Section 1

Questions 1 - 10

1. $\int \frac{x}{x^2+4}\,dx$

2. $\int \frac{x}{\sqrt{x^2+4}}\,dx$

3. $\int \frac{5x+2}{x^2-4}\,dx$

4. $\int \sin x \cos^3 x\,dx$

5. $\int \sin x \sec^3 x\,dx$

6. $\int \cos^2 \frac{x}{2}\,dx$

7. $\int x \sin x\,dx$

8. $\int x \sec^2 2x\,dx$

9. $\int \tan^{-1} 2x\,dx$

10. $\int \frac{x^3}{x^2+1}\,dx$

Questions 11 - 20

11. $\int \frac{x}{(x+2)(x+4)}\,dx$

12. $\int \frac{(x-1)(x+1)}{(x-2)(x-3)}\,dx$

13. $\int \frac{2x-1}{x^2+2x+3}\,dx$

14. $\int \frac{x^3}{2x-1}\,dx$

15. $\int \frac{1+x}{\sqrt{1-x-x^2}}\,dx$

16. $\int \frac{dx}{x^2(1-x^2)^{\frac{1}{2}}}$

17. $\int \frac{dx}{x\sqrt{a^2+x^2}}$

18. $\int \frac{dx}{x\sqrt{a^2-x^2}}$

19. $\int \frac{dx}{x\sqrt{x^2-a^2}}$

20. $\int \frac{x}{\sqrt{x}+1}\,dx$

Questions 21 - 30

21. $\displaystyle\int \frac{\cos^{-1} x}{\sqrt{1-x^2}}\,dx$

22. $\displaystyle\int \sqrt{\frac{x+1}{x-1}}\,dx$

23. $\displaystyle\int \frac{dx}{x(\ln x)^3}$

24. $\displaystyle\int \sec^4 3x\,dx$

25. $\displaystyle\int \frac{dx}{x^2(1-x)}$

26. $\displaystyle\int \frac{dx}{x^2(1+x^2)}$

27. $\displaystyle\int \frac{dx}{(1+x^2)^2}$

28. $\displaystyle\int \tan^3 x\,dx$

29. $\displaystyle\int \frac{dx}{5+3\cos x}$

30. $\displaystyle\int \frac{dx}{3+5\cos x}$

Questions 31 - 40

31. $\displaystyle\int \frac{\sin x}{5 + 3\cos x}\,dx$

32. $\displaystyle\int \frac{dx}{1 + \cos^2 x}$

33. $\displaystyle\int \frac{dx}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}$

34. $\displaystyle\int x^2 \sin x\,dx$

35. $\displaystyle\int \frac{x^2}{(x-1)(x-2)(x-3)}\,dx$

36. $\displaystyle\int \frac{e^x}{e^x - 1}\,dx$

37. $\displaystyle\int \frac{dx}{3\sin^2 x + 5\cos^2 x}$

38. $\displaystyle\int x^3 e^{5x^4 - 7}\,dx$

39. $\displaystyle\int x^5 \ln x\,dx$

40. $\displaystyle\int \frac{3x + 2}{x(x+1)^3}\,dx$

Questions 41 - 50

41. $\displaystyle\int \ln(x^3)\,dx$

42. $\displaystyle\int \frac{dx}{e^x + e^{-x}}$

43. $\displaystyle\int (5x^3 + 7x - 1)^{\frac{3}{2}} \cdot (15x^2 + 7)\,dx$

44. $\displaystyle\int \frac{dx}{(x^2+1)(x^2+4)}$

45. $\displaystyle\int (x^2 + x + 1)^{-1}\,dx$

46. $\displaystyle\int e^x \sin 2x\,dx$

47. $\displaystyle\int (x^2 + x - 1)^{-1}\,dx$

48. $\displaystyle\int (x^2 - x)^{-\frac{1}{2}}\,dx$

49. $\displaystyle\int \frac{1-2x}{3+x}\,dx$

50. $\displaystyle\int x^3(4+x^2)^{-\frac{1}{2}}\,dx$

Questions 51 - 60

51. $\int \frac{\sin 2x}{3\cos^2 x + 4\sin^2 x}\, dx$

52. $\int \frac{x^2}{1 - x^4}\, dx$

53. $\int \frac{dx}{\sin x \cos x}$

54. $\int \ln\sqrt{x-1}\, dx$

55. $\int \frac{dx}{e^x - 1}$

56. $\int \frac{\sec^2 x}{\tan^2 x - 3\tan x + 2}\, dx$

57. $\int \frac{x+1}{(x^2 - 3x + 2)^{\frac{1}{2}}}\, dx$

58. $\int \sin 2x \cos x\, dx$

59. $\int \frac{x}{1 + x^3}\, dx$

60. $\int x \tan^{-1} x\, dx$

Questions 61 - 70

61. $\int (1 + 3x + 2x^2)^{-1}\ dx$

62. $\int (9 - x^2)^{\frac{1}{2}}\ dx$

63. $\int (9 + x^2)^{\frac{1}{2}}\ dx$

64. $\int x(9 + x^2)^{\frac{1}{2}}\ dx$

65. $\int \sec^2 x \tan^3 x\ dx$

66. $\int x^2 e^{-x}\ dx$

67. $\int x e^{x^2}\ dx$

68. $\int \sin x \tan x\ dx$

69. $\int \sin^4 x \cos^3 x\ dx$

70. $\int \frac{x^3 + 1}{x^3 - x}\ dx$

Questions 71 - 80

71. $\int \ln(x + \sqrt{x^2 - 1})\ dx$

72. $\int \dfrac{dx}{(x+1)^{\frac{1}{2}} + (x+1)}$

73. $\int_0^4 \dfrac{x}{\sqrt{x+4}}\ dx$

74. $\int_1^2 \dfrac{dx}{x(1+x^2)}$

75. $\int_1^2 \dfrac{\ln x}{x}\ dx$

76. $\int_0^1 \cos^{-1} x\ dx$

77. $\int_1^2 \dfrac{x+1}{\sqrt{-2+3x-x^2}}\ dx$

78. $\int_0^{\frac{\pi}{2}} \dfrac{dx}{\cos^2 x + 2\sin^2 x}$

79. $\int_0^1 x\sqrt{1-x^2}\ dx$

80. $\int_2^4 x \ln x\ dx$

Section 9

Questions 81 - 90

81. $\int_1^2 \frac{dx}{x^2+5x+4}$

82. $\int_0^{\frac{\pi}{2}} \left(1+\frac{1}{2}\sin x\right)^{-1} dx$

83. $\int_0^1 x^2 e^{-x}\, dx$

84. $\int_0^1 \frac{7+x}{1+x+x^2+x^3}\, dx$

85. $\int_0^1 \frac{e^{-2x}}{e^{-x}+1}\, dx$

86. $\int_0^{\frac{a}{2}} \frac{y}{a-y}\, dy$

87. $\int_0^a \frac{(a-x)^2}{a^2+x^2}\, dx$

88. $\int_0^1 \frac{x+3}{(x+2)(x+1)^2}\, dx$

89. $\int_0^1 \frac{x^2}{x^6+1}\, dx$

90. $\int_0^{\pi} \cos^2 mx\ dx$, where m is an integer.

Questions 91 - 100

91. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \sin 2x \; dx$

92. $\int_{0}^{\frac{a}{2}} x^2\sqrt{a^2 - x^2} \; dx$

93. $\int_{0}^{\frac{\pi}{4}} \sec^2 x \tan x \; dx$

94. $\int_{0}^{1} (x+2)\sqrt{x^2+4x+5} \; dx$

95. $\int_{1}^{2} x(\ln x)^2 \; dx$

96. $\int_{3}^{4} \frac{x^2+4}{x^2-1} \; dx$

97. $\int_{1}^{4} \frac{x^2+4}{x(x+2)} \; dx$

98. $\int_{0}^{\frac{\pi}{2}} \frac{\cos x}{5 - 3\sin x} \; dx$

99. $\int_{0}^{1} \frac{1}{(4-x^2)^{\frac{3}{2}}} \; dx$

100. $\int_{0}^{\frac{\pi}{2}} 2\sin\theta\cos\theta(3\sin\theta - 4\sin^3\theta) \; d\theta$

Chapter 2

Quick Solutions

Questions 1 - 100

WAIT!

Are you sure you should be here?

We strongly advise you to attempt questions in blocks of ten. And dont check your answers until you finish the block!

Don't use the solutions to give you hints; you're only cheating yourself. Make sure you give each question your best attempt before coming to this section (or the next) to check your answers!

1. $\frac{1}{2}\ln|x^2+4|+C$
2. $\sqrt{x^2+4}+C$
3. $2\ln|x+2|+3\ln|x-2|+C$
4. $-\frac{\cos^4 x}{4}+C$
5. $\frac{1}{2}\sec^2 x+C$
6. $\frac{x}{2}+\frac{\sin x}{2}+C$
7. $-x\cos x+\sin x+C$
8. $\frac{x}{2}\tan 2x+\frac{1}{4}\ln|\cos 2x|+C$
9. $x\tan^{-1}2x-\frac{1}{4}\ln|1+4x^2|+C$
10. $\frac{x^2}{2}-\frac{1}{2}\ln|x^2+1|+C$

Questions 11 - 20

11. $2\ln|x+4| - \ln|x+2| + C$

12. $x + 8\ln|x-3| - 3\ln|x-2| + C$

13. $\ln|x^2+2x+3| - \dfrac{3}{\sqrt{2}}\tan^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right) + C$

14. $\dfrac{x^3}{6} + \dfrac{x^2}{8} + \dfrac{x}{8} + \dfrac{1}{16}\ln|2x-1| + C$

15. $-\sqrt{1-x-x^2} + \dfrac{1}{2}\sin^{-1}\left(\dfrac{2x+1}{\sqrt{5}}\right) + C$

16. $-\dfrac{\sqrt{1-x^2}}{x} + C$

17. $-\dfrac{1}{a}\ln\left|\dfrac{a}{x} + \dfrac{\sqrt{a^2+x^2}}{x}\right| + C$

18. $-\dfrac{1}{a}\ln\left|\dfrac{a}{x} + \dfrac{\sqrt{a^2-x^2}}{x}\right| + C$

19. $\dfrac{1}{a}\sec^{-1}\dfrac{x}{a} + C$

20. $\dfrac{2}{3}x^{\frac{3}{2}} - x + 2x^{\frac{1}{2}} - 2\ln\left(\sqrt{x}+1\right) + C$

Questions 21 - 30

21. $-\dfrac{(\cos^{-1}x)^2}{2}+C$

22. $\sqrt{x^2-1}+\ln\left|x+\sqrt{x^2-1}\right|+C$

23. $-\dfrac{1}{2(\ln x)^2}+C$

24. $\dfrac{\tan 3x}{3}+\dfrac{\tan^3 3x}{9}+C$

25. $\ln\left|\dfrac{x}{1-x}\right|-\dfrac{1}{x}+C$

26. $-\dfrac{1}{x}-\tan^{-1}x+C$

27. $\dfrac{\tan^{-1}x}{2}+\dfrac{x}{2(x^2+1)}+C$

28. $\dfrac{\tan^2 x}{2}+\ln|\cos x|+C$

29. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{\tan\frac{x}{2}}{2}\right)+C$

30. $\dfrac{1}{4}\ln\left|\dfrac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right|+C$

Questions 31 - 40

31. $-\frac{1}{3}\ln|5+3\cos x| + C$

32. $= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$

33. $\ln|\sec x + \tan x| + C$

34. $-x^2\cos x + 2x\sin x + 2\cos x + C$

35. $\frac{1}{2}\ln|x-1| - 4\ln|x-2| + \frac{9}{2}\ln|x-3| + C$

36. $\ln|e^x - 1| + C$

37. $\frac{\sqrt{15}}{15}\tan^{-1}\left(\sqrt{\frac{3}{5}}\tan x\right) + C$

38. $\frac{1}{20}e^{5x^4-7} + C$

39. $\frac{x^6}{6}\ln x - \frac{x^6}{36} + C$

40. $2\ln|x| - 2\ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C$

Questions 41 - 50

41. $3x \ln x - 3x + C$

42. $\tan^{-1}(e^x) + c$

43. $\frac{2}{5}(5x^3 + 7x - 1)^{\frac{5}{2}} + C$

44. $\frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right) + C$

45. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

46. $\frac{e^x}{5}\left(\sin 2x - 2\cos 2x\right) + C$

47. $\frac{1}{\sqrt{5}}\ln\left|\frac{2x+1-\sqrt{5}}{2x+1+\sqrt{5}}\right| + C$

48. $\ln\left|2x - 1 + 2\sqrt{x^2 - x}\right| + C$

49. $7\ln|x+3| - 2x + C$

50. $\frac{1}{3}(x^2 - 8)\sqrt{x^2 + 4} + C$

Questions 51 - 60

51. $\ln(3+\sin^2 x)+C$

52. $\dfrac{1}{4}\ln|1+x|-\dfrac{1}{4}\ln|1-x|-\dfrac{1}{2}\tan^{-1}x+C$

53. $-2\ln|\operatorname{cosec}x+\cot x|+C$ OR $=\ln|\tan x|+C$

54. $\dfrac{1}{2}(x-1)\ln(x-1)-\dfrac{1}{2}x+C$

55. $\ln|1-e^{-x}|+C$ OR $\ln|e^x-1|-x+C$

56. $\ln|\tan x-2|-\ln|\tan x-1|+C$

57. $\sqrt{x^2-3x+2}+\dfrac{5}{2}\ln\left|2x-3+2\sqrt{x^2-3x+2}\right|+C$

58. $-\dfrac{2}{3}\cos^3 x+C$ OR $-\dfrac{1}{6}\cos 3x-\dfrac{1}{2}\cos x+C$

59. $-\dfrac{1}{3}\ln|1+x|+\dfrac{1}{6}\ln|1-x+x^2|+\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right)+C$

60. $\dfrac{x^2}{2}\tan^{-1}x-\dfrac{1}{2}x+\dfrac{1}{2}\tan^{-1}x+C$

Questions 61 - 70

61. $\ln|2x+1| - \ln|x+1| + C$

62. $\frac{1}{2}\left[9\sin^{-1}\left(\frac{x}{3}\right) + x\sqrt{9-x^2}\right] + C$

63. $\frac{1}{2}x\sqrt{x^2+9} + \frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right| + C_2$

64. $\frac{1}{3}(x^2+9)^{\frac{3}{2}} + C$

65. $\frac{\tan^4 x}{4} + C$

66. $-e^{-x}(x^2+2x+2) + C$

67. $\frac{1}{2}e^{x^2} + C$

68. $\ln|\sec x + \tan x| - \sin x + C$

69. $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$

70. $x - \ln|x| + \ln|x-1| + C$

Questions 71 - 80

71. $x \ln\left(x + \sqrt{x^2 - 1}\right) - \sqrt{x^2 - 1} + C$

72. $2 \ln\left|\sqrt{x + 1} + 1\right| + C$

73. $\frac{16}{3}\left(2 - \sqrt{2}\right)$

74. $\frac{1}{2} \ln \frac{8}{5}$

75. $\frac{1}{2}(\ln 2)^2$

76. 1

77. $\frac{5\pi}{2}$

78. $\frac{\pi\sqrt{2}}{4}$

79. $\frac{1}{3}$

80. $14 \ln 2 - 3$

Questions 81 - 90

81. $\frac{1}{3}\ln\left(\frac{5}{4}\right)$

82. $\frac{2\pi}{3\sqrt{3}}$

83. $2 - 5e^{-1}$

84. $\frac{3}{2}\ln 2 + \pi$

85. $\ln\left(\frac{1+e}{2e}\right) - \frac{1}{e} + 1$

86. $\frac{a}{2}(\ln 4 - 1)$

87. $a(1 - \ln 2)$

88. $\ln\frac{3}{4} + 1$

89. $\frac{\pi}{12}$

90. $\frac{\pi}{2}$

Questions 91 - 100

91. $\frac{\pi}{4} - \frac{1}{4}$

92. $\frac{a^4}{192}\left(4\pi - 3\sqrt{3}\right)$

93. $\frac{1}{2}$

94. $\frac{5\sqrt{5}}{3}\left(2\sqrt{2} - 1\right)$

95. $2(\ln 2)^2 - 2\ln 2 + \frac{3}{4}$

96. $1 + \frac{5}{2}\ln\frac{6}{5}$

97. 3

98. $\frac{1}{3}\ln\frac{5}{2}$

99. $\frac{\sqrt{3}}{12}$

100. $\frac{2}{5}$

Chapter 3

Worked Solutions

When reading the worked solutions provided, ask yourself:

- Was my method the most efficient one?
- Why have the solutions done what they have done?
- What would I change if I were doing this integral again?
- How should I approach this type of integral in the future?

