2D Transient Heat Transfer

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Contents

1 Introduction

In this report, I am going to discuss 2D transient heat flow through a flat plate made of three different materials - copper, aluminum, and steel. Those three materials have different thermal properties, so I will compare the results. I will also calculate the time needed to achieve a steady state.

2 Heat Transfer Equation

The heat transfer equation (HTE) is a partial differential equation:

$$\frac{\partial T}{\partial t} - \alpha \nabla^2 T = 0 \tag{1}$$

In this case, I will solve this equation in a 2D Cartesian coordinate system. The HTE is given by:

$$\frac{\partial T}{\partial t} - \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = 0 \tag{2}$$

Where:

- T is the temperature
- t is time
- x and y are the Cartesian coordinates
- α is the diffusivity constant

The diffusivity constant is given by:

$$\alpha = \frac{\lambda}{\rho c_p} \tag{3}$$

Where:

- λ is the thermal conductivity
- ρ is the density
- c_p is the specific heat capacity

I assume that the materials are isotropic, so the thermal conductivity is equal in every direction. Also, there are no radiation, convection, or heat sources. There is only heat conduction.

3 Finite Difference Method

The Finite Difference Method (FDM) is a numerical method for solving differential equations by approximating derivatives with finite differences.
The derivative of a function $f(x)$ is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{4}$$

Where h is a small number approaching zero. In FDM, instead of using a limit, we use a difference. The formula is:

- FDM with a forward step

$$f'(x) = \frac{f(x+h) - f(x)}{h} \tag{5}$$

- FDM with a backward step

$$f'(x) = \frac{f(x) - f(x-h)}{h} \tag{6}$$

- FDM with symmetric step

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} \tag{7}$$

The case of FDM is to discretize the domain.

$$x_i = i\Delta x \tag{8}$$

$$y_j = j\Delta y \tag{9}$$

$$t_k = k\Delta t \tag{10}$$

It is easy to gather that i, j, k are steps for each difference for x, y and t respectively. What is wanted to get from HTE is the solution T, which is:

$$T(x, y, z) = T^k_{i,j} \tag{11}$$

The HTE can be rewritten using FDM. In this case, the FDM with the forward step is used for derivative with respect to time. Second-order derivatives with respect to x and y are calculated using FDM with a symmetric step.

$$\frac{T^{k+1}_{i,j} - T^k_{i,j}}{\Delta t} - \alpha \left(\frac{T^k_{i+1,j} - 2T^k_{i,j} + T^k_{i-1,j}}{\Delta x^2} + \frac{T^k_{i,j+1} - 2T^k_{i,j} + T^k_{i,j-1}}{\Delta y^2} \right) = 0 \tag{12}$$

By assuming that Δx equals Δy the equation is simplified:

$$T^{k+1}_{i,j} = \gamma \left(T^k_{i+1,j} + T^k_{i-1,j} + T^k_{i,j+1} + T^k_{i,j-1} - 4T^k_{i,j} \right) + T^k_{i,j} \tag{13}$$

Where:

$$\gamma = \alpha \frac{\Delta t}{\Delta x^2} \tag{14}$$

This is the explicit method to get a solution for HTE. It will be numerically stable if:

