Time Series Analysis: Homework 2 (20232863 Keywoong Bae)

1. Consider three processes:
   1) $Z_t = Z_{t-1} - 0.5Z_{t-2} + a_t$
   2) $Z_t = (1 - 0.8B + 0.2B^2)a_t$
   3) $(1 - 0.5B)Z_t = (1 - 0.8B)a_t$

(a) Represent each process in terms of AR form.

(b) Represent each process in terms of MA form.

(a) AR form

1) $(1 - B + 0.5B^2) Z_t = a_t$, $\pi_1 = -1$, $\pi_2 = -0.5$, $\pi_i = 0 \, (i = 3, 4, 5, \cdots)$

2) $(1 - 0.8B + 0.2B^2)^{-1} Z_t = a_t$, $\pi_1 = -0.8$, $\pi_2 = -0.44$, $\pi_3 = -0.192$

3) $(1 - 0.5B)(1 - 0.8B)^{-1} Z_t = a_t$

$\pi_1 = -0.3$, $\pi_2 = -0.24$, $\pi_3 = -0.192$

(b) MA form

1) $Z_t = (1 - B + 0.5B^2)^{-1} a_t$

$\psi_1 = -1$, $\psi_2 = -0.5$, $\psi_3 = 0, \cdots$

2) $Z_t = a_t - 0.8a_{t-1} + 0.2a_{t-2}$,

$\psi_1 = 0.8$, $\psi_2 = -0.2$, $\psi_i = 0 \, (i = 3, 4, \cdots)$

3) $Z_t = (1 - 0.8B)(1 - 0.5B)^{-1} a_t$

$= (1 - 0.8B)(1 + 0.5B + 0.25B^2 + \cdots) a_t$

$= (1 - 0.3B - 0.15B^2 - 0.075B^3 - \cdots) a_t$

$\psi_1 = 0.3$ $\quad \psi_2 = 0.15$ $\quad \psi_3 = 0.075$

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2. Obtain ACF and PACF of each process given in Problem #1.

1) $Z_t = Z_{t-1} - 0.5Z_{t-2} + a_t$
2) $Z_t = (1 - 0.8B + 0.2B^2)a_t$
3) $(1 - 0.5B)Z_t = (1 - 0.8B)a_t$

*ACF

$Z_t = \phi Z_{t-1} + a_t$라 할 때

$\rho(1) = \phi$, $\rho(2) = \phi^2, \ldots$

(1) ACF: $\phi_1 = 1$, $\phi_2 = -0.5$

$\rho(1) = \frac{1}{1.5} = 0.67$, $\rho(2) = \frac{1}{1.5 - 0.5} = 0.17$, $\rho(3) = 0.17 - 0.5 \times 0.67$
$= -0.155$

PACF: $\rho(1) = 0.67$, $\rho(2) = -0.5$, $\rho(k) = 0$ ($k = 3, 4, \ldots$)

(2) ACF: $\theta_1 = 0.8$, $\theta_2 = -0.2$

$\rho(1) = \frac{-0.8 - 0.16}{1 + 0.64 + 0.04} = -0.5714$, $\rho(2) = \frac{0.2}{1 + 0.64 + 0.04} = 0.1190$, $\rho(3) = 0$

PACF

$\rho(1) = -0.5714$, $\rho(2) = \frac{0.1190 - 0.5714^2}{1 - 0.5714^2} = -0.3081$,

$$\rho(3) = \frac{(-0.5714)^3 + 2 \times 0.5714 \times 0.119 - 0.5714 \times 0.119^2}{1 - 2 \times 0.5714^2 \times 0.119 - 0.119^2} = -0.1429$$

(3) ACF

$\phi_1 = 0.5$, $\theta_1 = 0.8$

$\rho(1) = \frac{(0.5 - 0.8) \times (1 - 0.4)}{1 + 0.64 - 2 \times 0.4} = 0.2143$, $\rho(2) = 0.5 \times 0.2143 = 0.10715$, $\rho(3) = 0.5^2 \times 0.2143$
$= 0.05357$

PACF

$\rho(1) = 0.2143$, $\rho(2) = \frac{0.10715 - 0.2143^2}{1 - 0.2143^2} = 0.0642$

$$\rho(3) = \frac{0.2143^3 - 2 \times 0.2143 \times 0.10715 + 0.2143 \times 0.10715^2}{1 - 2 \times 0.2143^2 + 2 \times 0.2143^2 \times 0.10715 - 0.10715^2} = -0.0371$$

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3. Time series $\{X_t, t \geq 1\}$ and $\{Y_t, t \geq 1\}$, respectively, follows a stationary process given below. Assume that the mean of each time series has zero.