This is the most important part when studying for Integration. Doing the integrals themselves does absolutely nothing. Your studying only truly begins once you carefully examine the worked solutions and reflect upon what you would do differently next time.

Questions 1 - 10

1. $\displaystyle\int \frac{x}{x^2+4}\,dx$

Method 1: Reverse Chain Rule

Observing that $\dfrac{d}{dx}(x^2+4) = 2x$:

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx$$

$$= \frac{1}{2}\ln|x^2+4| + C, \text{ applying the reverse chain rule.}$$

Method 2: Algebraic Substitution

Observing that $\dfrac{d}{dx}(x^2+4) = 2x$, $u = x^2+4$ is a suitable choice of substitution:

$$\int \frac{x}{x^2+4}\,dx = \int \frac{1}{u}\left(\frac{1}{2}\,du\right)$$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+4| + C$$

let $u = x^2+4$

$\dfrac{du}{dx} = 2x$

$x\,dx = \dfrac{1}{2}\,du$

2. $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx$

Method 1: Reverse Chain Rule

Observing that $\dfrac{d}{dx}(x^2+4) = 2x$:

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \frac{1}{2}\int 2x(x^2+4)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\cdot\frac{(x^2+4)^{\frac{1}{2}}}{\frac{1}{2}} + C, \text{ by the reverse chain rule.}$$

$$= \sqrt{x^2+4} + C$$

Method 2: Algebraic Substitution #1

Observing that $\dfrac{d}{dx}(x^2+4) = 2x$, $u = x^2+4$ is a suitable choice of substitution:

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \int u^{-\frac{1}{2}}\left(\frac{1}{2}\,du\right)$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= u^{\frac{1}{2}} + C$$

$$= \sqrt{x^2+4} + C$$

let $u = x^2+4$

$\dfrac{du}{dx} = 2x$

$x\,dx = \dfrac{1}{2}\,du$

Method 3: Algebraic Substitution #2

The goal of substitution is to eliminate difficult parts of the integrand, thus subbing $u = \sqrt{x^2+4}$ will also be suitable:

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \int \frac{u}{u}\,du$$

$$= \int du$$

$$= u + C$$

$$= \sqrt{x^2+4} + C$$

let $u = \sqrt{x^2+4}$

$$\frac{du}{dx} = \frac{x}{\sqrt{x^2+4}}$$

$$x\,dx = u\,du$$

Method 4: Trigonometric Substitution

We can make use of the identity $\tan^2\theta + 1 = \sec^2\theta$ to help simplify the integrand:

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \int \frac{2\tan\theta}{\sqrt{4\tan^2\theta+4}}(2\sec^2\theta)\,d\theta$$

$$= \int \frac{2\tan\theta}{2\sec\theta}(2\sec^2\theta)\,d\theta$$

$$= 2\int \tan\theta\sec\theta\,d\theta$$

$$= -2\int (\cos\theta)^{-2}(-\sin\theta)\,d\theta$$

$$= \frac{2(\cos\theta)^{-1}}{-1} \times (-1) + C$$

let $x = 2\tan\theta$

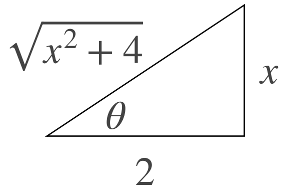
$$\frac{dx}{d\theta} = 2\sec^2\theta$$

$$dx = 2\sec^2\theta\,d\theta$$

$$= 2\sec\theta + C$$

We then substitute back in:

$$\tan\theta = \frac{x}{2} \implies \sec\theta = \frac{\sqrt{x^2+4}}{2}$$



$$= \sqrt{x^2+4} + C$$

3. $\displaystyle\int \frac{5x+2}{x^2-4}\,dx$

Method 1: Partial Fractions

Observing that the denominator has two linear factors, we shall use partial fraction decomposition.

Let $\dfrac{5x+2}{x^2-4} = \dfrac{A}{x+2} + \dfrac{B}{x-2}$.

$\therefore 5x+2 = A(x-2) + B(x+2)$

letting $x = 2$: $12 = 4B \implies B = 3$.

letting $x = -2$: $-8 = -4A \implies A = 2$.

Hence, $\displaystyle\int \frac{5x+2}{x^2-4}\,dx = \int \left(\frac{2}{x+2} + \frac{3}{x-2}\right)\,dx$

$$= 2\ln|x+2| + 3\ln|x-2| + C$$

4. $\displaystyle\int \sin x \cos^3 x\,dx$

Method 1: Reverse Chain Rule

Observing that $\dfrac{d}{dx}(\cos x) = -\sin x$:

$$\int \sin x \cos^3 x\,dx = -\int (-\sin x)\,\cos^3 x\,dx$$

$$= -\frac{\cos^4 x}{4} + C, \text{ by the reverse chain rule.}$$

Method 2: Substitution

Observing that $\dfrac{d}{dx}(\cos x) = -\sin x$, $u = \cos x$ is a suitable substitution:

$$\int \sin x \cos^3 x\,dx = \int u^3(-du)$$

$$= -\int u^3\,du$$

$$= -\frac{u^4}{4} + C$$

$$= -\frac{\cos^4 x}{4} + C$$

let $u = \cos x$

$\dfrac{du}{dx} = -\sin x$

$\sin x\,dx = -du$

5. $\int \sin x \sec^3 x \, dx$

Method 1: Reverse Chain Rule

Sine and cosine are very related functions, so we rewrite $\sec x$ in terms of $\cos x$.

$$\int \sin x \sec^3 x \, dx = \int \sin x \, (\cos x)^{-3} \, dx$$

Observing that $\frac{d}{dx}(\cos x) = -\sin x$

$$= -\int (-\sin x)(\cos x)^{-3} \, dx$$

$$= -\frac{(\cos x)^{-2}}{-2}$$

$$= \frac{1}{2}\sec^2 x + C$$

Method 2: Substitution

Note that $\sec x = \frac{1}{\cos x}$, and that $\frac{d}{dx}(\cos x) = -\sin x$, hence $u = \cos x$ will be a suitable substitution:

$$\int \sin x \sec^3 x \, dx = \int u^{-3} \, (-du)$$

$$= -\frac{u^{-2}}{-2} + C$$

let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$\sin x \, dx = -du$

Substituting back in:

$$= \frac{1}{2}\sec^2 x + C$$

6. $\int \cos^2 \frac{x}{2} \, dx$

Method 1: Half Angle Identity

Dealing with powers of trigonometric functions is not too nice, so we will use the half-angle identity to help turn the integrand into a single power:

As $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$:

$$\int \cos^2 \frac{x}{2} \, dx = \frac{1}{2}\int (1 + \cos x) \, dx$$

$$= \frac{1}{2}(x + \sin x) + C$$

$$= \frac{x}{2} + \frac{\sin x}{2} + C$$

7. $\int x \sin x \, dx$

Method 1: Integration by Parts

Notice that the integrand is a product of two functions, this hints that integration by parts will help.

Letting:

u	x	$-\cos x$	v
u'	1	$\sin x$	v'

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

8. $\int x \sec^2 2x \, dx$

Method 1: Integration by Parts

Notice that the integrand is a product of two functions, this hints that integration by parts will help.

Letting:

u	x	$\frac{1}{2} \tan 2x$	v
u'	1	$\sec^2 2x$	v'

$$\int x \sec^2 2x \, dx = \frac{x}{2} \tan 2x - \frac{1}{2} \int \tan 2x \, dx$$

$$= \frac{x}{2} \tan 2x - \frac{1}{2} \int \frac{\sin 2x}{\cos 2x} \, dx$$

We then note that $\frac{d}{dx}(\cos 2x) = -2 \sin 2x$ and we manipulate the integral such that we can apply the reverse chain rule:

$$= \frac{x}{2} \tan 2x + \frac{1}{4} \int \frac{(-2 \sin 2x)}{\cos 2x} \, dx$$

$$= \frac{x}{2} \tan 2x + \frac{1}{4} \ln |\cos 2x| + C$$

9. $\int \tan^{-1} 2x \, dx$

Method 1: Integration by Parts

This integral is a type of integral which doesn't succumb to normal integration techniques, thus we shall try integration by parts, where $v' = 1$.

Letting:

u	$\tan^{-1} 2x$	x	v
u'	$\dfrac{2}{1+4x^2}$	1	v'

$$\int \tan^{-1} 2x \, dx = x \tan^{-1} 2x - \int \frac{2x}{1+4x^2} \, dx$$

$$= x \tan^{-1} 2x - \frac{1}{4} \int \frac{8x}{1+4x^2} \, dx$$

$$= x \tan^{-1} 2x - \frac{1}{4} \ln |1 + 4x^2| + C$$

10. $\displaystyle\int \frac{x^3}{x^2+1} \, dx$

Method 1: Long Division

We observe the degree of the numerator is greater than the denominator, thus the best course of action is polynomial long division. This yields:

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

Hence, $\displaystyle\int \frac{x^3}{x^2+1} \, dx = \int x \, dx - \frac{1}{2} \int \frac{2x}{x^2+1} \, dx$

$$= \frac{x^2}{2} - \frac{1}{2} \ln |x^2 + 1| + C$$

Method 2: Algebraic Manipulation

Our goal with the manipulation is to obtain $x^2 + 1$ on the numerator so that we are able to simplify the integral:

$$\int \frac{x^3}{x^2+1} \, dx = \int \frac{x^3 + x - x}{x^2+1} \, dx$$

$$= \int \frac{x(x^2+1)}{x^2+1} \, dx - \int \frac{x}{x^2+1}$$

$$= \int x \, dx - \frac{1}{2} \int \frac{2x}{x^2+1} \, dx$$

$$= \frac{x^2}{2} - \frac{1}{2} \ln |x^2 + 1| + C$$

Questions 11 - 20

11. $\displaystyle\int \frac{x}{(x+2)(x+4)}\,dx$

Method 1: Partial Fractions

Observing that the denominator has two linear factors, we shall use partial fraction decomposition.

Let $\dfrac{x}{(x+2)(x+4)} = \dfrac{A}{x+2} + \dfrac{B}{x+4}$

$\therefore x = A(x+4) + B(x+2)$

letting $x = -2$, $2A = -2 \implies A = -1$.

letting $x = -4$, $-2B = -4 \implies B = 2$

$$\therefore \frac{x}{(x+2)(x+4)} = \frac{-1}{x+2} + \frac{2}{x+4}$$

$$\int \frac{x}{(x+2)(x+4)}\,dx = \int \left(\frac{2}{x+4} - \frac{1}{x+2}\right) dx$$

$$= 2\ln|x+4| - \ln|x+2| + C$$

12. $\displaystyle\int \frac{(x-1)(x+1)}{(x-2)(x-3)}\,dx$

Method 1: Partial Fractions

The degree of the numerator is the same as the denominator, so we first need to long divide. After that, our new fraction has linear factors in the denominator, so we apply partial fractions.

$$\int \frac{(x-1)(x+1)}{(x-2)(x-3)}\,dx = \int \frac{x^2-1}{x^2-5x+6}\,dx$$

Now, $\dfrac{x^2-1}{x^2-5x+6} = 1 + \dfrac{5x-7}{x^2-5x+6}$, by long division.

Using partial fractions, let $\dfrac{5x-7}{(x-2)(x-3)} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$.

$\therefore A(x-3) + B(x-2) = 5x-7$

letting $x = 3$, $B = 15-7 \implies B = 8$.

letting $x = 2$, $-A = 3 \implies A = -3$

$$\therefore \int \frac{(x-1)(x+1)}{(x-2)(x-3)}\,dx = \int 1\,dx + 8\int \frac{1}{x-3}dx - 3\int \frac{1}{x-2}\,dx$$

$$= x + 8\ln|x-3| - 3\ln|x-2| + C$$

13. $\int \frac{2x-1}{x^2+2x+3}\,dx$

Method 1: Algebraic Manipulation

Observing that $\frac{d}{dx}(x^2+2x+3) = 2x+2$, we shall split the numerator:

$$\int \frac{2x-1}{x^2+2x+3}\,dx = \int \left(\frac{2x+2}{x^2+2x+3} - \frac{3}{2+(x+1)^2}\right)dx$$

We then apply the reverse chain rule to the first fraction, and the standard integral result to the second fraction:

$$= \ln|x^2+2x+3| - \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

14. $\int \frac{x^3}{2x-1}\,dx$

Method 1: Long Division

Noticing that the degree of the numerator is greater than the degree of the denominator, we shall use long division.

$\frac{x^3}{2x-1} = \frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{8} + \frac{\frac{1}{8}}{2x-1}$, by long division.

$$\therefore \int \frac{x^3}{2x-1}\,dx = \int \left[\frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{8} + \frac{1}{8(2x-1)}\right] dx$$

$$= \int \left(\frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{8}\right)dx + \frac{1}{16}\int \frac{2}{2x-1}\,dx$$

$$= \frac{x^3}{6} + \frac{x^2}{8} + \frac{x}{8} + \frac{1}{16}\ln|2x-1| + C$$

15. $\displaystyle\int \frac{1+x}{\sqrt{1-x-x^2}}\,dx$

Method 1: Algebraic Manipulation

Observing that $\dfrac{d}{dx}(1-x-x^2) = -1-2x$, we can split the numerator as $1+x = -\dfrac{1}{2}(-2x-1)+\dfrac{1}{2}$:

$$\int \frac{1+x}{\sqrt{1-x-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x-1}{\sqrt{1-x-x^2}}dx + \frac{1}{2}\int \frac{1}{\sqrt{1-x-x^2}}dx$$

For our remaining integral, we then complete the square.

$$= -\frac{1}{2}\int(-1-2x)(1-x-x^2)^{-\frac{1}{2}}\,dx + \frac{1}{2}\int \frac{1}{\sqrt{\frac{5}{4}-\left(x+\frac{1}{2}\right)^2}}dx$$

