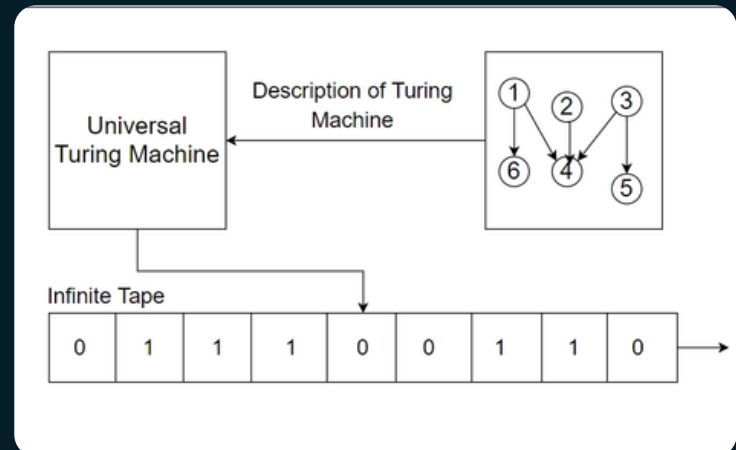


What is Turing Machines ?

- In the context of automata theory and the theory of computation, Turing machines are used to study the properties of algorithms and to determine what problems can and cannot be solved by computers.
- They provide a way to model the behavior of algorithms and to analyze their computational complexity, which is the amount of time and memory they require to solve a problem



Hilbert's 10th Problem

Hilbert's 10th problem, also known as the 10th problem on David Hilbert's list of 23 unsolved problems, asks whether there exists an algorithm to determine whether a given Diophantine equation (a polynomial equation with integer coefficients) has integer solutions.

Hilbert's 10th Problem Example

Diophantine Equation with No solution

You have asked to solve the linear equation:

$$4x + 6y = 13$$

Calculating GCD(4,6) gives:

$$6 = 1 * 4 + 2$$

$$4 = 2 * 2 + 0$$

Sorry, no solutions as 2 does not divide 13

Diophantine Equation with solution

You have asked to solve the linear equation:

$$3x + 5y = 15$$

Calculating GCD(3,5) gives:

$$5 = 1 * 3 + 2$$

$$3 = 1 * 2 + 1$$

$$2 = 2 * 1 + 0$$

Then applying the Extended Euclidean Algorithm:

$$1 = (1 * 3) + (-1 * 2)$$

$$= (-1 * 5) + (2 * 3)$$

A particular solution is:

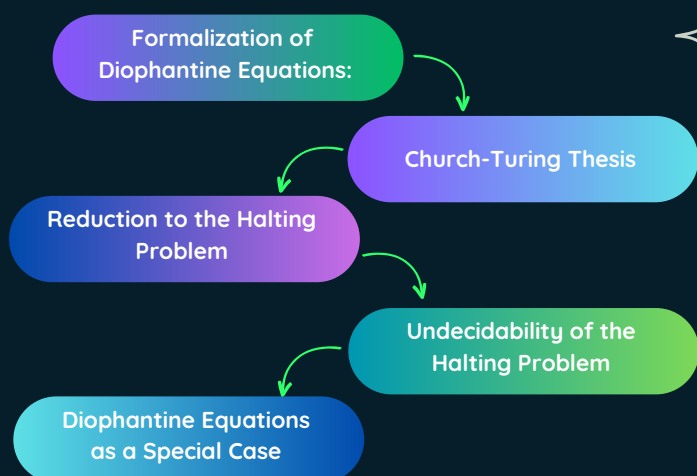
$$x = 30$$

$$y = -15$$

Definition of Turing Machines

- Q is a finite set of states
- T is the tape alphabet (symbols which can be written on Tape)
- B is blank symbol (every cell is filled with B except input alphabet initially)
- Σ is the input alphabet (symbols which are part of input alphabet)
- δ is a transition function which maps $Q \times T \rightarrow Q \times T \times \{L, R\}$. Depending on its present state and present tape alphabet (pointed by head pointer), it will move to new state, change the tape symbol (may or may not) and move head pointer to either left or right.
- q_0 is the initial state
- F is the set of final states. If any state of F is reached, input string is accepted.

Turing machines and the theoretical framework they provide for algorithmic computation played a central role in showing the undecidability of Hilbert's 10th problem



By demonstrating a reduction from Hilbert's 10th problem to the undecidable Halting Problem, mathematicians were able to conclude that there is no general algorithm to solve all Diophantine equations, as stated in Hilbert's 10th problem.

References:

- https://en.wikipedia.org/wiki/Hilbert%27s_tenth_problem
- <https://www.geeksforgeeks.org/turing-machine-in-toc/>
- https://sites.math.washington.edu/~morrow/336_15/papers/andrew.pdf