PAGE No.
DATE / /

MS MOD 2

- Random No's :-
studocu They are necessary basic ingredient in the simulation of almost all discrete systems.
Most computer languages have object, subroutine or function that will generate random no's.
Properties :-
A sequence of random no's w1, w2, ... must have 2 imp. stastistical properties, Independence i.e each generated rand no must be independent of other rand no's & Uniformity i.e generated rand no's are in continuous uniform distribution betw 0 & 1.

- Pseudo Random No's :-
studocu Means false no's generated, bcz generating no's using known method removes potential for true randomness.
Goal is to produce sequence of no's betw 0 & 1 that imitates ideal properties of rand no's.
Errors while generating rand no's :-
i) The no's may not be uniformly distributed
ii) May be discrete value instead of continous value.
iii) Mean may be too high / low.
iv) Variance → too high / low.
Characteristics of Good Random No's :-
i) Fast generation of RN's.
ii) Portable i.e same results must be produced when executed on diff. OS, PC's.
iii) Sufficient long cycle → no's must be replicated after long time.
iv) Given starting point, it should generate same set of rand no's.
v) Match closely uniformity & independence.

Scanned by PDF Scanner

$X_1, X_2, X_3, \ldots$ are called random no's.
$R_1, R_2, R_3, \ldots$ are called rand. integer values.

PAGE No.
DATE / /

- Linear Congruential Method :-

LCM produces sequences of integers $X_1, X_2, \ldots$ betⁿ 0 & m-1 according to recursive relationship :-

$X_{i+1} = (aX_i + c) \bmod m, \quad i = 0, 1, 2, \ldots$

where X_0 = seed value
a - constant multiplier
c - increment
m - modulus

if $c \neq 0$ the form is called mixed congruential method.
when $c = 0$, form is called multiplicative congruential method.

Maximal period based on condⁿs :-

1) For m, a power of 2 ($m = 2^b$) & $c \neq 0$, period $p = 2^b$ is achieved given that c is relatively prime to m &
$a = 1 + 4K, \quad K = 0, 1, 2, \ldots$

2) for $m = 2^b$ & $c = 0$, $p = 2^{b-2}$ achieved given X_0 is odd & multiplier $a = 3 + 8K$ or $a = 5 + 8K$, $K = 0, 1, 2, \ldots$

3) for m a prime no & $c = 0$, $p = m - 1$ achieved given that a has property that smallest integer is such that $a^k - 1$ is divisible by m is $k = m - 1$

Eg. Use LCM to generate sequence of RN's with $X_0 = 27$, $a = 17$, $c = 43$ & $m = 100$

→ $X_0 = 27$

$X_1 = (aX_0 + c) \bmod m = (17 \times 27 + 43) \bmod 100 = 2$

Random Integer Values $R_i = \frac{X_i}{m}$ $\therefore R_1 = \frac{X_1}{m} = \frac{2}{100} = 0.02$

$X_2 = (17 \times 2 + 43) \bmod 100 = 77$

$R_2 = \frac{X_2}{m} = \frac{77}{100} = 0.77$

$X_3 = (17 \times 77 + 43) \bmod 100 = 52$

$R_3 = X_3/m = 52/100 = 0.52$

Scanned by PDF Scanner

PAGE No.
DATE / /

yt • Tests for RN's :-
Uniformity → Kolmogorov Smirnov test, Chi-Square
Independence → runs test, autocorrelation, gap.

Testing for Uniformity :-
$H_0 : R_i \sim U[0,1]$ — failure to reject H_0 means evidence of non-uniformity hasn't detected
$H_1 : R_i \nsim U[0,1]$

Testing for Independence :-
$H_0 : R_i$ independent — " " " of dependence hasn't detected.
$H_1 : R_i \neg$ independent

yt • K-S Test :-
1) Arrange the given no in ascending order.
2) Compute :-

$$D^+ = \max_{1 \le i \le N} \left\{ \frac{i}{N} - R_i \right\}$$

$$D^- = \max_{1 \le i \le N} \left\{ R_i - \left(\frac{i-1}{N} \right) \right\}$$

N - total no. of random no
degree of freedom
Sample Size

3) Compute :-

$$D = \max(D^+, D^-)$$

4) Compute theoretical value D_α for a given level of significance α from standard KS Test table.
5) If $D < D_\alpha$ then accept hypothesis, else reject

O/P determined by Null Hypothesis.

