Proceedings of the 2007 IEEE/RSJ International Conference on Intelligent Robots and Systems
San Diego, CA, USA, Oct 29 - Nov 2, 2007

Force-Closure Analysis of General 6-DOF Cable Manipulators

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***Abstract*—This paper proposed a systematic method of verifying the force-closure condition for general 6-DOF cable manipulators with more than seven cables. Without explicitly solving the inverse dynamics problem, the method can determine, by examining the Jacobian matrix of the manipulator, whether a feasible force solution exists. The equivalence (i.e., necessity and sufficiency) of the proposed method to an existing force-closure theorem developed for the robot grasping problem is mathematically proven and hence, this work rigorously shows, for the first time, that the theory for grasping problem is also applicable to cable manipulators. Moreover, a convex-analysis based simplification of the method is discussed.**

I. Introduction

A cable-driven parallel manipulator (called *cable manipulator* for short in this paper) consists of an end-effector and a number of cables and actuators. The cables connect the end-effector in parallel to the actuators mounted on a fixed base and control the end-effector by exerting tensions in the cables. Cable manipulators have several advantages over their rigid-link counterparts [1], [2], [3]. First, for the same manipulator size, they can have larger workspaces because their joints can reel out a large amount of cables. Second, all of their actuators and transmission systems can always be mounted on the fixed base and thus, they have a higher payload-to-weight ratio, which makes them attractive for high-load and high-acceleration applications. Third, their special designs make them less expensive, modular, and easy to reconfigure. Finally, and also the most important characteristic for model-based controls, they have much simpler dynamics model than their rigid-link counterparts if the inertia of the cables is ignored because the mass of the cables is usually much smaller than those of the end-effector and payload.

A fundamental requirement for a cable manipulator to be fully controllable is that the cable forces must be able to balance any wrenches applied to the end-effector including the inertia wrench while no cables are required to be in compression at any working configuration. This non-compression condition for the cables means that all the cable forces must be nonnegative if a compression is represented by a negative force and a tension by a nonnegative force. Such a force feasibility problem is a *force-closure* or a vector-closure problem where the referred vectors are the row vectors of the Jacobian matrix of the manipulator. Precisely, the force-closure problem referred to here is indeed a *wrench-closure* problem because it deals with both force and moment in 3-D space. The term "force-closure" is still employed in the paper because it has been popularly used in the robotics community. Several researchers pointed out that the force-closure problem of cable manipulators is similar to that of multiple fingers grasping a frictionless rigid-body [4], [5]. In the former, all the cables must be in tension while in the latter all the fingers must be in compression. The grasping problem has been extensively studied in 80's and 90's of the last century. The most cited works for solving the rigid-body grasping problem are based on convex analysis [6], [7]. Handling the force-closure problem of cable manipulators based on the similarity between cable manipulators and multi-finger hands can be found in [5], [8], [9]. Previous research works mainly focused on the solution of nonnegative cable forces from statics or dynamics equations in terms of a given configuration with or without additional conditions (e.g., joint rate limits, joint force limits, etc.). Much less work has been reported in developing systematic methods of judging whether the force-closure condition is satisfied for a given configuration of a general 6-DOF cable manipulator with more than seven cables.

Ming and Higuchi [4] provided a proof of the design condition of keeping positive cable forces. However, the proof of the sufficient condition was based upon an assumption that the matrix $(\mathbf{1}-\mathbf{A}^{+}\mathbf{A})$ is always nonnegative, where $\mathbf{A}$ is the transpose of the Jacobian matrix and $\mathbf{1}$ is the identity matrix. Whether or not this assumption holds true for a general n-cable manipulator is not obvious and hence, more detailed proof is needed. Although the authors of [8] pointed out that the positive cable forces of a 6-DOF 7-cable manipulator can be obtained if the row vectors of the Jacobian matrix satisfy the force-closure condition as defined in [6], they missed a key step that is to mathematically show the relationship between the force-closure condition and the end-effector pose of such a manipulator. Moreover, it did not address the question of how to systematically judge whether or not the force-closure condition is satisfied from a given Jacobian matrix without explicitly solving the inverse

Manuscript received March 28, 2007. This work was supported in part by the 21st Century Aerospace Research Cluster, New Mexico State University.

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dynamics problem. Diao and Ma [10] proposed a method of checking whether the force-closure condition is satisfied for a configuration of a 6-DOF 7-cable manipulator. The method was mathematically proven. Gouttefarde and Gosselin [11] proved under what condition the force-closure exists for a configuration of a k-DOF cable manipulator with $k+1$ cables. Pham et al. [12] proposed a "recursive dimension reduction algorithm" to check the force-closure condition of cable manipulators. The algorithm was derived based on convex analysis presented in [13]. Although the algorithm is systematic, no mathematical proof was provided to show that it is equivalent to the original force-closure theorem as defined in their cited reference [13]. Moreover, the algorithm is inefficient in computation because it has a computational complexity of $O(n^{k-1})$ for a k-DOF n-cable manipulator.

