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Expanded Optimization for Discovering Optimal Lateral Handling Bicycles

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Abstract

Previously, we introduced a method of optimizing four primary geometric parameters of a bicycle's design to minimize its Handling Quality Metric. Here we expand that method to optimize 23 geometric and inertial parameters in the benchmark parameterimization of the linear Whipple-Carvallo bicycle model. To ensure physically realizable bicycle designs we include 7 equality constraints, 21 inequality constraints, and maximal and minamal bounds on each free opmization parameter. This improves over the prior work by expanding the search space with many more parameters and the guarntee of realizability. We close by presenting five bicycle designs that have optimal later handling qualities. The bicycles are not generally self-stable and have some ununusal characteristics.

1 Introduction

Physical design features of ground vehicles can affect their lateral handling qualities. Geometry, mass, and mass distribution of the vehicle's components as well as tire characteristics are primary contributors to poor and good handling due to their important influence on the vehicle's dynamics. In past work, we have presented a theoretical and computational framework for assessing the lateral task-independent handling qualities of simplified single track vehicle designs [1, 4]. In subsequent work, we showed that minimizing our proposed handling quality metric (HQM) can produce theoretically optimal handling designs when only four geometric parameters are explored as the optimization variables [3]. The present work's goal is to expand this optimization problem to all of the geometry, mass, and inertial parameters present in the linear Whipple-Carvallo bicycle model [2]. This broadens the search space considerably but we constrain it so that only realizable optimal bicycle designs are discovered. To do so, we formulate a constrained optimization problem and use derivative-free optimization to discover optimal, yet realizable, bicycle designs.

2 Bicycle Model Parameterization

Our problem formulation relies on a new bicycle model parameterization that reflects both a reformulation of and addition to the benchmark parameterization of the linear Whipple-Carvallo bicycle model [2]. We call this the "principal parameterizatoin" as opposed to the "benchmark parameterization" in [2]. This parameterization differs from the benchmark parameterization in three ways. Firstly, the person and rear frame are treated as separate rigid bodies each with their on inertial parameters. Secondly, we express the inertial parameters of each rigid body in terms of central principal radii of gyration to decouple the mass from the inertia terms. Lastly, we introduce two simple dimensional parameters that define the geometric extents of the person which are used to constrain the location of the person's body. Table 1 provides the parameters and the reference values which are derived from the measurements of a Batavus Browser Bicycle and the rider "Jason" presented in [4]. This parameterization can be transformed into the benchmark parameterization readily, but not vice versa.

Table 1. Full set of 47 principal parameters and their default values derived from the measurements in [4] of the Batavus Browser bicycle and rider "Jason".

Variable	Value	Units	Description
c	0.068581	m	Trail
w	1.1210	m	Wheelbase
λ	0.39968	rad	Steer axis tilt
g	9.81	$\mathrm{m\,s^{-2}}$	Acceleration due to gravity
v	3.0, 5.0, 7.0, 9.0	$\mathrm{m\,s^{-1}}$	Forward speed
Rear Wheel [R]			
m_R	3.11	kg	Mass
r_R	0.34096	m	Radius
x_R	0.0	m	X mass center coordinate
y_R	0.0	m	Y mass center coordinate
z_R	-0.34096	m	Z mass center coordinate
k_{Raa}	0.17050	m	Central principal radii of gyration about A_R
k_{Rbb}	0.17050	m	Central principal radii of gyration about B_R
k_{Ryy}	0.22136	m	Central principal radii of gyration about Y
Front Wheel [F]			
m_F	2.02	kg	Mass
r_F	0.34353	m	Radius
x_F	1.1210	m	X mass center coordinate
y_F	0.0	m	Y mass center coordinate
z_F	-0.34353	m	Z mass center coordinate
k_{Faa}	0.20917	m	Central principal radii of gyration about A_F
k_{Fbb}	0.20917	m	Central principal radii of gyration about B_F
k_{Fyy}	0.27179	m	Central principal radii of gyration about Y
Person [P]			
l_P	1.728	m	Body length
w_P	0.483	m	Body width
m_P	83.5	kg	Mass
x_P	0.31577	m	X mass center coordinate
y_P	0.0	m	Y mass center coordinate
z_P	-1.0990	m	Z mass center coordinate
k_{Paa}	0.36797	m	Central principal radii of gyration about A_P
k_{Pbb}	0.15276	m	Central principal radii of gyration about B_P
k_{Pyy}	0.36717	m	Central principal radii of gyration about Y
α_P	0.18618	rad	Angle about Y between A_P and X
Front Frame [H]			
m_H	3.2200	kg	Mass
x_H	0.86695	m	X mass center coordinate
y_H	0.0	m	Y mass center coordinate
z_H	-0.74824	m	Z mass center coordinate
k_{Haa}	0.29556	m	Central principal radii of gyration about A_H
k_{Hbb}	0.14493	m	Central principal radii of gyration about B_H
k_{Hyy}	0.27630	m	Central principal radii of gyration about Y
α_H	0.36995	rad	Angle about Y between A_H and X
Rear Frame [D]			
m_D	9.8600	kg	Mass
x_D	0.27595	m	X mass center coordinate
y_D	0	m	Y mass center coordinate
z_D	-0.53784	m	Z mass center coordinate
k_{Daa}	0.28587	m	Central principal radii of gyration about A_D
k_{Dbb}	0.22079	m	Central principal radii of gyration about B_D
k_{Dyy}	0.36539	m	Central principal radii of gyration about Y
α_D	1.1722	rad	Angle about Y between A_D and X