These integrals are now all standard integrals:

$$= -\frac{1}{2}\left(\frac{(1-x-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right) + \frac{1}{2}\sin^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right) + C$$

$$= -\sqrt{1-x-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + C$$

16. $\displaystyle\int \frac{dx}{x^2(1-x^2)^{\frac{1}{2}}}$

Method 1: Trigonometric Substitution

We can use the Pythagorean Identity of $1-\sin^2\theta = \cos^2\theta$ to help simplify the integral:

$$\int \frac{dx}{x^2(1-x^2)^{\frac{1}{2}}} = \int \frac{\cos\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}}d\theta$$

$$= \int \frac{1}{\sin^2\theta}d\theta$$

$$= \int \operatorname{cosec}^2\theta\ \mathrm{d}\theta$$

$$= -\cot\theta + C$$

$$= -\frac{\cos\theta}{\sin\theta} + C$$

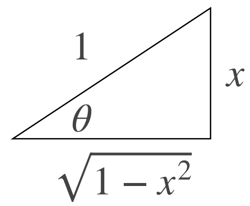
let $x = \sin\theta$

$$\frac{dx}{d\theta} = \cos\theta$$

$$dx = \cos\theta\ d\theta$$

We then substitute back in:

$$\sin\theta = \frac{x}{1} \implies \cos\theta = \sqrt{1-x^2}$$



$$= -\frac{\sqrt{1-x^2}}{x} + C$$

17. $\displaystyle\int \frac{dx}{x\sqrt{a^2+x^2}}$

Method 1: Trigonometric Substitution

We can use the Pythagorean Identity of $1 + \tan^2\theta = \sec^2\theta$ to help simplify the integral:

$$\int \frac{dx}{x\sqrt{a^2+x^2}} = \int \frac{a\sec^2\theta}{a\tan\theta\sqrt{a^2+a^2\tan^2\theta}}d\theta$$

let $x = a\tan\theta$

$$\frac{dx}{d\theta} = a\sec^2\theta$$

$$dx = a\sec^2\theta \; d\theta$$

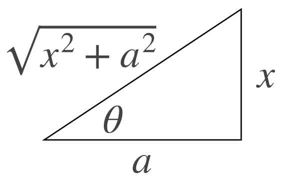
$$= \int \frac{\sec\theta}{a\tan\theta}\; d\theta$$

$$= \frac{1}{a}\int \operatorname{cosec}\theta \, d\theta$$

$$= -\frac{1}{a}\ln|\operatorname{cosec}\theta + \cot\theta| + C$$

By drawing a triangle, we can find $\operatorname{cosec}\theta$ and $\cot\theta$:

$$\frac{x}{a} = \tan\theta$$



$$= -\frac{1}{a}\ln\left|\frac{a}{x} + \frac{\sqrt{a^2+x^2}}{x}\right| + C$$

18. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}}$

Method 1: Trigonometric Substitution

We can use the Pythagorean Identity of $1 - \sin^2\theta = \cos^2\theta$ to help simplify the integral:

$$\int \frac{dx}{x\sqrt{a^2-x^2}} = \int \frac{a\cos\theta}{a\sin\theta\sqrt{a^2-a^2\sin^2\theta}}d\theta$$

let $x = a\sin\theta$

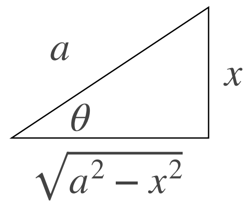
$$dx = a\cos\theta \; d\theta$$

$$= \frac{1}{a}\int \operatorname{cosec}\theta \, d\theta$$

$$= -\frac{1}{a}\ln|\operatorname{cosec}\theta + \cot\theta| + C$$

By drawing a triangle, we can find $\operatorname{cosec}\theta$ and $\cot\theta$:

$$\frac{x}{a} = \sin\theta$$



$$= -\frac{1}{a}\ln\left|\frac{a}{x} + \frac{\sqrt{a^2-x^2}}{x}\right| + C$$

19. $\int \frac{dx}{x\sqrt{x^2 - a^2}}$

Method 1: Trigonometric Substitution

We can use the Pythagorean Identity of $\sec^2\theta - 1 = \tan^2\theta$ to help simplify the integral:

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}}d\theta$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + C$$

let $x = a\sec\theta$

$dx = a\sec\theta\, d\theta$

Now $x = a\sec\theta \implies \theta = \sec^{-1}\frac{x}{a}$:

$$= \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

20. $\int \frac{x}{\sqrt{x}+1}\,dx$

Method 1: Algebraic Substitution

We can use the substitution $u = \sqrt{x}$ to simplify the integrand.

$$\int \frac{x}{\sqrt{x}+1}\,dx = 2\int \frac{u^3}{u+1}du$$

let $u = \sqrt{x}$

$dx = 2u\,du$

Now, $\frac{u^3}{u+1} = u^2 - u + 1 + \frac{-1}{u+1}$, by long division

$$\therefore \int \frac{x}{\sqrt{x}+1}\,dx = 2\int \left(u^2 - u + 1 - \frac{1}{u+1}\right)du$$

$$= 2\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right] + C$$

Now substituting back in:

$$= \frac{2}{3}x^{\frac{3}{2}} - x + 2x^{\frac{1}{2}} - 2\ln\left(\sqrt{x}+1\right) + C$$

Questions 21 - 30

21. $\displaystyle\int \frac{\cos^{-1} x}{\sqrt{1-x^2}}\, dx$

Method 1: Substitution

$$\int \frac{\cos^{-1} x}{\sqrt{1-x^2}}\, dx = -\int u\, du$$

$$= -\frac{u^2}{2} + C$$

let $u = \cos^{-1} x$

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Now substituting back in:

$$= -\frac{(\cos^{-1} x)^2}{2} + C$$

Method 2: Reverse Chain Rule

Observe that $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$. Hence:

$$\int \frac{\cos^{-1} x}{\sqrt{1-x^2}}\, dx = -\int \cos^{-1} x \left(-\frac{1}{\sqrt{1-x^2}}\right) dx$$

$$= -\frac{(\cos^{-1} x)^2}{2} + C$$

22. $\displaystyle\int \sqrt{\frac{x+1}{x-1}}\, dx$ (no longer on standard integrals)

Method 1: Rationalising the Numerator

We rationalise the numerator form a quadratic on the denominator:

$$\int \sqrt{\frac{x+1}{x-1}}\, dx = \int \sqrt{\frac{x+1}{x-1}} \times \frac{\sqrt{x+1}}{\sqrt{x+1}}\, dx$$

$$= \int \frac{x+1}{\sqrt{x^2-1}}\, dx$$

$$= \int \frac{x}{\sqrt{x^2-1}}\, dx + \int \frac{1}{\sqrt{x^2-1}}\, dx$$

The rightmost integral used to be a standard integral, but has been removed from the formula sheet. Instead, we multiply top and bottom by $x + \sqrt{x^2-1}$.

$$= \sqrt{x^2-1} + \int \frac{\left(\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)}{x+\sqrt{x^2-1}}\, dx$$

Observe that the numerator is the derivative of the denominator:

$$= \sqrt{x^2-1} + \ln\left|x + \sqrt{x^2-1}\right| + C$$

23. $\int \frac{dx}{x(\ln x)^3}$

Method 1: Reverse Chain Rule

Observing that $\frac{d}{dx}(\ln x) = \frac{1}{x}$:

$$\int \frac{dx}{x(\ln x)^3} = \int (\ln x)^{-3} \cdot \frac{1}{x}\, dx$$

$$= \frac{(\ln x)^{-2}}{-2} + C$$

$$= -\frac{1}{2(\ln x)^2} + C$$

Method 2: Substitution

Observing that $\frac{d}{dx}(\ln x) = \frac{1}{x}$, subbing $u = \ln x$ will be suitable:

$$\int \frac{dx}{x(\ln x)^3} = \int u^{-3}\, du$$

$$= -\frac{u^{-2}}{2} + C$$

let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dx}{x} = du$$

Now substituting back in:

$$= -\frac{1}{2(\ln x)^2} + C$$

24. $\int \sec^4 3x\, dx$

Method 1: Trigonometric Identity and Reverse Chain Rule

We can use the Pythagorean Identity that $\sec^2\theta = 1 + \tan^2\theta$ to make use of the reverse chain rule:

$$\int \sec^4 3x\, dx = \int \sec^2 3x(1 + \tan^2 3x)\, dx$$

$$= \int \sec^2 3x\, dx + \int \sec^2 3x \tan^2 3x\, dx$$

The leftmost integral is now a standard result, so we now manipulate the remaining integral. Noting that $\frac{d}{dx}(\tan 3x) = 3\sec^2 3x$, we can apply the reverse chain rule:

$$= \int \sec^2 3x\, dx + \frac{1}{3}\int (\tan 3x)^2(3\sec^2 3x)\, dx$$

$$= \frac{\tan 3x}{3} + \frac{(\tan 3x)^3}{9} + C$$

25. $\displaystyle\int \frac{dx}{x^2(1-x)}$

Method 1: Partial Fractions

Observing that the denominator consists of several factors, we can use partial fraction decomposition.

Let $\dfrac{1}{x^2(1-x)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{1-x}$.

$\therefore Ax(1-x) + B(1-x) + Cx^2 = 1$

letting $x = 1$, $C = 1$

letting $x = 0$, $B = 1$

letting $x = -1$, $-2A + 2B + C = 1 \implies A = 1$

$$\therefore \int \frac{dx}{x^2(1-x)} = \int \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{1-x}\right) dx$$

$$= \ln|x| - \frac{1}{x} - \ln|1-x| + C$$

$$= \ln\left|\frac{x}{1-x}\right| - \frac{1}{x} + C$$

26. $\displaystyle\int \frac{dx}{x^2(1+x^2)}$

Method 1: Trigonometric Substitution

We can use the Pythagorean Identity $1 + \tan^2\theta = \sec^2\theta$ to help simplify the integrand.

$$\int \frac{dx}{x^2(1+x^2)} = \int \frac{\sec^2\theta}{\tan^2\theta(1+\tan^2\theta)} d\theta$$

$$= \int \cot^2\theta \ d\theta$$

$$= \int (\operatorname{cosec}^2\theta - 1)\, d\theta$$

$$= \int \operatorname{cosec}^2\theta \ d\theta - \int d\theta$$

$$= -\cot\theta - \theta + C$$

let $x = \tan\theta$

$\dfrac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta \ d\theta$

Substituting back in:

$$= -\frac{1}{x} - \tan^{-1}x + C$$

27. $\int \frac{dx}{(1+x^2)^2}$

Method 1: Trigonometric Substitution

We can use the Pythagorean Identity $1 + \tan^2\theta = \sec^2\theta$ to help simplify the integrand.

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\, d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{2}(\theta + \cos\theta\sin\theta) + C$$

let $x = \tan\theta$

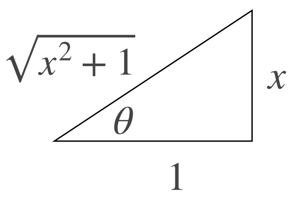
$$\frac{dx}{d\theta} = \sec^2\theta$$

$$dx = \sec^2\theta\ d\theta$$

By drawing a triangle, we can find $\cos\theta$ and $\sin\theta$:

$$\frac{x}{1} = \tan\theta$$

$$\implies \sin\theta = \frac{x}{\sqrt{x^2+1}},\ \cos\theta = \frac{1}{\sqrt{x^2+1}}$$



$$= \frac{\tan^{-1}x}{2} + \frac{x}{2(x^2+1)} + C$$

28. $\int \tan^3 x\, dx$

Method 1: Trigonometric Identity

We can use the Pythagorean Identity $1 + \tan^2\theta = \sec^2\theta$ to help simplify the integrand.

$$\int \tan^3 x\, dx = \int \tan x(\sec^2 x - 1)\ dx$$

$$= \int \tan x \sec^2 x\ dx - \int \tan x\ dx$$

$$= \int \tan x \sec^2 x\ dx - \int \frac{\sin x}{\cos x}\, dx$$

$$= \int \tan x \sec^2 x\ dx + \int \frac{(-\sin x)}{\cos x}\, dx$$

$= \frac{\tan^2 x}{2} + \ln|\cos x| + C$, by the reverse chain rule.

29. $\displaystyle\int \frac{dx}{5 + 3\cos x}$

Method 1: t-Substitution

There are no obvious/nice manipulations such substitutions, so we will apply the t-substitution to convert the trigonometric terms into algebraic terms:

$$\int \frac{dx}{5+3\cos x} = \int \frac{1}{5 + \frac{3(1-t^2)}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{1+t^2}{5+5t^2+3-3t^2} \cdot \frac{2}{1+t^2}\,dt$$

$$= 2\int \frac{dt}{2t^2+8}$$

$$= \int \frac{dt}{t^2+4}$$

$$= \frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

let $t = \tan\dfrac{x}{2}$

$$dx = \frac{2}{1+t^2}\,dt$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Subbing back in:

$$= \frac{1}{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right) + C$$

30. $\displaystyle\int \frac{dx}{3 + 5\cos x}$

Method 1: t-Substitution

There are no obvious/nice manipulations such substitutions, so we will apply the t-substitution to convert the trigonometric terms into algebraic terms:

$$\int \frac{dx}{3+5\cos x} = \int \frac{1}{3 + \frac{5-5t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{1+t^2}{3+3t^2+5-5t^2} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{8-2t^2}\,dt$$

$$= \int \frac{1}{4-t^2}\,dt$$

let $t = \tan\dfrac{x}{2}$

$$dx = \frac{2}{1+t^2}\,dt$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

From here we need to decompose the integrand into partial fractions. Let $\dfrac{1}{4-t^2} = \dfrac{A}{2+t} + \dfrac{B}{2-t}$.

$\therefore A(2-t) + B(2+t) = 1$

letting $t = 2$, $4B = 1$, $B = \dfrac{1}{4}$

letting $t = -2$, $4A = 1$, $A = \dfrac{1}{4}$

$$= \frac{1}{4}\int\left(\frac{1}{2+t}+\frac{1}{2-t}\right)dt$$

$$= \frac{1}{4}\left(\ln|2+t| - \ln|2-t|\right) + C$$

$$= \frac{1}{4}\ln\left|\frac{2+t}{2-t}\right| + C$$

Now subbing back in:

$$= \frac{1}{4}\ln\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right| + C$$

Questions 31 - 40

31. $\displaystyle\int \frac{\sin x}{5+3\cos x}\,dx$

Method 1: Reverse Chain Rule

Observing that $\dfrac{d}{dx}(5+3\cos x) = -3\sin x$:

$$\int \frac{\sin x}{5+3\cos x}\,dx = -\frac{1}{3}\int \frac{-3\sin x}{5+3\cos x}\,dx$$

$$= -\frac{1}{3}\ln|5+3\cos x| + C, \text{ by the reverse chain rule.}$$

Method 2: Substitution

$u=\cos x,\ u=3\cos x,\ u=5+3\cos x$ are all effective substitutions.

However, $u=5+3\cos x$ will simplify the integrand the most:

$$\int \frac{\sin x}{5+3\cos x}\,dx = \int \frac{1}{u}\left(-\frac{1}{3}du\right)$$

$$= -\frac{1}{3}\ln|u| + C$$

$$= -\frac{1}{3}\ln|5+3\cos x| + C$$

let $u = 5+3\cos x$

$$\frac{du}{dx} = -3\sin x$$

$$\sin x\,dx = -\frac{1}{3}du$$

32. $\displaystyle\int \frac{dx}{1+\cos^2 x}$

Method 1: t-Substitution

There are no nice substitutions, so we apply the t-substitution, but before we do so, we need to remove any powers in order to simplify the integrand:

Applying the double result $\cos^2 x = \dfrac{1}{2}(1+\cos 2x)$:

$$\int \frac{dx}{1+\cos^2 x} = \int \frac{dx}{1+\frac{\cos 2x}{2}+\frac{1}{2}}$$

$$= \int \frac{2}{\cos 2x + 3}\,dx$$

$$= \int \frac{2}{\frac{1-t^2}{1+t^2}+3} \times \frac{dt}{1+t^2}$$

let $t = \tan x$

$$dx = \frac{1}{1+t^2}\,dt$$

$$\cos 2x = \frac{1-t^2}{1+t^2}$$

$$= \int \frac{2}{\frac{1-t^2+3+3t^2}{1+t^2}} \times \frac{dt}{1+t^2}$$

$$= \int \frac{2}{2t^2+4}\ dt$$

$$= \int \frac{1}{t^2+2}\ dt$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

Method 2: Algebraic Manipulation

We divide the top and bottom by $\cos^2 x$:

$$\int \frac{dx}{1+\cos^2 x} = \int \frac{\sec^2 x}{\sec^2 x + 1}\, dx$$

Applying the Pythagorean identity $1 + \tan^2\theta = \sec^2\theta$:

$$= \int \frac{\sec^2 x}{2+\tan^2 x}\, dx$$

Noting that $\frac{d}{dx}(\tan x) = \sec^2 x$, we can apply the reverse chain rule:

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

33. $\displaystyle\int \frac{dx}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$

Method 1: Double Angle Identity

We observe that we can simplify the denominator using the double angle identity for cosine: $\cos 2\theta = \cos^2\theta - \sin^2\theta$:

$$\int \frac{dx}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} = \int \frac{dx}{\cos x}$$

$$= \int \sec x\, dx$$

This is now an integral we have encountered before, and the trick is to multiply the top and bottom by $\sec x + \tan x$:

$$= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\, dx$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\, dx$$

Observe that $\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x$, thus we apply the reverse chain rule:

$$= \ln|\sec x + \tan x| + C$$

34. $\int x^2 \sin x \, dx$

Method 1: Integration by Parts

The integrand is a product of two factors, so we integrate by parts.

