Applied Stochastic Processes: Homework 1 (20232863 Keywoong Bae)

1.1. Let N denote a nonnegative integer-valued random variable. Show that

$$E[N] = \sum_{k=1}^{\infty} P\{N \geq k\} = \sum_{k=0}^{\infty} P\{N > k\}.$$

In general show that if X is nonnegative with distribution F, then

$$E[X] = \int_0^{\infty} \overline{F}(x)\, dx$$

and

$$E[X^n] = \int_0^{\infty} n x^{n-1} \overline{F}(x)\, dx.$$

1-1

N : nonnegative integer-valued random variable.

Let $I_k = \begin{cases} 1 & \text{if } N \geq k \\ 0 & \text{else.} \end{cases}$ and $J_k = \begin{cases} 1 & \text{if } N > k \\ 0 & \text{else.} \end{cases}$

Then, $N = \sum_{k=1}^{\infty} I_k = \sum_{k=0}^{\infty} J_k$.

$$E[N] = \sum_{k=1}^{\infty} E[I_k] = \sum_{k=0}^{\infty} E[J_k]$$

Since $E[I_k] = P[N \geq k]$ and $E[J_k] = E[N > k]$,

$$E[N] = \sum_{k=1}^{\infty} P\{N \geq k\} = \sum_{k=0}^{\infty} P\{N > k\}.$$

$$E[X] = \int_0^{\infty} x \cdot dF(x) = -\int_0^{\infty} x \cdot d\overline{F}(x)$$

$$E[X] = -x\,\overline{F}(x)\Big]_0^{\infty} + \int_0^{\infty} \overline{F}(x)\, dx$$

Since $E[X] < \infty$, $\lim_{x \to \infty} x \cdot \overline{F}(x) = 0$, giving

$$E[X] = \int_0^{\infty} \overline{F}(x)\, dx \quad \blacksquare$$

1.2. If X is a continuous random variable having distribution F show that.

(a) $F(X)$ is uniformly distributed over $(0, 1)$,

(b) if U is a uniform $(0, 1)$ random variable, then $F^{-1}(U)$ has distribution F, where $F^{-1}(x)$ is that value of y such that $F(y) = x$

#1-2 - (a), (b)

Since $F_X(x)$ is a distribution function, $F_X(x) \in [0,1]$ for all $x \in \mathbb{R}$ so that $Y = F_X(X)$ takes one of the values in $[0,1]$.

Thus for $y \geq 1$, $F_Y(y) = P(Y \leq y) = 1$ and for $y < 0$, $F_Y(y) = P(Y \leq y) = 0$.

Now for $0 \leq y < 1$,

$$F_Y(y) = P(Y \leq y) = P(0 \leq Y \leq y) = P(F_X(X) \in [0,y])$$
$$= P(X \in F_X^{-1}([0,y])).$$

Note that

$$F_X^{-1}([0,y]) = \{x \in \mathbb{R} \mid F_X(x) \leq y\} = (-\infty, z].$$

for some $z \in \mathbb{R}$, since $F_X(x)$ is a monotone continuous function on $\mathbb{R}$.

Note that $F_X(z) = y$.

Thus, $F_Y(y) = P(X \in F^{-}([0,y])) = P(X \leq z) = F_X(z) = y$.

and then Y is uniformly distributed over $(0,1)$

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1.8. Let X_1 and X_2 be independent Poisson random variables with means λ_1 and λ_2.

(a) Find the distribution of $X_1 + X_2$

(b) Compute the conditional distribution of X_1 given that $X_1 + X_2 = n$

1-8 - (a)

$X_1 \sim P(\lambda_1)$, $X_2 \sim P(\lambda_2)$

$$P(X_1 + X_2 = n) = \sum_{i=0}^{n} P(X+Y=n, X=i)$$

$$= \sum_{i=0}^{n} P(Y=n-i, X=i) = \sum_{i=0}^{n} P(Y=n-i)\, P(X=i)$$

$$= \sum_{i=0}^{n} e^{-\lambda_1} \cdot \frac{\lambda_1^{n-i}}{(n-i)!} \times e^{-\lambda_2} \cdot \frac{\lambda_2^{i}}{i!}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{1}{n!} \sum_{i=0}^{n} \frac{n!}{(n-i)!\, i!} \cdot \lambda_1^{n-i} \cdot \lambda_2^{i}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{1}{n!} \cdot \sum_{i=0}^{n} \binom{n}{i} \cdot \lambda_1^{n-i} \cdot \lambda_2^{i}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!}$$