In this paper a systematic method of verifying the force-closure condition for 6-DOF cable manipulators with more than seven cables is developed. With this method, one only needs to form and check at most C_n^5 vectors in R^6 to determine the force-closure status of a 6-DOF n-cable manipulator. Since no assumptions on the design or the architecture of cable manipulators are imposed in the modeling and analysis, the method is generally applicable to any 6-DOF n-cable manipulator as long as its Jacobian matrix has a full rank. The paper also provided a convex-analysis based discussion on how to verify the force-closure condition by checking less than C_n^5 vectors in R^6.

The remaining of the paper is organized as follows: The modeling of cable manipulators is presented in Section II, which is followed by introducing the systematic method of checking force-closure condition in Section III. A convex-analysis based simplification of the proposed method is discussed in Section IV. The paper is concluded in Section V.

II. MODELING OF CABLE MANIPULATORS

The kinematics and dynamics models of a general 6-DOF n-cable $(n > 7)$ manipulator are derived based on the architecture shown in Figs. 1 and 2, respectively. The end-effector (moving platform) of such a manipulator is assumed to be controlled by more than seven cables with their driving actuators mounted to the fixed base.

In Fig. 1, $\mathbf{q}_i \in R^3$ $(i = 1,2,\cdots,n)$ is the vector along the ith cable and has the same length as the cable. The length of the ith cable is represented by scalar q_i which is also considered as the manipulator's joint variable. $\mathbf{u}_i$ is the unit vector along the ith cable. A_i and B_i are the two attaching points of the ith cable on the base and the end-effector, respectively. The positions of the two attaching points are represented by vectors $\mathbf{a}_i$ and $\mathbf{b}_i$, respectively. Obviously, $\mathbf{a}_i$ is a constant vector in the base frame F_o and $\mathbf{b}_i$ is a constant vector in the end-effector frame F_e. The origin of frame F_e is fixed at a reference point P of the end-effector, which is used to define the position of the end-effector. Based on the kinematics notation defined in Fig. 1, the position of the end-effector can be described as

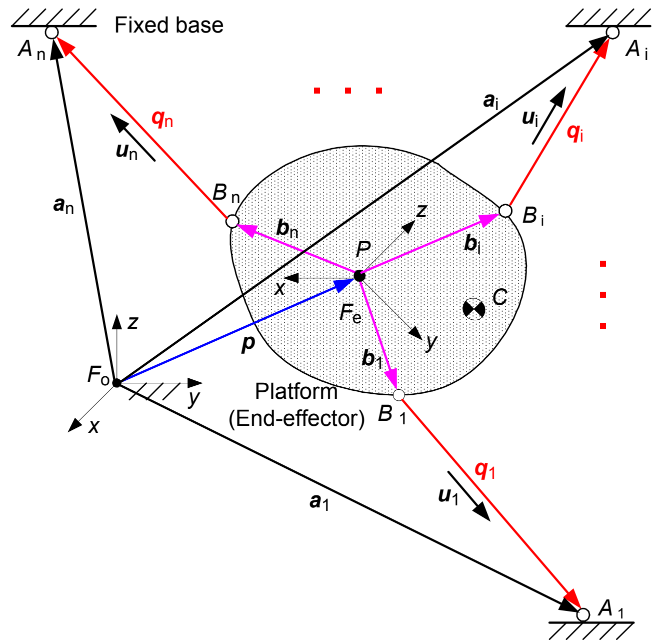

Fig. 1. Kinematics notation of a general 6-DOF cable manipulator

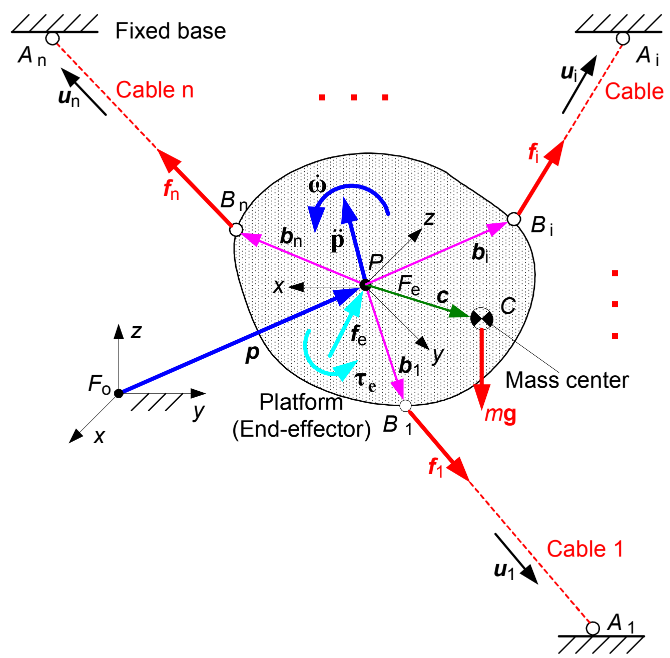

Fig. 2. Dynamics notation of a general 6-DOF cable manipulator

$$\mathbf{p} = \mathbf{a}_i - \mathbf{q}_i - \mathbf{b}_i \quad \text{for } i = 1,2,\cdots,n \tag{1}$$