3 Bounds and Constraints

The optimal principal parameters are subject to a set of constraints designed to ensure that a physically realizable bicycle is obtained from the opimization procedure. These constraints are made up of bounds on the free parameters and both equality and inequality constraints among the parameters. Below the basic constraint concepts presented and grouped by the associated rigid body or collection thereof:

Total T The resulting combination of the five rigid bodies.

- The likely physical extents of the rigid bodies must exist above the ground plane.
- Both bicycle and rider are symmetric about the rider's sagittal plane.
- The total mass is below a reasonably human lift-able amount.
- The wheels cannot overlap.
- The bicycle cannot topple forward during hard braking or backward during hard acceleration.
- The closed loop path tracking system [1] must be stable. This is required to obtain a valid HQM value.

Rider P A single rigid body represents the rider.

- Rider mass is that of an average person.
- The rider's joint angles are fixed in a nominal configuration typical of upright bicycling and the resulting mass distribution is derived from standard body segment estimation methods.
- The rider cannot penetrate the ground.

Frames H**,** D Front frame (handlebar + fork) and rear frame

- The rear frame is planar in nature and the front frame's moments of inertia are consistently dependent.
- The mass and inertia of the frames are positive and large enough to be constructed from a steel space frame.

Wheels F**,** R Both front and rear wheels have identical constraints.

- Wheel radius and mass must positive and be greater than a minimum value.
- Wheels are inertially wheel-like, i.e. symmetric about each plane and most of the mass is at the rim.

These bounds, equality, and inequality constraints are presented mathematically in tables 2, 3, 4 respectively and explained in more detail in the following sections.

3.1 Person [P]

We assume that the person's joint configurations are such that they are in a nominal configuration for pedaling, i.e. an average normal everyday riding position on a typical bicycle, i.e. they stayed in the same configuration as they were seated on the Batavus Browser bicycle. The person is assumed to be symmetric about the XZ plane. We allow the rider to be rotated about the Y axis and positioned anywhere within the plane of symmetry above the ground.

To prevent the rider from being positioned and oriented such that their body is not penetrating the ground we introduce two dimensions that define a cross whose apex is at the center of mass of the

Table 2. Free parameter upper and lower bounds

Min		Parameter		Max
$-\infty$	$\leq$	w	$\leq$	∞
$-\infty$	$\leq$	c	$\leq$	∞
$-\pi/2$	$\leq$	λ	$\leq$	$\pi/2$
1.0	$\leq$	m_D	$\leq$	∞
$-\infty$	$\leq$	x_D	$\leq$	∞
$-\infty$	$\leq$	z_D	$\leq$	0.0
0.0	$\leq$	k_{Daa}	$\leq$	∞
0.0	$\leq$	k_{Dbb}	$\leq$	∞
$-\pi/2$	$\leq$	α_D	$\leq$	$\pi/2$
$-\infty$	$\leq$	x_P	$\leq$	∞
$-\infty$	$\leq$	z_P	$\leq$	0.0
$-\pi/2$	$\leq$	α_P	$\leq$	$\pi/2$
0.25	$\leq$	m_H	$\leq$	∞
$-\infty$	$\leq$	x_H	$\leq$	∞
$-\infty$	$\leq$	z_H	$\leq$	0.0
0.0	$\leq$	k_{Haa}	$\leq$	∞
0.0	$\leq$	k_{Hbb}	$\leq$	∞
0.0	$\leq$	k_{Hyy}	$\leq$	∞
$-\pi/2$	$\leq$	α_H	$\leq$	$\pi/2$
0.127	$\leq$	r_R	$\leq$	∞
1.0	$\leq$	m_R	$\leq$	∞
0.127	$\leq$	r_F	$\leq$	∞
1.0	$\leq$	m_F	$\leq$	∞

