1. 三角恒等式

(1) 形如 $\sqrt{1 \pm \cos(x)}$

例.

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos(2x)}\,\mathrm{d}x$$

推导过程:

由 $\cos(2x) = 1 - 2\sin^2(x)$, 得:

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos(2x)}\,\mathrm{d}x = \sqrt{2}\int_0^{\frac{\pi}{2}} |\sin(x)|\,\mathrm{d}x$$

由于 $\sin(x)$ 在定义域区间 $[0, \frac{\pi}{2}]$ 内为正, 所以:

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos(2x)}\,\mathrm{d}x = -\sqrt{2}\cos(x)\Big|_0^{\frac{\pi}{2}} = 0 - (-\sqrt{2}) = \sqrt{2}$$

(2) 形如 $\sqrt{1-\sin^2(x)}/\sqrt{1+\tan^2(x)}$

例.

$$\int_0^{\pi} \sqrt{1-\cos^2(x)}\,\mathrm{d}x$$

推导过程:

由 $\sin^2(x) = 1 - \cos^2(x)$, 得:

$$\int_0^{\pi} \sqrt{1-\cos^2(x)}\,\mathrm{d}x = \int_0^{\pi} \sqrt{sin^2(x)}\,\mathrm{d}x = \int_0^{\pi} |\sin(x)|\,\mathrm{d}x$$

由于 $\sin(x)$ 在区间 $[0, \pi]$ 内为正, 所以:

$$\int_0^{\pi} \sqrt{1-\cos^2(x)}\,\mathrm{d}x = \int_0^{\pi} \sin(x)\,\mathrm{d}x = -\cos(x)\Big|_0^{\pi} = 1 - (-1) = 2$$

(3) 形如 $\dfrac{1}{\sec(x)-1}$

例.

$$\int \frac{1}{\sec(x)-1}\,\mathrm{d}x$$

推导过程:

$$\begin{aligned}\int \frac{1}{\sec(x)-1}\,\mathrm{d}x = \int \frac{1}{\sec(x)-1}\times\frac{\sec(x)+1}{\sec(x)+1}\,\mathrm{d}x = \int \frac{\sec(x)+1}{\sec^2(x)-1}\,\mathrm{d}x\\ = \int \frac{\sec(x)}{\tan^2(x)}\,\mathrm{d}x + \int \frac{1}{\tan^2(x)}\,\mathrm{d}x = \int \frac{\cos(x)}{\sin^2(x)}\,\mathrm{d}x + \int \frac{1}{\tan^2(x)}\,\mathrm{d}x\end{aligned}$$

设 $t=\sin(x)$, 则 $\mathrm{d}t=\cos(x)\,\mathrm{d}x$, 得:

$$\int \frac{\cos(x)}{\sin^2(x)}\,\mathrm{d}x = \int \frac{1}{t^2}\,\mathrm{d}t = -\frac{1}{t}+C = -\csc(x)+C$$

由 $\frac{\mathrm{d}}{\mathrm{d}x}\cot(x) = -\csc^2(x)$, 得:

$$\int \frac{1}{\tan^2(x)}\,\mathrm{d}x = \int \cot^2(x)\,\mathrm{d}x = \int \csc^2(x)\,\mathrm{d}x - \int \mathrm{d}x = -\cot(x) - x + C$$

$$\therefore \int \frac{1}{\sec(x)-1}\,\mathrm{d}x = -\csc(x)-\cot(x)-x+C$$

(4) 形如 $\sin(\alpha)\cos(\beta)$

例.

$$\int \sin(19x)\cos(3x)\,\mathrm{d}x$$

推导过程:

和/差角公式:

$$\boxed{\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)}$$

$$\boxed{\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)}$$

$$\boxed{\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B)}$$

$$\boxed{\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)}$$

可推断出:

$$\boxed{\sin(A)\cos(B)=\frac{1}{2}(\sin(A+B)+\sin(A-B))}$$

$$\boxed{\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B))}$$

$$\boxed{\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))}$$

$$\begin{aligned}
\therefore \int \sin(19x)\cos(3x)\,\mathrm{d}x &= \frac{1}{2}\int (\sin(19x+3x)+\sin(19x-3x))\,\mathrm{d}x \\
&= \frac{1}{2}\int (\sin(22x)+\sin(16x))\,\mathrm{d}x \\
&= \frac{1}{2}\left(-\frac{\cos(22x)}{22}-\frac{\cos(16x)}{16}\right)+C \\
&= -\frac{\cos(22x)}{44}-\frac{\cos(16x)}{32}+C
\end{aligned}$$