$$\Delta t \leq \frac{\Delta x^2}{4\alpha} \tag{15}$$

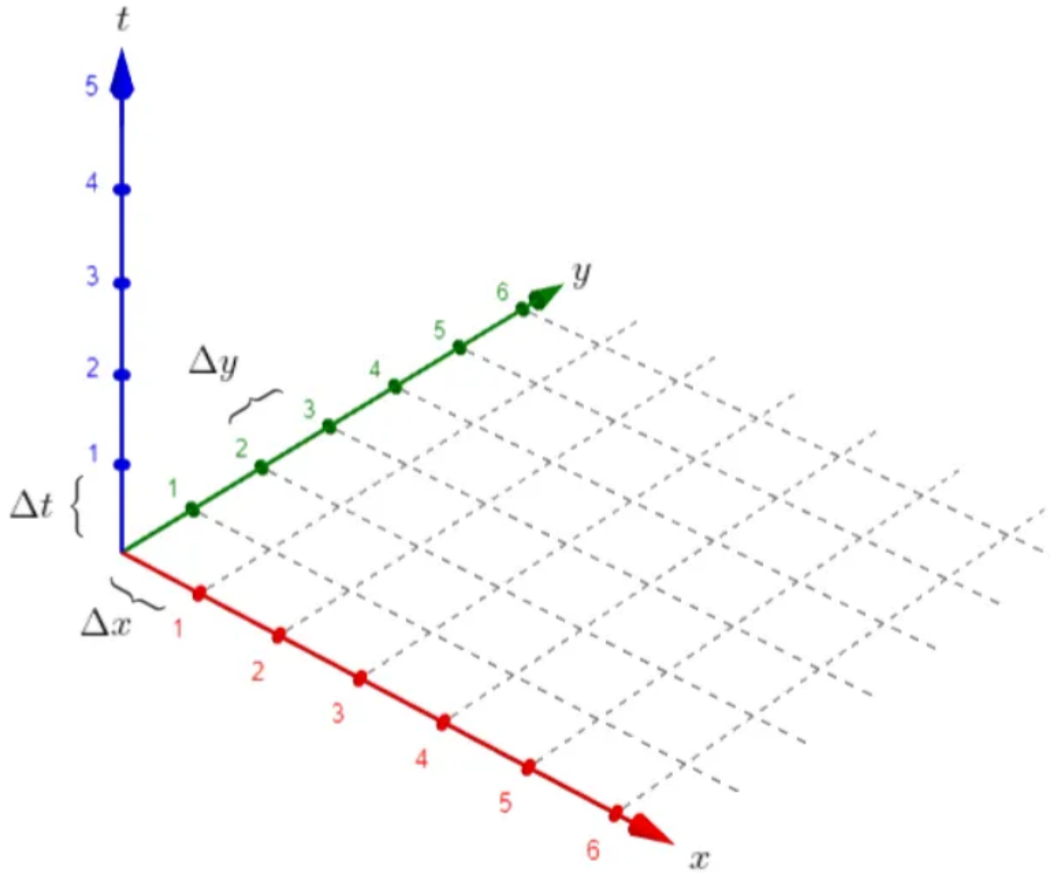


Figure 1: Cartesian coordinate, where x and y axis are for spatial variables, and t for temporal variable.

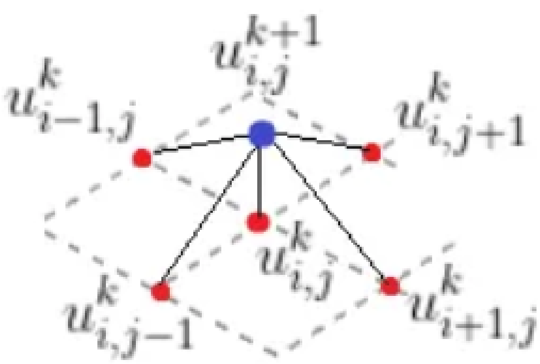


Figure 2: Explicit method stencil.

4 My Analysis

4.1 Material Properties

Materials used for analysis are copper, steel, and aluminum.

Material	Thermal Conductivity [W/(m·K)]	Density [kg/m^3]	Specific Heat Capacity [J/(kg·K)]	Diffusivity Constant [m^2/s]
Copper	385	8960	390	1.1×10^{-4}
Steel	17	7900	482	4.5×10^{-6}
Aluminum	200	2700	1029	7.2×10^{-5}

Table 1: Thermal and physical properties of copper, steel, and aluminum

4.2 Solved Problem

The problem is the square flat plate with the length of the edges equals 50 mm. Boundary conditions are:

- $T = 100\,^\circ$C at the top edge
- $T = 20\,^\circ$C at the remaining edges
- $T = 20\,^\circ$C inside the square

I assume that

$$\Delta x = 1mm \tag{16}$$

The maximal time steps which result from inequality (15) are equal:

- $\Delta t = 0.0023s$ for copper
- $\Delta t = 0.056s$ for steel
- $\Delta t = 0.0035s$ for aluminum

That is why I assume the time step

$$\Delta t = 0.001s \tag{17}$$

for every calculation.
To calculate the time to reach the steady state I will iterate through the temperature field in every time step. If the maximum temperature difference between k and $k+1$ time steps is less than $0.001\,^\circ$C, I will assume that I found the steady state.