Table 3. Equality constraints

Constraint	Equation	Description
g_1	$I_{Dyy} = \sqrt{I_{Dxx}^2 + I_{Dzz}^2}$	Rear frame is planar.
g_2	$k_{Ryy} = r_R$	Rear wheel is a ring
g_3	$k_{Raa} = k_{Ryy}/2$	Rear wheel is a ring
g_4	$k_{Rbb} = k_{Ryy}/2$	Rear wheel is a ring
g_5	$k_{Fyy} = r_F$	Front wheel is a ring
g_6	$k_{Faa} = k_{Fyy}/2$	Front wheel is a ring
g_7	$k_{Fbb} = k_{Fyy}/2$	Front wheel is a ring

Table 4. Inequality constraints

Constraint	Equation	Description
c_1	$\sqrt{I_{Hxx}^2 + I_{Hzz}^2} \geq I_{Hyy}$	Consistent moments of inertia.
c_2	$0 \geq z_P + l_P/2 \cos \alpha_P$	Person cannot penetrate ground.
c_3	$0 \geq z_P + w_P/2 \sin \alpha_P$	Person cannot penetrate ground.
c_4	$0 \geq z_P - l_P/2 \cos \alpha_P$	Person cannot penetrate ground.
c_5	$0 \geq z_P - w_P/2 \sin \alpha_P$	Person cannot penetrate ground.
c_6	$x_T \geq \lvert z_T \rvert/4$	Maximum acceleration of $1/4g$.
c_7	$w - x_T \geq 3/4\lvert z_T \rvert$	Maximum deceleration of $3/4g$.
c_8	$2k_{Hyy} \geq \sqrt{(x_H - w)^2 + (z_H + r_F)^2}$	Minimal inertial spread.
c_9	$2k_{Dyy} \geq \sqrt{(x_D - 0)^2 + (z_D + r_R)^2}$	Minimal inertial spread.
c_{10}	$w \geq r_F + r_R$	Non-overlapping wheels.
c_{11}	$25\text{kg} \geq m_D + m_H + m_R + m_F$	Maximum bicycle mass.
c_{12}	$-z_D \geq 1.4k_{Dyy}$	Rear frame cannot penetrate ground.
c_{13}	$-z_H \geq 1.4k_{Hyy}$	Front frame cannot penetrate ground.
$c_{14,\ldots,21}$	$0 \geq s_1, \ldots, s_8$	Closed loop stability.

person and the cross axes are parallel to the principal axes in the XZ plane. $l_P/2$ is the distance along the principal axis to the tip of the toes and w_P is the distance along the second principal axes to the tip of the hands. Constraints $c_2, \ldots, c_5$.

3.2 Front Frame [H]

The front frame is symmetric about the XZ plane so $I_{Hxy}, I_{Hyz} = 0$. We allow for any angular orientation of the XZ principal directions but limit the angle to $-\frac{\pi}{2} \leq \alpha_H \leq \frac{\pi}{2}$. We prevent the rear frame from penetrating the ground by limiting the inertial spread with respect to its mass center, c_{13}, but also set a minimum inertial spread to ensure a frame can span from the rear wheel to the mass center of the rear frame, c_8. The spread factor in c_{13} of 1.4 is based on the ratio of geometrical spread of a typical bicycle frame and its radius of gyration. The front frame is not planar but is narrow with respect to the XZ plane, which is enforced by constraint c_1.

3.3 Rear Frame [D]

Several constraints are set for the rear frame. We constrain the rear frame to be planar, g_1, and symmetric with respect to the XZ plane. We prevent the rear frame from penetrating the ground by limiting the inertial spread with respect to its mass center, c_{12}, but also set a minimum inertial spread to ensure a frame can span from the rear wheel to the mass center of the rear frame, c_9. The spread factor in c_{12} of 1.4 is based on the ratio of geometrical spread of a typical bicycle frame and its radius of gyration. Several parameters are bounded. We require the rear frame mass to be positive, the center of mass not penetrate the ground, and we allow for any angular orientation of the XZ principal directions but limit the angle to $-\frac{\pi}{2} \leq \alpha_D \leq \frac{\pi}{2}$ as angle beyond that are redundant.