Scanned by PDF Scanner

PAGE No.
DATE / /

Q. Using KS Test, check for property of uniformity for the i/p set of random no's.
0.54, 0.73, 0.98, 0.11 & 0.68. Assume level of significance to be 0.05

→

i	1	2	3	4	5
R_i	0.11	0.54	0.68	0.73	0.98

} Step 1

Formula:- $D^+_{(i)} = \left(\frac{i}{N} - R_i \right)$ (Note:- If -ve value found, ignore it.)

$D^-_{(i)} = \left\{ R_i - \frac{(i-1)}{N} \right\}$.

D^+	0.09	–	–	0.07	0.02
D^-	0.11	0.34	0.28	0.13	0.18

} Step 2

Among D^+, max^m is 0.09 & among D^-, max^m is 0.34.

$D = max(max\ D^+, max\ D^-)$
$= max(0.09, 0.34)$
$= 0.34$ ← step 4.

from KS Test Table D_α for $\alpha = 0.05$ is 0.565.
∴ $D < D_\alpha$
∴ Null Hypothesis is accepted. ← Step 5

Scanned by PDF Scanner

- **Chi-Square Test :-**

yt It is used when no. of i/p's are huge to check uniformity.

Q. Using chi-square test with $\alpha = 0.05$, check for uniformity of given random no's.

Given :- 100 random no's (Range :- 0 to 1)

→ Step 1 :- Define Hypothesis.

Step 2 :- n, such that $E_i \geq 5$. & $E_i = N/n$.

$\therefore \frac{100}{n} \geq 5 \quad \therefore n \leq 20$.

Step 3 :- Lets take $n = 10$,

Step 4 :-

Interval	O_i	$E_i = N/n$	$(O_i - E_i)^2 / E_i$
0 – 0.1	8	10	0.4
0.1 – 0.2	9	10	0.1
0.2 – 0.3	10	10	0
0.3 – 0.4	6	10	1.6
0.4 – 0.5	13	10	0.9
0.5 – 0.6	8	10	0.4
0.6 – 0.7	11	10	0.1
0.7 – 0.8	12	10	0.4
0.8 – 0.9	7	10	0.9
0.9 – 1.0	16	10	3.6

Step 5 :- $\chi_0^2 = 0.4 + 0.1 + 0 + 1.6 + 0.9 + 0.4 + 0.1 + 0.4 + 0.9 + 3.6 = 8.4$

Step 6 :- $\alpha = 0.05$, $n - 1 = 9$ $\therefore \chi^2_{\alpha, n-1} = 16.9$ (From chi-square Table)

Step 7 :- $\chi_0^2 < \chi^2_{\alpha, n-1}$ $\therefore H_0$ is accepted

$\therefore$ No's are uniformly distributed

Scanned by PDF Scanner

- Monte-Carlo Simulation :- (Numericals from YT).
Mathematical technique that simulates range of possible outcomes for an uncertain event. These predictions are based on estimated range of values instead of fixed set of values & evolve randomly. Computers use this method to analyze data & predict future outcome based on action.
Usecases :- finance, Project management, healthcare.

- Runs Test :-
It is a statistical procedure which determines whether sequence of data within given distribution have been derived with random process or not.

- Runs Up & Runs Down Algo :-

S1 :- Define Hypothesis as -
H_0 :- $R_i \sim$ independently
H_1 :- $R_i \nsim$ independently

S2 :- Write sequence of runs up & runs down

S3 :- Count total no. of runs (a) present in sequence.

S4 :- Count mean & variance a :-

$$\mu_a = \frac{2N-1}{3} \qquad \sigma_a^2 = \frac{16N-29}{90}$$

S5 :- Std normal statistics :-

$$Z_0 = \frac{a-\mu_a}{\sigma_a} \; ; \; Z_0 \sim N[0,1]$$

S6 :- Determine critical value, $-Z_{\alpha/2} \le Z_0 \le Z_{\alpha/2}$
If above cond^n checked
$\Rightarrow$ H_0 not rejected.

Scanned by PDF Scanner

PAGE No.

DATE / /

Q. Use RURD for $\alpha = 0.05$ $\therefore Z_{\alpha/2} = 0.025 = 1.96$

Test nos :- 0.12, 0.01, 0.23, 0.28, 0.89, 0.31, 0.64, 0.28, 0.33, 0.93

→ S1 :- Define Hypothesis. (Ho :- $R_i \sim$ independent
H1 :- $R_i \nsim$ independent)

S2 :- Sequence :-

subtract " " " " " " " "

0.12, 0.01, 0.23, 0.28, 0.89, 0.31, 0.64, 0.28, 0.33

If after subtraction, result is -ve then put '−' else '+'

$$\underbrace{-}_{1}\ \underbrace{+\ +\ +}_{2}\ \underbrace{-}_{3}\ \underbrace{+}_{4}\ \underbrace{-}_{5}\ \underbrace{+\ +}_{6}$$

(Note :- actual no's in qn 10 but seq is 9)

$\therefore$ Total no. of runs $a = 6$.