Letting:

u	x^2	$-\cos x$	v
u'	$2x$	$\sin x$	v'

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2\int x \cos x \; dx$$

The next integral we have to deal with is another product of two factors, so we integrate by parts again.

Letting:

u	x	$\sin x$	v
u'	1	$\cos x$	v'

$$\int x \cos x \; dx = x \sin x - \int \sin x \; dx$$

$$= x \sin x + \cos x + C$$

Hence, $\int x^2 \sin x \, dx = -x^2 \cos x + 2(x \sin x + \cos x) + C$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C$$

35. $\int \dfrac{x^2}{(x-1)(x-2)(x-3)} \, dx$

Method 1: Partial Fractions

Observing that the degree of the denominator is greater than the numerator, and that denominator has three linear factors, we shall use partial fraction decomposition.

Let $\dfrac{x^2}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$

$$\therefore A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) = x^2$$

letting $x = 2$: $0 - B + 0 = 4 \implies B = -4$

letting $x = 1$: $2A + 0 + 0 = 1 \implies A = \dfrac{1}{2}$

letting $x = 3$: $0 + 0 + 2C = 9 \implies C = \dfrac{9}{2}$

Hence, $\dfrac{x^2}{(x-1)(x-2)(x-3)} = \dfrac{1}{2(x-1)} - \dfrac{4}{x-2} + \dfrac{9}{2(x-3)}$.

Thus: $\int \dfrac{x^2}{(x-1)(x-2)(x-3)} \, dx$

$$= \int \left(\frac{1}{2(x-1)} - \frac{4}{x-2} + \frac{9}{2(x-3)} \right) dx$$

$$= \frac{1}{2} \ln|x-1| - 4\ln|x-2| + \frac{9}{2} \ln|x-3| + C$$

36. $\int \frac{e^x}{e^x - 1}\, dx$

Method 1: Reverse Chain Rule

Observing that $\frac{d}{dx}(e^x - 1) = e^x$

$\int \frac{e^x}{e^x - 1}\, dx = \ln|e^x - 1| + C$, by the reverse chain rule.

Method 2: Substitution

Observe that $\frac{d}{dx}(e^x - 1) = e^x$, so $u = e^x - 1$ will be a suitable substitution:

$$\int \frac{e^x}{e^x - 1}\, dx = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|e^x - 1| + C$$

let $u = e^x - 1$

$$\frac{du}{dx} = e^x$$

$$e^x\, dx = du$$

37. $\int \frac{dx}{3\sin^2 x + 5\cos^2 x}$

Method 1: t-Substitution

There are no “nice substitutions”, so we will use the t-substitution. Before we do so, we need to remove all powers from the integrand. To do so, we need to manipulate before we apply the double angle identity.

$$\int \frac{dx}{3\sin^2 x + 5\cos^2 x} = \int \frac{dx}{3(1 - \cos^2 x) + 5\cos^2 x}$$

$$= \int \frac{dx}{3 + 2\cos^2 x}$$

$$= \int \frac{dx}{3 + 2\left(\frac{\cos 2x + 1}{2}\right)}$$

$$= \int \frac{dx}{\cos 2x + 4}$$

We are now able to carry out the t-substitution:

$$= \int \frac{1}{\frac{1 - t^2}{1 + t^2} + 4} \times \frac{dt}{1 + t^2}$$

$$= \int \frac{1}{\frac{1 - t^2 + 4 + 4t^2}{1 + t^2}} \times \frac{dt}{1 + t^2}$$

$$= \int \frac{1 + t^2}{3t^2 + 5} \times \frac{dt}{1 + t^2}$$

let $t = \tan x$

$$dx = \frac{1}{1 + t^2}\, dt$$

$$\cos 2x = \frac{1 - t^2}{1 + t^2}$$

$$= \int \frac{1}{3t^2 + 5}\ dt$$

$$= \frac{1}{3} \int \frac{1}{t^2 + \frac{5}{3}}\ dt$$

$$= \frac{1}{3} \left(\frac{\sqrt{3}}{\sqrt{5}} \right) \tan^{-1} \sqrt{\frac{3}{5}} t + C$$

Substituting back in:

$$= \frac{\sqrt{15}}{15} \tan^{-1} \left(\sqrt{\frac{3}{5}} \tan x \right) + C$$

Method 2: Algebraic Manipulation

We divide the top and bottom by $\cos^2 x$:

$$\int \frac{dx}{3 \sin^2 x + 5 \cos^2 x} = \int \frac{\sec^2 x}{3 \tan^2 x + 5}\, dx$$

Noting that $\frac{d}{dx}\left(\sqrt{3} \tan x\right) = \sqrt{3} \sec^2 x$, we manipulate such that we can apply the reverse chain rule:

$$= \frac{1}{\sqrt{3}} \int \frac{\sqrt{3} \sec^2 x}{\left(\sqrt{3} \tan x\right)^2 + 5}\, dx$$

$$= \frac{1}{\sqrt{15}} \tan^{-1} \left(\sqrt{\frac{3}{5}} \tan x \right) + C$$

38. $\int x^3 e^{5x^4 - 7}\, dx$

Method 1: Reverse Chain Rule

Observe that $\frac{d}{dx}(5x^4 - 7) = 20x^3$, thus we manipulate such that we can apply the reverse chain rule:

$$\int x^3 e^{5x^4-7}\, dx = \frac{1}{20} \int (20x^3) e^{5x^4-7}\, dx$$

$$= \frac{1}{20} e^{5x^4-7} + C$$

Method 2: Algebraic Substitution

Observing that $\frac{d}{dx}(5x^4 - 7) = 20x^3$, $u = 5x^4 - 7$ will be a suitable substitution:

$$\int x^3 e^{5x^4-7}\, dx = \int e^u \left(\frac{1}{20}\, du \right)$$

$$= \frac{1}{20} e^u + C$$

let $u = 5x^4 - 7$

$$\frac{du}{dx} = 20x^3$$

$$x^3\, dx = \frac{1}{20}\, du$$

Substituting back in:

$$= \frac{1}{20} e^{5x^4-7} + C$$

39. $\int x^5 \ln x \, dx$

Method 1: Integration by Parts

Our integrand is a product of two functions, hence we integrate by parts:

Letting:

u	$\ln x$	$\frac{x^6}{6}$	v
u'	$\frac{1}{x}$	x^5	v'

$$\int x^5 \ln x \, dx = \frac{x^6}{6} \ln x - \frac{1}{6} \int x^5 \, dx$$

$$= \frac{x^6}{6} \ln x - \frac{1}{6} \cdot \frac{x^6}{6} + C$$

$$= \frac{x^6}{6} \ln x - \frac{x^6}{36} + C$$

40. $\int \frac{3x+2}{x(x+1)^3} \, dx$

Method 1: Partial Fractions

Observing that the degree of the denominator is greater than the numerator can contains linear factors, we decompose the integrand into partial fractions.

Let $\frac{3x+2}{x(x+1)^3} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$

$\therefore A(x+1)^3 + Bx(x+1)^2 + Cx(x+1) + Dx = 3x + 2$

letting $x = 0$: $A + 0 + 0 + 0 = 2 \implies A = 2.$

letting $x = -1$: $0 + 0 + 0 - D = -1 \implies D = 1.$

letting $x = 1$: $8A + 4B + 2C + D = 5$

Substituting our previous values: $4B + 2C = -12$

letting $x = 2$: $27A + 18B + 6C + 2D = 8$

Substituting our previous values: $18B + 6C = -48$

We then solve:

$2B + C = -6$ and $3B + C = -8$ simultaneously.

Subtracting these equations:

$(3B + C) - (2B + C) = -8 - (-6) \implies B = -2$

Substituting this back in: $C = -2.$

Hence: $\dfrac{3x+2}{x(x+1)^3} = \dfrac{2}{x} - \dfrac{2}{x+1} - \dfrac{2}{(x+1)^2} + \dfrac{1}{(x+1)^3}$

Thus:

$$\int \frac{3x+2}{x(x+1)^3}\,dx = \int \left(\frac{2}{x} - \frac{2}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}\right)\,dx$$

$$= 2\ln|x| - 2\ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C$$

Questions 41 - 50

41. $\int \ln(x^3)\,dx$

Method 1: Integration by Parts

We can simplify the integral using the log law: $\log_a(x^b) = b\log_a(x)$:

$$\int \ln(x^3)\,dx = 3\int \ln x\,dx$$

We then compute the integral using integration by parts.

Letting:

u	$\ln x$	x	v
u'	$\frac{1}{x}$	1	v'

$$= 3\left(x\ln x - \int x\cdot\frac{1}{x}\,dx\right)$$

$$= 3\left(x\ln x - x\right) + C$$

$$= 3x\ln x - 3x + C$$

42. $\displaystyle\int \frac{dx}{e^x + e^{-x}}$

Method 1: Algebraic Manipulation and Reverse Chain Rule

We multiply the top and bottom by e^x:

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{(e^x)^2 + 1}\,dx$$

Observing $\frac{d}{dx}(e^x) = e^x$, this integral is now in a standard result:

$$= \tan^{-1}(e^x) + C$$

Method 2: Algebraic Manipulation and Substitution

After the manipulation, instead of applying the reverse chain rule, we could instead apply the substitution $u = e^x$.

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{(e^x)^2 + 1}\,dx$$

$$= \int \frac{du}{u^2 + 1}$$

$$= \tan^{-1} u + C$$

let $u = e^x$

$$\frac{du}{dx} = e^x$$

$$e^x\,dx = du$$

Substituting back in:

$$= \tan^{-1}(e^x) + c$$

43. $\int (5x^3+7x-1)^{\frac{3}{2}} \cdot (15x^2+7)\,dx$

Method 1: Reverse Chain Rule

Observe that $\frac{d}{dx}(5x^3+7x-1) = 15x^2+7$, hence we can apply the reverse chain rule directly.

$$\int (5x^3+7x-1)^{\frac{3}{2}} \cdot (15x^2+7)\,dx = \frac{(5x^3+7x-1)^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= \frac{2}{5}(5x^3+7x-1)^{\frac{5}{2}} + C$$

Method 2: Algebraic Substitution

Observing that $\frac{d}{dx}(5x^3+7x-1) = 15x^2+7$, we let $u = 5x^3+7x-1$ as our substitution.

$$\int (5x^3+7x-1)^{\frac{3}{2}} \cdot (15x^2+7)\,dx = \int u^{\frac{3}{2}}du$$

$$= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= \frac{2}{5}u^{\frac{5}{2}} + C$$

let $u = 5x^3+7x-1$

$\frac{du}{dx} = 15x^2+7$

$du = (15x^2+7)\,dx$

Substituting back in:

$$= \frac{2}{5}(5x^3+7x-1)^{\frac{5}{2}} + C$$

44. $\int \frac{dx}{(x^2+1)(x^2+4)}$

Method 1: Partial Fractions (faster method)

The denominator consists of two quadratic factors, so we decompose the integrand into partial fractions. While the method of substitution appears to be quite slow, we can use complex numbers to speed up the process.

Let $\frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$

$\therefore (Ax+B)(x^2+4) + (Cx+D)(x^2+1) = 1$

letting $x = i$: $(Ai+B)(-1+4)+0 = 1 \implies Ai+B = \frac{1}{3}$

Equating real and imaginary parts: $B = \frac{1}{3}$, $A = 0$

letting $x = 2i$: $0 + (2Ci+D)(-4+1) = 1 \implies 2Ci+D = -\frac{1}{3}$

Equating real and imaginary parts: $D = -\frac{1}{3}$, $C = 0$

Hence: $\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)}$.

Thus $\int \frac{dx}{(x^2+1)(x^2+4)} = \frac{1}{3}\int \frac{dx}{x^2+1} - \frac{1}{3}\int \frac{dx}{x^2+4}$

$$= \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Method 2: Partial Fractions

We decompose the integrand into partial fractions:

Let $\dfrac{1}{(x^2+1)(x^2+4)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+4}$

$\therefore (Ax+B)(x^2+4) + (Cx+D)(x^2+1) = 1$

For non-4U students, we can still solve for the constants by either equating coefficients/substituting real values. Here we choose the method of substitution.

letting $x = 0$: $4B + D = 1$ (1)

letting $x = 1$: $5A + 5B + 2C + 2D = 1$ (2)

letting $x = -1$: $-5A + 5B - 2C + 2D = 1$ (3)

letting $x = 2$: $16A + 8B + 10C + 5D = 1$ (4)

We then solve simultaneously:

Adding equations (2) and (3): $10B + 4D = 2$ (5)

Performing (5) $-4 \times$ (1): $-6B = -2 \implies B = \dfrac{1}{3}$.

Subbing $B = \dfrac{1}{3}$ back into (1): $\dfrac{4}{3} + D = 1 \implies D = -\dfrac{1}{3}$.

If we then sub these values of B and D into (2) and (4):

$5A + 2C = 0$ and $16A + 10C = 0$

Solving these two new equations, we get that $A = C = 0$.

Hence: $\dfrac{1}{(x^2+1)(x^2+4)} = \dfrac{1}{3(x^2+1)} - \dfrac{1}{3(x^2+4)}$.

Thus $\displaystyle\int \frac{dx}{(x^2+1)(x^2+4)} = \frac{1}{3}\int \frac{dx}{x^2+1} - \frac{1}{3}\int \frac{dx}{x^2+4}$

$$= \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right) + C$$

45. $\int (x^2 + x + 1)^{-1}\, dx$

Method 1: Completing the Square

Observe that we have an irreducible quadratic in the denominator, so we complete the square and then apply our standard integral result.

$$\int (x^2 + x + 1)^{-1}\, dx = \int \frac{dx}{x^2 + x + 1}$$

$$= \int \frac{dx}{x + x + \frac{1}{2}^2 + 1 - \frac{1}{2}^2}$$

$$= \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + C$$

46. $\int e^x \sin 2x\, dx$

Method 1: Integration by Parts

Our integrand is a product of two functions, so we apply integration by parts.

Letting:

u	$\sin(2x)$	e^x	v
u'	$2\cos(2x)$	e^x	v'

$$\int e^x \sin 2x\, dx = e^x \sin(2x) - 2\int \cos(2x) e^x\ dx$$

Our new integral is another product of two functions, so we apply integration by parts again:

Letting:

u	$\cos(2x)$	e^x	v
u'	$-2\sin(2x)$	e^x	v'

$$= e^x \sin(2x) - 2\left[e^x \cos(2x) + 2\int e^x \sin(2x)\right]\ dx$$

$$= e^x \sin(2x) - 2e^x \cos(2x) - 4\int e^x \sin(2x)\, dx$$

Notice that we have reformed our original integral. Rearranging:

$$5\int e^x \sin(2x) = e^x \sin(2x) - 2e^x \cos(2x) + C_1$$

Thus, $\int e^x \sin 2x = \frac{e^x}{5}\left(\sin 2x - 2\cos 2x\right) + C_2$

47. $\int (x^2+x-1)^{-1}\,dx$

Method 1: Partial Fractions

After some algebraic manipulations, the quadratic in our denominator is a product of two linear factors, so we decompose the integrand into partial fractions.

$$\int (x^2+x-1)^{-1}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2-\frac{5}{4}}dx$$

$$= \int \frac{dx}{\left(x+\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(x+\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}$$

We now decompose into partial fractions.

