1. 三角恒等式

(1) 形如 $\sqrt{1 \pm \cos(x)}$

例.

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos(2x)}\,\mathrm{d}x$$

推导原理:

将一个数与三角函数的运算, 转化为该三角函数半角的三角函数平方, 便于开根号

推导过程:

由 $\cos(2x) = 1 - 2\sin^2(x)$, 得:

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos(2x)}\,\mathrm{d}x = \sqrt{2}\int_0^{\frac{\pi}{2}} |\sin(x)|\,\mathrm{d}x$$

由于 $\sin(x)$ 在定义域区间 $[0, \frac{\pi}{2}]$ 内为正, 所以:

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos(2x)}\,\mathrm{d}x = -\sqrt{2}\cos(x)\Big|_0^{\frac{\pi}{2}} = 0 - (-\sqrt{2}) = \sqrt{2}$$

(2) 形如 $\sqrt{1-\sin^2(x)}/\sqrt{1+\tan^2(x)}$

例.

$$\int_0^{\pi} \sqrt{1-\cos^2(x)}\,\mathrm{d}x$$

推导原理:

将根号下一个数字与三角函数平方的运算, 转化为该三角函数角的另一个三角函数的平方, 便于开根号

推导过程:

由 $\sin^2(x) = 1 - \cos^2(x)$, 得:

$$\int_0^{\pi} \sqrt{1-\cos^2(x)}\,\mathrm{d}x = \int_0^{\pi} \sqrt{sin^2(x)}\,\mathrm{d}x = \int_0^{\pi} |\sin(x)|\,\mathrm{d}x$$

由于 $\sin(x)$ 在区间 $[0, \pi]$ 内为正, 所以:

$$\int_0^{\pi} \sqrt{1-\cos^2(x)}\,\mathrm{d}x = \int_0^{\pi} \sin(x)\,\mathrm{d}x = -\cos(x)\Big|_0^{\pi} = 1 - (-1) = 2$$

(3) 形如 $\dfrac{1}{\sec(x)-1}$

例.

$$\int \frac{1}{\sec(x)-1}\,\mathrm{d}x$$

推导原理:

将分子分母同时乘以分母的共轭式

推导过程:

$$\begin{aligned}\int \frac{1}{\sec(x)-1}\,\mathrm{d}x &= \int \frac{1}{\sec(x)-1}\times\frac{\sec(x)+1}{\sec(x)+1}\,\mathrm{d}x = \int \frac{\sec(x)+1}{\sec^2(x)-1}\,\mathrm{d}x \\ &= \int \frac{\sec(x)}{\tan^2(x)}\,\mathrm{d}x + \int \frac{1}{\tan^2(x)}\,\mathrm{d}x = \int \frac{\cos(x)}{\sin^2(x)}\,\mathrm{d}x + \int \frac{1}{\tan^2(x)}\,\mathrm{d}x\end{aligned}$$

设 $t=\sin(x)$, 则 $\mathrm{d}t=\cos(x)\,\mathrm{d}x$, 得:

$$\int \frac{\cos(x)}{\sin^2(x)}\,\mathrm{d}x = \int \frac{1}{t^2}\,\mathrm{d}t = -\frac{1}{t}+C = -\csc(x)+C$$

由 $\dfrac{\mathrm{d}}{\mathrm{d}x}\cot(x) = -\csc^2(x)$, 得:

$$\int \frac{1}{\tan^2(x)}\,\mathrm{d}x = \int \cot^2(x)\,\mathrm{d}x = \int \csc^2(x)\,\mathrm{d}x - \int \mathrm{d}x = -\cot(x)-x+C$$

$$\therefore \int \frac{1}{\sec(x)-1}\,\mathrm{d}x = -\csc(x)-\cot(x)-x+C$$

(4) 形如 $\sin(\alpha)\cos(\beta)$

例.

$$\int \sin(19x)\cos(3x)\,\mathrm{d}x$$

推导原理:

利用和/差角公式:

$$\boxed{\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)}$$

$$\boxed{\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)}$$

$$\boxed{\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B)}$$

$$\boxed{\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)}$$

可推断出:

$$\boxed{\sin(A)\cos(B)=\frac{1}{2}(\sin(A+B)+\sin(A-B))}$$

$$\boxed{\cos(A)\cos(B)=\frac{1}{2}(\cos(A+B)+\cos(A-B))}$$

$$\boxed{\sin(A)\sin(B)=\frac{1}{2}(\cos(A-B)-\cos(A+B))}$$

推导过程:

$$
\begin{aligned}
\therefore \int \sin(19x)\cos(3x)\,\mathrm{d}x &= \frac{1}{2}\int(\sin(19x+3x)+\sin(19x-3x))\,\mathrm{d}x \\
&= \frac{1}{2}\int(\sin(22x)+\sin(16x))\,\mathrm{d}x \\
&= \frac{1}{2}\left(-\frac{\cos(22x)}{22}-\frac{\cos(16x)}{16}\right)+C \\
&= -\frac{\cos(22x)}{44}-\frac{\cos(16x)}{32}+C
\end{aligned}
$$

二、关于三角函数的幂的积分

(1)sin 或 cos 的幂

I、至少一个乘积项为奇次幂

例.