$\therefore\ X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$ ▨

1-8 - (b)

$X_1 + X_2 = n$.

$$P(X_2 = x_2 \mid X_1 + X_2 = n) = \frac{P(X_2 = x_2 \text{ and } X_1 + X_2 = n)}{P(X_1 + X_2 = n)}$$

$$= \frac{P(X_2 = x_2 \text{ and } X_2 = n - x_1)}{P(X_1 + X_2 = n)} = \frac{P(X_2 = x_2)\, P(X_2 = n - x_2)}{P(X_1 + X_2 = n)}$$

$$\therefore\ P(X_2 = x_2 \mid X_1 + X_2 = n) = \frac{(\lambda_1^{x_1} e^{-\lambda_1}/x_1!) \cdot (\lambda_2^{n-x_2} \cdot e^{-\lambda_2}/(n-x_1)!)}{\lambda_1^n \cdot \lambda_2^n \cdot e^{-(\lambda_1+\lambda_2)}/n!}$$ ▨

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1.11. If X is a nonnegative integer-valued random variable then the function $P(z)$, defined for $|z| \le 1$ by

$$P(z) = E[z^X] = \sum_{j=0}^{\infty} z^j P\{X = j\},$$

is called the probability generating function of X

(a) Show that

$$\frac{d^k}{dz^k} P(z)_{|z=0} = k!\, P\{X = k\}.$$

(b) With 0 being considered even, show that

$$P\{X \text{ is even}\} = \frac{P(-1) + P(1)}{2}$$

(c) If X is binomial with parameters n and p, show that

$$P\{X \text{ is even}\} = \frac{1 + (1 - 2p)^n}{2}.$$

(d) If X is Poisson with mean λ, show that

$$P\{X \text{ is even}\} = \frac{1 + e^{-2\lambda}}{2}$$

(e) If X is geometric with parameter p, show that

$$P\{X \text{ is even}\} = \frac{1 - p}{2 - p}.$$

(f) If X is a negative binomial random variable with parameters r and p, show that

$$P\{X \text{ is even}\} = \frac{1}{2}\left[1 + (-1)^r \left(\frac{p}{2 - p}\right)^r\right].$$

1-11-(a)

$$P(z) = \sum_{j=0}^{\infty} z^j P(k=j) = P(k=0) + z \cdot P(k=1) + z^2 P(k=2) + \cdots + z^k P(k=k) + \cdots$$

$$\frac{d}{dz} P(z)\Big|_{z=0} = P(k=1), \quad \frac{d^2}{dz^2} P(z)\Big|_{z=0} = 2\, P(k=2)$$

$$\frac{d^3}{dz^3} P(z)\Big|_{z=0} = 3 \times 2 \times P(k=3) = 3! \times P(k=3)$$

$$\therefore \frac{d^k}{dz^k} P(z)\Big|_{z=0} = k!\, P(k=k) \quad ▥$$

1-11-(b)

$p(X=\text{even})$

$= p(X=0) + p(X=2) + p(X=4) + \cdots$

$= \dfrac{p(X=0) + p(X=1) + p(X=2) + p(X=3) + \cdots}{2}$

$= \dfrac{p(X=0) - p(X=1) + p(X=2) - p(X=3) + \cdots}{2}$

$= \frac{1}{2} \sum_{j=0}^{\infty} 1^j \, p(X=j) + \frac{1}{2} \sum_{j=0}^{\infty} (-1)^j \, p(X=j)$

$= \frac{1}{2} P(1) + \frac{1}{2} P(-1) = \dfrac{P(1) + P(-1)}{2}.$ $\blacksquare$

1-11-(c)

$X \sim N \cdot B(n, p)$

$P(z) = E[z^X] = \sum_{k=0}^{n} z^k \binom{n}{k} p^k \cdot (1-p)^{n-k}$

$= (1-p+zp)^n = (1+(z-1)p)^n$

$p(x=\text{even}) = \dfrac{P(1) + P(-1)}{2} = \dfrac{1 + (1-2p)^n}{2}$ $\blacksquare$