S3 :- $$\mu_a = \frac{2N-1}{3} = \frac{2 \times 10 - 1}{3} = 6.33$$

$$\sigma_a^2 = \frac{16N - 29}{90} = \frac{16 \times 10 - 29}{90} = 1.45.$$

S4 :- $$Z_0 = \frac{a - \mu_a}{\sigma_a} = \frac{6 - 6.33}{\sqrt{1.45}} = -0.27$$

S5 :- $Z_{0.025} = 1.96$

$-1.96 \le Z_0 \le 1.96$

$\therefore$ Cond^n satisfies.

Ho is accepted.

Scanned by PDF Scanner

PAGE No.
DATE / /

- Runs Above & Below Mean Test :-

S1 :- Define Hypothesis for independence

S2 :- seq of runs above & below mean

S3 :- Count no. of obs above mean (n_1) & below mean (n_2) & b = Total no. of runs.

S4 :- Compare mean & variance of b,

$$\mu_b = \frac{2n_1 . n_2}{N} + \frac{1}{2}$$

$$\sigma_b^2 = \frac{2n_1 . n_2(2n_1 . n_2 - N)}{N^2(N-1)}$$

S5 :- Compute std. normal

$$Z_0 = \frac{b - \mu_b}{\sigma_b} , \quad Z_0 \sim N[0, 1]$$

S6 :- same as RURD

S7 :- Same as RURD.

Q. Test no's by RABM,

0.11, 0.23, 0.45, 0.08, 0.11, 0.50, 0.09, 0.60, 0.81

($\alpha = 0.05$ $\therefore$ $Z_{\alpha/2} = Z_{0.025} = 1.96$)

→ S1 :- Define Hypothesis.

S2 :- for decimal no's take mean 0.495 & for whole no's take mean 49.5.

Now, we take mean as 0.495.

If no's are < 0.495 put '−' else '+'

$\therefore$ $\underbrace{- \quad - \quad - \quad - \quad -}_{1} \quad \underbrace{+}_{2} \quad \underbrace{-}_{3} \quad \underbrace{+ \quad +}_{4.}$

b = 4.

$n_1 = 3$, $n_2 = 6$

S3 :- $\mu_b = \frac{2n_1 . n_2}{N} + \frac{1}{2} = \frac{2 . 3 . 6}{9} + \frac{1}{2} = 4.5$.

Scanned by PDF Scanner

$$\sigma_b^2 = \frac{2n_1 n_2 (2n_1 n_2 - N)}{N^2(N-1)}$$

$$= \frac{2\times3\times6\,(2\times3\times6-9)}{9^2(9-1)} = 1.5.$$

S4 : $Z_0 = \frac{b-\mu_b}{\sigma_b} = \frac{4-4.5}{\sqrt{1.5}} = -0.4.$

S5 :- $-Z_{\alpha/2} \le Z_0 \le Z_{\alpha/2}$

$-1.96 \le -0.4 \le 1.96$

Cond^n satisfies.

$\Rightarrow H_0$ accepted.

- Length of Runs by RABM or RURD :- (Numerical from 4T)

S1 :- Define Hypo.

S2 :- sequence of RURD/RABM

S3 :- find total length of runs in seq.

S4 :- Prepare table for obs. runs :-

length of Runs	1	2	3	...	n
obs no. of runs.	-	-	-	...	-

S5 :- Compute exp value of Y_i

1] RURD, $E(Y_i) = \frac{2}{(i+3)!} \times \left[N^2(i^2+3i+1) - (i^3+3i^2-i-4)\right]$

only if $i \le N-2$

else.