$\int \sin^7(x)\cos^4(x)\,\mathrm{d}x$

解题思路:

将奇次幂分解为 1 次方和 n-1 次方, 并且将剩下的 n-1 偶次幂利用 $\cos^2(x)+\sin^2(x)=1$ 进行转化

推导过程:

设 $t=\cos(x)$, 则 $\mathrm{d}t=-\sin(x)\,\mathrm{d}x$, 得:

$$
\begin{aligned}
\int \sin^7(x)\cos^4(x)\,\mathrm{d}x &= -\int \sin^6(x)\cos^4(x)(-\sin(x)\,\mathrm{d}x) \\
&= -\int(1-\cos^2(x))^3\cos^4(x)(-\sin(x)\,\mathrm{d}x) \\
&= -\int(1-t^2)t^4\,\mathrm{d}t \\
&= -\int(1-3t^2+3t^4-t^6)t^4\,\mathrm{d}t \\
&= -\int(t^4-3t^6+3t^8-t^{10})\,\mathrm{d}t \\
&= -\frac{1}{5}t^5+\frac{3}{7}t^7-\frac{1}{3}t^9+\frac{1}{11}t^{11}+C \\
&= -\frac{1}{5}\cos^5(x)+\frac{3}{7}\cos^7(x)-\frac{1}{3}\cos^9(x)+\frac{1}{11}\cos^{11}(x)+C
\end{aligned}
$$

II、两个乘积项都为偶次幂

例.

$\int \cos^2(x)\sin^4(x)\,\mathrm{d}x$

解题思路:

利用倍角公式降低幂次

推导过程:

$$\begin{aligned}
\int \cos^2(x)\sin^4(x)\,\mathrm{d}x &= \int \frac{1}{2}(1+\cos(2x))(\frac{1}{2}(1-\cos(2x)))^2\,\mathrm{d}x \\
&= \frac{1}{8}\int (1+\cos(2x))(1-\cos(2x))^2\,\mathrm{d}x \\
&= \frac{1}{8}\int (1-\cos(2x)-\cos^2(2x)+\cos^3(2x))\,\mathrm{d}x \\
&= \frac{1}{8}\int 1\,\mathrm{d}x - \frac{1}{8}\int \cos(2x)\,\mathrm{d}x - \frac{1}{8}\int \cos^2(2x)\,\mathrm{d}x + \frac{1}{8}\int \cos^3(2x)\,\mathrm{d}x \\
&= \frac{1}{8}x - \frac{1}{16}\sin(2x) - \frac{1}{16}\int (1+\cos(4x))\,\mathrm{d}x + \frac{1}{8}\int (1-\sin^2(2x))\cos(2x)\,\mathrm{d}x \\
&= \frac{1}{8}x - \frac{1}{16}\sin(2x) - \frac{1}{16}(x+\frac{\sin(4x)}{4}) + \frac{1}{8}(\frac{\sin(2x)}{2} - \frac{\sin^3(2x)}{6}) + C \\
&= \frac{1}{16}x - \frac{1}{48}\sin^3(2x) - \frac{1}{64}\sin(4x) + C
\end{aligned}$$

(2)tan 的幂 (cot 类似)

I、当幂为 1

例.

$\int \tan(x)\,\mathrm{d}x$

解题思路:

转化为 $\dfrac{\sin(x)}{\cos(x)}$ 格式

推导过程:

假设 $t=\cos(x)$, 则 $\mathrm{d}t = -\sin(x)\,\mathrm{d}x$

$$
\begin{aligned}
\int \tan(x)\,\mathrm{d}x &= \int \frac{\sin(x)}{\cos(x)}\,\mathrm{d}x \\
&= -\int \frac{1}{t}\,\mathrm{d}t \\
&= -\ln|t| + C \\
&= -\ln|\cos(x)| + C
\end{aligned}
$$

II、当幂为 2

例.

$\int \tan^2(x)\,\mathrm{d}x$

解题思路:

将 $\tan^2(x)$ 转化为 $\sec^2(x)-1$

推导过程:

$$
\begin{aligned}
\int \tan^2(x)\,\mathrm{d}x &= \int (\sec^2(x)-1)\,\mathrm{d}x \\
&= \int \sec^2(x)\,\mathrm{d}x - \int 1\,\mathrm{d}x \\
&= \tan(x) - x + C
\end{aligned}
$$

III、当幂大于等于 3

例.