Let $\dfrac{1}{x^2+x-1} = \dfrac{A}{x+\frac{1}{2}+\frac{\sqrt{5}}{2}} + \dfrac{B}{x+\frac{1}{2}-\frac{\sqrt{5}}{2}}$

$$\therefore A\left(x+\frac{1}{2}-\frac{\sqrt{5}}{2}\right) + B\left(x+\frac{1}{2}+\frac{\sqrt{5}}{2}\right) = 1$$

letting $x = -\dfrac{1}{2}+\dfrac{\sqrt{5}}{2}$: $\sqrt{5}B = 1 \implies B = \dfrac{1}{\sqrt{5}}$

letting $x = -\dfrac{1}{2}-\dfrac{\sqrt{5}}{2}$: $-\sqrt{5}A = 1 \implies A = -\dfrac{1}{\sqrt{5}}$

Hence: $\dfrac{1}{x^2+x-1} = -\dfrac{\frac{1}{\sqrt{5}}}{x+\frac{1}{2}+\frac{\sqrt{5}}{2}} + \dfrac{\frac{1}{\sqrt{5}}}{x+\frac{1}{2}-\frac{\sqrt{5}}{2}}$.

Thus:

$$\int (x^2+x-1)^{-1}\,dx = -\frac{1}{\sqrt{5}}\int \frac{dx}{x+\frac{1}{2}+\frac{\sqrt{5}}{2}} + \frac{1}{\sqrt{5}}\int \frac{dx}{x+\frac{1}{2}-\frac{\sqrt{5}}{2}}$$

$$= -\frac{1}{\sqrt{5}}\ln\left|x+\frac{1}{2}+\frac{\sqrt{5}}{2}\right| + \frac{1}{\sqrt{5}}\ln\left|x+\frac{1}{2}-\frac{\sqrt{5}}{2}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{x+\frac{1}{2}-\frac{\sqrt{5}}{2}}{x+\frac{1}{2}+\frac{\sqrt{5}}{2}}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2x+1-\sqrt{5}}{2x+1+\sqrt{5}}\right| + C$$

48. $\int (x^2 - x)^{-\frac{1}{2}}\, dx$ (no longer on standard integrals)

Method 1: Completing the Square

We complete the square for the quadratic in the denominator

$$\int (x^2 - x)^{-\frac{1}{2}}\, dx = \int \frac{1}{\sqrt{x^2 - x}}\, dx$$

$$= \int \frac{1}{\sqrt{(x-1)^2 - \frac{1}{4}}}\, dx$$

This integral is now in the form of $\int \frac{dx}{\sqrt{x^2 - a^2}}$, which was previously a standard integral but is now removed with the addition of the formula sheet.

$$= \ln\left| x - \frac{1}{2} + \sqrt{x^2 - x} \right| + C_1$$

$$= \ln\left| 2x - 1 + 2\sqrt{x^2 - x} \right| + C_2$$

49. $\int \frac{1 - 2x}{3 + x}\, dx$

Method 1: Algebraic Manipulation

Our goal is to manipulate the numerator into a multiple of the denominator.

$$1 - 2x = 1 - 2(x + 3 - 3)$$

$$= 7 - 2(x + 3)$$

Hence: $\int \frac{1 - 2x}{3 + x}\, dx = \int \frac{7 - 2(x+3)}{3 + x}\, dx$

$$= \int \frac{7}{3 + x}\, dx - 2\int dx$$

$$= 7\ln|x + 3| - 2x + C$$

50. $\int x^3(4+x^2)^{-\frac{1}{2}}\ dx$

Method 1: Algebraic Substitution #1

Although a slightly less obvious substitution, observe that $\frac{d}{dx}(4+x^2) = 2x$, and that $x^3 = x \times x^2$. Hence we choose $u = 4 + x^2$ as our substitution.

$$\begin{aligned}\int x^3(4+x^2)^{-\frac{1}{2}}\,dx &= \int (u-4)u^{-\frac{1}{2}}\left(\frac{1}{2}\,du\right)\\ &= \frac{1}{2}\int\left(u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\right)\,du\\ &= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}}\right) + C\end{aligned}$$

let $u = x^2 + 4$

$$\iff x^2 = u - 4$$

$$\frac{du}{dx} = 2x$$

$$x\,dx = \frac{1}{2}du$$

Substituting back in:

$$\begin{aligned}&= \frac{1}{3}(x^2+4)^{\frac{3}{2}} - 4(x^2+4)^{\frac{1}{2}} + C\\ &= \frac{\sqrt{x^2+4}}{3}\left(x^2+4-12\right) + C\\ &= \frac{1}{3}(x^2-8)\sqrt{x^2+4} + C\end{aligned}$$

Method 2: Algebraic Substitution #2

Another suitable substitution is $u = \sqrt{x^2+4}$. Notice that this removes the “most difficult” part of the integrand, which is the radical in the denominator.

$$\begin{aligned}\int x^3(4+x^2)^{-\frac{1}{2}}\,dx &= \int \frac{x^3}{\sqrt{4+x^2}}\,dx\\ &= \int \frac{x}{\sqrt{4+x^2}}\cdot x^2\,dx\\ &= \int (u^2-4)\,du\\ &= \frac{u^3}{3} - 4u + C\\ &= \frac{1}{3}u(u^2-12) + C\end{aligned}$$

let $u = \sqrt{x^2+4}$

$$\iff x^2 = u^2 - 4$$

$$\frac{du}{dx} = \frac{x}{\sqrt{x^2+4}}$$

$$du = \frac{x}{\sqrt{x^2+4}}\,dx$$

Substituting back in:

$$\begin{aligned}&= \frac{1}{3}\sqrt{x^2+4}(x^2+4-12) + C\\ &= \frac{1}{3}(x^2-8)\sqrt{x^2+4} + C\end{aligned}$$

Method 3: Trigonometric Substitution

Another potential substitution is a trigonometric substitution, utilising the identity $1+\tan^2\theta = \sec^2\theta$. Although not the most efficient, it is still a perfectly valid method.

$$\int x^3(4+x^2)^{-\frac{1}{2}}\,dx = \int \frac{x^3}{\sqrt{4+x^2}}\,dx$$

let $x = 2\tan\theta$

$$\frac{dx}{d\theta} = 2\sec^2\theta$$

$$dx = 2\sec^2\theta\,d\theta$$

$$= \int \frac{8\tan^3\theta}{\sqrt{4+4\tan^2\theta}}(2\sec^2\theta)\,d\theta$$

$$= \int \frac{16\tan^3\theta\sec^2\theta}{2\sec\theta}\,d\theta$$

$$= 8\int \tan^3\theta\sec\theta\,d\theta$$

$$= 8\int \frac{\sin^3\theta}{\cos^4\theta}\,d\theta$$

$$= 8\int \frac{\sin\theta(1-\cos^2\theta)}{\cos^4\theta}\,d\theta$$

$$= 8\int \frac{\sin\theta}{\cos^4\theta}\,d\theta - 8\int \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

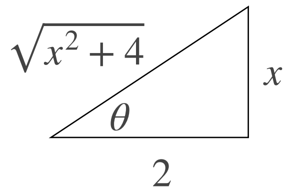
$$= -\frac{8}{3}(\cos\theta)^{-3} + 8(\cos\theta)^{-1} + C$$

$$= \frac{8}{\cos\theta} - \frac{8}{3\cos^3\theta} + C$$

We then use a triangle to find the value of $\cos\theta$ to substitute back in:

$$\tan\theta = \frac{x}{2}$$

$$\implies \cos\theta = \frac{2}{\sqrt{x^2+4}}$$



Hence:

$$\int x^3(4+x^2)^{-\frac{1}{2}}\,dx = 8\cdot\frac{\sqrt{x^2+4}}{2} - \frac{8}{3}\cdot\frac{(x^2+4)\sqrt{x^2+4}}{8} + C$$

$$= 4\sqrt{x^2+4} - \frac{1}{3}(x^2+4)\sqrt{x^2+4} + C$$

$$= \frac{1}{3}(x^2-8)\sqrt{x^2+4} + C$$

Questions 51 - 60

51. $\displaystyle\int \frac{\sin 2x}{3\cos^2 x + 4\sin^2 x}\,dx$

Method 1: Manipulation and Reverse Chain Rule

Observe that $\frac{d}{dx}(\sin^2 x) = 2\sin x\cos x = \sin 2x$, and that we can change $\cos^2 x = 1 - \sin^2 x$.

$$\int \frac{\sin 2x}{3\cos^2 x + 4\sin^2 x}\,dx = \int \frac{2\sin x\cos x}{3(1-\sin^2 x) + 4\sin^2 x}\,dx$$

$$= \int \frac{2\sin x\cos x}{3+\sin^2 x}\,dx$$

$= \ln|3+\sin^2 x| + C$, by the reverse chain rule.

Now as $3 + \sin^2 x > 0$ for all real x, we can remove the absolute values:

$$= \ln(3+\sin^2 x) + C$$

Method 2: Manipulation and Substitution

Observing that $\frac{d}{dx}(\sin^2 x) = 2\sin x\cos x = \sin 2x$, we can apply the same manipulation as in Method 1, but instead apply the substitution $u = \sin^2 x$.

$$\int \frac{\sin 2x}{3\cos^2 x + 4\sin^2 x}\,dx = \int \frac{2\sin x\cos x}{3(1-\sin^2 x) + 4\sin^2 x}\,dx$$

$$= \int \frac{2\sin x\cos x}{3+\sin^2 x}\,dx$$

$$= \int \frac{du}{3+u}$$

$$= \ln|3+u| + C$$

Substituting back in:

$$= \ln|3+\sin^2 x| + C$$

Now as $3 + \sin^2 x > 0$ for all real x, we can remove the absolute values:

$$= \ln(3+\sin^2 x) + C$$

52. $\displaystyle\int \frac{x^2}{1-x^4}\,dx$

Method 1: Partial Fractions

We observe that the denominator can be factorised into linear and quadratic factors, which we can then decompose into partial fractions.

$$\int \frac{x^2}{1-x^4}\,dx = \int \frac{x^2}{(1-x^2)(1+x^2)}\,dx$$

$$= \int \frac{x^2}{(1-x)(1+x)(1+x^2)}\,dx$$

Let $\dfrac{x^2}{(1-x^2)(1+x^2)} = \dfrac{A}{1-x} + \dfrac{B}{1+x} + \dfrac{Cx+D}{x^2+1}$

$\therefore x^2 = A(1+x)(x^2+1) + B(1-x)(x^2+1) + (Cx+D)(1-x^2)$

letting $x = i$: $-1 = 0 + 0 + 2Ci + 2D$

Equating real and imaginary parts: $D = -\dfrac{1}{2}, C = 0.$

letting $x = 1$: $1 = 4A \implies A = \dfrac{1}{4}$

letting $x = -1$: $1 = 0 + 4B \implies B = \dfrac{1}{4}$

Hence: $\dfrac{x^2}{(1-x^2)(1+x^2)} = \dfrac{\frac{1}{4}}{1-x} + \dfrac{\frac{1}{4}}{1+x} - \dfrac{\frac{1}{2}}{x^2+1}.$

Thus:

$$\int \frac{x^2}{1-x^4}\,dx = \frac{1}{4}\int \frac{dx}{1-x} + \frac{1}{4}\int \frac{dx}{1+x} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{4}\ln|1+x| - \frac{1}{4}\ln|1-x| - \frac{1}{2}\tan^{-1}x + C$$

53. $\displaystyle\int \frac{dx}{\sin x \cos x}$

Method 1: Double Angle Formula

We use the double angle for sine: $\sin 2x = 2\sin x \cos x$ to simplify the integral:

$$\int \frac{dx}{\sin x \cos x} = \int \frac{2}{2\sin x \cos x}\,dx$$

$$= \int \frac{2\ dx}{\sin 2x}$$

$$= 2\int \operatorname{cosec} x\,dx$$

The integral of $\operatorname{cosec} x$ is another difficult one, but the manipulation we use is to multiply the top and bottom by $\operatorname{cosec} x + \cot x$.

$$= 2\int \frac{(\operatorname{cosec} x + \cot x)\operatorname{cosec} x}{\operatorname{cosec} x + \cot x}\,dx$$

Observe that $\dfrac{d}{dx}(\operatorname{cosec} x + \cot x) = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$, so we can apply the reverse chain rule.

$$= -2\ln|\operatorname{cosec} x + \cot x| + C$$

Method 2: Algebraic Manipulation

If we divide the top and bottom by $\cos^2 x$:

$$\int \frac{dx}{\sin x \cos x} = \int \frac{\sec^2 x}{\tan x}\,dx$$

Note that $\dfrac{d}{dx}(\tan x) = \sec^2 x$, so we apply the reverse chain rule:

$$= \ln|\tan x| + C$$

54. $\int \ln\sqrt{x-1}\,dx$

Method 1: Integration by Parts

We make use of the log law $\log_a(x^b) = b\log_a(x)$, and then much like how we evaluate $\int \ln x\,dx$, we integrate by parts.

$$\int \ln\sqrt{x-1}\,dx = \frac{1}{2}\int \ln(x-1)\,dx$$

Letting:

u	$\ln(x-1)$	x	v
u'	$\frac{1}{x-1}$	1	v'

$$= \frac{1}{2}\left(x\ln(x-1) - \int \frac{x}{x-1}\,dx\right)$$

To evaluate the new integral, we will use an algebraic manipulation:

$$= \frac{1}{2}x\ln(x-1) - \frac{1}{2}\int \frac{x-1+1}{x-1}\,dx$$

$$= \frac{1}{2}x\ln(x-1) - \frac{1}{2}\int \left(1 + \frac{1}{x-1}\right)\,dx$$

$$= \frac{1}{2}x\ln(x-1) - \frac{1}{2}x - \frac{1}{2}\ln|x-1| + C$$

As x is restricted to $x-1>0$ due to the domain of the integrand, we can omit the absolute value of the logarithm:

$$= \frac{1}{2}(x-1)\ln(x-1) - \frac{1}{2}x + C$$

55. $\int \frac{dx}{e^x-1}$

Method 1: Algebraic Manipulation

We divide the top and bottom of the integrand by e^x:

$$\int \frac{dx}{e^x-1} = \int \frac{e^{-x}}{1-e^{-x}}\,dx$$

Observe $\frac{d}{dx}(1-e^{-x}) = e^{-x}$, thus we apply the reverse chain rule.

$$= \ln|1-e^{-x}| + C$$

Method 2: Substitution and Partial Fractions

The best substitution here is $u = e^x$, in order to eliminate the "most difficult" part of the integrand.

$$\int \frac{dx}{e^x-1} = \int \frac{1}{u(u-1)}du$$

let $u = e^x$

$$\frac{du}{dx} = e^x$$

$$dx = \frac{1}{u}du$$

Our denominator now consists of two linear factors, so we decompose the integrand into partial fractions

Let $\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$.

$$\therefore A(u-1) + Bu = 1$$

letting $u=1$: $B=1$ and letting $u=0$: $A=-1$

Hence: $\dfrac{1}{u(u-1)} = -\dfrac{1}{u} + \dfrac{1}{u-1}$.

Thus:

$$\int \frac{dx}{e^x - 1} = \int \frac{du}{u-1} - \int \frac{du}{u}$$

$$= \ln|u-1| - \ln|u| + C$$

Substituting back in:

$$= \ln|e^x - 1| - \ln|e^x| + C$$

$$= \ln|e^x - 1| - x + C$$

While this result appears different from the one obtained in Method 1, note that these two forms are equivalent when suitable log laws are applied.

56. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x - 3\tan x + 2}\,dx$

Method 1: Substitution and Partial Fractions

Note that $\dfrac{d}{dx}(\tan x) = \sec^2 x$, thus we choose $u = \tan x$ as our substitution.

$$\int \frac{\sec^2 x}{\tan^2 x - 3\tan x + 2}\,dx = \int \frac{du}{u^2 - 3u + 2}$$

$$= \int \frac{du}{(u-2)(u-1)}$$

Our denominator now consists of two linear factors, so we decompose the integrand into partial fractions.

Let $\dfrac{1}{(u-2)(u-1)} = \dfrac{A}{u-2} + \dfrac{B}{u-1}$.

$\therefore A(u-1) + B(u-2) = 1$

letting $u = 1$: $-B = 1 \implies B = -1$

letting $u = 2$: $A = 1$

Hence: $\dfrac{1}{(u-2)(u-1)} = \dfrac{1}{u-2} - \dfrac{1}{u-1}$.