$\frac{2}{N!}$; $i \le N-1$

2] RABM, $E(Y_i) = \frac{N \times \omega_i}{E(I)}$, $N > 20$

$$\omega_i = \left(\frac{n_1}{N}\right)^i \left(\frac{n_2}{N}\right) + \left(\frac{n_1}{N}\right)\left(\frac{n_2}{N}\right)^i$$

DATE / /

$$E(I) = \frac{n_1}{n_2} + \frac{n_2}{n_1}$$

S6 :- Mean for RURD $\Rightarrow \mu_a = \frac{2N-1}{3}$

Exp total no of runs (RABM) $\Rightarrow E(A) = \frac{N}{E(I)}$

S7 :- Compute exp no of runs $\geq$ max. length

for RURD = $\mu_a - \sum_{i=1}^{m} E(Y_i)$

RABM = $E(A) - \sum_{i=1}^{m} E(Y_i)$

S8 :- Apply Chi-Square Test

S9 :- $\chi^2_{\alpha, n-1}$

S10 :- $\chi_0 < \chi^2_{\alpha, n-1}$. If this condn satisfies then H_0 is accepted.

- Auto correlation Test :-

$i + (M+1)m = N \longrightarrow$ find M.

$$\hat{\rho} = \frac{1}{M+1} \left[\sum_{k=0}^{M} R_{i+km} \cdot R_{i+(k+1)m} \right] - 0.25$$

$$\hat{\sigma} = \sqrt{\frac{13M+7}{12(M+1)}}$$

$Z_0 = \frac{\hat{\rho}}{\hat{\sigma}}$ If $Z_0 < Z_{critical} \Rightarrow H_0$ accepted.

i.e $-Z_{\alpha/2} \leq Z_0 \leq Z_{\alpha/2}$

Scanned by PDF Scanner

PAGE No.	
DATE	/ /

Q. Consider seq of 30 no's. Test whether 3rd, 8th, 13th, ... numbers in seq are auto correlated with $\alpha = 0.05$ & $Z_{0.025} = 1.96$.

Rand. no :-

0.12, 0.01, 0.23, 0.28, 0.89,
0.31, 0.64, 0.28, 0.83, 0.93,
0.99, 0.15, 0.33, 0.35, 0.91,
0.41, 0.60, 0.27, 0.75, 0.88,
0.69, 0.49, 0.05, 0.43, 0.95,
0.58, 0.19, 0.36, 0.69, 0.87

→ S1:- H_0 :- assuming no's are independent correlated.

S2:- $i + (M+1)m = N$.

$3 + (M+1)5 = 30$.

$\therefore M = 4.4 \approx 4$.

S3:- $\hat{\rho} = \frac{1}{4+1} \left[R_3 \cdot R_8 + R_8 \cdot R_{13} + R_{13} \cdot R_{18} + R_{18} \cdot R_{23} + R_{23} \cdot R_{28} \right] - 0.25$

$\hat{\rho} = \frac{1}{5} \left[0.23 \times 0.28 + 0.28 \times 0.33 + 0.33 \times 0.27 + 0.27 \times 0.05 + 0.05 \times 0.36 \right] - 0.25$

$\hat{\rho} = -0.194$

$\hat{\sigma} = \sqrt{\frac{13M+7}{12(M+1)}} = 0.128$

$Z_0 = \frac{\hat{\rho}}{\hat{\sigma}} = \frac{-0.194}{0.128} = -1.516$.

$-Z_{\alpha/2} \leq Z_0 \leq Z_{\alpha/2} = -1.96 \leq -1.56 \leq 1.96$

Cond^n satisfies

$\therefore$ H_0 accepted

Scanned by PDF Scanner

DATE / /

- Gap Test :-

It is used to count no. of digits betwn successive occurences of same digits.

The probability of gap is determined by

P(m followed by exactly x non-m digits)

$= (0.9)^x (0.1)\ ,\ x = 0, 1, 2, \ldots$

Algo :-

S1 :- Define Hypo. H_0 :- $R_i \sim$ indep.

H_1 :- $R_i \nsim$ "

S2 :- Determine no. of gaps & length of each gap associated with each digit (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

S3 :- Select interval width based on no. of gaps & generate freq. distribution table for sample of gaps & apply KS Test

Gap leng	freq	Rel. freq.	Acc. rel. freq. (or cumulative) $S_N(x)$	CDF $F(x)$	$\lvert F(x) - S_N(x) \rvert$

S4 :- Compute test stats D, which is max^m deviation betwn $F(x)$ & $S_N(x)$

$$D = \max[F(x) - S_N(x)]$$

S5 :- Determine cri. value, D_α for size N for LOS, α from table. $\left(D_\alpha = \frac{1.36}{\sqrt{N}}\right)$

S6 :- If $D > D_\alpha \Rightarrow H_0$ rejected.

CDF :- $F(x) = 1 - 0.9^{x+1}$ where x is upper limit

Numerical from YT.

Scanned by PDF Scanner