< 시계열 분석 1장: 시계열 예측을 위한 평활기법 >

20232863 배기웅

1.2 이동 평균법

* Simple Moving Average

- $X_t = c + a_t$, c는 상수이며 a_t는 white noise로 분리.
- $\min_c Q = \sum_{t=T-N+1}^{T} (X_t - c)^2$, $\hat{c} = \frac{1}{N}\sum_{t=T-N+1}^{T} X_t$ 이며 $\hat{c} = M_t$로 표기
- $M_T = \frac{1}{N}\sum_{t=T-N+1}^{T} X_t$, $E[M_T] = E[X_t] = c$, $Var[M_T] = \frac{1}{N^2}(N \cdot Var[X_t]) = \frac{\sigma_a^2}{N}$

* One-step-ahead forecast (Simple MA)

- $f_{T:1} = E[X_{T+1} \mid X_T, \dots] = c$

  $\hat{f}_{T:1} = \hat{c} = M_T$

- $\hat{f}_{T:k} = \hat{c} = M_T$, $k = 1, 2, 3, \dots$ (k-step ahead)

- 예측오차: $e_{T:1} = X_{T+1} - \hat{f}_{T:1} = X_{T+1} - M_T$

  예측오차의 분산: $Var[e_{T:1}] = Var[X_{T+1} - M_T] = Var[X_{T+1}] + Var[M_T]$

  $= \sigma_a^2 + \frac{1}{N}\sigma_a^2 = (1 + \frac{1}{N})\sigma_a^2$

* Double Moving Average

- $\underbrace{X_t = c + bt + a_t}_{\text{선형추세}}$, $M_T^{(2)} = \frac{1}{N}\sum_{i=T-N+1}^{T} M_i$

- 이중이동평균: 최근 N개의 단순이동평균이 다시 이동평균을 취한 값

- $E[M_T] = \frac{1}{N}\sum_{t=T-N+1}^{T} E[X_t] = \frac{1}{N}\sum_{t=T-N+1}^{T} E[c + bt + a_t]$

  $= c + b \cdot T - \frac{N-1}{2} b$

- $E[M_T^{(2)}] = \frac{1}{N}\sum_{t=T-N+1}^{T} E[M_i] = \frac{1}{N}\sum E[c + bT - \frac{N-1}{2} b]$

  $= c + bT - (N-1) b$

· $E[M_T]$와 $E[M_T^{(2)}]$에 대한 연립방정식의 결과

$\Rightarrow c = 2 \cdot E[M_T] - E[M_T^{(2)}] - bT$

$b = \frac{2}{N-1}(E[M_T] - E[M_T^{(2)}])$

· c와 b에 대한 추정치

$\Rightarrow \hat{c} = 2M_T - M_T^{(2)} - bT$

$\hat{b} = \frac{2}{N-1}(M_T - M_T^{(2)})$

$\Rightarrow E[M_T] - E[M_T^{(2)}] = \frac{N-1}{2} b$

* One-step-ahead forecast (double MA)

· $f_{T,1} = E[X_{T+1} \mid X_T, \dots] = c + b(T+1)$

· $\hat{f}_{T,1} = \hat{c} + \hat{b}(T+1) = 2M_T - M_T^{(2)} + b$

· $\underbrace{\hat{f}_{T,k} = \hat{c} + \hat{b}(T+k)}_{k\text{-step-ahead forecast}} = 2M_T - M_T^{(2)} + k\hat{b}$, $k = 1, 2, \dots$

1.3 지수평활법

* 단순지수평활법 (Simple Exponential Smoothing, EWMA)

- 가중치 λ를 곱하여 시간적 효과를 고려
- $\min_c Q = \sum_{t=1}^{T} \lambda^{T-t} (X_t - c)^2$ (과거로 갈수록 가중치 λ 작아짐)
- $\hat{c} = \frac{1-\lambda}{1-\lambda^T} \sum_{t=1}^{T} \lambda^{T-t} \cdot X_t$

$T \to \infty \sim \hat{c} = (1-\lambda) \cdot \sum_{i=0}^{\infty} \lambda^i \cdot X_{T-i} = S_T$

- 여기서 $\alpha = 1-\lambda$로 정의하면 다음과 같음

$S_T = \alpha \cdot X_T + (1-\alpha) S_{T-1}$, α는 smoothing constant.

