CompSci 161
Spring 2021 Discussion 05:
Divide and Conquer:
The Master Theorem

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The Master Theorem

$\log_2 4 = 2$

A common running time for a D&C algorithm:
$T(n) = aT(n/b) + f(n)$
($a \geq 1$, $b > 1$, $f(n)$ is asymptotically positive)

$a = 4$
$b = 2$
$f(n) = n$

- If there is a small constant $\varepsilon > 0$ such that $f(n)$ is $\mathcal{O}(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
- For example, express $T(n) = 4T(n/2) + n$

Case 1: $\exists \varepsilon > 0$ n^1 is $O(n^{2-\varepsilon})$?

$1 \leq 2 - \varepsilon$

$0 < \varepsilon \leq 1$ ✓

$T(n)$ is $\Theta(n^2)$

n is $O(n)$ // $\varepsilon = 1$

n is $O(n^{1.99})$ // $\varepsilon = .01$

n is $O(n^{1.5})$ // $\varepsilon = 0.5$

3 ## The Master Theorem

$\log_2 2 = 1$

$\log^2 n \neq \log(n^2)$ $\quad \log^2 n = (\log n)^2$

A common running time for a D&C algorithm:
$T(n) = aT(n/b) + f(n)$
($a \geq 1$, $b > 1$, $f(n)$ is asymptotically positive)

$a = 2$
$b = 2$
$f(n) = n\log n$

- If there is a constant $k \geq 0$, such that $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
- For example, express $T(n) = 2T(n/2) + n\log n$

$\exists k$

one

$n \log n$ is $\Theta(n^1 \cdot \log^k n)$? Yes, $k=1$

$T(n)$ is $\Theta(n^1 \cdot \log^2 n)$

4 ## The Master Theorem

$\log_3 1 = 0$

A common running time for a D&C algorithm:
$T(n) = aT(n/b) + f(n)$
($a \geq 1$, $b > 1$, $f(n)$ is asymptotically positive)

$a = 1$
$b = 3$
$f(n) = n$

- If there is a small constant $\varepsilon > 0$ such that $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$.
- For example, express $T(n) = T(n/3) + n$

$-\exists \varepsilon$ n is $\Omega(n^{0+\varepsilon})$? $\quad 0 < \varepsilon \leq 1$

$-\exists \varepsilon$ $1 \geq 0 + \varepsilon$

$T(n)$ is $\Theta(n)$

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The Master Theorem

$a = 9$ $\log_3 9 = 2$

$b = 3$

- $T(n) = 9T(n/3) + n^{2.5}$

$f(n) = n^{2.5}$

Case 1? $\exists \varepsilon$ $n^{2.5}$ is $O(n^{2-\varepsilon})$?

Case 2? $\exists k$ $n^{2.5}$ is $\Theta(n^{2} \cdot \log^k n)$

Case 3? $\exists \varepsilon$ $n^{2.5}$ is $\Omega(n^{2+\varepsilon})$?

$0 < \varepsilon \le 1/2$

So $T(n)$ is $\Theta(n^{2.5})$

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Master Theorem: A Challenge!

- $T(n) = 2T(\cancel{\sqrt{n}}\, n^{1/2}) + \log n$

$k = \log n \longleftrightarrow n = 2^k$

$\sqrt{n}$ is $(2^{\log n})^{1/2}$

$S(k) = T(n) = T(2^k) = 2\underbrace{T(2^{k/2})}\, + k$

$S(k) = 2S(k/2) + k$

which is $S(k)$ is $\Theta(k \log k)$

$\Rightarrow \Theta(\log n \cdot \log \log n)$

Master Theorem: Additional Practice

$\log_b a = 1$

- $T(n) = 2T(n/2) + 1$

  $0 \le 1 - \varepsilon$

  $\varepsilon \le 1$

  Case 1 $\exists \varepsilon$ 1 is $O(n^{1-\varepsilon})$? Yes.

  So $T(n)$ is $\Theta(n)$

  $\log^0 n = 1$

- $T(n) = 2T(n/2) + n$

  case 1: $\exists \varepsilon$ n is $O(n^{1-\varepsilon})$? NO

  case 2: $\exists^k$ n is $\Theta(n \cdot \log^k n)$? yes, $k = 0$

  so $T(n)$ is $\Theta(n \log n)$

- $T(n) = 2T(n/2) + n^2$

  Case 1: $\exists \varepsilon$ n^2 is $O(n^{1-\varepsilon})$? X

  Case 2: $\exists k$ n^2 is $\Theta(n \cdot \log^k n)$? X

  Case 3: $\exists \varepsilon$ n^2 is $\Omega(n^{1+\varepsilon})$? $0 < \varepsilon \le 1$ ✓

  So $T(n)$ is $\Theta(n^2)$

Warning: spoilers ahead. The rest of the slides have answers to the rest of the handout. Please attempt them on your own first. :)

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Master Theorem: Additional Practice

- $T(n) = 2T(n/4) + 1$ $\Theta(\sqrt{n})$
- $T(n) = 2T(n/4) + \sqrt{n}$ $\Theta(\sqrt{n} \cdot \log n)$
- $T(n) = 2T(n/4) + n$ $\Theta(n)$

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Master Theorem: Additional Practice

- $T(n) = 9T(n/3) + n$ $\Theta(n^2)$
- $T(n) = T(2n/3) + 1$ $\Theta(\log n)$
- $T(n) = 3T(n/4) + n \log n$ $\Theta(n \log n)$

10 Master Theorem: Additional Practice

- $T(n) = 2T(n/4) + n^2$ $\Theta(n^2)$
- $T(n) = 2T(n/4) + n^4$ $\Theta(n^4)$
- $T(n) = T(7n/10) + n$ $\Theta(n)$

11 Master Theorem: Additional Practice

- $T(n) = 16T(n/4) + n^2$ $\Theta(n^2 \log n)$
- $T(n) = 7T(n/3) + n^2$ $\Theta(n^2)$
- $T(n) = 7T(n/2) + n^2$ $\Theta(n^{\log_2 7})$

OK to write this way.