3.4 Front [F] and Rear [R] Wheels

We enforce the assumption that both wheels have moments of inertia of that of a simple ring, $g_2 \ldots g_7$ and that mass and radius should be greater than a minimal size based on small purchasable spoked wheel with tire.

3.5 Total Bike [T]

The trail and wheelbase can take on any real values. The steer axis tilt is limited to 180 degrees. We introduce a constraint c_{10} that prevents the wheels from physically overlapping and require that the bicycle be lift-able by an average person, c_{11}. Finally, we require that the bicycle not topple forward during hard breaking or backward during hard acceleration.

$$-\frac{3g}{4} < \text{acceleration} < \frac{g}{4} \tag{1}$$

This translates to two constraints, c_6, c_7 that bound the total center of mass (x_T, z_T) in a triangle in the XZ plane. Lastly, we constrain the eight closed loop eigenvalues associated with the controller in [1] to be stable, i.e. have negative real parts. These are expressed in constraints $c_{14}, \ldots, c_{21}$. Closed loop stability is required for the HQM to provide a meaningful result.

4 Optimization

The above constraints leaves 23 of the 47 parameters free for optimizing which we collect in the vector $\mathbf{p} \in \mathbb{R}^{23}$ and define as:

$$\begin{aligned}\mathbf{p} = [w \quad c \quad \lambda \quad m_D \quad x_D \quad z_D \quad k_{Daa} \quad k_{Dbb} \quad \alpha_D \quad x_P \quad z_P \quad \alpha_P \\ m_H \quad x_H \quad z_H \quad k_{Haa} \quad k_{Hbb} \quad k_{Hyy} \quad \alpha_H \quad r_R \quad m_R \quad r_F \quad m_F]\end{aligned} \tag{2}$$

Our objective in the optimization is to minimize the peak HQM value subject to the bounds, $\mathbf{p}^L, \mathbf{p}^U$, and the constraints $\mathbf{g}(\mathbf{p}), \mathbf{c}(\mathbf{p})$. Given a set of bicycle model parameter values we generate a bandwidth limited human-like controller using the methods in [4]. Once the closed loop stable controller is constructed, the HQM can be computed as per the definition in [1] and the scaler peak value returned as the objective J. This problem is presented as a non-linear programming problem in the following equation.

$$\begin{aligned}&\underset{\mathbf{p}}{\text{minimize}} \quad J(\mathbf{p}) = max(\text{HQM}(\mathbf{p})) \\ &\text{subject to} \\ &\qquad \mathbf{g}(\mathbf{p}) \leq \mathbf{0} \\ &\qquad \mathbf{c}(\mathbf{p}) = \mathbf{0} \\ &\qquad \mathbf{p}^L \leq \mathbf{p} \leq \mathbf{p}^U\end{aligned} \tag{3}$$

We make use of the derivative-free optimizer CMA-ES [**?**] to find solutions to this problem. The optimization supports parameter bounds and equality constraints but does not support inequality constraints. To get around this limitation we move the inequality constraints into the objective function an penalize the objective if the constraints are violated with the following rules:

$$J(\mathbf{p}) = \begin{cases} max(\text{HQM}(\mathbf{p})) & \text{if} \quad any(\mathbf{g}(\mathbf{p})) < 0 \\ 30 + ||\mathbf{g}_+(\mathbf{p})||/10 & \text{if} \quad any(\mathbf{g}(\mathbf{p})) \geq 0 \text{ and } ||\mathbf{g}_+(\mathbf{p})|| < 30 \\ ||\mathbf{g}_+(\mathbf{p})|| & \text{if} \quad any(\mathbf{g}(\mathbf{p})) \geq 0 \text{ and } ||\mathbf{g}_+(\mathbf{p})|| \geq 30 \end{cases} \tag{4}$$

where $||\mathbf{g}_+||$ is the norm of the positive elements of $\mathbf{g}$.

This creates a discontinous objective function but in practice the CMS-ES algorthim is able to move into the parameter space where all the constraints are satisfied and find a (local) minima. For our purposes, this sufficiently finds parameter values that produce an optimally handling design.

