CompSci 161
Spring 2021 Lecture 8:
Divide and Conquer III:
Multiplication Algorithms

2

Integer Multiplication

- Given two n-bit integers X, Y, compute $X \cdot Y$
- Example: What is $13 \cdot 11$?

$$\left.\begin{array}{r} 13 \\ \times\ 11 \\ \hline 13 \\ 13\text{X} \\ \hline 143 \end{array}\right\} \Theta(n^2) \text{ if both } n\text{-digits}$$

$(12+1)(12-1) = 12^2 - 1^2$
$= 144 - 1 =$
143

3

What is a Computer Anyway?

- Example: What is $13 \cdot 11$?

$$13 \cdot 11$$
$$26 \cdot 5$$
$$\cancel{52 \cdot 2}$$
$$104 \cdot 1$$
$$\boxed{143}$$

4

Al-Khwarizmi's algorithm

Why does al-Khwarizmi's algorithm work?

- Example: What is $13 \cdot 11$?

$$13: \quad 1101$$
$$11: \quad \times 1011$$

$$\begin{array}{rl} 1101 & \longleftarrow 13 \\ 1101\text{x} & \longleftarrow 26 \\ 0000\text{xx} & //52, \text{if}\ldots \\ 1101\text{xxx} & \longleftarrow 104 \\ \hline \end{array}$$

5

Starting D&C for Integer Multiplication

$$X \times Y = (X_H \times 2^{n/2} + X_L) \times (Y_H \times 2^{n/2} + Y_L)$$
$$= \underbrace{X_H \cdot Y_H}_{A} \times 2^n + (\underbrace{X_H Y_L}_{B} + \underbrace{X_L Y_H}_{C}) \times 2^{n/2} + \underbrace{X_L Y_L}_{D}$$

$T(n) = 4T(n/2) + \Theta(n)$ is $\Theta(n^2)$

dec? inc?

ex: $13 = \underset{H}{11} : \underset{L}{01}$

$13 = 3 \cdot 2^2 + 1$

$11 = 2 \cdot 2^2 + 3$

$11 = \underset{H}{10} \mid \underset{L}{11}$

Algorithm Mult(X, Y)

Create X_H, X_L, Y_H, Y_L

A = Mult(X_H, Y_H) // 3.2

B = Mult(X_H, Y_L) // 3.3

C = Mult(X_L, Y_H) // 1.2

D = Mult(X_L, Y_L) // 1.3

return $A \times 2^n + (B+C) \cdot 2^{n/2} + D$

6

Improved Integer Multiplication

$\log_2 3 < \log_2 4$

- Do we really need to compute $X_H \cdot Y_L$?

$$2^N \cdot X_H Y_H + 2^{N/2}(X_H Y_L + Y_H X_L) + X_L Y_L$$

$$(X_L + X_H)(Y_H + Y_L) = X_L Y_H + X_L Y_L + X_H Y_H + X_H Y_L$$

A = mult(X_H, Y_H)

D = mult(Y_L, X_L)

E = mult($X_H + X_L$, $Y_H + Y_L$)

F = E − A − D; return $A \cdot 2^N + D + F \cdot 2^{N/2}$

$3T(n/2) + \Theta(n)$ is $\Theta(n^{\log_2 3})$

7 # Integer Multiplication Example

- Example: What is $13 \cdot 11$?

X_H	X_L	Y_H	Y_L
11	01	10	11
3	1	2	3

$X_H \cdot Y_H = 6$

$Y_L \, X_L = 3$

$(X_H + X_L) = 4$

$(Y_H + Y_L) = 5$

$F = 20 - 6 - 3 = 11$

Result: $6 \cdot 2^4 + 11 \cdot 2^2 + 3$
$= 96 + 44 + 3$
$= 143$

8 # Matrix Multiplication

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} I & J \\ K & L \end{bmatrix}$$

- $I = AE + BG$
- $J = AF + BH$
- $K = CE + DG$
- $L = CF + DH$

$T(n) = 8T\left(\frac{n}{2}\right) + \Theta(n^2)$
is $\Theta(n^3)$

- Running time if basis for D & C algorithm?
- Why must algorithms to solve this be $\Omega(n^2)$?

9

Strassen's Algorithm

$T(n) = 7T(n/2) + \Theta(n^2)$
$\rightarrow \Theta(n^{\log_2 7})$

- $S_1 = A(F - H)$
- $S_2 = (A + B)H$
- $S_3 = (C + D)E$
- $S_4 = D(G - E)$
- $S_5 = (A + D)(E + H)$
- $S_6 = (B - D)(G + H)$
- $S_7 = (A - C)(E + F)$

$$\begin{aligned} I &= S_5 + S_6 + S_4 - S_2 \\ &= (A + D)(E + H) + (B - D)(G + H) \\ &\quad + D(G - E) - (A + B)H \\ &= AE + BG \end{aligned}$$

$$\begin{aligned} J &= S_1 + S_2 \\ &= A(F - H) + (A + B)H \\ &= AF - AH + AH + BH \\ &= AF + BH \end{aligned}$$

$$K = S_3 + S_4$$
$$L = S_1 - S_7 - S_3 + S_5$$