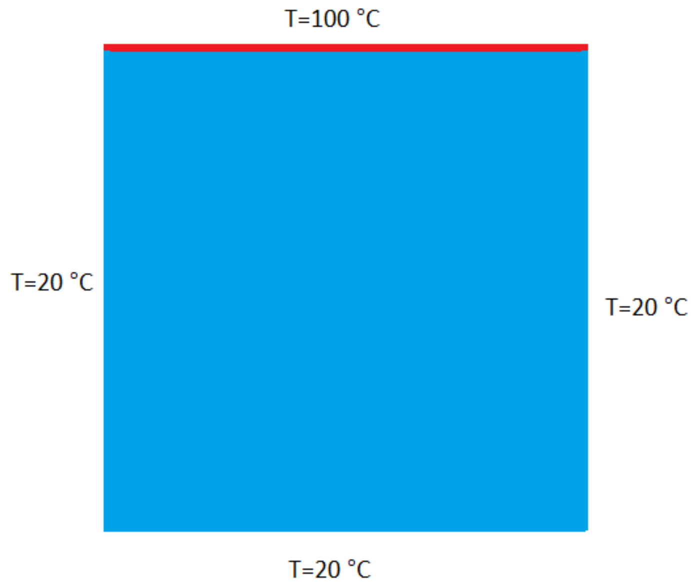
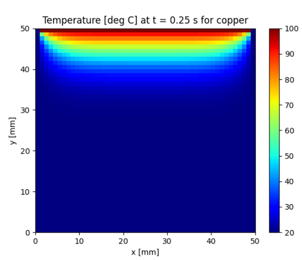


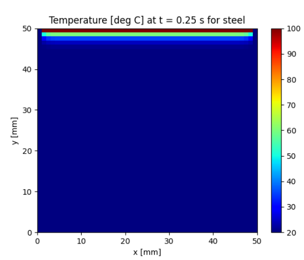
Figure 3: Boundary conditions for the exercise.

5 Results

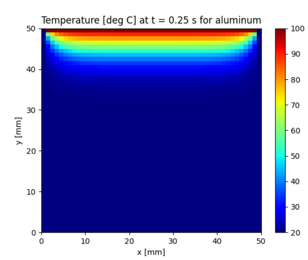
Achieved results are presented in this section below.



(a) Copper

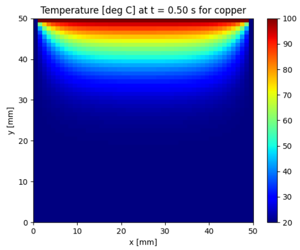


(b) Steel

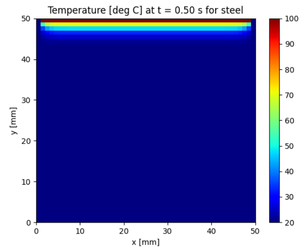


(c) Aluminum

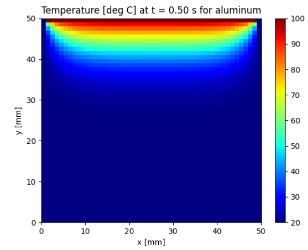
Figure 4: Temperature distribution for plates made of different materials at t=0.25s.



(a) Copper

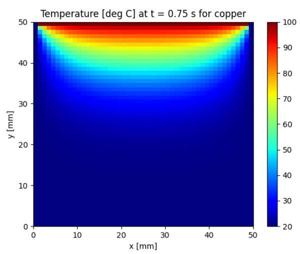


(b) Steel

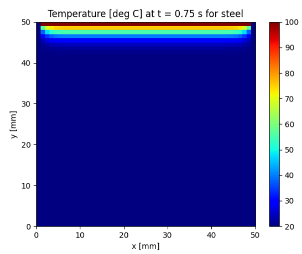


(c) Aluminum

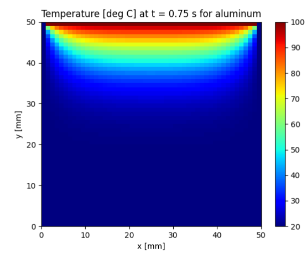
Figure 5: Temperature distribution for plates made of different materials at t=0.50s.



(a) Copper

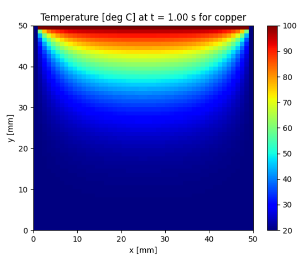


(b) Steel

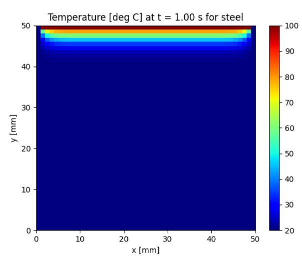


(c) Aluminum

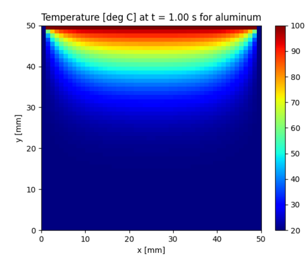
Figure 6: Temperature distribution for plates made of different materials at t=0.75s.