$X_t = a_t - \theta a_{t-1}$

$Y_t = \phi Y_{t-1} + a_t$

where a_t's are white noises (same ones related to X_t's and Y_t's) having mean 0, and variance σ_a^2. Define

$Z_t = X_t + Y_t$

(a) Obtain $Var[Z_t]$

(b) Obtain $Cov[Z_t, Z_{t-1}]$

(c) Obtain $Corr[Z_t, Z_{t-1}]$

(d) What type of model does $\{Z_t, t \geq 1\}$ follow ?

(a) $Var[Z_t] = Var[X_t] + Var[Y_t] + 2 \cdot Cov[X_t, Y_t]$

$Var[X_t] = Var[a_t - \theta a_{t-1}] = (1+\theta^2)\sigma_a^2$

$Var[Y_t] = \phi^2 Var[Y_{t-1}] + \sigma_a^2 = \dfrac{\sigma_a^2}{1-\phi^2}$

$Cov[X_t, Y_t] = Cov[a_t - \theta a_{t-1}, \phi Y_{t-1} + a_t] = Cov[a_t, a_t] - \theta\phi \cdot Cov[a_{t-1}, Y_{t-1}]$

$= \sigma_a^2 - \phi\, Cov[a_{t-1}, a_{t-1}] = (1 - \theta \cdot \phi)\sigma_a^2$

$\therefore Var[Z_t] = \sigma_a^2 \left[1 + \theta^2 + \dfrac{1}{1-\phi^2} + 2(1-\theta\phi)\right]$

(b) $Z_t = (1-\theta B)a_t + (1-\phi B)^{-1} a_t = (2 + (\phi - \theta)B + \phi^2 B^2 + \phi^3 B^3 + \cdots)a_t$

$Cov[Z_t, Z_{t-1}] = \sigma_a^2 [2(\phi-\theta) + \phi^2(\phi - \theta) + \phi^5 + \phi^7 + \cdots]$

(c) $Corr[Z_t, Z_{t-1}] = \dfrac{Cov[Z_t, Z_{t-1}]}{Var[Z_t]}$

(d) ARMA(1,2)

4. Consider a times series given by

$Z_t = (1/3)Z_{t-1} + (2/9)Z_{t-2} + a_t$

(a) Is this satisfying the stationarity condition ? How about invertibility condition ?

(b) Show that ACF is given by

$$\rho(k) = \frac{16}{21}\left(\frac{2}{3}\right)^k + \frac{5}{21}\left(-\frac{1}{3}\right)^k, \quad k = 1, 2, \ldots$$

(a) AR(2) model $\phi_1 = \frac{1}{3}$, $\phi_2 = \frac{2}{9}$ satisfying the stationarity condition and invertibility condition

(b) $\phi_2(z) = 1 - \frac{1}{3}z - \frac{2}{9}z^2 = (\frac{1}{3}z+1)(-\frac{2}{3}z+1) = 0$

$r = \frac{3}{2}, -3$

$$p(k) = \frac{\frac{1}{3/2} \times (1-(-\frac{1}{3})^2)(\frac{1}{3/2})^k - (-\frac{1}{3})(1-(1/\frac{3}{2})^2)(-\frac{1}{3})^k}{(\frac{1}{3/2} + \frac{1}{3})(1+\frac{1}{3/2}\times(-\frac{1}{3}))}$$

$$= \frac{16}{21} \times (\frac{2}{3})^k + \frac{5}{21} \times (-\frac{1}{3})^k$$

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5. Let P_t be the price of a commodity at time t. It is known that a series of $\ln(P_t/P_{t-1})$ follows AR(1) model.

(a) What type of model does $\ln(P_t)$ follows ?

(b) Find the stationarity and invertibility conditions for the parameters in the model of $\ln(P_t)$.

(a) $\ln \frac{P_t}{P_{t-1}} = \phi \cdot \ln \frac{P_{t-1}}{P_{t-2}} + a_t$

$\ln P_t - \ln P_{t-1} = \phi [\ln P_{t-1} - \ln P_{t-2}] + a_t$

$\ln P_t = (1+\phi) \ln P_{t-1} - \phi \ln P_{t-2} + a_t$

AR(2) model

(b) stationarity (x), Invertibility (x)