Table 5. Peak HQM values for the reference bicycle and the optimal bicycles at each speed.

Speed [m/s]	Reference Peak HQM	Optimal Peak HQM	Percent Improvement
3	13.0753	2.0118	85%
5	4.5213	0.0115	100%
7	3.0434	0.0220	99%
9	2.3377	0.8386	64%

5 Results

We discover four bicycles for four different design speeds (3, 5, 7, and 9 m/s) that have an optimally low HQM, see Table 5 and satisify all constraints and parameter boundaries. We belive these bicycles to be physically realizable with minor differences. The most striking feature is that all of the bicycle are larger that the reference bicycle in some way. The 3 m/s bicycle places both the rider's mass center and the rear frame's mass center close to 3 meters above the ground plane. The 5 and 7 m/s bicycles have wheelbase values that are simlar to the reference bicycle, but 3 and 9 m/s have very large wheel bases. The 9 m/s bicycle is very large overall. Only the 5 m/s bicycle is of a scale close to a typical bicycle. It is noteable that the 9 m/s bicycle has signficant negative trail.

3 - Rider is over 3 m in the air and leaned forward. - 3 m wheel base with 0.7 meter positive trail - wheels are over double the diameter of browser wheels - minimum front frame mass

5 m/s - rider is laid back a 0.5 meter off ground - < 1 m wheelbase - essentially zero trail (5 mm negative) - minimum front frame mass

7 m/s - miminal rotatoinal inertia about the steer axis - tall slender fork/handlebar - no trail (1 mm postive) - half sized wheels (12 inches) - normal wheelbase (1.157) - rider tipped forward and low (0.6 m) - vertical steer axis - 3 m high frame center of mass

9 - very large 5.6 m wheelbase

6 Discussion and Conclusion

References

[1] HESS, R., MOORE, J. K., AND HUBBARD, M. Modeling the Manually Controlled Bicycle. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans 42*, 3 (Feb. 2012), 545–557.

[2] MEIJAARD, J. P., PAPADOPOULOS, J. M., RUINA, A., AND SCHWAB, A. L. Linearized dynamics equations for the balance and steer of a bicycle: A benchmark and review. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 463*, 2084 (Aug. 2007), 1955–1982.

[3] MOORE, J., HUBBARD, M., AND HESS, R. A. An Optimal Handling Bicycle. In *Proceedings of the 2016 Bicycle and Motorcycle Dynamics Conference* (Sept. 2016), Figshare.

[4] MOORE, J. K. *Human Control of a Bicycle*. Doctor of Philosophy, University of California, Davis, CA, Aug. 2012.