from which one has

$$q_i^2 = [\mathbf{a}_i - \mathbf{p} - \mathbf{b}_i]^T[\mathbf{a}_i - \mathbf{p} - \mathbf{b}_i] \quad \text{for } i = 1,2,\cdots,n \tag{2}$$

Differentiating (2) with respect to time, and then organizing the n resulting equations into a matrix form, one obtains

$$\dot{\mathbf{q}} = \mathbf{J}\mathbf{t} \tag{3}$$

where

$$\dot{\mathbf{q}} \equiv [\dot{q}_1 \quad \dot{q}_2 \quad \cdots \quad \dot{q}_n]^T \tag{4}$$

$$\mathbf{J} \equiv -\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \\ \mathbf{b}_1 \times \mathbf{u}_1 & \mathbf{b}_2 \times \mathbf{u}_2 & \cdots & \mathbf{b}_n \times \mathbf{u}_n \end{bmatrix}^T \tag{5}$$

$$\mathbf{t} \equiv \begin{bmatrix} \dot{\mathbf{p}} \\ \boldsymbol{\omega} \end{bmatrix} \equiv [\dot{x} \quad \dot{y} \quad \dot{z} \quad \omega_x \quad \omega_y \quad \omega_z]^T \tag{6}$$

In the above equations, vector $\dot{\mathbf{p}}$ represents the linear velocity of point P on the end-effecotr; vector $\boldsymbol{\omega}$ is the angular velocity of the end-effector; and vector $\mathbf{t}$ represents the twist vector in R^6 which consists of both the linear and angular velocities of the end-effector. Moreover, $\mathbf{J}$ is the $n \times 6$ Jacobian matrix of the cable manipulator. For a cable manipulator, the inertia of the cables is usually negligible compared to those of the end-effector and the payload. Therefore, one can ignore the inertia of the cables, which will significantly simplify the dynamics model of the manipulator. In other words, each cable can be modeled as a massless string. Based on the dynamics notation defined in Fig. 2, one can derive the equations of motion of a cable manipulator with respect to point P as follows

$$\mathbf{J}^T \mathbf{f} = \mathbf{w}_e + \mathbf{w}_g - \mathbf{M}\dot{\mathbf{t}} - \mathbf{N}\mathbf{t} \tag{7}$$

which can be rewritten into a compact form as

$$\mathbf{A}\mathbf{f} = \mathbf{w} \tag{8}$$

where

$$\begin{aligned} &\mathbf{A} \equiv \mathbf{J}^T,\ \mathbf{f} \equiv [f_1 \quad f_2 \quad \cdots \quad f_n]^T,\ \mathbf{w} \equiv \mathbf{w}_e + \mathbf{w}_g - \mathbf{M}\dot{\mathbf{t}} - \mathbf{N}\mathbf{t} \\ &\mathbf{M} \equiv \begin{bmatrix} m\mathbf{1} & -m\mathbf{c}^\times \\ m\mathbf{c}^\times & \mathbf{I} \end{bmatrix},\ \mathbf{N} \equiv \begin{bmatrix} \mathbf{0} & -m(\boldsymbol{\omega} \times \mathbf{c})^\times \\ m(\boldsymbol{\omega} \times \mathbf{c})^\times & -(\mathbf{I}\boldsymbol{\omega})^\times \end{bmatrix} \\ &\mathbf{w}_e \equiv \begin{bmatrix} \mathbf{f}_e \\ \boldsymbol{\tau}_e \end{bmatrix},\ \mathbf{w}_g \equiv \begin{bmatrix} m\mathbf{g} \\ \mathbf{c} \times m\mathbf{g} \end{bmatrix} \end{aligned} \tag{9}$$

In the above equations, vectors $\mathbf{w}_e \in R^6$ and $\mathbf{w}_g \in R^6$ are the external wrench and the gravity wrench exerted on point P, respectively; m is the mass of the end-effector including any attached payload; $\mathbf{I}$ is the 3×3 inertia tensor of the end-effector about point P; $\mathbf{g} \in R^3$ is the gravity acceleration vector; vectors $\mathbf{f}_e \in R^3$ and $\boldsymbol{\tau}_e \in R^3$ are external force and moment applied to point P; $\mathbf{f} \in R^n$ is a vector consisting of all individual cable forces; scalar f_i is the cable force of the ith cable. In addition, $\mathbf{0}$ and $\mathbf{1}$ are the 3×3 zero matrix and identity matrix, respectively; $\mathbf{c} \in R^3$ is the position vector of the mass center of the end-effector in frame F_e; and $(\ )^\times$ is a operator representing the cross product $(\ )\times$.