1-11-(d)

$x \sim P(\lambda)$

$P(z) = E[z^k] = \sum_{k=0}^{\infty} z^k \cdot e^{-\lambda} \cdot \dfrac{\lambda^k}{k!}$

$= e^{-\lambda} \cdot \sum_{k=0}^{\infty} \dfrac{(z\lambda)^k}{k!} = e^{-\lambda} \times e^{z\lambda} = e^{\lambda(z-1)}$

$p(x=\text{even}) = \dfrac{P(1) + P(-1)}{2} = \dfrac{1 + e^{-2\lambda}}{2}$ $\blacksquare$

#1-11-(e)

$X \sim \text{geom}(p)$

$$p(X=x) = p \cdot (1-p)^{x-1}, \quad x = 1, 2, 3, \cdots$$

$$P(z) = E[z^X] = \sum_{k=1}^{\infty} z^k \cdot p(1-p)^{k-1}$$

$$= p \cdot z \cdot \sum_{k=1}^{\infty} (z - zp)^{k-1} = \frac{p \cdot z}{1 - z + zp}$$

$$P(X = \text{even}) = \frac{P(1) + P(-1)}{2}$$

$$= \frac{\frac{p}{p} + \frac{-p}{2-p}}{2} = \frac{1}{2}\left(1 + \frac{-p}{2-p}\right)$$

$$= \frac{1}{2} \times \frac{2-p-p}{2-p} = \frac{1-p}{2-p} \quad \blacksquare$$

#1-11-(f)

$X \sim NB(r, p)$

$$p(X=x) = \binom{x-1}{r-1} p^r \cdot (1-p)^{x-r}, \quad x = r, r+1, \cdots$$

$$P(z) = E[z^k] = \sum_{k=r}^{\infty} z^k \cdot \binom{k-1}{r-1} p^k \cdot (1-p)^{k-r}$$

$$= (pz)^r \cdot \sum_{k=r}^{\infty} \binom{x-1}{r-1} (z - zp)^{k-r} = \frac{(pz)^r}{(1 - z + zp)^r}$$

$$P(X = \text{even}) = \frac{P(1) + P(-1)}{2}$$

$$= \frac{p + \left(\frac{-p}{2-p}\right)^r}{2} = \frac{1}{2}\left[1 + (-1)^r \cdot \left(\frac{p}{2-p}\right)^r\right] \quad \blacksquare$$

1.16. Let $f(x)$ and $g(x)$ be probability density functions, and suppose that for some constant c, $f(x) \le cg(x)$ for all x. Suppose we can generate random variables having density function g, and consider the following algorithm.

Step 1: Generate Y, a random variable having density function g.

Step 2: Generate U, a uniform (0, 1) random variable.

Step 3: If $U \le \dfrac{f(Y)}{cg(Y)}$ set $X = Y$. Otherwise, go back to Step 1.

Assuming that successively generated random variables are independent, show that:

(a) X has density function f

(b) the number of iterations of the algorithm needed to generate X is a geometric random variable with mean c.

#1-16-(a)

$$P(Y_n \le x \mid E_n) = P(Y_n \le x, E_n) / P(E_n)$$

$$= c \cdot E[P(Y_n \le x, E_n \mid Y_n)]$$

$$= c \cdot \int_{-\infty}^{x} \frac{f(y)}{c\,g(y)} g(y)\, dy = F(x),$$

where $F(x) = \int_{-\infty}^{x} f(u)\, du$. On the nth iteration, conditional on the algorithm terminating at that iteration, Y_n has conditional CDF, so X has density f ▨

#1-16-(b)

Define infinite sequences $\{Y_n, n \ge 1\}$ and $\{U_n, n \ge 1\}$ where Y_n are iid with density g and U_n are iid uniform $(0,1)$. Define $E_n = \{U_n \le f(Y_n)/c \cdot g(Y_n)\}$. E_n is a sequence of independent trials with and the algorithm stops at the first "success". Thus the number of iterations is geometric with parameter

$$p = P(E_n) = E[P(E_n \mid Y_n)] = \int_{-\infty}^{\infty} \frac{f(y)}{c \cdot g(y)} g(y)\, dy = \frac{1}{c}, \quad \frac{1}{p} = c$$

