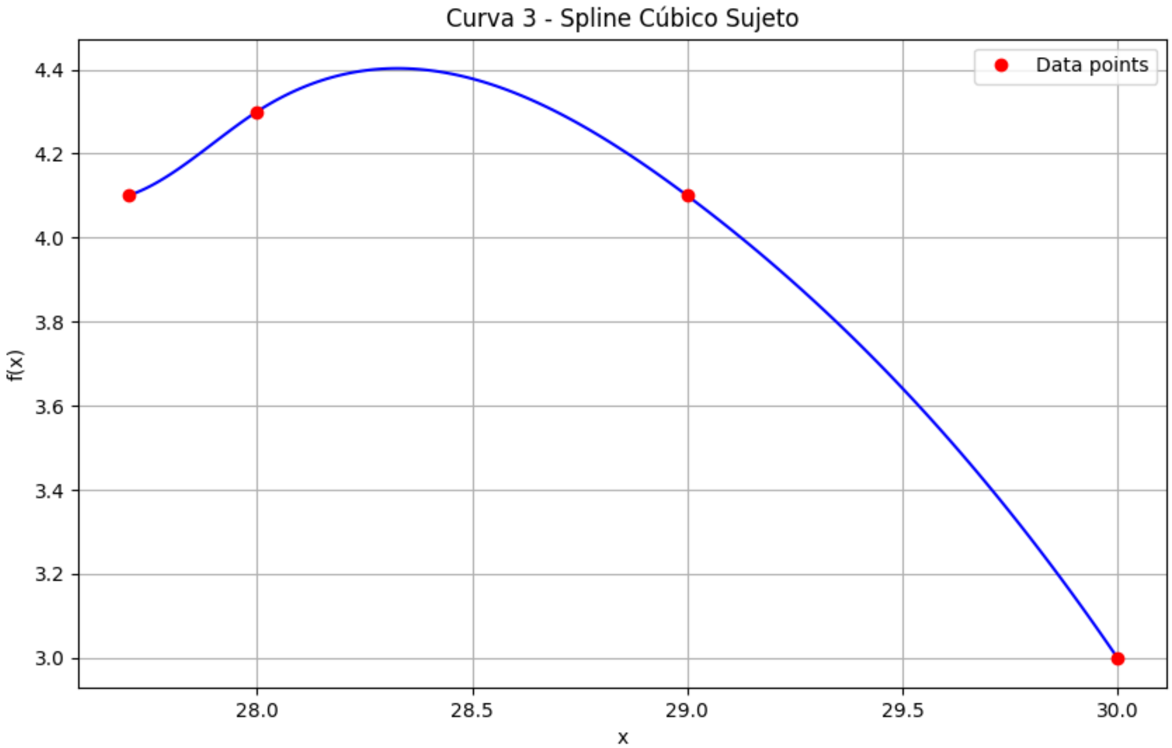
- Graficas de splines
- Curva 1


Curva 1 - Spline Cúbico Sujeto
Data points
f(x)
x
7
6
5
4
3
2
4
6
8
10
12
14
16

- Curva 2


Curva 2 - Spline Cúbico Sujeto
Data points
7.0
6.5
6.0
5.5
5.0
4.5
4.0
f(x)
18
20
22
24
26
28
x

- Curva 3


Curva 3 - Spline Cúbico Sujeto
Data points
f(x)
x
4.4
4.2
4.0
3.8
3.6
3.4
3.2
3.0
28.0
28.5
29.0
29.5
30.0