For any given motion condition (i.e., pose, velocities and accelerations of the end-effector) and external wrench, one can find a set of cable forces from (8), which is indeed the inverse dynamics problem of the manipulator. Since a cable can support a tension (i.e., a nonnegative force) only, the solution of the inverse dynamics problem will not always be feasible. This force feasibility problem is indeed a force-closure problem which will be addressed in the next section.

III. Method of Checking Force-Closure

A cable manipulator is said to have a force-closure in a particular configuration if and only if any external wrench applied to the end-effector can be sustained through a set of nonnegative cable forces. In other words, the force-closure condition is satisfied if and only if the inverse dynamics problem has a feasible solution regardless the external wrench applied to the end-effector. Such a force-closure condition can be mathematically described as

$$\forall\, \mathbf{w} \in R^6,\ \exists\, \mathbf{f} \geq \mathbf{0},\ \ni \mathbf{A}\mathbf{f} = \mathbf{w} \tag{10}$$

where $\mathbf{f} \geq \mathbf{0}$ means that each component of vector $\mathbf{f}$ is greater than or equal to zero. Equation (10) indicates that the force-closure condition is satisfied if and only if the column vectors of matrix $\mathbf{A}$, denoted by $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, can positively span R^6. In other words, any vector $\mathbf{v} \in R^6$ can be expressed as $\mathbf{v} = f_1\mathbf{a}_1 + f_2\mathbf{a}_2 + \cdots + f_n\mathbf{a}_n$, where $f_i \geq 0$ $(i = 1, 2, \cdots, n)$. According to [14], the force-closure condition is equivalent to the following theorem.

Theorem 1: Equation (10) has a solution if and only if the nonzero projections of all the n column vectors of matrix $\mathbf{A}$ on every direction in R^6 do not have the same sign. In other words, for any nonzero vector $\mathbf{v} \in R^6$, the nonzero dot products of vector $\mathbf{v}$ and the column vectors of matrix $\mathbf{A}$ must have different signs (i.e., some of them are positive and some are negative).

Note that the wording of Theorem 1 has been modified from its original form in [14] for easier understanding of its geometric interpretation. As pointed out by [14], the proof of the theorem can be found in [13]. The geometric interpretation of the theorem is: for any hyperplane passing through the origin of R^6, at least one of the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ points to one side of the hyperplane and another one points to the other side of the hyperplane. In other words, there should not exist one hyperplane passing through the origin of R^6 and having all the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ except those on the hyperplane point to only one side of the hyperplane.

Theorem 1 states that, if and only if the set of nonzero dot products $\mathbf{v}^T\mathbf{a}_i$ $(i = 1, 2, \cdots, n)$ have different signs for <u>every</u> nonzero vector $\mathbf{v}$ in R^6, then matrix $\mathbf{A}$ satisfies the force-closure condition. This theorem, used as is, is inconvenient for verifying force-closure because it requires one to check the sign condition for each and every nonzero vector in R^6. A computationally more attractive and also systematic method is to check only a number of vectors in R^6 which are formed from the column vectors of matrix $\mathbf{A}$. Such a method is introduced and proven next.

Method 1: Let $\mathbf{n} \in R^6$ be a nonzero vector perpendicular to any set of 5 linearly independent column vectors of matrix $\mathbf{A}$. The necessary and sufficient condition for (10) to have a solution is that the nonzero dot products of vector $\mathbf{n}$ and the remaining column vectors of matrix $\mathbf{A}$ (i.e., those column vectors not used to form vector $\mathbf{n}$) have different signs.

Since the $\mathbf{n}$ vector is indeed the normal vector of the hyperplane (in R^6) formed by the 5 linearly independent column vectors used to form vector $\mathbf{n}$, it is called *normal vector* for the convenience of discussions in the rest of the paper. Algorithmically, Method 1 can be implemented as described in the following procedure:

1) Select a set of 5 linearly independent column vectors of matrix $\mathbf{A}$ to form a normal vector $\mathbf{n}$. This is always possible because $\mathbf{A}$ has been assumed to have a full rank. For example, if $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_5$ are the 5 selected column vectors. Then $\mathbf{n}$ can be determined from a generalized cross product defined as [2]:

$$\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_5 \tag{11}$$

In fact, normal vector $\mathbf{n}$ is a candidate for vector $\mathbf{v}$ in Theorem 1. The components of normal vector $\mathbf{n}$ can be calculated as

$$n_i = (-1)^{i+1} \det[\mathbf{a}_1^i \ \mathbf{a}_2^i \ \cdots \mathbf{a}_5^i], \quad i = 1, 2, \cdots, 6 \tag{12}$$

where $\mathbf{a}_j^i$ is equal to $\mathbf{a}_j$ with its *i*th component removed.