Thus: $\displaystyle\int \frac{\sec^2 x}{\tan^2 x - 3\tan x + 2}\,dx = \int \frac{du}{u-2} - \int \frac{du}{u-1}$

$$= \ln|u-2| - \ln|u-1| + C$$

Substituting back in:

$$= \ln|\tan x - 2| - \ln|\tan x - 1| + C$$

57. $\int \frac{x+1}{(x^2-3x+2)^{\frac{1}{2}}}\,dx$ (no longer on standard integrals)

Method 1: Algebraic Manipulation and Reverse Chain Rule

Our goal is to manipulate the numerator into a multiple of the derivative of the denominator. As $\frac{d}{dx}(x^2-3x+2) = 2x-3$, we manipulate $x+1$ as so: $x+1 = \frac{1}{2}(2x-3)+\frac{3}{2}+1$.

$$\int \frac{x+1}{(x^2-3x+2)^{\frac{1}{2}}}\,dx = \int \frac{\frac{1}{2}(2x-3)+\frac{5}{2}}{\sqrt{x^2-3x+2}}\,dx$$

$$= \frac{1}{2}\int \frac{2x-3}{\sqrt{x^2-3x+2}}\,dx + \frac{5}{2}\int \frac{dx}{\sqrt{x^2-3x+2}}$$

We apply the reverse chain rule to the leftmost integral, and complete the square for the rightmost integral.

$$= \sqrt{x^2-3x+2} + \frac{5}{2}\int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}}$$

$$= \sqrt{x^2-3x+2} + \frac{5}{2}\ln\left|x-\frac{3}{2}+\sqrt{x^2-3x+2}\right| + C_1$$

$$= \sqrt{x^2-3x+2} + \frac{5}{2}\ln\left|2x-3+2\sqrt{x^2-3x+2}\right| + C_2$$

58. $\int \sin 2x \cos x\,dx$

Method 1: Double Angle Identity

We apply the double angle formula for sine: $\sin 2x = 2\sin x\cos x$.

$$\int \sin 2x \cos x\,dx = 2\int \sin x \cos^2 x\,dx$$

We now observe that $\frac{d}{dx}(\cos x) = -\sin x$, so we apply the reverse chain rule: (alternatively you can make a substitution $u = \cos x$)

$$= -\frac{2}{3}\cos^3 x + C$$

Method 2: Product to Sums

Our integrand consists of a product of trigonometric functions, so we use the product to sum identity:

$\sin A\cos B = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$, setting $A = 2x$ and $B = x$.

$$\int \sin 2x\cos x\,dx = \frac{1}{2}\int \left(\sin 3x + \sin x\right)\,dx$$

$$= -\frac{1}{6}\cos 3x - \frac{1}{2}\cos x + C$$

59. $\displaystyle\int \frac{x}{1+x^3}\,dx$

Method 1: Partial Fractions

We can factorise the denominator into a quadratic and linear factor, allowing us to decompose the integrand into partial fractions.

$$\int \frac{x}{1+x^3}\,dx = \int \frac{x}{(1+x)(1-x+x^2)}\,dx$$

Now, let $\displaystyle\frac{x}{(1+x)(1-x+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2}$

$\therefore A(1-x+x^2) + (Bx+C)(1+x) = x$

letting $x = -1$: $3A = -1 \implies A = -\dfrac{1}{3}$

letting $x = 0$: $A + C = 0 \implies C = \dfrac{1}{3}$

letting $x = 1$: $A + 2B + 2C = 1 \implies B = \dfrac{1}{3}$

Hence: $\displaystyle\frac{x}{(1+x)(1-x+x^2)} = -\frac{1}{3(1+x)} + \frac{x+1}{3(1-x+x^2)}$

Thus: $\displaystyle\int \frac{x}{1+x^3}\,dx = -\frac{1}{3}\int \frac{dx}{1+x} + \frac{1}{3}\int \frac{x+1}{1-x+x^2}\,dx$

In order to simplify the right integral, our goal is to express the numerator is a multiple of the derivative of the denominator.

We do so through: $x+1 = \dfrac{1}{2}(2x-1) + \dfrac{3}{2}$.

Hence:

$$\int \frac{x}{1+x^3}\,dx = -\frac{1}{3}\int \frac{dx}{1+x} + \frac{1}{6}\int \frac{2x-1}{1-x+x^2}\,dx + \frac{1}{2}\int \frac{dx}{1-x+x^2}\,dx$$

We then complete the square for the right most integral.

$$= -\frac{1}{3}\int \frac{1}{1+x}dx + \frac{1}{6}\int \frac{2x-1}{1-x+x^2}\,dx + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx$$

$$= -\frac{1}{3}\ln|1+x| + \frac{1}{6}\ln|1-x+x^2| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= -\frac{1}{3}\ln|1+x| + \frac{1}{6}\ln|1-x+x^2| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

60. $\int x \tan^{-1} x \, dx$

Method 1: Integration by Parts

Our integrand is a product of two functions, hence we apply integration by parts.

Letting:

u	$\tan^{-1} x$	$\frac{1}{2}x^2$	v
u'	$\frac{1}{1+x^2}$	x	v'

$$\int x \tan^{-1} x \, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$$

Questions 61 - 70

61. $\int (1+3x+2x^2)^{-1}\,dx$

Method 1: Partial Fractions

Our denominator can be factorised into two linear factors, so we decompose the integrand into partial fractions.

$$\int (1+3x+2x^2)^{-1}\,dx = \int \frac{1}{(2x+1)(x+1)}dx$$

Let $\dfrac{1}{(2x+1)(x+1)} = \dfrac{A}{2x+1} + \dfrac{B}{x+1}$.

$\therefore A(x+1) + B(2x+1) = 1$

letting $x = -1$: $-B = 1 \implies B = -1$

letting $x = -\frac{1}{2}$: $\frac{1}{2}A = 1 \implies A = 2$

Hence: $\dfrac{1}{(2x+1)(x+1)} = \dfrac{2}{2x+1} - \dfrac{1}{x+1}$

Thus: $\displaystyle\int (1+3x+2x^2)^{-1}\,dx = \int \frac{2}{2x+1}\,dx - \int \frac{dx}{x+1}$

$$= \ln|2x+1| - \ln|x+1| + C$$

62. $\int (9-x^2)^{\frac{1}{2}}\,dx$

Method 1: Trigonometric Substitution

We make use of the Pythagorean identity $1-\sin^2\theta = \cos^2\theta$ in order to simplify the integral.

$$\int (9-x^2)^{\frac{1}{2}}\,dx = \int \sqrt{9-9\sin^2\theta}(3\cos\theta)\,d\theta$$

$$= \int \sqrt{9\cos^2\theta}(3\cos\theta)\,d\theta$$

$$= 9\int \cos^2\theta\,d\theta$$

$$= \frac{9}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C$$

let $x = 3\sin\theta$

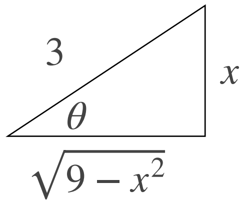
$\dfrac{dx}{d\theta} = 3\cos\theta$

$dx = 3\cos\theta\,d\theta$

We then substitute back in:

$$\frac{x}{3} = \sin\theta$$

$$\implies \cos\theta = \frac{\sqrt{9-x^2}}{3}$$



$$= \frac{9}{2}\sin^{-1}\frac{x}{3} + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$

$$= \frac{1}{2}\left[9\sin^{-1}\left(\frac{x}{3}\right) + x\sqrt{9-x^2}\right] + C$$

63. $\displaystyle\int (9+x^2)^{\frac{1}{2}}\,dx$

Method 1: Trigonometric Substitution

We can make use of the Pythagorean identity $1+\tan^2\theta = \sec^2\theta$ to help simplify the integral.

$$\int (9+x^2)^{\frac{1}{2}}\,dx = \int \sqrt{9+9\tan^2\theta}(3\sec^2\theta)\,d\theta$$

$$= \int \sqrt{9\sec^2\theta}(3\sec^2\theta)\,d\theta$$

$$= 9\int \sec^3\theta\,d\theta$$

let $x = 3\tan\theta$

$$\frac{dx}{d\theta} = 3\sec^2\theta$$

$$dx = 3\sec^2\theta\,d\theta$$

This integral we evaluate using integration by parts.

Letting:

u	$\sec\theta$	$\tan\theta$	v
u'	$\sec\theta\tan\theta$	$\sec^2\theta$	v'

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\,d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$$

$$2\int \sec^3\theta\,d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$$

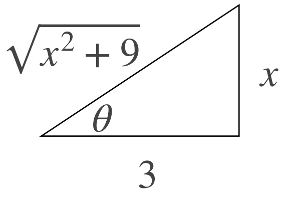
Hence:

$$\int (9+x^2)^{\frac{1}{2}}\,dx = \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta+\tan\theta| + C$$

Substituting back in:

$$\frac{x}{3} = \tan\theta$$

$$\implies \sec\theta = \frac{\sqrt{x^2+9}}{3}$$



$$= \frac{9}{2}\cdot\frac{\sqrt{x^2+9}}{3}\cdot\frac{x}{3} + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right| + C_1$$

$$= \frac{1}{2}x\sqrt{x^2+9} + \frac{9}{2}\ln\left|\frac{x+\sqrt{x^2+9}}{3}\right| + C_1$$

$$= \frac{1}{2}x\sqrt{x^2+9} + \frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right| + C_2$$

64. $\int x(9+x^2)^{\frac{1}{2}}\,dx$

Method 1: Reverse Chain Rule

Observe that $\frac{d}{dx}(x^2+9) = 2x$, hence we can apply the reverse chain rule.

$$\int x(9+x^2)^{\frac{1}{2}}\,dx = \frac{1}{2}\int (2x)(x^2+9)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\cdot\frac{(x^2+9)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{1}{3}(x^2+9)^{\frac{3}{2}} + C$$

65. $\int \sec^2 x \tan^3 x \ dx$

Method 1: Reverse Chain Rule

Observe that $\frac{d}{dx}(\tan x) = \sec^2 x$, hence we can immediately apply the reverse chain rule.

$$\int \sec^2 x \tan^3 x \ dx = \frac{\tan^4 x}{4} + C$$

66. $\int x^2 e^{-x} \ dx$

Method 1: Integration by Parts

Our integrand is a product of two functions, so we integrate by parts. Letting:

u	x^2	$-e^{-x}$	v
u'	$2x$	e^{-x}	v'

$$\int x^2 e^{-x} \ dx = -x^2 e^{-x} + 2\int xe^{-x} \, dx$$

Our new integral is again a product of two functions, so we integrate by parts again. Letting:

u	x	$-e^{-x}$	v
u'	1	e^{-x}	v'

$$= -x^2 e^{-x} + 2\left[-xe^{-x} + \int e^{-x} \, dx\right]$$

$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$= -e^{-x}(x^2 + 2x + 2) + C$$

67. $\displaystyle\int xe^{x^2}\ dx$

Method 1: Reverse Chain Rule

Observe that $\dfrac{d}{dx}(x^2) = 2x$, so we can manipulate in order to apply the reverse chain rule.

$$\int xe^{x^2}\ dx = \frac{1}{2}\int (2x)e^{x^2}\,dx$$

$$= \frac{1}{2}e^{x^2} + C$$

68. $\displaystyle\int \sin x \tan x\ dx$

Method 1: Pythagorean Identity

We rewrite $\tan x = \dfrac{\sin x}{\cos x}$ and then apply the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1.$

$$\int \sin x \tan x\ dx = \int \frac{\sin^2 x}{\cos x}dx$$

$$= \int \frac{1-\cos^2 x}{\cos x}dx$$

$$= \int (\sec x - \cos x)\ dx$$

$$= \ln|\sec x + \tan x| - \sin x + C$$

69. $\int \sin^4 x \cos^3 x \ dx$

Method 1: Pythagorean Identity and Reverse Chain Rule

We observe that $\frac{d}{dx}(\sin x) = \cos x$, and that the power of $\cos x$ is odd, so if we use the Pythagorean identity $\cos^2 x = 1 - \sin^2 x$, we will be in a good position to apply the reverse chain rule.

$$\int \sin^4 x \cos^3 x \ dx = \int \sin^4 x(1 - \sin^2 x)\cos x \ dx$$

$$= \int \sin^4 x \cos x \, dx - \int \sin^6 x \cos x \, dx$$

$$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C,$$

by the reverse chain rule.

70. $\int \frac{x^3 + 1}{x^3 - x} \ dx$

Method 1: Partial Fractions

We use an algebraic manipulation in order to simplify the integrand before we decompose into partial fractions.

$$\int \frac{x^3 + 1}{x^3 - x} \ dx = \int \frac{x^3 - x + x + 1}{x^3 - x} \, dx$$

$$= \int \left(1 + \frac{x + 1}{x(x^2 - 1)}\right) dx$$

$$= \int \left(1 + \frac{1}{x(x - 1)}\right) dx$$

Now, let $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$.

$\therefore 1 = A(x - 1) + Bx$

letting $x = 1$: $B = 1$

letting $x = 0$: $A = -1$

Therefore $\frac{1}{x(x-1)} = -\frac{1}{x} + \frac{1}{x-1}$.

Hence, $\int \frac{x^3 + 1}{x^3 - x} \ dx = \int \left(1 - \frac{1}{x} + \frac{1}{x - 1}\right) dx$

$$= x - \ln|x| + \ln|x - 1| + C$$

Questions 71 - 80

71. $\displaystyle\int \ln\left(x+\sqrt{x^2-1}\right)\ dx$

Method 1: Integration by Parts

Much like how we evaluate $\displaystyle\int \ln x\, dx$, we will integrate by parts.

Letting:

u	$\ln\left(x+\sqrt{x^2-1}\right)$	x	v
u'	$\dfrac{1}{\sqrt{x^2-1}}$	1	v'

$$\int \ln\left(x+\sqrt{x^2-1}\right)\ dx = x\ln\left(x+\sqrt{x^2-1}\right) - \int \frac{x}{\sqrt{x^2-1}}dx$$

We manipulate such that we can apply the reverse chain rule:

$$= x\ln\left(x+\sqrt{x^2-1}\right) - \frac{1}{2}\int (2x)(x^2-1)^{-\frac{1}{2}}\,dx$$

$$= x\ln\left(x+\sqrt{x^2-1}\right) - \sqrt{x^2-1} + C$$

72. $\displaystyle\int \frac{dx}{(x+1)^{\frac{1}{2}}+(x+1)}$

Method 1: Algebraic Substitution

There are no obvious substitutions, instead we replace the "most difficult" part of the integrand, which in this case is $\sqrt{x+1}$.

$$\int \frac{dx}{(x+1)^{\frac{1}{2}}+(x+1)} = \int \frac{2u}{u+u^2}\,du$$

let $u = \sqrt{x+1}$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$$

$$dx = 2u\,du$$

$$= 2\int \frac{du}{u+1}$$

$$= 2\ln|u+1| + C$$

Substituting back in:

$$= 2\ln\left|\sqrt{x+1}+1\right| + C$$

73. $\int_0^4 \frac{x}{\sqrt{x+4}}\,dx$

Method 1: Algebraic Manipulation

Our goal is to rewrite the numerator in terms of $x+4$.

$$\int_0^4 \frac{x}{\sqrt{x+4}}\,dx = \int_0^4 \frac{x+4-4}{\sqrt{x+4}}\,dx$$

$$= \int_0^4 \left(\sqrt{x+4} - \frac{4}{\sqrt{x+4}}\,dx\right)$$

$$= \left[\frac{2}{3}(x+4)^{\frac{3}{2}} - 8\sqrt{x+4}\right]_0^4$$

$$= \left(\frac{2}{3}\cdot 8^{\frac{3}{2}} - 8\sqrt{8}\right) - \left(\frac{2}{3}\cdot 4^{\frac{3}{2}} - 8\sqrt{4}\right)$$

$$= \frac{32\sqrt{2}}{3} - 16\sqrt{2} - \frac{16}{3} + 16$$

$$= \frac{32}{3} - \frac{16\sqrt{2}}{3}$$

$$= \frac{16}{3}\left(2-\sqrt{2}\right)$$

Method 2: Algebraic Substitution

While we could substitute $u = x+4$, $u = \sqrt{x+4}$ will be a much better substitution.

let $u = \sqrt{x+4}$

$$\iff x = u^2 - 4$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x+4}}$$

$$dx = 2u\,du$$

when $x = 0$, $u = 2$

when $x = 4$, $u = 2\sqrt{2}$

$$\int_0^4 \frac{x}{\sqrt{x+4}}\,dx = \int_2^{2\sqrt{2}} \frac{u^2-4}{u}(2u\,du)$$

$$= 2\int_2^{2\sqrt{2}} (u^2-4)\,du$$

$$= 2\left[\frac{1}{3}u^3 - 4u\right]_2^{2\sqrt{2}}$$

$$= 2\left[\left(\frac{16}{3}\sqrt{2} - 8\sqrt{2}\right) - \left(\frac{8}{3} - 8\right)\right]$$

$$= 2\left(\frac{16}{3} - \frac{8}{3}\sqrt{2}\right)$$

$$= \frac{16}{3}\left(2-\sqrt{2}\right)$$

74. $\displaystyle\int_1^2 \frac{dx}{x(1+x^2)}$

Method 1: Partial Fractions

Our denominator consists of a linear and quadratic factor, so we use partial fractions.