$\int \tan^6(x)\,\mathrm{d}x$

解题思路:

首先, 从中提取一个 $\tan^2(x)$ 变化为 $\sec^2(x)-1$, 然后被积分部分分成两部分. 第一部分为关于 $t=\tan^2(x)$ 的积分; 第二部分为 $\tan(x)$ 的更低次幂, 继续循环当前操作推导过程:

$$\int \tan^6(x)\,\mathrm{d}x = \int \tan^4(x)(\sec^2(x)-1)\,\mathrm{d}x = \int \tan^4(x)\sec^2(x)\,\mathrm{d}x - \int \tan^4(x)\,\mathrm{d}x$$

设 $t=\tan(x)$, 则 $\mathrm{d}t=\sec^2(x)\,\mathrm{d}x$, 得:

$$\begin{aligned}\int \tan^4(x)\sec^2(x)\,\mathrm{d}x &= \int t^4\,\mathrm{d}t\\ &= \frac{1}{5}t^5+C\\ &= \frac{1}{5}\tan^5(x)+C\end{aligned}$$

设 $t=\tan(x)$, 则 $\mathrm{d}t=\sec^2(x)\,\mathrm{d}x$, 得:

$$\begin{aligned}\int \tan^4(x)\,\mathrm{d}x &= \int \tan^2(x)(\sec^2(x)-1)\,\mathrm{d}x\\ &= \int \tan^2(x)\sec^2(x)\,\mathrm{d}x-\int \tan^2(x)\,\mathrm{d}x\\ &= \int \tan^2(x)\sec^2(x)\,\mathrm{d}x-\int (\sec^2(x)-1)\,\mathrm{d}x\\ &= \int \tan^2(x)\sec^2(x)\,\mathrm{d}x-\int \sec^2(x)\,\mathrm{d}x+\int 1\,\mathrm{d}x\\ &= \int t^2\,\mathrm{d}t-\int 1\,\mathrm{d}t+\mathrm{d}1\,\mathrm{d}x\\ &= \frac{1}{3}t^3-t+x+C\\ &= \frac{1}{3}\tan^3(x)-\tan(x)+x+C\end{aligned}$$

合并结果, 得:

$$\int \tan^6(x)=\frac{1}{5}\tan^5(x)-\frac{1}{3}\tan^3(x)+\tan(x)-x+C$$

(3)sec 的特征 (csc 类似)

I、当幂等于 1

例.

$$\int \sec(x)\,\mathrm{d}x$$

解题思路:

分子与分母同时乘以 $\sec(x)+\tan(x)$, 得到形如 $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$ 的结果

推导过程:

设 $t=\sec(x)+\tan(x)$, 则 $\mathrm{d}t=\sec(x)\tan(x)+\sec^2(x)\,\mathrm{d}x$, 得:

$$\begin{aligned}\int \sec(x)\,\mathrm{d}x &= \int \sec(x) \times \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,\mathrm{d}x \\ &= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,\mathrm{d}x \\ &= \int \frac{1}{t}\,\mathrm{d}t \\ &= \ln|t| + C \\ &= \ln|\sec(x)+\tan(x)| + C\end{aligned}$$

II、当幂等于 2

例.

$\int \sec^2(x)\,\mathrm{d}x$

解题思路:

$\int \sec^2(x)\,\mathrm{d}x = \tan(x) + C$

III、当幂大于等于 3

例.

$\int \sec^6(x)\,\mathrm{d}x$

解题思路:

提取出 $\sec^2(x)$, 利用分部积分公式: $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$

推导过程:

$\int \sec^6(x)\,\mathrm{d}x = \int \sec^4(x)\sec^2(x)\,\mathrm{d}x$

可得到以下结论:

$$\begin{aligned} u &= \sec^4(x) & v &= \tan(x) \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= 4\sec^4(x)\tan(x) & \frac{\mathrm{d}v}{\mathrm{d}x} &= \sec^2(x)\end{aligned}$$

利用分部积分公式:

$$\int \sec^4(x)\sec^2(x)\,\mathrm{d}x = \sec^4(x)\tan(x) - 4\int \sec^4(x)\tan^2(x)\,\mathrm{d}x \tag{1}$$

$$\begin{aligned}\int \tan^2(x)\sec^4(x) &= \int (\sec^2(x)-1)\sec^4(x)\,\mathrm{d}x \\ &= \int \sec^6(x)\,\mathrm{d}x - \int \sec^4(x)\,\mathrm{d}x \end{aligned} \tag{2}$$

将(2)代入(1), 得:

$\int \sec^6(x)\,\mathrm{d}x = \dfrac{1}{5}\sec^4(x)\tan(x) + \dfrac{4}{5}\int \sec^4(x)\,\mathrm{d}x$