2) Check the signs of the nonzero dot products of normal vector $\mathbf{n}$ and the remaining column vectors of matrix $\mathbf{A}$ (i.e., $\mathbf{a}_6$, $\mathbf{a}_7, \cdots, \mathbf{a}_n$ in the example). If they have the same sign, one can conclude that (10) has no solution. Otherwise, go to step 3).

3) Select another set of 5 linearly independent column vectors of matrix $\mathbf{A}$ and repeat steps 1) and 2). There will be up to C_n^5 sets of 5 linearly independent column vectors of matrix $\mathbf{A}$ to form up to C_n^5 normal vectors. If all of these C_n^5 normal vectors have passed Step 2), then (10) has a solution.

The difference of Method 1 from Theorem 1 is that the former requires one to form and check at most C_n^5 normal vectors while the latter requires one to check the sign condition for <u>all</u> nonzero vectors in R^6. Hence, Method 1 is algorithmically more convenient to use. References [9] and [15] also mentioned this method in different forms. However, they did not present a proof that the method provides a necessary and sufficient condition for force-closure. In other words, it has not been proven that the examination of only C_n^5 normal vectors (i.e., the $\mathbf{n}$ vectors) as defined in Method 1 is equivalent to the examination of <u>all</u> the nonzero vectors (i.e., the $\mathbf{v}$ vectors) in R^6 as required by Theorem 1. To fill up this gap, a proof of the equivalence between Method 1 and Theorem 1 using convex analysis is provided in the rest of this section.

Proof of necessary condition:

What needs to be proven is that, if the nonzero dot products of every nonzero vector $\mathbf{v}$ in R^6 and the column vectors of matrix $\mathbf{A}$ have different signs (Theorem 1), then the nonzero dot products of every normal vector $\mathbf{n}$ and the remaining column vectors of matrix $\mathbf{A}$ (i.e., those column vectors not used to form the $\mathbf{n}$ vector) must also have different signs (Method 1).

Since the nonzero dot products of every nonzero vector $\mathbf{v}$ in R^6 and the column vectors of matrix $\mathbf{A}$ have different signs, it can be concluded that, for each of the C_n^5 normal vectors formed in Method 1, the nonzero dot products of the normal vector and the column vectors of matrix $\mathbf{A}$ must also have different signs because all the normal vectors are nonzero vectors in R^6 by definition. Without loss of generality, assume that normal vector $\mathbf{n}$ is one of the C_n^5 normal vectors, which is perpendicular to 5 linearly independent column vectors of matrix $\mathbf{A}$, say, $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_5$, then one can obtain

$$\mathbf{n}^T \mathbf{A} = \mathbf{n}^T [\mathbf{a}_1 \cdots \mathbf{a}_5 \ \ \mathbf{a}_6, \cdots \mathbf{a}_n] = [0 \cdots 0 \ \ \mathbf{n}^T \mathbf{a}_6, \cdots \mathbf{n}^T \mathbf{a}_n] \tag{13}$$

which indicates that only the dot products of normal vector $\mathbf{n}$ and the remaining column vectors, $\mathbf{a}_6, \mathbf{a}_7, \cdots, \mathbf{a}_n$, can be nonzero and have a sign. Thus, one can deduce that for each of the C_n^5 normal vectors, the nonzero dot products of the normal vector and the remaining column vectors of matrix $\mathbf{A}$ must have different signs. Hence, the necessary condition is proven.

Proof of sufficient condition:

Sufficient condition is more difficult to prove. What needs to be proven is that, if the nonzero dot products of each of the C_n^5 normal vectors and the remaining column vectors of matrix $\mathbf{A}$ have different signs (Method 1), then one can ensure that the nonzero dot products of every nonzero vector in R^6 and the column vectors of matrix $\mathbf{A}$ will also have different signs (Theorem 1). This will be done by a strategy of disproof. That is first to assume that the sufficient condition is met but the conclusion is opposite; then a consequent condition being contradictive to the assumed sufficient condition can be drawn. Specifically, assume that the sufficient condition holds true and there does exist a nonzero vector $\mathbf{v}$ in R^6 meeting the opposite conclusion that all the nonzero dot products of vector $\mathbf{v}$ and the column vectors of matrix $\mathbf{A}$ have the same sign. Then this will consequently lead to a condition which contradicts the sufficient condition, i.e., the nonzero dot products of each of the C_n^5 normal vectors and the remaining column vectors of matrix $\mathbf{A}$ have different signs.

