CompSci 161
Spring 2021 Lecture 10:
Divide and Conquer V:
Closest Pair of Points

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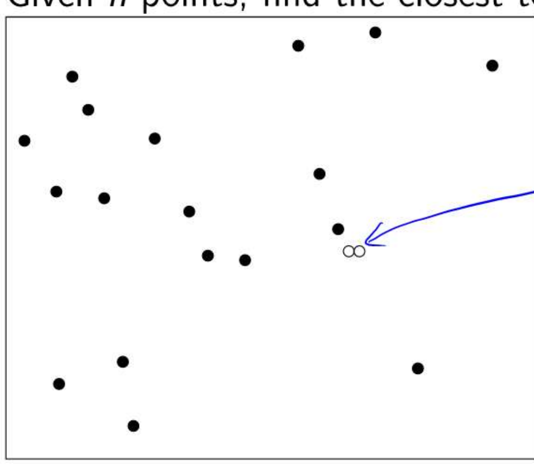
Closest Pair of Points

Given *n* points, find the closest together.



found them

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Closest Pair of Points: Brute Force

`Closest-Pair`
Input: n points in $2D$-space
Output: The closest pair of points.

min $= \infty$
for $i = 2 \to n$ **do**
  for $j = 1 \to i - 1$ **do**
    if $(x_j - x_i)^2 + (y_j - y_i)^2 <$ min **then**
      min $= (x_j - x_i)^2 + (y_j - y_i)^2$
      closestPair $= ((x_i, y_i), (x_j, y_j))$
return closestPair

Checks all pairs, $\mathcal{O}(1)$ each, $\mathcal{O}(n^2)$

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Closest Pair of Points: Starting D&C

Incorrect but reasonable starting point:

`Closest-Pair`
Input: n points in $2D$-space
Output: The closest pair of points.

If P is sufficiently small, use brute force. // $\mathcal{O}(1)$
$x_m \leftarrow$ median x-value from P
$L \leftarrow$ any points from P with x-coordinate $\leq x_m$
$R \leftarrow$ any points from P with x-coordinate $> x_m$
Let l_1 and l_2 be the closest pair of points in L
Let r_1 and r_2 be the closest pair of points in R
return whichever pair is closer together

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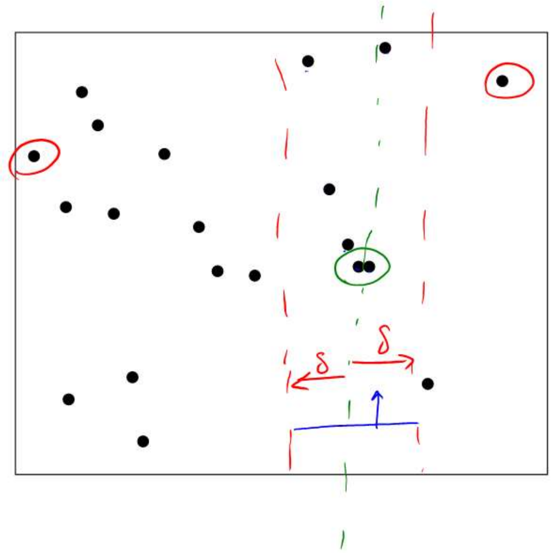
Visualizing the D&C



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Closest Pair of Points: Idea One

`Closest-Pair`
Input: n points in $2D$-space
Output: The closest pair of points.

- If P is sufficiently small, use brute force. // $\mathcal{O}(1)$
- $x_m \leftarrow$ median x-value from P
- $L \leftarrow$ any points from P with x-coordinate $\leq x_m$
- $R \leftarrow$ any points from P with x-coordinate $> x_m$
- Let l_1 and l_2 be the closest pair of points in L
- Let r_1 and r_2 be the closest pair of points in R
- Check all pairs (a, b) with $a \in L$ and $b \in R$

OK, but $\frac{n}{2}$ on each half

checks $O(n^2)$ pairs

$T(n) = 2T(n/2) + n^2$ is $O(n^2)$

need to find a way to do this in better than n^2 time

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Closest Pair of Points: Idea Two

Sort by y-coordinate prior to first call.

Idea two. Let $\delta =$ and partition the two...

If P is sufficiently small, use brute force. $//\mathcal{O}(1)$

$\Theta(n)$ $x_m \leftarrow$ median x-value from P ← use selection here

$\Theta(n)$ $L \leftarrow$ any points from P with x-coordinate $\leq x_m$

$R \leftarrow$ any points from P with x-coordinate $> x_m$

$2T(n/2)$ Let l_1 and l_2 be the closest pair of points in L

Let r_1 and r_2 be the closest pair of points in R

Let $\delta = \min(d(l_1, l_2), d(r_1, r_2))$

Let $M =$ set of points that could be closer than δ

$\Theta(n)$ steps Observation: only need those within δ of X_m by x-coordinate

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Finding closest pair in M

Now we have:

- $\delta =$ closest pair from L or R
- a set M of points, sorted by y-coordinate,

Determine one of two pieces of information:

- The closest pair of points in M but I only care if dist $< \delta$
- Nothing, if closest pair in M more than δ apart

9 ## How close can points in M be?

In order to finish the conquer step efficiently, I need to prove this claim:
Claim: if two points in M are distance $< \delta$ apart, then their indices are within a constant ← does not increase with #points
Why would I care? all points within δ of i are indexed $[i-c, i+c]$
TBD: value of c: 11

$\min = \delta$

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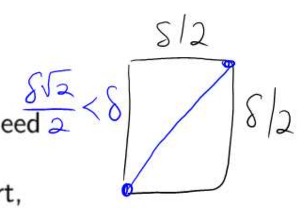
O(n) { for i=1 to |M|
         → for j=i+1 to min(i+c, |M|)     } O(1)
              if d(p_i, p_j) < min
                 min = d(p_i, p_j)
                 closest = (p_i, p_j)
```

$T(n) = 2T(n/2) + n$ is $O(n \log n)$

9 ## How close can points in M be?



In order to finish the conquer step efficiently, I need to prove this claim:
Claim: if two points in M are distance $< \delta$ apart, then their indices are within a constant
Proof: Look at points between $x_m + \delta$ and $x_m - \delta$