1.22. The conditional variance of X, given Y, is defined by

$$\mathrm{Var}(X|Y) = E[(X - E[X|Y])^2|Y]$$

Prove the conditional variance formula, namely,

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y]).$$

Use this to obtain Var(X) in Example 1 5(B) and check your result by differentiating the generating function

1-22

By the definition of Variance,

$$\begin{aligned}\mathrm{Var}[X] &= E[(X-E[X])^2] = E[(X-E[X|Y]+E[X|Y]-E[X])^2] \\ &= E[(X-E[X|Y])^2] + E[(E[X|Y]-E[X])^2] \\ &\quad + 2\cdot \underline{E[(X-E[X|Y])(E[X|Y]-E[X])]}\end{aligned}$$

$$\begin{aligned}&E[(X-E[X|Y])(E[X|Y]-E[X])] \\ &= E[E[(X-E[X|Y])]\cdot E[E[X|Y]-E[X]]|Y] \\ &= E[(E[X|Y]-E[X])\,E[(X-E[X|Y]|Y]] \\ &= E[(E[X|Y]-E[X])(E[X|Y]-E[X|Y])] \\ &= 0\end{aligned}$$

Law of iterated expectation

$E[X]$ is a constant

$E[X|Y]$: function of Y.

$E[E[X|Y]|Y] = E[X|Y]$

$$\begin{aligned}\therefore \mathrm{Var}[X] &= E[(X-E[X])^2] + E[(E[X|Y]-E[X])^2] \\ &= E[\underline{E[(X-E[X|Y])^2|Y]}] + E[(E[X|Y]-E[E[X|Y]])^2] \\ &= E[\mathrm{Var}[X|Y]] + \mathrm{Var}[E[X|Y]] \quad \blacksquare\end{aligned}$$

1.36. Use Jensen's inequality to prove that the arithmetic mean is at least as large as the geometric mean. That is, for nonnegative x_i, show that

$$\sum_{i=1}^{n} x_i/n \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}.$$

*** Jensen Inequality?**

$$E[f(x)] \geq f(E[X]).$$

 # 1-36.

Jensen's inequality says that for a convex function $f(x)$ with $c_i \geq 0$, $\sum_{i=1}^{n} c_i = 1$ if we have arbitrary $x_1, x_2, \ldots, x_n$ then

$$f(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n) \leq c_1 f(x_1) + c_2 f(x_2) + \cdots c_n f(x_n)$$

(If f is concave $\leq$ turns to $\geq$)

For an explanation, let $f(x)$ $\log x$ and $c_1 = c_2 = \cdots = c_n = \frac{1}{n}$.

$$\log(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n) = \log\left(\frac{1}{n}x_1 + \frac{1}{n}x_2 + \cdots + \frac{1}{n}x_n\right)$$

$$\geq \frac{1}{n}\log x_1 + \frac{1}{n}\log x_2 + \cdots + \frac{1}{n}\log x_n \geq \frac{1}{n}(\log x_1 + \log x_2 + \cdots + \log x_n)$$

$$\log\left(\frac{1}{n}(\log x_1 + \log x_2 + \cdots + \log x_n)\right) \geq \frac{1}{n}\log(x_1 \cdot x_2 \cdots x_n)$$

$$\log\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \geq \log(x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}$$

$$\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right] \geq (x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{\frac{1}{n}}$$

Thus, $\displaystyle\sum_{i=1}^{n} \frac{x_i}{n} \geq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$ ∎

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1.43. For a nonnegative random variable X, show that for $a > 0$,

$$P\{X \geq a\} \leq E[X^t]/a^t$$

Then use this result to show that $n! \geq (n/e)^n$

By Markov Inequality, $P\{X \geq a\} \leq \frac{E[X^t]}{a^t}$. , $(X \geq 0)$

Too difficult... $\underline{n! \geq (n/e)^n}$.

$$E\left[\frac{n^n}{e^n}\right] = \frac{E[n^n]}{e^n} \sim \frac{E[n^n]}{e^n}$$

$$P(X \geq e) \leq \frac{E[X^n]}{e^n} \quad ? \leq x!$$