Suppose the sufficient condition of Method 1 holds true and there does exist a nonzero vector $\mathbf{v}$ in R^6 such that all the nonzero dot products of vector $\mathbf{v}$ and the column vectors of matrix $\mathbf{A}$ have the same sign. Without loss of generality, assume this same sign being positive, i.e.,

$$\mathbf{v}^T\mathbf{a}_i \geq 0 \quad \text{for } i = 1,2,\cdots,n \tag{14}$$

The vector $\mathbf{v}$ can define a hyperplane passing through the origin of R^6, i.e.,

$$H = \{\mathbf{y} \in R^6 : \mathbf{v}^T\mathbf{y} = 0\} \tag{15}$$

From (14) one knows that all the $\mathbf{a}_i$ vectors except those on hyperplane H point to only one side of hyperplane H.

Let vector $\mathbf{o} = [0\,0\,0\,0\,0\,0]^T$ denote the origin of R^6 and $\mathrm{co}(\mathbf{o},\mathbf{a}_1,\cdots,\mathbf{a}_n)$ denote the convex hull formed by the ends of the vectors $\mathbf{o},\mathbf{a}_1,\cdots,\mathbf{a}_n$. By the definition of convex hull, the origin of R^6 must be either inside or on the boundary of $\mathrm{co}(\mathbf{o},\mathbf{a}_1,\cdots,\mathbf{a}_n)$. With this in mind and also note that hyperplane H passes through the origin of R^6 and that all the $\mathbf{a}_i$ vectors except those on the hyperplane point to only one side of the hyperplane, one can conclude that the origin of R^6 must be on the boundary of $\mathrm{co}(\mathbf{o},\mathbf{a}_1,\cdots,\mathbf{a}_n)$. Otherwise, hyperplane H must cut through $\mathrm{co}(\mathbf{o},\mathbf{a}_1,\cdots,\mathbf{a}_n)$, which means at least one of the vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n$ points to one side of the hyperplane and another one points to the other side of the hyperplane. As a result, $\mathrm{co}(\mathbf{o},\mathbf{a}_1,\cdots,\mathbf{a}_n)$ must have a facet containing the origin of R^6 together with five linearly independent $\mathbf{a}_i$ vectors. Let H_c denote the hyperplane containing this facet. Then hyperplane H_c can be uniquely determined by the vertices of the facet. Without loss of generality, assume that vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_5$ are a set of five linearly independent $\mathbf{a}_i$ vectors on hyperplane H_c. Since hyperplane H_c passes through the origin of R^6 and a facet of $\mathrm{co}(\mathbf{o},\mathbf{a}_1,\cdots,\mathbf{a}_n)$, it is a supporting hyperplane and hence, the remaining $\mathbf{a}_i$ vectors (i.e., $\mathbf{a}_6,\mathbf{a}_7,\cdots,\mathbf{a}_n$) except those on hyperplane H_c must point to only one side of hyperplane H_c. As a result, the nonzero dot products of the normal vector $\mathbf{n}$ of hyperplane H_c and the remaining $\mathbf{a}_i$ vectors must have the same sign because all of these $\mathbf{a}_i$ vectors point to only one side of this hyperplane. Based on the n column vectors of matrix $\mathbf{A}$, one can form at most C_n^5 different normal vectors. Thus, the normal vector $\mathbf{n}$ of hyperplane H_c must be one of the C_n^5 normal vectors. This contradicts the sufficient condition of Method 1, i.e., the nonzero dot products of each of the C_n^5 normal vectors and the remaining column vectors of matrix $\mathbf{A}$ must have different signs. Therefore, there is no such a nonzero vector $\mathbf{v}$ in R^6 as assumed in (14). This completes the proof of the sufficient condition.

IV. SIMPLIFICATION OF THE METHOD

From Section III, one knows that Method 1 is equivalent to Theorem 1 and thus, it can be employed to check the force-closure condition for any 6-DOF cable manipulators with more than seven cables, as long as the jacobian matrix has a full rank. Although Method 1 requires one to check all the C_n^5 normal vectors, in theory one can check less normal vectors as discussed next.