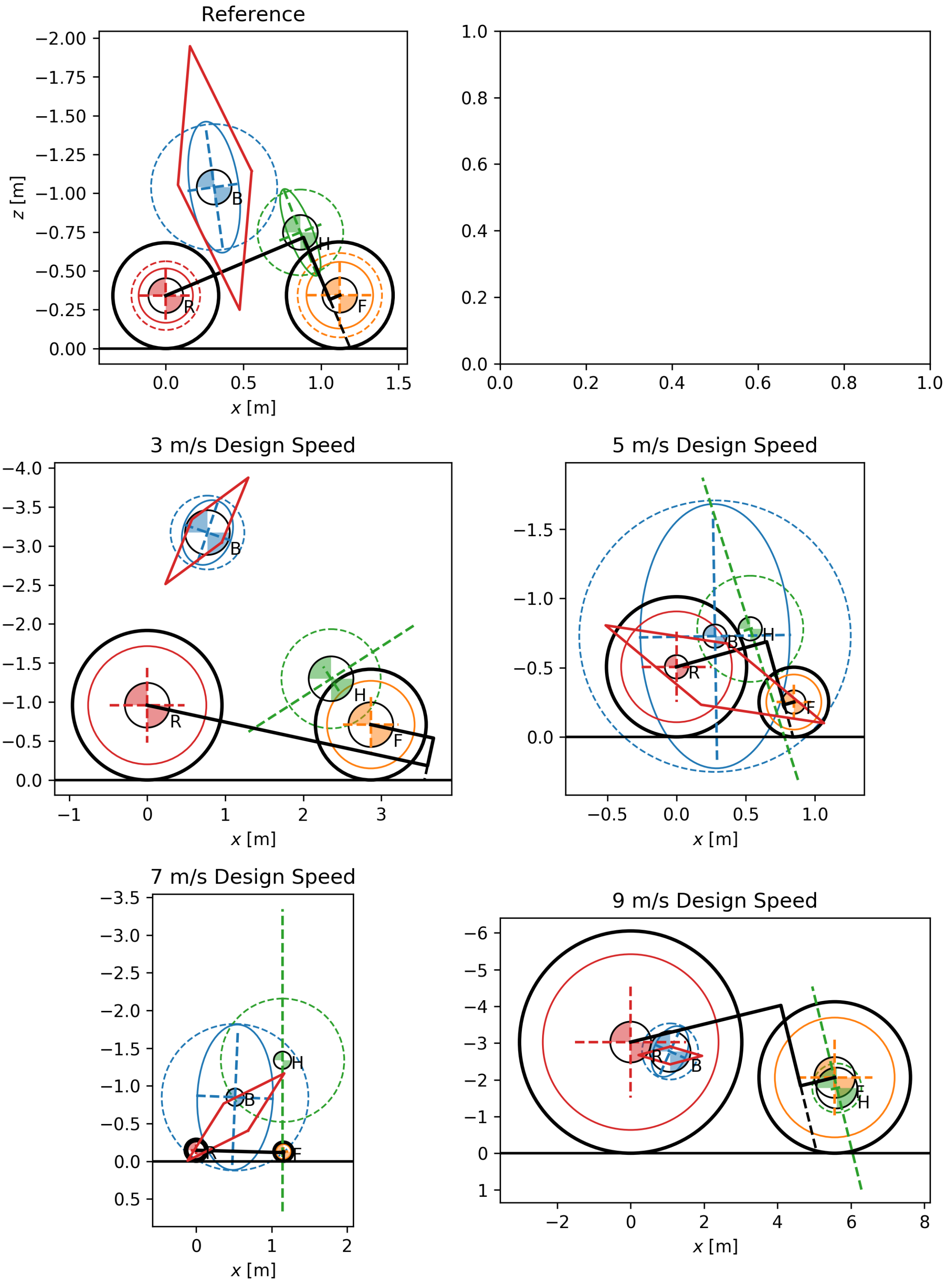

Figure 1. Depcitions of the bicycle geometry and geometric representations of the inertial quanties for the reference bicycle and four optimal solutions at 3, 5, 7, and 9 m/s. Five rigid bodies are shown for each bicycle: front wheel (oragne), rear wheel (purple), rear frame (blue), front frame (green), and person (red). The solid black lines represent the essential bicycle geometry. The dotted black line represents the steer axis. The solid colored curves represent the contours of solid ellipsoids with equivalent inertia as the principal inertia of the associated rigid body. The dotted colored lines represent the extents of the centerial radii of gyration of each rigid body.