We decompose into partial fractions.

Let $\dfrac{1}{x(1+x^2)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$.

$\therefore 1 = A(x^2+1) + (Bx+C)x$

letting $x = 0$: $A = 1$

letting $x = i$: $1 = (Bi + C)i \implies -B + Ci = 1$

Equating real and imaginary parts: $B = -1$ and $C = 0$.

Therefore $\dfrac{1}{x(x^2+1)} = \dfrac{1}{x} - \dfrac{x}{x^2+1}$

Hence $\displaystyle\int_1^2 \frac{dx}{x(1+x^2)} = \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$

$$= \left[\ln|x| - \frac{1}{2}\ln|x^2+1|\right]_1^2$$

$$= \left(\ln 2 - \frac{1}{2}\ln 5\right) - \left(0 - \frac{1}{2}\ln 2\right)$$

$$= \frac{3}{2}\ln 2 - \frac{1}{2}\ln 5$$

$$= \frac{1}{2}\ln\frac{8}{5}$$

Method 2: Algebraic Substitution

A far more obscure substitution, but one that is very succinct is the substitution $u = \dfrac{1}{x}$.

$$\int_1^2 \frac{dx}{x(1+x^2)} = \int_1^{\frac{1}{2}} \frac{u}{1+\frac{1}{u^2}}\left(-\frac{1}{u^2}\right) du$$

$$= \int_{\frac{1}{2}}^1 \frac{u^3}{u^2+1}\cdot\frac{1}{u^2}\, du$$

$$= \int_{\frac{1}{2}}^1 \frac{u}{u^2+1}\, du$$

let $u = \dfrac{1}{x}$

$\dfrac{du}{dx} = -\dfrac{1}{x^2}$

$dx = -\dfrac{1}{u^2}\, du$

when $x = 1$, $u = 1$

when $x = 2$, $u = \dfrac{1}{2}$

Manipulating such that we can apply the reverse chain rule:

$$= \frac{1}{2}\int_{\frac{1}{2}}^1 \frac{2u}{u^2+1}\, du$$

$$= \frac{1}{2}\Big[\ln|u^2+1|\Big]_{\frac{1}{2}}^1$$

$$= \frac{1}{2}\left(\ln 2 - \ln\frac{5}{4}\right)$$

$$= \frac{1}{2}\ln\frac{8}{5}$$

75. $\displaystyle\int_1^2 \frac{\ln x}{x}\, dx$

Method 1: Reverse Chain Rule

Observe $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$, thus we apply the reverse chain rule.

$$\int_1^2 \frac{\ln x}{x}\, dx = \left[\frac{1}{2}(\ln x)^2\right]_1^2$$

$$= \frac{1}{2}(\ln 2)^2 - 0$$

$$= \frac{1}{2}(\ln 2)^2$$

76. $\displaystyle\int_0^1 \cos^{-1} x\, dx$

Method 1: Integration by Parts

This integral we deal with by integrating by parts, where $v' = 1$.

Letting:

u	$\cos^{-1} x$	x	v
u'	$-\dfrac{1}{\sqrt{1-x^2}}$	1	v'

$$\int_0^1 \cos^{-1} x\, dx = \left[x \cos^{-1} x\right]_0^1 + \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$$

We then manipulate such that we can apply the reverse chain rule:

$$= \left[x \cos^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{(-2x)}{\sqrt{1-x^2}}\, dx$$

$$= \left[x \cos^{-1} x\right]_0^1 - \left[\sqrt{1-x^2}\right]_0^1$$

$$= (0-0) - (0-1)$$

$$= 1$$

77. $\displaystyle\int_1^2 \frac{x+1}{\sqrt{-2+3x-x^2}}\,dx$

Method 1: Algebraic Manipulation

Our goal is to manipulate the numerator and express it in terms of the derivative of the bottom.

As $x+1 = -\dfrac{1}{2}(-2x+3)+\dfrac{5}{2}$:

$$\int_1^2 \frac{x+1}{\sqrt{-2+3x-x^2}}\,dx = -\frac{1}{2}\int_1^2 \frac{2x}{\sqrt{-2+3x-x^2}} + \frac{5}{2}\int \frac{dx}{\sqrt{-2+3x-x^2}}$$

We apply the reverse chain rule to the left most integral, and complete the square to the right most integral.

$$= -\left[\sqrt{-2+3x-x^2}\right]_1^2 + \frac{5}{2}\int \frac{1}{\sqrt{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx$$

$$= -(0-0) + \frac{5}{2}\left[\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{1}{2}}\right)\right]_1^2$$

$$= \frac{5}{2}\left[\sin^{-1}(2x-3)\right]_1^2$$

$$= \frac{5}{2}\left(\sin^{-1}1 - \sin^{-1}(-1)\right)$$

$$= \frac{5\pi}{2}$$

78. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2 x + 2\sin^2 x}$

Method 1: t-Substitution

Before we apply the t-substitution, we need to simplify. We do so using the Pythagorean identity and double angle formula.

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2 x + 2\sin^2 x} = \lim_{a\to\frac{\pi}{2}^-}\int_0^a \frac{dx}{\cos^2 x + 2\sin^2 x}\,dx$$

$$= \lim_{a\to\frac{\pi}{2}^-}\int_0^a \frac{dx}{1+\frac{1}{2}(1-\cos 2x)}\,dx$$

$$= 2\lim_{a\to\frac{\pi}{2}^-}\int_0^a \frac{dx}{3-\cos 2x}$$

Now we are able to apply the t-substitution.

$$= 2\lim_{a\to\frac{\pi}{2}^-}\int_0^{\tan a} \frac{\frac{1}{1+t^2}}{3-\frac{1-t^2}{1+t^2}}\,dt$$

let $t = \tan x$

$dx = \dfrac{1}{1+t^2}\,dt$

when $x=0,\ t=0$

when $x=a,\ t=\tan a$

$$= 2\lim_{a\to\frac{\pi}{2}^-}\int_0^{\tan a} \frac{1}{2+4t^2}\,dt$$

$$= \lim_{a\to\frac{\pi}{2}^-}\int_0^{\tan a} \frac{1}{1+2t^2}\,dt$$

$$= \lim_{a\to\frac{\pi}{2}^-}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}t\right)\right]_0^{\tan a}$$

$$= \frac{1}{\sqrt{2}} \lim_{a \to \frac{\pi}{2}^-} \tan^{-1}\left(\sqrt{2}\tan a\right)$$

Now as $a \to \frac{\pi}{2}^-$, $\sqrt{2}\tan a \to +\infty$, hence $\tan^{-1}\left(\sqrt{2}\tan a\right) \to \frac{\pi}{2}^-$.

$$= \frac{1}{\sqrt{2}} \cdot \frac{\pi}{2}$$

$$= \frac{\pi\sqrt{2}}{4}$$

Method 2: Algebraic Manipulation

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2 x + 2\sin^2 x} = \lim_{a \to \frac{\pi}{2}^-} \int_0^a \frac{dx}{\cos^2 x + 2\sin^2 x}\, dx$$

We divide the top and bottom by $\cos^2 x$.

$$= \lim_{a \to \frac{\pi}{2}^-} \int_0^a \frac{\sec^2 x}{1 + 2\tan^2 x}\, dx$$

$$= \lim_{a \to \frac{\pi}{2}^-} \frac{1}{\sqrt{2}} \int_0^a \frac{\sqrt{2}\sec^2 x}{1 + \left(\sqrt{2}\tan x\right)^2}\, dx$$

$$= \lim_{a \to \frac{\pi}{2}^-} \frac{1}{\sqrt{2}} \left[\tan^{-1}\left(\sqrt{2}\tan x\right)\right]_0^a$$

$$= \frac{1}{\sqrt{2}} \lim_{a \to \frac{\pi}{2}^-} \tan^{-1}\left(\sqrt{2}\tan a\right)$$

$= \frac{\pi\sqrt{2}}{4}$ (by the same argument above)

79. $\int_0^1 x\sqrt{1-x^2}\, dx$

Method 1: Reverse Chain Rule

Observe that $\frac{d}{dx}(1-x^2) = -2x$, so we manipulate such that we can apply the reverse chain rule.

$$\int_0^1 x\sqrt{1-x^2}\, dx = -\frac{1}{2}\int_0^1 (-2x)\sqrt{1-x^2}\, dx$$

$$= -\frac{1}{2} \cdot \frac{2}{3}\left[(1-x^2)^{\frac{3}{2}}\right]_0^1$$

$$= -\frac{1}{3}(0-1)$$

$$= \frac{1}{3}$$

80. $\int_2^4 x \ln x \; dx$

Method 1: Integration by Parts

Our integrand is a product of two integrals, so we apply integration by parts. Letting:

u	$\ln x$	$\frac{1}{2}x^2$	v
u'	$\frac{1}{x}$	x	v'

$$\int_2^4 x \ln x \; dx = \left[\frac{1}{2}x^2 \ln x\right]_2^4 - \frac{1}{2}\int_2^4 x\,dx$$

$$= \left(\frac{16}{2}\ln 4 - \frac{4}{2}\ln 2\right) - \frac{1}{2}\left[\frac{x^2}{2}\right]_2^4$$

$$= (16\ln 2 - 2\ln 2) - \frac{1}{2}(8-2)$$

$$= 14\ln 2 - 3$$

Questions 81 - 90

81. $\displaystyle\int_1^2 \frac{dx}{x^2+5x+4}$

Method 1: Partial Fractions

Factorising the denominator into two linear factors, we then decompose the integrand into partial fractions.

$$\int_1^2 \frac{dx}{x^2+5x+4} = \int_1^2 \frac{dx}{(x+4)(x+1)}$$

Let $\dfrac{1}{(x+4)(x+1)} = \dfrac{A}{x+4} + \dfrac{B}{x+1}$.

$\therefore\ A(x+1) + B(x+4) = 1$

letting $x = -1$: $3B = 1 \implies B = \dfrac{1}{3}$

letting $x = -4$: $-3A = 1 \implies A = -\dfrac{1}{3}$

Therefore $\dfrac{1}{(x+4)(x+1)} = -\dfrac{1}{3(x+4)} + \dfrac{1}{3(x+1)}$.

Hence:

$$\begin{aligned}\int_1^2 \frac{dx}{x^2+5x+4} &= \int_1^2 \left(-\frac{1}{3(x+4)} + \frac{1}{3(x+1)}\right) dx \\ &= \frac{1}{3}\left[\ln|x+1| - \ln|x+4|\right]_1^2 \\ &= \frac{1}{3}\left[(\ln 3 - \ln 2) - (\ln 6 - \ln 5)\right] \\ &= \frac{1}{3}\ln\left(\frac{3\times 5}{2\times 6}\right) \\ &= \frac{1}{3}\ln\left(\frac{5}{4}\right)\end{aligned}$$

82. $\int_0^{\frac{\pi}{2}} \left(1 + \frac{1}{2}\sin x\right)^{-1} dx$

Method 1: t-Substitution

There are no immediate substitutions or manipulations, and our integrand contains a single trigonometric function, hence we apply the t-substitution.

$$\int_0^{\frac{\pi}{2}} (1 + \frac{1}{2}\sin x)^{-1}\, dx = \int_0^{\frac{\pi}{2}} \frac{2}{2 + \sin x}\, dx$$

$$= \int_0^1 \frac{2}{2 + \frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\, dt$$

$$= 2\int_0^1 \frac{dt}{1 + t + t^2}$$

We then complete the square for the quadratic in the denominator:

$$= \int_0^1 \frac{2}{\frac{3}{4} + \left(t + \frac{1}{2}\right)^2} dt$$

$$= \frac{4}{\sqrt{3}} \left[\tan^{-1}\left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1$$

$$= \frac{4}{\sqrt{3}} \left[\tan^{-1}\left(\frac{2t + 1}{\sqrt{3}}\right)\right]_0^1$$

$$= \frac{4}{\sqrt{3}} \left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$= \frac{4}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{2\pi}{3\sqrt{3}}$$

83. $\int_0^1 x^2 e^{-x}\,dx$

Method 1: Integration by Parts

Our integrand is a product of two functions, so we apply integration by parts. Letting:

u	x^2	$-e^{-x}$	v
u'	$2x$	e^{-x}	v'

$$\int_0^1 x^2 e^{-x}\,dx = \left[-x^2e^{-x}\right]_0^1 + 2\int_0^1 xe^{-x}\,dx$$

Our new integrand is again a product of two functions, so we integrate by parts again. Letting:

u	x	$-e^{-x}$	v
u'	1	e^{-x}	v'

$$= \left[-x^2e^{-x}\right]_0^1 + 2\left[\left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx\right]$$

$$= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^1$$

$$= -\left[e^{-x}(x^2+2x+2)\right]_0^1$$

$$= 2 - 5e^{-1}$$

84. $\int_0^1 \frac{7+x}{1+x+x^2+x^3}\,dx$

Method 1: Partial Fractions

Factorising the denominator into a linear and quadratic factor, we then decompose the integrand into partial fractions.

$$\int_0^1 \frac{7+x}{1+x+x^2+x^3}\,dx = \int_0^1 \frac{7+x}{(x+1)(x^2+1)}\,dx$$

Now, let $\frac{7+x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$.

$\therefore A(x^2+1) + (Bx+C)(x+1) = 7+x$

letting $x = -1$: $2A = 6 \implies A = 3$

letting $x = 0$: $A + C = 7 \implies C = 4$

letting $x = 1$: $2A + 2B + 2C = 8 \implies B = -3$

Therefore $\frac{7+x}{(x+1)(x^2+1)} = \frac{3}{x+1} + \frac{-3x+4}{x^2+1}$

Hence: $\int_0^1 \frac{7+x}{1+x+x^2+x^3}\,dx = \int_0^1 \left(\frac{3}{x+1} - \frac{3x}{x^2+1} + \frac{4}{x^2+1}\right)dx$

$$= \left[3\ln|x+1| - \frac{3}{2}\ln|x^2+1| + 4\tan^{-1}x\right]_0^1$$

$$= \frac{3}{2}\ln 2 + \pi$$

85. $\displaystyle\int_0^1 \frac{e^{-2x}}{e^{-x}+1}\,dx$

Method 1: Substitution and Partial Fractions

There are no immediate substitutions/manipulations, so a good substitution to try would be $u = e^x$.

$$\int_0^1 \frac{e^{-2x}}{e^{-x}+1}\,dx = \int_1^e \frac{u^{-2}}{u^{-1}+1}\cdot u^{-1}\,du$$

$$= \int_1^e \frac{u^{-3}}{u^{-1}+1}\,du$$

$$= \int_1^e \frac{du}{u^3+u^2}$$

$$= \int_1^e \frac{du}{u^2(1+u)}$$

let $u = e^x$

$$\frac{dx}{du} = e^x$$

$$dx = \frac{1}{u}\,du$$

when $x = 0,\ u = 1$

when $x = 1,\ u = e$

Our denominator consists of a linear and quadratic factor, so we decompose into partial fractions.