Let $\mathrm{co}(\mathbf{a}_1,\cdots,\mathbf{a}_n)$ denote the convex hull formed by the ends of the column vectors $\mathbf{a}_1,\cdots,\mathbf{a}_n$ of matrix $\mathbf{A}$. By the definition of convex hull, the ends of the column vectors $\mathbf{a}_1,\cdots,\mathbf{a}_n$ must be either inside or on the boundary of $\mathrm{co}(\mathbf{a}_1,\cdots,\mathbf{a}_n)$. Thus, the column vectors of matrix $\mathbf{A}$ can be divided into two groups: Column Vector Group 1 (CVG1) and Column Vector Group 2 (CVG2). CVG1 consists of all the column vectors of matrix $\mathbf{A}$ whose ends are on the boundary of $\mathrm{co}(\mathbf{a}_1,\cdots,\mathbf{a}_n)$ while CVG2 consists of the column vectors of matrix $\mathbf{A}$ whose ends are inside $\mathrm{co}(\mathbf{a}_1,\cdots,\mathbf{a}_n)$. Recall that each of the C_n^5 normal vectors in Method 1 is formed by a set of 5 linearly independent column vectors of matrix $\mathbf{A}$. Thus, the C_n^5 normal vectors formed in Method 1 can also be divided into two groups: Normal Vector Group 1 (NVG1) and Normal Vector Group 2 (NVG2). If a normal vector is formed by a set of 5 linearly independent column vectors with at least one of which comes from CVG2, then it belongs to NVG2, otherwise, a normal vector belongs to NVG1. As already mentioned in Section III, the five linearly independent column vectors forming a normal vector $\mathbf{n}$ in Method 1 also form a unique 5-dimensional hyperplane passing through the origin and normal to vector $\mathbf{n}$. It is not difficult to understand from geometrical point of view that the hyperplane corresponding to any normal vector in NVG2 must cut through $\mathrm{co}(\mathbf{a}_1,\cdots,\mathbf{a}_n)$ because the hyperplane, by definition, contains at least one column vector whose end is inside the convex hull. Therefore, one can deduce that at least one of the column vectors $\mathbf{a}_1,\cdots,\mathbf{a}_n$ points to one side of the hyperplane and another one points to the other side of the hyperplane because the vertices of $\mathrm{co}(\mathbf{a}_1,\cdots,\mathbf{a}_n)$ must be the ends of column vectors. Hence, one can conclude that all the normal vectors in NVG2 must meet the sufficient condition of Method 1 and thus, do not need to be checked at all. As a result, what one really needs to check is only the normal vectors in NVG1.

Since graphic representations can be made in 1-D, 2-D or 3-D spaces only, the idea discussed above is illustrated using a 2-D example here. Assume that vectors $\mathbf{a}_1,\cdots,\mathbf{a}_5$ are the column vectors of matrix $\mathbf{A}$ of a 2-DOF 5-cable manipulator. The $\mathbf{a}_i$ vectors corresponding to two specific poses of the manipulator are shown in Figs. 3(a) and 3(b), respectively. Formed by the ends of the column vectors

$\mathbf{a}_1,\cdots,\mathbf{a}_5$, the convex hulls of the two specific poses of the manipulator are represented by the triangles in dotted lines in Fig. 3. The 5 (i.e., C_5^1) normal vectors formed according to Method 1 are represented by vectors $\mathbf{n}_1,\cdots,\mathbf{n}_5$ and the 5 hyperplanes corresponding to the 5 normal vectors are represented by $H_1,\cdots,H_5$. Note that the hyperplanes are straight lines in this case. Since each convex hull has the column vectors $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3$ and $\mathbf{a}_5$ in CVG1, one only needs to apply Method 1 to the 4 column vectors $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3$ and $\mathbf{a}_5$ in CVG1 rather than all the 5 column vectors of matrix $\mathbf{A}$, namely, one only needs to check the 4 normal vectors $\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3$ and $\mathbf{n}_5$ in NVG1 rather than all the 5 (i.e., C_5^1) normal vectors as required by Method 1. For the pose in Fig. 3(a), one can find that the nonzero dot products of each of the normal vectors $\mathbf{n}_1$ (or $\mathbf{n}_5$) and the corresponding remaining column vectors of matrix $\mathbf{A}$ have the same sign. Thus, it can be concluded that this pose does not satisfy the force-closure condition. On the contrary, for the pose in Fig. 3(b), one finds that the nonzero dot products of each of the normal vectors $\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3$ and $\mathbf{n}_5$ and the corresponding remaining column vectors of matrix $\mathbf{A}$ have different signs. Therefore, this pose satisfies the force-closure condition.