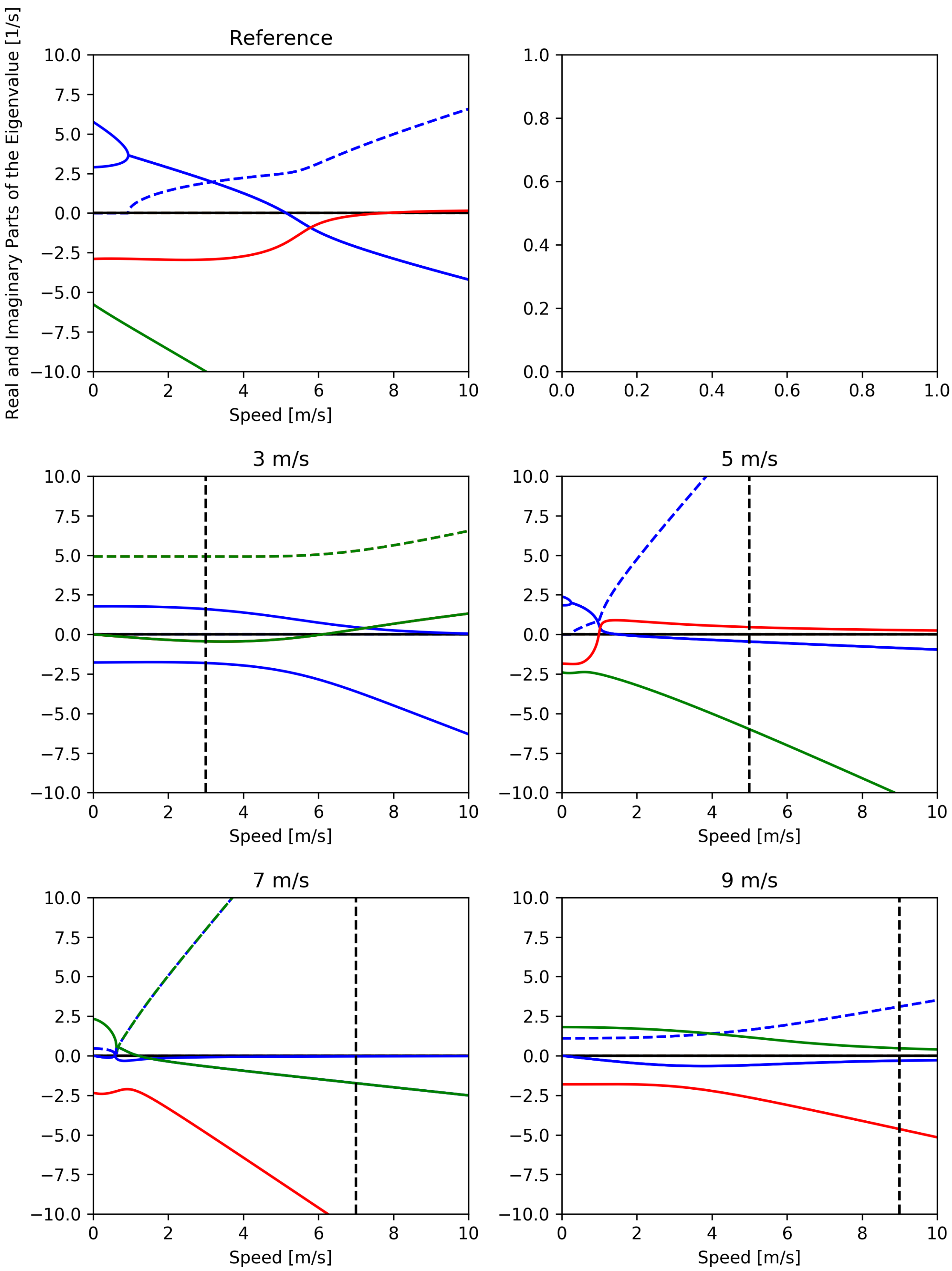
Real and Imaginary Parts of the Eigenvalue [1/s]
Reference
3 m/s
5 m/s
7 m/s
9 m/s
Speed [m/s]

Figure 2.

Table 6. Optimal principal parameter values for each design speed.

Parameter	$3\ \mathrm{m\,s^{-1}}$	$5\ \mathrm{m\,s^{-1}}$	$7\ \mathrm{m\,s^{-1}}$	$9\ \mathrm{m\,s^{-1}}$
c	0.688	-0.005	0.001	-0.484
w	2.866	0.847	1.157	5.557
λ	-0.213	0.271	-0.028	0.239
m_D	3.22	11.53	10.32	7.02
x_D	0.958	0.287	0.427	1.020
z_D	-2.605	-2.719	-2.976	-4.065
α_D	0.663	0.915	1.123	1.077
k_{Daa}	1.449	1.218	1.324	0.861
k_{Dbb}	0.724	1.387	1.290	2.014
k_{Dyy}	1.471	1.559	1.555	2.031
m_H	0.25	0.25	0.49	3.54
x_H	2.356	0.532	1.145	5.615
z_H	-1.298	-0.781	-1.340	-1.776
α_H	0.572	-1.265	1.571	0.238
k_{Haa}	0.186	0.0491	0.000	2.846
k_{Hbb}	1.259	1.145	2.006	0.168
k_{Hyy}	0.636	0.383	0.818	0.669
m_F	1.62	4.40	2.64	5.51
r_F	0.710	0.252	0.127	2.063
k_{Faa}	0.355	0.126	0.064	1.031
k_{Fyy}	0.710	0.252	0.127	2.063
x_P	0.765	0.276	0.526	1.079
z_P	-3.194	-0.453	-0.586	-2.662
α_P	-0.661	1.150	-0.836	1.558
m_R	12.63	8.36	6.31	2.25
r_R	0.958	0.506	0.146	3.027
k_Raa	0.479	0.253	0.073	1.514
k_Ryy	0.958	0.506	0.146	3.027