$E[S_T] = \alpha \cdot \sum_{i=0}^{\infty} (1-\alpha)^i E[X_{T-i}] = c \cdot \alpha \cdot \sum_{i=0}^{\infty} (1-\alpha)^i = c$

$Var[S_T] = \alpha^2 \cdot \sum_{i=0}^{\infty} (1-\alpha)^{2i} \cdot Var[X_{T-i}] = \alpha^2 \cdot \sigma_a^2 \cdot \sum_{i=0}^{\infty} (1-\alpha)^{2i}$

$= \frac{\alpha}{2-\alpha} \sigma_a^2$

- $\hat{f}_{T,k} = \hat{c} = S_T$, $k = 1, 2, 3, \dots$ (k-step-ahead forecast)

$e_{T,k} = X_{T+k} - \hat{f}_{T,k} = X_{T+k} - S_T$

$Var[e_{T,k}] = Var[X_{T+k} - S_T] = Var[X_{T+k}] + Var[S_T]$

$= \frac{2}{2-\alpha} \cdot \sigma_a^2$

* Brown 이중지수평활법 (Double Exponential Smoothing)

- $S_T^{(2)} = \alpha \cdot \sum_{i=0}^{\infty} (1-\alpha)^i S_{T-i}$

$= \alpha \cdot S_T + (1-\alpha) S_{T-1}^{(2)}$

- $E[S_T] = c + bT - \frac{1-\alpha}{\alpha} b = E[X_T] - \frac{1-\alpha}{\alpha} b$

· $E[S_T^{(2)}] = E[S_T] - \frac{1-\alpha}{\alpha} b$

↳ Simple ES의 기댓값과 Double ES의 기댓값 간에 격차 존재.

· $\hat{b} = \frac{\alpha}{1-\alpha}(S_T - S_T^{(2)})$, $\hat{c} = 2 \cdot S_T - S_T^{(2)} - \hat{b} T.$

· $\hat{f}_{T,1} = \hat{c} + \hat{b}(T+1) = (2 + \frac{\alpha}{1-\alpha}) S_T - (1 + \frac{\alpha}{1-\alpha}) S_T^{(2)}$

$\hat{f}_{T,k} = \hat{c} + \hat{b}(T+k) = 2 \cdot S_T - S_T^{(2)} + k \cdot \hat{b}$, $k = 1, 2, \ldots$

* Holt의 선형추세 지수평활법

· 수준(L_t)와 추세(b_t)를 갱신

$L_t = \alpha \cdot X_t + (1-\alpha)(L_{T-1} + b_{t-1})$

$b_t = \beta(L_t - L_{t-1}) + (1-\beta) b_{t-1}$

$0<\alpha<1$: 수준에 대한 평활 상수, $0<\beta<1$: 추세에 대한 평활 상수

초깃값 : $L_1 = X_1$, $b_1 = X_1 - X_0$

· $\hat{f}_{T,k} = L_T + k \cdot b_T$

* 계절성을 고려한 Winters 모형.

· Holt 모형 + 계절성(Seasonality)

· $L_t = \alpha \cdot \frac{X_t}{S_{t-m}} + (1-\alpha)(L_{t-1} + b_{t-1})$

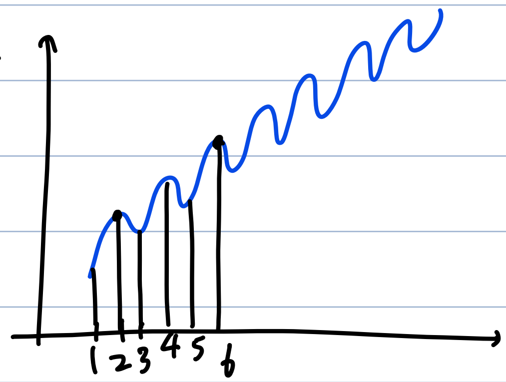
$b_t = \beta(L_t - L_{t-1}) + (1-\beta) b_{t-1}$

$S_t = \gamma \cdot \frac{X_t}{L_t} + (1-\gamma) S_{t-m}$ (S_t 계절성 지수, m: 계절성 주기, γ : 평활상수)

· $\hat{f}_{T,k} = (L_T + k \cdot b_T) S_{T-m+k}$, $k = 1, 2, \ldots$

1.4 분해법을 이용한 예측

* Decomposition Method - Addictive

- trend와 계절성을 분리
- $X_t = b_t + S_t + \varepsilon_t$, $S_t = S_{t-m}$, $\sum_{i=1}^{m} s_i = 0$
- m : 주기
- 

⇒ trend를 제거시켜야 함 (detrend)

↓

CM_t 이용 ⇒ $DX_t^{(T)} = X_t - \hat{CM}_t$

-

t	X_t	$M_t\,(N=4)$	CM_t
1	x_1		
2	x_2		
3	x_3		
4	x_4		
5	x_5		
6	x_6		
7	x_7		
8	x_8		

$$CM_4 = \frac{0.5X_1 + X_2 + X_3 + 0.5X_4}{4}$$

$$M_4 = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

- 짝수인 경우 CM_t $(m = 2q)$
  - $\hat{CM}_t = \frac{1}{m}\left(0.5X_{t-q} + X_{t-q+1} + \cdots + 0.5X_{t+q}\right)$
- 홀수인 경우 CM_t $(m = 2q+1)$
  - $\hat{CM}_t = \frac{1}{m}\left(X_{t-q} + X_{t-q+1} + \cdots + X_{t+q}\right)$

* Decomposition method - Multiplicative

- $X_t = b_t \times S_t \times \varepsilon_t$, $S_t = S_{t-m}$, $\sum_{i=1}^{m} S_i = m$

1.5 예측성능척도

- MSE, RMSE, MAD, MAPE