Let $\dfrac{1}{u^2(1+u)} = \dfrac{A}{u} + \dfrac{B}{u^2} + \dfrac{C}{u+1}$.

$$\therefore 1 = Au(u+1) + B(u+1) + Cu^2$$

letting $u = 0$: $B = 1$

letting $u = -1$: $C = 1$

letting $u = 1$: $1 = 2A + 2B + C \implies A = -1$

Therefore: $\dfrac{1}{u^2(1+u)} = -\dfrac{1}{u} + \dfrac{1}{u^2} + \dfrac{1}{u+1}$

Hence: $\displaystyle\int_0^1 \frac{e^{-2x}}{e^{-x}+1}\,dx = \int_1^e \left(-\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u+1}\right) du$

$$= \left[-\ln u - \frac{1}{u} + \ln|1+u|\right]_1^e$$

$$= \left[\ln\left|\frac{1+u}{u}\right| - \frac{1}{u}\right]_1^e$$

$$= \left(\ln\left|\frac{1+e}{e}\right| - \frac{1}{e}\right) - \left(\ln 2 - 1\right)$$

$$= \ln\left(\frac{1+e}{2e}\right) - \frac{1}{e} + 1$$

86. $\int_0^{\frac{a}{2}} \frac{y}{a-y}\, dy$

Method 1: Algebraic Manipulation

Our goal is to express the numerator in terms of the denominator.

As $y = -(a-y) + a$:

$$\int_0^{\frac{a}{2}} \frac{y}{a-y}\, dy = -\int_0^{\frac{a}{2}} \frac{a-y}{a-y}\, dy + \int_0^{\frac{a}{2}} \frac{a}{a-y}\, dy$$

$$= -\int_0^{\frac{a}{2}} 1\, dy - \Big[a \ln|a-y|\Big]_0^{\frac{a}{2}}$$

$$= -\Big[y\Big]_0^{\frac{a}{2}} - \Big[a \ln|a-y|\Big]_0^{\frac{a}{2}}$$

$$= -\frac{a}{2} - a\left(\ln\left|\frac{a}{2}\right| - \ln|a|\right)$$

$$= -\frac{a}{2} - a\ln\left|\frac{\left(\frac{a}{2}\right)}{a}\right|$$

$$= -\frac{a}{2} - a\ln\left(\frac{1}{2}\right)$$

$$= -\frac{a}{2} + a\ln 2$$

$$= \frac{a}{2}(\ln 4 - 1)$$

87. $\int_0^{a} \frac{(a-x)^2}{a^2+x^2}\, dx$

Method 1: Algebraic Manipulation

Expanding the numerator, we can manipulate the integral such that we can easily apply the reverse chain rule.

$$\int_0^{a} \frac{(a-x)^2}{a^2+x^2}\, dx = \int_0^{a} \frac{a^2-2ax+x^2}{a^2+x^2}\, dx$$

$$= \int_0^{a} \left(1 - \frac{2ax}{a^2+x^2}\right) dx$$

$$= \Big[x - a\ln|a^2+x^2|\Big]_0^{a}$$

$$= \left(a - a\ln|2a^2|\right) - (0 - a\ln|a^2|)$$

$$= a - a\ln\left|\frac{2a^2}{a^2}\right|$$

$$= a(1 - \ln 2)$$

88. $\displaystyle\int_0^1 \frac{x+3}{(x+2)(x+1)^2}\,dx$

Method 1: Partial Fractions

Our denominator consists of a linear and quadratic factor, so we decompose the integrand into partial fractions.

Let $\dfrac{x+3}{(x+2)(x+1)^2} = \dfrac{A}{x+2} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$.

$\therefore A(x+1)^2 + B(x+1)(x+2) + C(x+2) = x+3$

letting $x = -1$: $C = 2$

letting $x = -2$: $A = 1$

letting $x = 0$: $A + 2B + 2C = 3 \implies B = -1$

Therefore $\dfrac{x+3}{(x+2)(x+1)^2} = \dfrac{1}{x+2} - \dfrac{1}{x+1} + \dfrac{2}{(x+1)^2}$

Hence: $\displaystyle\int_0^1 \frac{x+3}{(x+2)(x+1)^2}\,dx = \int_0^1 \left(\frac{1}{x+2} - \frac{1}{x+1} + \frac{2}{(x+1)^2}\right) dx$

$$= \left[\ln|x+2| - \ln|x+1| - \frac{2}{x+1}\right]_0^1$$

$$= \left(\ln 3 - \ln 2 - 1\right) - (\ln 2 - 0 - 2)$$

$$= \ln\frac{3}{4} + 1$$

89. $\displaystyle\int_0^1 \frac{x^2}{x^6+1}\,dx$

Method 1: Reverse Chain Rule

Observe that $\dfrac{d}{dx}(x^3) = 3x^2$, thus we can manipulate such that we can apply the reverse chain rule.

$$\int_0^1 \frac{x^2}{x^6+1}\,dx = \frac{1}{3}\int_0^1 \frac{3x^2}{(x^3)^2+1}\,dx$$

$$= \frac{1}{3}\left[\tan^{-1}(x^3)\right]_0^1$$

$$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{12}$$

90. $\displaystyle\int_0^{\pi} \cos^2 mx\ dx$, where m is an integer.

Method 1: Half Angle Formula

We aim to simplify the integral by reducing the power of $\cos(mx)$.

We do so by applying the half angle formula $\cos^2 x = \dfrac{1}{2}(1 + \cos 2x)$.

$$\int_0^{\pi} \cos^2 mx\ dx = \frac{1}{2}\int_0^{\pi} (1 + \cos 2mx)\ dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2mx}{2m}\right]_0^{\pi}$$

$$= \frac{1}{2}\left(\pi - \frac{1}{2m}\sin(2m\pi)\right) - 0$$

Now the sine of any integer multiple of π is zero, hence:

$$\int_0^{\pi} \cos^2 mx\ dx = \frac{\pi}{2}$$

Section 10

Questions 91 - 100

91. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \sin 2x \; dx$

Method 1: Integration by Parts

Our integrand is a product of two functions, hence we integrate by parts. Letting:

u	x	$-\frac{1}{2}\cos 2x$	v
u'	1	$\sin 2x$	v'

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \sin 2x \; dx = \left[-\frac{1}{2}x\cos 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2x \, dx$$

$$= \left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{4} + 0\right) - \left(0 - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} - \frac{1}{4}$$

92. $\displaystyle\int_{0}^{\frac{a}{2}} x^2\sqrt{a^2 - x^2} \; dx$

Method 1: Trigonometric Substitution

We can use the Pythagorean identity $1 - \sin^2\theta = \cos^2\theta$ to help simplify the integral.

let $x = a\sin\theta$ — when $x = 0,\ \theta = 0$

$\dfrac{dx}{d\theta} = a\cos\theta$ — when $x = a,\ \theta = \dfrac{\pi}{6}$

$dx = a\cos\theta\, d\theta$

$$\int_{0}^{\frac{a}{2}} x^2\sqrt{a^2 - x^2} \; dx = \int_{0}^{\frac{\pi}{6}} (a\sin\theta)^2\sqrt{a^2 - a^2\sin^2\theta}(a\cos\theta)\, d\theta$$

$$= a^4\int_{0}^{\frac{\pi}{6}} \sin^2\theta\cos^2\theta\, d\theta$$

We then use the double and half angle formula to help simplify the integral.

$$= \frac{a^4}{4} \int_0^{\frac{\pi}{6}} (2 \sin \theta \cos \theta)^2 \, d\theta$$

$$= \frac{a^4}{4} \int_0^{\frac{\pi}{6}} \sin^2 2\theta \, d\theta$$

$$= \frac{a^4}{8} \int_0^{\frac{\pi}{6}} (1 - \cos 4\theta) \, d\theta$$

$$= \frac{a^4}{8} \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\frac{\pi}{6}}$$

$$= \frac{a^4}{8} \left(\frac{\pi}{6} - \frac{1}{4} \sin \frac{2\pi}{3} \right) - 0$$

$$= \frac{a^4}{8} \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8} \right)$$

$$= \frac{a^4}{192} \left(4\pi - 3\sqrt{3} \right)$$

93. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x \tan x \ dx$

Method 1: Reverse Chain Rule

Observe that $\dfrac{d}{dx}(\tan x) = \sec^2 x$, hence we can immediately apply the reverse chain rule.

$$\int_0^{\frac{\pi}{4}} \sec^2 x \tan x \ dx = \left[\frac{1}{2} \tan^2 x \right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{2} - 0 \right)$$

$$= \frac{1}{2}$$

94. $\displaystyle\int_0^1 (x+2)\sqrt{x^2+4x+5}\ dx$

Method 1: Reverse Chain Rule

Observe that $\dfrac{d}{dx}(x^2+4x) = 2x+4$, hence we manipulate such that we can apply the reverse chain rule.

$$\int_0^1 (x+2)\sqrt{x^2+4x+5}\ dx = \frac{1}{2}\int_0^1 (2x+4)\sqrt{x^2+4x+5}\, dx$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left[(x^2+4x+5)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{1}{3}\left(10^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}\left(10\sqrt{10} - 5\sqrt{5}\right)$$

$$= \frac{5\sqrt{5}}{3}\left(2\sqrt{2}-1\right)$$

95. $\displaystyle\int_1^2 x(\ln x)^2\ dx$

Method 1: Integration by Parts

Our integrand is a product of two functions, hence we integrate by parts. Letting:

u	$(\ln x)^2$	$\dfrac{x^2}{2}$	v
u'	$\dfrac{2\ln x}{x}$	x	v'

$$\int_1^2 x(\ln x)^2\ dx = \left[\frac{x^2}{2}(\ln x)^2\right]_1^2 - \int_1^2 x\ln x\, dx$$

Our new integrand is again a product of two functions, hence we integrate by parts again. Letting:

u	$\ln x$	$\dfrac{x^2}{2}$	v
u'	$\dfrac{1}{x}$	x	v'

$$= \left[\frac{x^2}{2}(\ln x)^2\right]_1^2 - \left(\left[\frac{1}{2}x^2\ln x\right]_1^2 - \frac{1}{2}\int_1^2 x\, dx\right)$$

$$\int_1^2 x(\ln x)^2\, dx = \left[\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ \ln x - \frac{1}{4}x^2\right]_1^2$$

$$= \left[\frac{1}{2}\cdot 4(\ln 2)^2 - \frac{1}{2}\cdot 4\ln 2 - \frac{1}{4}\cdot 4\right] - \left(0 - 0 - \frac{1}{4}\right)$$

$$= 2(\ln 2)^2 - 2\ln 2 + \frac{3}{4}$$

96. $\displaystyle\int_3^4 \frac{x^2+4}{x^2-1}\, dx$

Method 1: Partial Fractions

We simplify by rewriting the numerator in terms of the denominator, and then decomposing into partial fractions.

$$\int_3^4 \frac{x^2+4}{x^2-1}\, dx = \int_3^4 \left(\frac{x^2-1}{x^2-1} + \frac{5}{x^2-1}\right) dx$$

Now, let $\dfrac{1}{x^2-1} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$.

$\therefore A(x-1) + B(x+1) = 1$

letting $x = 1$: $2B = 1 \implies B = \dfrac{1}{2}$

letting $x = -1$: $-2A = 1 \implies A = -\dfrac{1}{2}$

Therefore $\dfrac{1}{x^2-1} = -\dfrac{1}{2(x+1)} + \dfrac{1}{2(x-1)}$

Hence: $\displaystyle\int_3^4 \frac{x^2+4}{x^2-1}\, dx = \int_3^4 \left(1 - \frac{1}{2(x+1)} + \frac{1}{2(x-1)}\right) dx$

$$= \left[x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1|\right]_3^4$$

$$= \left(4 - \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3\right) - \left(3 - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 2\right)$$

$$= 1 + \frac{5}{2}\ln\frac{6}{5}$$

97. $\displaystyle\int_1^4 \frac{x^2+4}{x(x+2)}\,dx$

Method 1: Partial Fractions

We first simplify by expressing the numerator in terms of the denominator x^2+2x:

$$\int_1^4 \frac{x^2+4}{x(x+2)}\,dx = \int_1^4 \frac{x^2+2x-2x+4}{x(x+2)}\,dx$$

$$= \int_1^4 \left(1 - \frac{2x-4}{x(x+2)}\right)\,dx$$

We then decompose the integrand into partial fractions.

Let $\dfrac{2x-4}{x(x+2)} = \dfrac{A}{x} + \dfrac{B}{x+2}$

$\therefore\ A(x+2)+Bx = 2x-4$

letting $x=0$: $2A=-4 \implies A=-2$

letting $x=-2$: $-2B=-8 \implies B=4$

Therefore $\dfrac{2x-4}{x(x+2)} = -\dfrac{2}{x} + \dfrac{4}{x+2}$

Hence:

$$\int_1^4 \frac{x^2+4}{x(x+2)}\,dx = \int_1^4 \left(1+\frac{2}{x}-\frac{4}{x+2}\right)\,dx$$

$$= \Big[x + 2\ln|x| - 4\ln|x+2|\Big]_1^4$$

$$= \big(4 + 2\ln 4 - 4\ln 6\big) - (1 + 0 - 4\ln 3)$$

$$= 3 + \ln\left(\frac{4^2\times 3^4}{6^4}\right)$$

$$= 3 + \ln 1$$

$$= 3$$

98. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{5 - 3\sin x}\, dx$

Method 1: Reverse Chain Rule

Observe that $\frac{d}{dx}(5 - 3\sin x) = -3\cos x$, hence we manipulate

such that we can apply the reverse chain rule.

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{5 - 3\sin x}\, dx = -\frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{-3\cos x}{5 - 3\sin x}\, dx$$

$$= \left[-\frac{1}{3}\ln|5 - 3\sin x|\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{3}\left(\ln 2 - \ln 5\right)$$

$$= \frac{1}{3}\ln\frac{5}{2}$$

99. $\int_0^1 \frac{1}{(4 - x^2)^{\frac{3}{2}}}\, dx$

Method 1: Trigonometric Substitution

We can make use of the Pythagorean identity $1 - \sin^2\theta = \cos^2\theta$ to help simplify the integrand:

$$\int_0^1 \frac{1}{(4 - x^2)^{\frac{3}{2}}}\, dx = \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{(4 - 4\sin^2\theta)^{\frac{3}{2}}}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{8\cos^3\theta}\, d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \sec^2\theta\, d\theta$$

$$= \frac{1}{4}\Big[\tan\theta\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}} - 0\right)$$

$$= \frac{\sqrt{3}}{12}$$

let $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

when $x = 0$, $\theta = 0$

when $x = 1$, $\theta = \frac{\pi}{6}$

100. $\int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta(3\sin\theta - 4\sin^3\theta)\ d\theta$

Method 1: Double/Triple Angle Identity and Product to Sums

We simplify the integrand by recognising that:

$$\sin(2\theta) = 2\sin\theta\cos\theta \text{ and } \sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$

$$\int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta(3\sin\theta - 4\sin^3\theta)\ d\theta = \int_0^{\frac{\pi}{2}} \sin 2\theta \sin 3\theta\ d\theta$$

Our integrand is now a product of two trigonometric functions, so we apply the product to sum identities.

$$\cos(A - B) - \cos(A + B) = 2\sin A \sin B$$

$$\implies \sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

Letting $A = 3\theta$ and $B = 2\theta$ in our identity above:

$$\int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta(3\sin\theta - 4\sin^3\theta)\ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\cos\theta - \cos 5\theta\right)\ d\theta$$

$$= \frac{1}{2}\left[\sin\theta - \frac{1}{5}\sin 5\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(1 - \frac{1}{5}\right)$$

$$= \frac{2}{5}$$