Suppose we have an even number of people who are going to play (doubles) tennis, forming teams of size two. Each player has a *tennis rating*, (a positive number, where a higher number can be interpreted to mean a better player). The *quality* of each team is the *lower* tennis rating of a member of the team.

Our goal is to maximize the *sum* of the quality of teams formed.

1. (4 points) Consider the following **greedy heuristic**: pair the best player (highest tennis rating) with the worst player (lowest tennis rating) and solve recursive with unpaired players until everyone has a partner. Give a counter-example to demonstrate that this heuristic *does not* achieve the optimal solution.

   A complete counter-example includes an input, the output the algorithm will give, and a better output for the same data. Examples with malformed input (such as, in this example, an odd number of players) would not qualify as a counter-example, although they would be a good test case for an implementation (to make sure a program handled the error case correctly).

   Here is a counter-example; it is far from the only one.

   Input: we have $n = 4$ players. Their ratings are, respectively, 1,2,3, and 4.

   This algorithm will pair 1 and 4 and also 2 and 3, for a total value of 3 (1+2).

   However, if we pair 4 and 3 and also 2 and 1, we get a total value of 4 (3 + 1).

2. (5 points) Prove the following claim. The standard for how formal the proof needs to be is the same as it was in the relevant lecture and the homework

   Consider the following algorithm. We pair the two best players together, then the third and fourth best players, and so on. This will maximize the sum of the quality of teams formed.

   Claim: There is an optimal algorithm for this problem that makes the same decision I described in the previous paragraph.

   Hint: if the two best aren't paired together, who are they paired with?

   Thought process: Since this question uses the full input, the proof will look a lot more like scheduling with deadlines from lecture, or the associated homework problems.

   Suppose we have some alternate solution that *doesn't* follow this. Then there is some player who is the best player with whom we disagree for who their partner should be; because of this, we know that the second-best player (compared to this one) is also one for whom we disagree. That best player is paired with someone else (call that person P) and the second-best is paired with someone else (call that person Q). Note that the ratings of P and Q are the lower of the two partnerships in this solution.

   We have points equal to $P+Q$ for these two pairings. If we swap them, so P and Q are paired (as are the best two players), this only affects the sum of points from these two pairings, so it is enough to show that this does not decrease our solution quality.

   If we swap them, the best two are paired together as are P and Q. We end up with points equal to the second-best player's quality plus the smaller of P and Q. However, we know the second-best player is better than the *better* of P and Q (as neither is the best player), so this set of points is an improvement.