(a) Copper

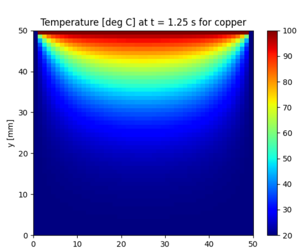


(b) Steel

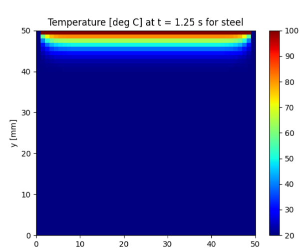


(c) Aluminum

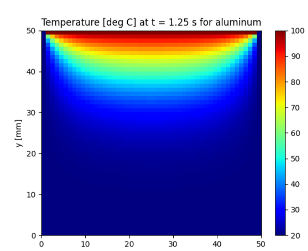
Figure 7: Temperature distribution for plates made of different materials at t=1.00s.



(a) Copper

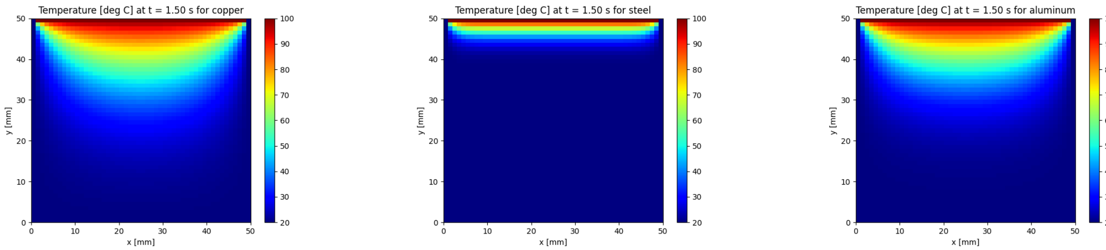


(b) Steel



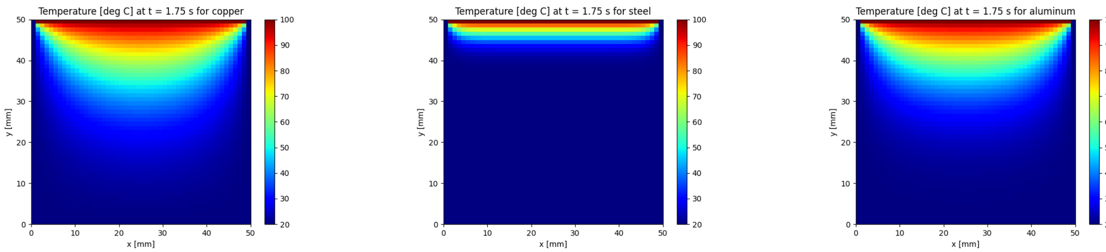
(c) Aluminum

Figure 8: Temperature distribution for plates made of different materials at t=1.25s.



(a) Copper (b) Steel (c) Aluminum

Figure 9: Temperature distribution for plates made of different materials at t=1.50s.



(a) Copper (b) Steel (c) Aluminum

Figure 10: Temperature distribution for plates made of different materials at t=1.75s.

In addition, the times required to achieve steady state equal:

- $t_{copper} = 3.73s$
- $t_{steel} = 17.40s$
- $t_{aluminum} = 4.99s$

6 Summary and conclusions

According to the achieved results, copper is the best thermal conductor among those three. The second best is aluminum, the third is steel. However, aluminum may be worth considering while designing radiators when low mass is required. Aluminium's thermal properties are worse than copper's but aluminum's density is over 3 times smaller than copper's. Also aluminum is much cheaper. From the calculations, we also see that steel is a poor thermal conductor and almost as heavy as copper, so there is no point in using steel as a material for radiators.

7 Bibliography

1. Stefan Wiśniewski, Tomasz S. Wiśniewski - "Wymiana Ciepła", WNT, 5th edition
2. Suhas V. Patankar - "Numerical Heat Transfer and Fluid Flow" - Series in computational methods in mechanics and thermal sciences
3. Jaan Kiusalaas - "Numerical Methods in Engineering with Python 3"