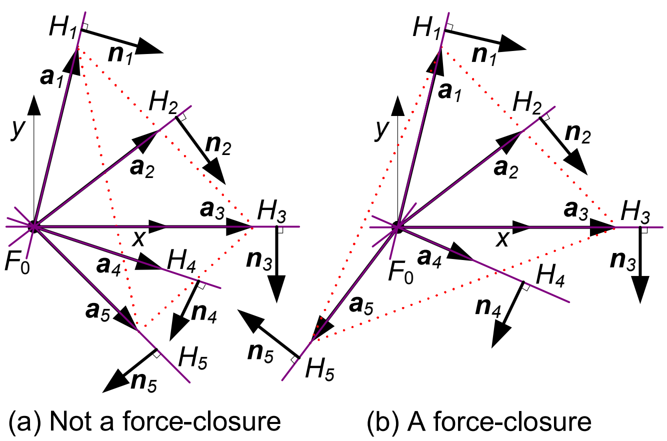

Fig. 3. A 2-D example of using convexity to determine the force-closure status at different poses

In fact, for complicated cable manipulators with more degrees of freedom and cables, the number of normal vectors really having to be checked can be significantly reduced in this way. However, it should be pointed out that, the above discussion of checking less normal vectors is of theoretical interest. Practically, it may not be so appealing because extra computations are required for identifying which of the column vectors $\mathbf{a}_1,\cdots,\mathbf{a}_n$ are in CVG1.

V. Conclusion

A systematic method was presented for verifying whether feasible cable-force solution always exists for a general cable-driven parallel manipulator with six degrees of freedom and more than seven cables. The verification is based upon the examination of the Jacobian matrix to see if it satisfies the force-closure condition (i.e., vector-closure or wrench-closure condition). It was proven in the paper for the first time that the proposed method is equivalent to the necessary and sufficient condition of force-closure stated in an existing force-closure theorem which was originally developed for applications other than cable manipulators. Therefore, this work rigorously connects the well-developed force-closure theory with the analysis of cable manipulators. It was also shown that the proposed method can be simplified if one can identify the convex hull formed by the row vectors of the Jacobian matrix.

References

[1] S.E. Landsberger and T.B. Sheridan, "A minimal, minimal linkage: the tension-compression parallel link manipulator", *Robotics, Mechatronics and Manufacturing Systems*, Vol. 2, 1993, pp. 81-88.

[2] R.G. Roberts, T.Graham and T.Lippitt, "On the inverse kinematics, statics, and fault tolerance of cable-suspended manipulators", *Journal of Robotic Systems*, Vol. 15, No. 10, 1998, pp. 581-597.

[3] P.M. Bosscher, "Disturbance robustness measures and wrench-feasible workspace generation techniques for cable-driven manipulators", *Doctoral Dissertation*, Georgia Institute of Technology, 2004.

[4] A. Ming and T. Higuchi, "Study on multiple degree-of-freedom positioning mechanism using wires (part 1)", *Int. Journal of the Japan Society of Precision Engineering*, Vol. 28, No. 2, 1994, pp. 131-138.

[5] I. Ebert-Uphoff and P.A. Voglewede, "On the Connections Between Cable-Driven Manipulators, Parallel Manipulators and Grasping", *Proc. of IEEE Int. Conf. on Robotics & Automation*, New Orleans, LA, 2004, pp. 4521-4526.

[6] V.-D. Nguyen, "Constructing force-closure grasps", *Int. Journal of Robotics Research*, Vol. 7, No. 3, 1988, pp. 3-16.

[7] M.R. Cutkosky, "On grasp, choice, grasp model, and the design of hands for manufacturing tasks", *IEEE Trans. On Robotics & Automation*, Vol. 5, No. 3, 1989, pp. 269-279.

[8] S. Kawamura, H. Kino, and C. Won, "High-speed manipulation by using a parallel wire-driven manipulators", *Robotica*, Vol. 18, 2000, pp. 13-21.

[9] G. Yang et al, "Kinematic design of a 7-DOF cable-driven humanoid arm: a solution-in-nature approach", *Proc. of IEEE/ASME Int. Conf. on Advanced Intelligent Mechatronics*, Monterey, CA, 2005, pp. 444-449.

[10] X. Diao and O. Ma, "A Method of Verifying the Force-Closure Condition for General Cable Manipulators with Seven Cables", Mechanism and Machine Theory, 2007. (to appear)

[11] M. Gouttefarde and C. Gosselin, "On the Properties and the Determination of the Wrench-Closure Workspace of Planar Parallel Cable-Driven Mechanisms", *Proc. Of the ASME int. Design Engineering Technical Conf. and Computer Information Engineering Conf.*, Salt Lake City, UT, 2004, pp. 337-346.

[12] C.B. Pham, S.H. Yeo, G. Yang, M.S. Kurbanhusen, and I-M. Chen, "Force-closure workspace analysis of cable-driven parallel mechanisms", *Mech. and Machine Theory*, Vol. 41, 2006, pp. 53-69.

[13] R.T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, 1970.

[14] R. Murray, Z. Li, and S.S. Sastry, *A Mathematical Introduction to Robotic Manipulation*, Boca Raton, FL, CRC Press, 1994.

[15] E. Stump and V. Kumar, "Workspace delineation of cable-actuated parallel manipulators", *Proc. Of the ASME int. Design Engineering Technical Conf. and Computer Information Engineering Conf.*, Salt Lake City, UT, Sep. 2004.